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A relativistic hypergeometric function

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Abstract

We survey our work on a function generalizing ${}_2F_1$. This function is a joint eigenfunction of four Askey–Wilson-type hyperbolic difference operators, reducing to the Askey–Wilson polynomials for certain discrete values of the variables. It is defined by a contour integral generalizing the Barnes representation of ${}_2F_1$. It has various symmetries, including a hidden D_4 symmetry in the parameters. By means of the associated Hilbert space transform, the difference operators can be promoted to self-adjoint operators, provided the parameters vary over a certain polytope in the parameter space Π. For a dense subset of Π, parameter shifts give rise to an explicit evaluation in terms of rational functions of exponentials ('hyperbolic' functions and plane waves).

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1. Introduction

In the following, we review various papers concerned with a function $R(a_+, a_-, \mathbf{c}; v, \hat{v})$ generalizing the hypergeometric function ${}_2F_1(a, b, c; w)$, namely, Refs. [11,17,18] (referred to as I, II and III) and Ref. [20]. As is well known, the ${}_2F_1$-function can be used to diagonalize the nonrelativistic Schrödinger operator (2.12), which arises in the context of nonrelativistic Calogero–Moser systems. In [10] we

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introduced the R-function to diagonalize a generalization of (2.12) arising in the context of relativistic Calogero–Moser systems. The pertinent relativistic quantum operator amounts to an analytic difference operator (AΔO) of hyperbolic Askey–Wilson type.

Even though we do consider the nonrelativistic limit $R \to {}_2F_1$ in this survey, it is beyond our present scope to elaborate on the physical setting and Calogero–Moser context for the R-function. For information on these aspects we refer to our lecture notes [10]. The results obtained in I have been reviewed before from various complementary viewpoints in [12,14,15], viz., integrable systems, special functions, and sine-Gordon theory, resp. Accordingly, our account of results from I is terse and biased towards subjects that we need to sketch our more recent work in II, III and [20].

In the above-mentioned articles we have included a great many references to related work, pertinent to the context at issue. Since we are focusing on our results concerning the R-function (which, to our knowledge, has not been studied by other authors), we only mention here various papers where non-polynomial functions have been considered that are also solutions to an Askey–Wilson-type difference equation [7,3,5,24,8,6,22]. It is an open problem to make their relation to the R-function more explicit (cf. in this connection Section 6.6 in [14]).

We proceed to sketch the organization of this review. In Section 2, we recall some known lore on ${}_2F_1$, in a form that suits our later requirements. Section 3 has an auxiliary character, too. Here we collect some salient features of the hyperbolic gamma function from [9], which is the building block of the R-function.

This prepares the ground for Section 4, in which the R-function is defined. We also specify its analyticity properties and collect some manifest symmetries. In Section 5, we detail and discuss the most prominent feature of the R-function, namely its being a joint eigenfunction of four independent hyperbolic Askey–Wilson-type AΔOs.

Just as ${}_2F_1$ can be specialized to the Jacobi polynomials, the R-function can be specialized to the Askey–Wilson polynomials [1,4]. This is sketched in Section 6.

The results mentioned thus far date back to I. Section 7 is concerned with the main results obtained in II. As it turns out, the R-function has a hidden D_4 symmetry in the four coupling parameters $\mathbf{c} \in \mathbb{C}^4$. This symmetry is best understood in terms of a similarity transform $\mathcal{E}(a_+, a_-, \gamma; v, \hat{v})$, where γ is linear in a_+, a_- and $\mathbf{c}$, cf. (7.2). Indeed, the $\mathcal{E}$-function is D_4 invariant, cf. (7.16), whereas the R-function is only D_4 covariant. The $\mathcal{E}$-function also has plane wave asymptotics for $\mathrm{Re}\, v \to \infty$, cf. (7.27)–(7.28).

In Section 8, we obtain the nonrelativistic limits of the R- and $\mathcal{E}$-functions and the four associated AΔOs, tying this in with the preparatory material in Section 2.

The Hilbert space eigenfunction transform corresponding to the $\mathcal{E}$-function is studied in III and surveyed in Sections 9 and 10. Section 9 concerns a sketch of our solution to the Plancherel problem (orthogonality and completeness). Along the way, the normalization integrals of the bound states arise in explicit form. For the ground state this gives rise to a hyperbolic analog of the (trigonometric) Askey–Wilson integral. Since this spin-off of our completeness proof is of considerable interest in itself, we have isolated it in Section 10. (See Stokman's preprint [23] for a quite different derivation of the relevant integral.)

