Omar Khayyam, René Descartes and Solutions to Algebraic Equations

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Abstract

Omar Khayyam is the only person who is remembered equally as a great poet and a great mathematician. We describe here the work of Omar Khayyam in the 12th century in solving algebraic equations and describe how his work may have influenced René Descartes in the 17th century. In particular, we discuss his solutions of cubic equations; these equations absorbed mathematicians from 9th to 16th century. Omar made a significant contribution to the finding of positive root through geometrical argument and thus foreshadowed the analytical geometry of Descartes. We give a preliminary background of the equations for non-mathematicians and describe Omar's classification of cubic equations. His method of solving cubic equations is then described. We also highlight continuing interest in his work, especially by the Omar Khayyam Club in London.[2]

1 Introduction

It must be very rare indeed for a personality to be remembered for his poetical work as well as for his mathematical work. Omar Khayyam is the only person who is remembered equally as a great poet and a great mathematician. George Boole, James Clerk Maxwell and so on wrote poetry but they are not known for their poetical work, whereas Henry Wordsworth Longfellow was an amateur mathematician.

We here describe the work of Omar in the 12th century in solving algebraic equations (in particular, the cubic) and describe how his work may have influenced René Descartes in the 17th century. Indeed, Descartes, who was a theologian and the rector of the University of Utrecht, was accused of being an atheist. Omar did not seem to have an easy life either, he was under constant danger of persecution for his beliefs (Kasir, 1931, p.3).

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2 Background

We will only discuss here a solution of cubic equations. These equations absorbed mathematicians from the 9th to the 16th century. The Italian mathematicians Cardano and Tartaglia finally found a complete solution in 16th century. On the other hand Omar made a significant contribution to the finding of a positive root (positive roots) through geometrical arguments, but it was not until the 17th century that Descartes established a relationship between geometry and algebra. Omar might have foreshadowed the analytical geometry of Descartes. Indeed, Descartes himself was primarily interested in solutions of equations as well. Descartes' methods are similar to those of Omar but Descartes realized that certain intersection points represented imaginary roots (Katz, 1998, p.448) and thus gave a complete solution for negative, positive and imaginary roots. As far as the equations of higher degree than three are concerned, Omar's work, not surprisingly, is very limited (Kasir 1931, p.34).

3 Preliminaries

For non-mathematicians, let us attempt to define a few terms in everyday language. According to the Oxford Dictionary,

"algebra = investigation of the properties of numbers and quantities by means of general symbols".

It also gives its etymology as in Arabic

"aljebr = reunion of broken parts (jabara = reunite)".

Algebra is the "soul" of modern mathematics. The name algebra is derived from "algibr w'al mukhabela" the title of every Saracen work on the subject since the time of Abu Jafar Muhammed ibn Musa al Khwarizmi (circa A.D. 825).

Another term is 'equation'. Again in the Oxford Dictionary,

"equation = formula affirming equivalence of two expressions connected by the sign =".

We will describe work from Omar's existing manuscripts on algebra - the one used by Woepcke (1851) and Kasir (1931). Kasir (1931, p.9) claims that the manuscript which he worked on as being in the personal library of Professor David Eugene Smith, Columbia University, which is substantially similar to a manuscript in Leyden library. The Leyden's library manuscript was used by Woepcke (1851) for his translation. Also, this forms, according to Kasir (p.9), "the backbone of Woepcke's Arabic text"; the first two pages of this Arabic text are reproduced here with a translation.

The simplest equation is a linear equation in one variable x, namely

$$2x - 3 = 4x + 2$$

which has a single solution $x = -5/2$. This is a linear equation because the highest power of x is one. The next equation is a quadratic equation

$$ax^2 + bx + c = 0$$

مقالة

فى الجبر والمقابلة

للحكيم الاوحد

ابى الفتح عمر بن ابراهيم

الخيامى

Figure 1a: Title page of Omar's Algebra in Arabic. (Leyden's manuscript)

A Treatise on
Algebra with Comparative Studies
By the Iranian Scholar
Abol Fath Omar Khayyam
(the son of Ebrahim Khayyam)

Figure 1b: Title page of Omar's Algebra, translated. (Leyden's manuscript)

رسالة الحكيم الفاضل

. غياث الدين ابى الفتح عمر بن ابراهيم الخيّامى النيشابورى (١)

قدّس الله روحه العزيز

فى البراهين على مسائل الجبر والمقابلة

بسم الله الرحمن الرحيم

الحمد لله رب العالمين والعاقبة للمتقين ولا عدوان الا على الظالمين
والصلوة على الانبياء وخصوصا على محمد واله الطاهرين اجمعين (٢) ان
احد المعانى التعليمية المحتاج اليها فى جزء الحكمة المعروف بالرياضى
هو صناعة الجبر والمقابلة الموضوعة لاستخراج المجهولات العددية

Figure 2a: Opening page of Omar's Algebra in Arabic. (Leyden's manuscript)

A Treatise by the Iranian Scholar
Ghiasedin Abol Fath Omar Khayyam
(the son of Ebrahim Khayyam)
(Prayer to Omar Khayyam)
The solution to the problems of Algebra with Comparative Studies
In the name of God, gracious and merciful!

Praise be to God, lord of all Worlds, a happy end to those who are pious, and ill-will to none but the merciless. May blessings repose upon the prophets, especially Mohammed and all his holy descendants.

One of the branches of knowledge needed is that division of philosophy known as Algebra/Mathematics. Algebra deals with extracting unknown numerical variables.

Figure 2b: Opening page of Omar's Algebra, translated. (Leyden's manuscript) where a, b and c are constants. Here the highest power of x is 2. Such an equation was solved a long time ago and has two solutions. Indeed, the solution as we know it today was presented by the great Indian mathematician Brahmagupta (6th century) with a single solution and Bháskara in (12th century) with both solutions (see, Katz, 1998, pp.226-227). These two solutions could be both real or both imaginary. Note that for example, the equation

$$x^2 = -1$$

has what are called imaginary roots - the square root of a negative number! Extending the idea of these equations further, a cubic equation is naturally of the form

$$ax^3 + bx^2 + cx + d = 0,$$

where a, b, c and d are constants. By a simple linear transformation $x = y - b/(3a)$, it reduces to the simple form (canonical form)

$$y^3 + py + q = 0.$$

where p and q are constants. This will have three solutions of which at least one is real.

Following the ancient Greeks, Omar used the successive powers of the unknown as the "root" or "side" "square" "cube" and so on. A statement that "A cube and square are equal to roots and numbers" in the above terminology would be in modern notation

$$x^3 + ax^2 = bx + c$$

which corresponds to the statement

$$\text{(cube)} + a \text{ (square)} = b \text{ (root)} + \text{number}.$$

Further, by a numerical solution, Omar meant a <u>whole number</u> satisfying the equation. On the other hand, a geometric solution meant the determination of an unknown <u>measurable quantity</u> through a line segment (Kasir, 1931, p.23).

4 Omar's Classification

Omar Khayyam was the first to classify equations comprehensively although in modern terms he was primarily checking the degree of an equation.

The first set of equations are with two terms or binomial equations, called by him <u>simple equations</u>. These are

1. $a = x$; 2. $a = x^2$; 3. $a = x^3$; 4. $bx = x^2$; 5. $cx^2 = x^3$; 6. $bx = x^3$.

The second set is what are called by him <u>compound equations</u> divided into trinomial (i.e. with three terms) or tetranomial (i.e. with four terms)

A. Trinomial quadratic equations:

7. $x^2 + bx = a$; 8. $x^2 + a = bx$; 9. $bx + a = x^2$.