A large amount of additional information can be obtained via an algebra of 32 parameter shifts. In particular, it can be shown that the R- and $\mathcal{E}$-functions have an elementary character (involving solely plane waves and hyperbolic functions) for a D_4 invariant dense set in the natural parameter space. We obtained these results in our recent paper [20] and review them in Section 11.

2. Preliminaries on ${}_2F_1$

We begin by recalling that the hypergeometric function ${}_2F_1$ admits three distinct representations. In historical order, these are Euler's integral representation, Gauss' series representation, and Barnes' integral representation, cf. e.g. [25,2]. The ${}_2F_1$-generalization at issue is defined by an integral representation of Barnes form, and no analogs of the Gauss and Euler forms are known. Thus we need only invoke Barnes' formula

$$ {}_2F_1(a,b,c;w) = \int_{\mathscr{C}} \exp(-\mathrm{i}z\ln(-w)) \cdot \frac{\Gamma(\mathrm{i}z)\Gamma(c)}{2\pi\Gamma(c-\mathrm{i}z)} \cdot \frac{\Gamma(a-\mathrm{i}z)\Gamma(b-\mathrm{i}z)}{\Gamma(a)\Gamma(b)}\,\mathrm{d}z. \tag{2.1} $$

Here the contour $\mathscr{C}$ runs parallel to the real axis, with indentations to avoid the upward pole sequence $z = \mathrm{i}n$, $n \in \mathbb{N}$, and the downward sequences $z = -\mathrm{i}a - \mathrm{i}n$, $-\mathrm{i}b - \mathrm{i}n$, $n \in \mathbb{N}$. Also, w belongs to the cut plane $|\mathrm{Arg}(-w)| < \pi$ and $\ln(-w)$ is chosen positive for negative w. On account of Stirling's formula, the integrand has exponential decay for $|\mathrm{Re}\, z| \to \infty$, and so the integral yields an analytic function of w in the cut plane.

Next, we reparametrize ${}_2F_1$ by introducing

$$ \psi_{\mathrm{nr}}(d,\tilde{d};v,\hat{v}) \equiv {}_2F_1((d+\tilde{d}+\mathrm{i}\hat{v})/2, (d+\tilde{d}-\mathrm{i}\hat{v})/2, d+1/2; -\sinh^2 v). \tag{2.2} $$

Then the hypergeometric differential equation implies that ψ_{nr} satisfies the eigenvalue equation

$$ H_v\psi_{\mathrm{nr}} = \hat{v}^2\psi_{\mathrm{nr}}, \tag{2.3} $$

where

$$ H_v \equiv -\frac{\mathrm{d}^2}{\mathrm{d}v^2} - 2[d\,\coth(v) + \tilde{d}\,\tanh(v)]\frac{\mathrm{d}}{\mathrm{d}v} - (d+\tilde{d})^2. \tag{2.4} $$

Moreover, using the contiguous relations for ${}_2F_1$, one can verify that ψ_{nr} also satisfies a 'dual' equation, to wit,

$$ A_{\hat{v}}\psi_{\mathrm{nr}} = 2\cosh(2v)\psi_{\mathrm{nr}}, \tag{2.5} $$

where

$$ A_{\hat{v}} \equiv \frac{[\hat{v}-\mathrm{i}(d+\tilde{d})]}{\hat{v}}\frac{[\hat{v}-\mathrm{i}(d-\tilde{d}+1)]}{\hat{v}-\mathrm{i}}(T_{2\mathrm{i}}^{\hat{v}}-1) + (\mathrm{i}\to-\mathrm{i}) + 2. \tag{2.6} $$

Here and below, the translation T_η^y acts as

$$ (T_\eta^y f)(y) \equiv f(y-\eta), \qquad \eta \in \mathbb{C}^* \tag{2.7} $$

on functions analytic in y; moreover, an expression of the form $F(\mathrm{i}) + (\mathrm{i} \to -\mathrm{i})$ is shorthand for $F(\mathrm{i}) + F(-\mathrm{i})$, it always being clear from context how to substitute.

For our later purposes it is important to point out that it is possible to verify both the differential equation (2.3) and the analytic difference equation (2.5) directly (but with due effort) from the Barnes representation (2.1). Indeed, this verification can serve as a paradigm for obtaining the analytic difference equations satisfied by the R-function, cf. Section 5.

Anticipating the similarity transformation of the R-function to the $\mathscr{E}$-function, cf. Section 7, we proceed to specify the analogous transformation for (2.2). It reads

$$\mathscr{E}_{\mathrm{nr}}(d,\tilde{d};v,\hat{v}) \equiv 2w_{\mathrm{nr}}(d,\tilde{d};v)^{1/2}\psi_{\mathrm{nr}}(d,\tilde{d};v,\hat{v})\frac{1}{\hat{c}_{\mathrm{nr}}(d,\tilde{d};\hat{v})}, \tag{2.8}$$

where

$$w_{\mathrm{nr}}(d,\tilde{d};v) \equiv [2\sinh\, v]^{2d}[2\cosh\, v]^{2\tilde{d}}, \quad \mathrm{Re}\, v>0, \tag{2.9}$$

$$\hat{c}_{\mathrm{nr}}(d,\tilde{d};\hat{v}) \equiv 2^{d+\tilde{d}}\pi^{-1/2}\Gamma(d+1/2)\frac{\Gamma(\mathrm{i}\hat{v}/2)\Gamma((\mathrm{i}\hat{v}+1)/2)}{\Gamma((\mathrm{i}\hat{v}+d+\tilde{d})/2)\Gamma((\mathrm{i}\hat{v}+d-\tilde{d}+1)/2)}. \tag{2.10}$$

Note that these functions are normalized so that

$$w_{\mathrm{nr}}(0,0;v) = \hat{c}_{\mathrm{nr}}(0,0;\hat{v}) = 1. \tag{2.11}$$

The corresponding transforms of (2.4) and (2.6) are then

$$\mathscr{H}_v = -\frac{\mathrm{d}^2}{\mathrm{d}v^2} + \frac{d(d-1)}{\sinh^2 v} - \frac{\tilde{d}(\tilde{d}-1)}{\cosh^2 v}, \tag{2.12}$$

$$\mathscr{A}_{\hat{v}} = T_{2\mathrm{i}}^{\hat{v}} + V_a(d,\tilde{d};\hat{v})T_{-2\mathrm{i}}^{\hat{v}} + V_b(d,\tilde{d};\hat{v}), \tag{2.13}$$

where

$$V_a(y) \equiv \frac{[y+\mathrm{i}(d+\tilde{d})][y+\mathrm{i}(d-\tilde{d}+1)][y-\mathrm{i}(d+\tilde{d}-2)][y-\mathrm{i}(d-\tilde{d}-1)]}{y(y+\mathrm{i})^2(y+2\mathrm{i})}, \tag{2.14}$$

$$V_b(y) \equiv \frac{2(d-\tilde{d})(d+\tilde{d}-1)}{y^2+1}. \tag{2.15}$$

In particular, H_v (2.4) turns into the nonrelativistic Schrödinger operator $\mathscr{H}_v$ (2.12).

3. The hyperbolic gamma function

The role of Euler's gamma function $\Gamma(z)$ in the ${}_2F_1$-representation (2.1) is played by the hyperbolic gamma function $G(a_+,a_-;z)$ in the Barnes-type integral representation for the R-function. We proceed to summarize some properties of $G(a_+,a_-;z)$, fixing

$$a_+, a_- > 0 \tag{3.1}$$

from now on. We also introduce

$$a \equiv (a_+ + a_-)/2, \quad \alpha \equiv 2\pi/a_+a_-. \tag{3.2}$$

With these conventions, the hyperbolic gamma function can be defined by the integral representation

$$G(z) = \exp\left(\mathrm{i}\int_0^\infty \frac{\mathrm{d}y}{y}\left(\frac{\sin 2yz}{2\sinh(a_+y)\sinh(a_-y)} - \frac{z}{a_+a_-y}\right)\right), \quad |\mathrm{Im}\; z| < a. \tag{3.3}$$

(We often suppress the dependence on the parameters a_+, a_- when this causes no ambiguity.) It extends to a meromorphic function satisfying the analytic difference equation (AΔE)

$$\frac{G(z+\mathrm{i}a_+/2)}{G(z-\mathrm{i}a_+/2)} = 2\cosh(\pi z/a_-). \tag{3.4}$$