B. Trinomial cubic equations reducible to quadratic equations:

$$10.\ x^3+cx^2=bx; \quad 11.\ x^3+bx=cx^2; \quad 12.\ cx^2+bx=x^3.$$

C. Trinomial cubic equations:

$$13.\ x^3+bx=a; \quad 14.\ x^3+a=bx; \quad 15.\ bx+a=x^3;$$

$$16.\ x^3+cx^2=a; \quad 17.\ x^3+a=cx^2; \quad 18.\ cx^2+a=x^3.$$

D. Tetranomial equations in which the sum of three terms is equal to the fourth term:

$$19.\ x^3+cx^2+bx=a; \quad 20.\ x^3+cx^2+a=bx;$$

$$21.\ x^3+bx+a=cx^2; \quad 22.\ cx^2+bx+a=x^3.$$

E. Tetranomial equations in which the sum of two terms is equal to the sum of the other two:

$$23.\ x^3+cx^2=bx+a; \quad 24.\ x^3+bx=cx^2+a; \quad 25.\ x^3+a=cx^2+bx.$$

Omar gave a history of which equations were solved already and further described his method of solving them according to these 25 categories. Algebraically, some of these categories are identical although geometrical constructions will differ.

5 Omar's Solutions

The method of Omar's solution can be described in modern terminology as follows. Let

$$x^3+px=q \quad , \quad p,q>0 \tag{1}$$

be the canonical form of the general equation. Define

$$y=(p)^{-\frac{1}{2}}x^2 \tag{2}$$

which is a parabola. Now we can rewrite (1) on multiplying by x as

$$x^4+px^2=qx$$

or by (2) we have

$$py^2+px^2=qx$$

i.e.

$$(x-\frac{q}{2p})^2+y^2=(\frac{q}{2p})^2 \tag{3}$$

which is a circle with centre $(q/2p,0)$ and radius $q/2p$ or the diameter q/p. Hence the positive root of the cubic given by (1) is the x-coordinate of the intersection of the circle given by (3) and the parabola given by (2). The geometrical construction will be described fully below.

We will consider a particular cubic equation with $p=4$ and $q=8$, i.e. here the parabola is $y=x^2/2$ and the circle has centre at $(1,0)$ and is of radius 1. The underlying cubic is

$$x^3+4x-8=0$$

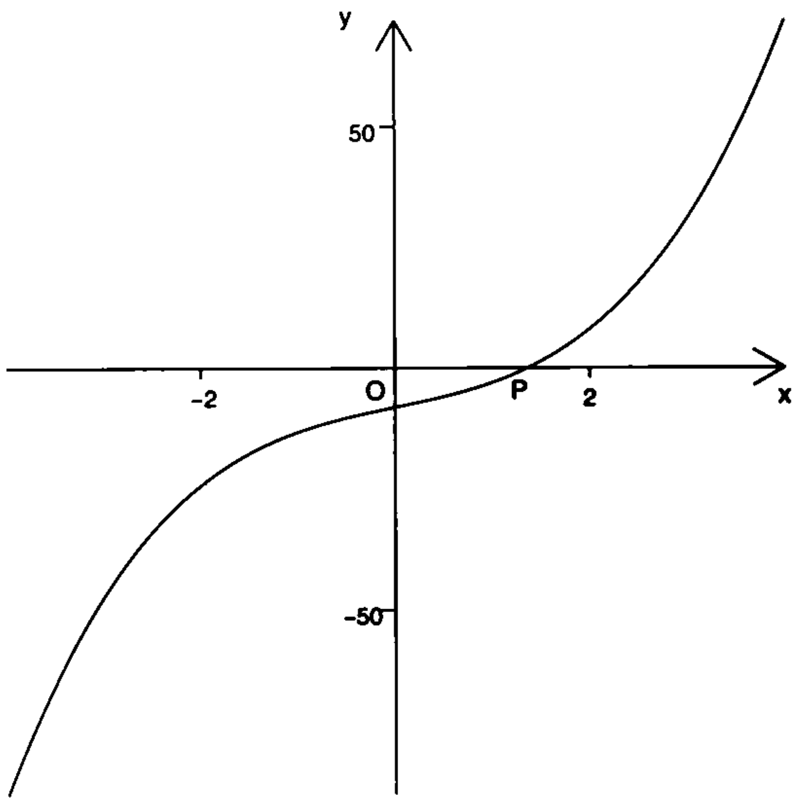

Figure 3: The curve $y = x^3 + 4x - 8$ in modern representation. The length OQ is the real solution of the cubic $x^3 + 4x - 8 = 0$.

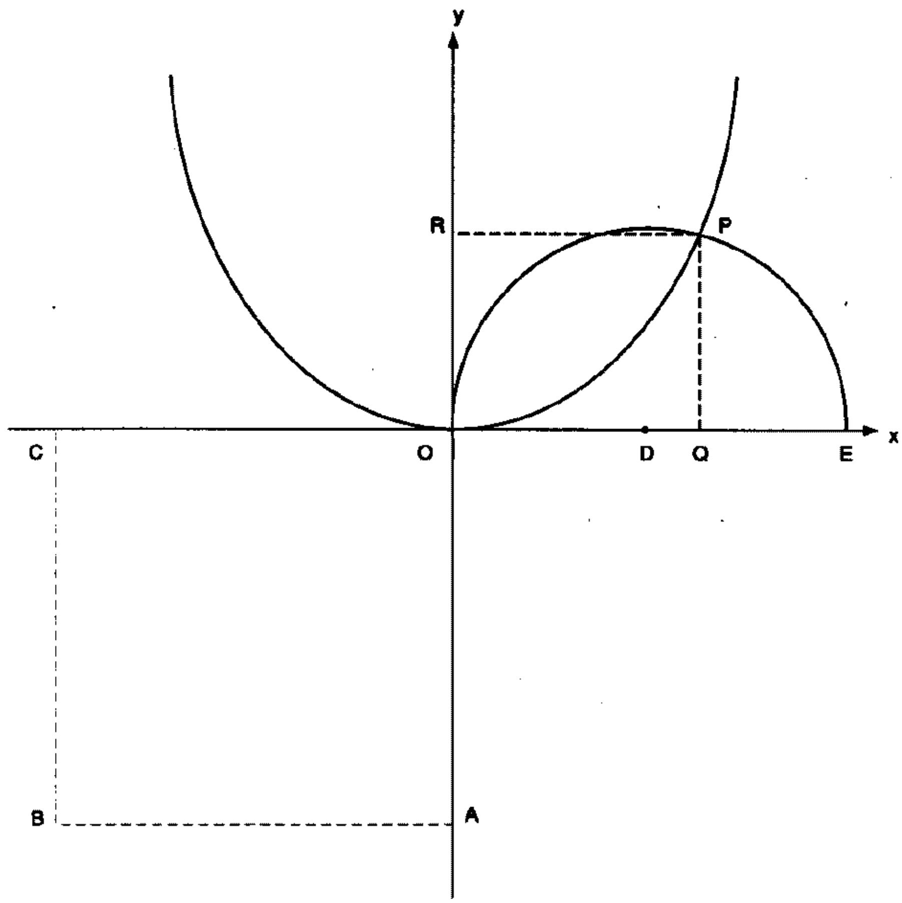

Figure 4: Omar's method of solving the cubic equation $x^3 + 4x - 8 = 0$; the segment OQ gives the positive root. See text for the method of construction of the square $ABCD$ to the semi-circle and the parabola. The solution to the cubic is the length OQ.

which has the positive root $x = 1.365$. A modern geometrical method is to plot the function $y = x^3 + 4x - 8$ (see Fig.3) and note the point of intersection of the function with the x-axis which gives the root/roots. In this case, there is only one real root (namely OP in Fig.3) and the other roots are imaginary because there are no other points of intersection with x-axis.

Of course, Omar's method was geometrical. Namely, the cubic equation is regarded as an equation between solids so that x represents a side of a cube so that p must represent an area ($p > 0$) which is expressible as a square geometrically whereas q itself must represent a solid.

We now describe the geometrical construction using (2) and (3). Draw a square of area p so that the sides are of length $\sqrt{p}$, say the segment OA in Fig.4. Draw a line perpendicular to OA at O and draw a circle of diameter q/p along this line (say OE) with centre D. Taking O as the vertex, and axis OR draw a parabola with parameter OA. (A parabola was defined then using the work of Apollonius (210 B.C.), see Katz, 1998, pp.119-120.) From the point P of intersection of the parabola and the circle, draw a perpendicular to OE and let Q be the foot of the perpendicular, then the length OQ gives the desired solution. Note that the axes are marked here for convenience of the modern reader and has nothing to do with the geometrical construction.

It should be noted that Cardano's formula (1545) given in his treatise in *Ars magna*, provides exactly the positive root for this case as

$$\left[\frac{q}{2} + \left(\frac{q^2}{4} + \frac{p^3}{27}\right)^{\frac{1}{2}}\right]^{\frac{1}{3}} + \left[\frac{q}{2} - \left(\frac{q^2}{4} + \frac{p^3}{27}\right)^{\frac{1}{2}}\right]^{\frac{1}{3}}.$$

Further, note that there are three solutions to a cubic and all three roots could be positive. One equation considered by Omar (Kasir, 1931, pp.92-93)

$$x^3 + 13\frac{1}{2}x + 5 = 10x^2$$

has two positive roots and one negative root. Namely, $x = 2, x = 4 \pm \frac{1}{2}\sqrt{74}$. Thus in Fig.5, there are 3 points of intersection with x-axis P, Q and R of the curve
$y = x^3 - 10x^2 + 13\frac{1}{2}x + 5 = 0$. Omar would have found only one positive root by his method of geometric construction.

In general, Omar's method would find one or two positive roots through two intersecting conics. All 25 cases described in Section 4 were dealt with by him systemmatically. Woepcke (1851) has given the appropriate pair of conics leading to various cases of cubic equations cited in Section 4 (see, for a translation, Kasir, 1931, pp.35-36). Thus, Woepcke(1851) has examined the inverse problem.

6 The Omar Khayyam Club

It is worth mentioning that a complete and exhaustive bibliography of manuscripts, editions, translations, parodies, ephemaral material related to Omar Khayyam upto 1928 is given by Potter (1929); the number of items listed are more than thirteen hundred! Subsequently, Halbach (1975) has given a sketchy

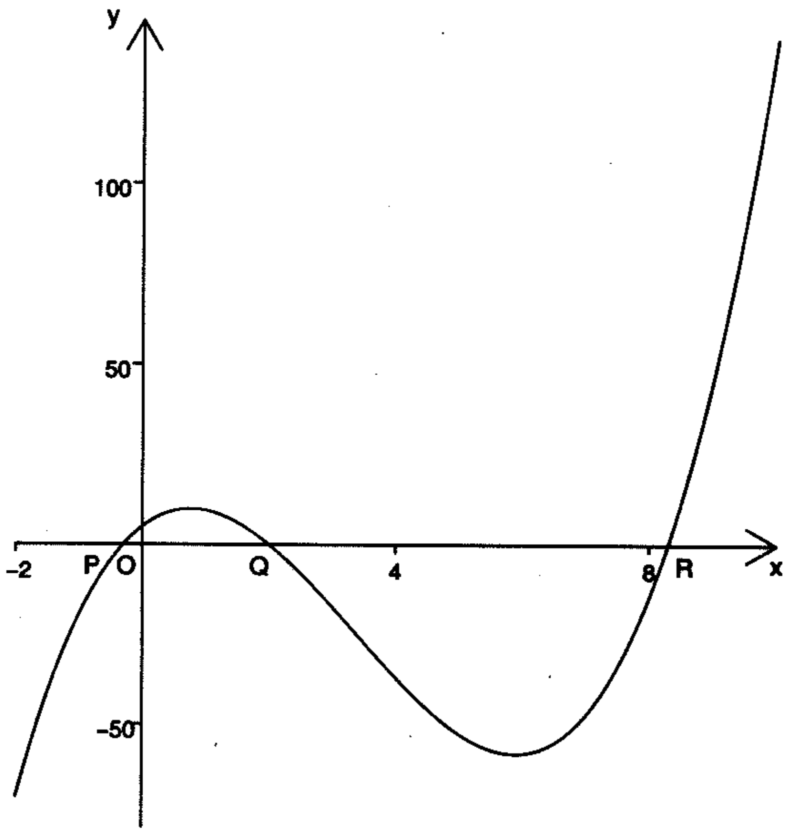

Figure 5: The curve $y = x^3 - 10x^2 + 13\frac{1}{2}x + 5$ and the three real solutions OP, OQ, QR of the Omar's cubic $x^3 - 10x^2 + 13\frac{1}{2}x + 5 = 0$.

bibliography, mainly on Rubaiyats. In fact William Edward Story (1918) presented a lecture on "Omar Khayyam as a Mathematician" at the American Omar Khayyam Club in 1918 which was published limited to 200 copies only. Note that the American Omar Khayyam Club does not exist any more; it closed down in 1930's.

One of the places where Omar Khayyam works are regularly remembered is the Omar Khayyam club in London which was founded on 13th October 1892 and it is still an active club. A drawing from its Millennium Club meeting of 18th November 1999 is shown in Fig.6. This depicts the frustration of an Omar's fan due to delay on the Jubilee line extension to the Millennium Dome in London and its effect on his abode!

Among literary figures, its early members included H.G. Wells, Aldous Huxley etc. and there were guests such as Thomas Hardy. There were also academics such as Sir Oliver Lodge and Professor Sir Walter Raleigh. The meetings of the club commemorate his work as well as Edward FitzGerald's contributions. In fact, there have been Iranian Ambassadors to the UK on its guest list. Important is the following letter to the club from the Iranian Embassy when the club (the OK club Vol. 1, 1910) celebrated Edward FitzGerald's birth centenary on 31st March 1909. Edward FitzGerald's translation of the Rubaiyat of Omar Khayyam in 1859 made the Western world aware of Omar's work; his translation is still regarded as the best translation in English.

> March 1909
>
> May I be allowed to offer the expression of my veneration and respect for the memory of the distinguished Poet, the centenary of whose birth you are celebrating to-day? His services to our literature and his claim on our gratitude are my excuse for intruding on a scene of so purely national a character. As a token of my feelings of admiration and respect for the immortal Poet who has so eloquently interpreted the Quatrains of our distinguished Poet, I should like to lay a wreath on his tomb, and should be glad to have your advice as to whether this would be convenient, and when I may do so.
>
> Yours faithfully
> The First Secretary of the Persian Embassy

The following club's "bidding prayer", since its first meeting of 1892, continues to celebrate his memory:

> *O, my friends, when I am sped, appoint a meeting,*
> *and when ye have met together be ye glad thereof,*
> *and when the cup-bearer holds in her hand a flagon of old wine,*
> *then think upon old Khayyam and drink to his memory.*

Note the word "Khayyam" in the last line!

7 Concluding Remarks

We begin this last section with the following single verse from the poem "Omar in The Third Millennium" by Paul Ableman, presented to the Omar Khayyam Club's Millennium meeting of 18th November 1999, in London which highlights

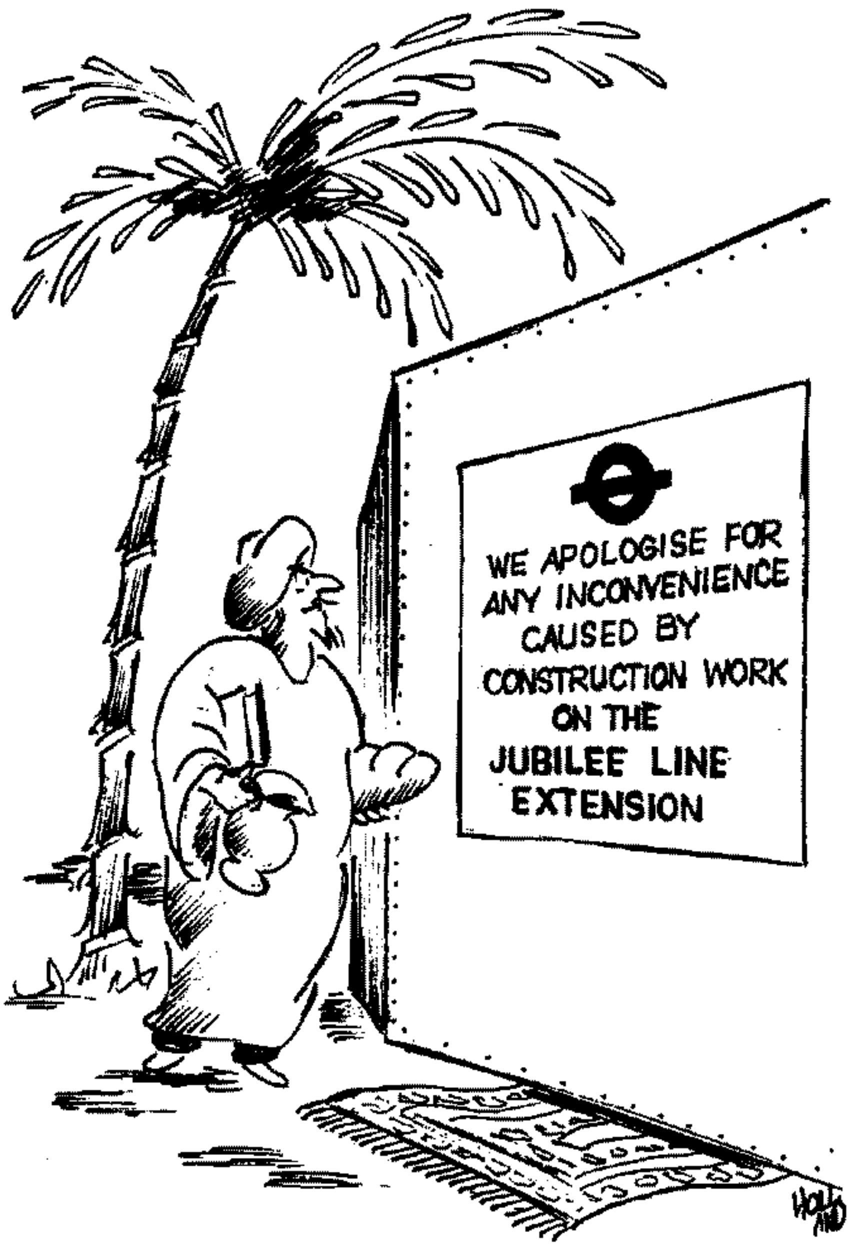

Figure 6: A drawing from the Omar Khayyam Club Millennium dinner menu of 19th November 1999 (The Jubilee line is a tube line leading now to the Millennium Dome, London).

the continuing interest in Omar of its members. (Note that here "wine" is again as symbolic as in the Omar Khayyam's Rubaiyat.)

So, brothers, fill the cup and drain the wine
Toast the millennium. It will be fine
For making friends and cash and high-tech toys.
But Omar's words will still give deeper joys.

To sum up, Omar classified cubic equations systematically and derived its algebraic solutions by geometric constructions and vice-versa. In fact, he seems to be the first to initiate unification of algebra and geometry, foreshadowing the analytical geometry of Descartes. Omar works on cubic equations definitely ranks him as one of the most original and the greatest mathematician and visionary of his time, and his work is remembered regularly at least at the Omar Khayyam Club in London!

Acknowledgements

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References

[1] Halbach,H. (1975) *Romance of the Rubaiyat.* Helen Halbach, Santa Barbara.

[2] Kasir,D.S. (1931) *The Algebra of Omar Khayyam.* Teachers College, Columbia University, NY.

[3] Katz,V.J. (1998) *A History of Mathematics*, 2nd edition. Addison Wesley Longman Inc., Harlow, England.

[4] Potter,A.G. (1929) *A Bibliography of the Rubaiyat of Omar Khayyam* . Igpen and Grant, London.

[5] Story,W.E. (1918) *Omar Khayyam as a Mathematician.* Rosemary Press, Boston.

[6] The Book of the Omar Khayyam Club (1910), 1892-1910. London. Printed for the members for private circulation.

[7] Woepcke,F. (1851) *L'Algèbre d'Omar Al Khayyâmî.* Paris.