The manifest symmetry of (3.3) under $a_+ \leftrightarrow a_-$ entails that $G(a_+, a_-; z)$ also obeys the AΔE

$$\frac{G(z+\mathrm{i}a_-/2)}{G(z-\mathrm{i}a_-/2)} = 2\cosh(\pi z/a_+). \tag{3.5}$$

From these features it is easy to see that $G(z)$ has poles p_{kl} and zeros z_{kl} given by

$$p_{kl} = -\mathrm{i}a - \mathrm{i}ka_+ - \mathrm{i}la_-, \quad z_{kl} = -p_{kl}, \quad k, l \in \mathbb{N}. \tag{3.6}$$

Likewise, the reflection equation

$$G(-z) = 1/G(z) \tag{3.7}$$

the complex conjugation relation

$$\overline{G(z)} = G(-\bar{z}) \tag{3.8}$$

and the scale invariance

$$G(\lambda a_+, \lambda a_-; \lambda z) = G(a_+, a_-; z), \qquad \lambda > 0 \tag{3.9}$$

are evident from (3.3) and (3.1).

We also have occasion to invoke some less conspicuous features of $G(z)$. These can all be found in [9], where we introduced and studied $G(z)$ (cf. also I, Appendix A). Specifically, we need the duplication formula

$$G(a_+, a_-; 2z) = \prod_{l,m=+,-} G(a_+, a_-; z + \mathrm{i}(la_+ + ma_-)/4) \tag{3.10}$$

cf. [9, (3.24)–(3.25)], and the limits

$$\lim_{\lambda \downarrow 0} \frac{G(\pi, \lambda; z - \mathrm{i}\lambda\kappa)}{G(\pi, \lambda; z - \mathrm{i}\lambda\mu)} = \exp[(\mu - \kappa)\ln(2\cosh(z))], \tag{3.11}$$

where $\kappa, \mu \in \mathbb{R}$, and z belongs to the cut plane $\mathbb{C}\backslash\{\pm \mathrm{i}[\pi/2, \infty)\}$ (cf. [9, (3.91)]), and

$$\lim_{\lambda \downarrow 0} G(\pi, \lambda; \mathrm{i}\pi/2 - \mathrm{i}\lambda\kappa)\exp[\kappa\ln(2\lambda)] = \frac{(2\pi)^{1/2}}{\Gamma(\kappa + 1/2)}, \tag{3.12}$$

where $\kappa \in \mathbb{C}$, cf. [9, Proposition III.6]. Moreover, the asymptotics

$$G(a_+, a_-; z) \sim \exp\left(\mp \frac{\mathrm{i}\pi}{2a_+a_-}\left[z^2 + \frac{1}{12}(a_+^2 + a_-^2)\right]\right), \quad \mathrm{Re}\, z \to \pm\infty \tag{3.13}$$

(cf. [9, Proposition III.4]) plays the same role for the R-function as the Stirling formula does for ${}_2F_1$ in its Barnes representation (2.1). Finally, we need the explicit evaluations

$$G(a_+, a_-; -\mathrm{i}a_\delta/2) = 2^{-1/2}, \qquad \delta = +, - \tag{3.14}$$

for normalization purposes. (To check (3.14), set $z = 0$ in (3.4)–(3.5) and use (3.7).)

4. The R-function: first steps

In order to define $R(a_+, a_-, \mathbf{c}; v, \hat{v})$, it is convenient to introduce parameters

$$s_1 \equiv c_0 + c_1 - a_-/2, \quad s_2 \equiv c_0 + c_2 - a_+/2, \quad s_3 \equiv c_0 + c_3, \tag{4.1}$$

$$\hat{c}_0 \equiv (c_0 + c_1 + c_2 + c_3)/2 \tag{4.2}$$

and functions

$$F(b; y, z) \equiv \frac{G(z + y + \mathrm{i}b - \mathrm{i}a)}{G(y + \mathrm{i}b - \mathrm{i}a)} \frac{G(z - y + \mathrm{i}b - \mathrm{i}a)}{G(-y + \mathrm{i}b - \mathrm{i}a)}, \tag{4.3}$$

$$K(\mathbf{c}; z) \equiv \frac{1}{G(z + \mathrm{i}a)} \prod_{j=1}^{3} \frac{G(\mathrm{i}s_j)}{G(z + \mathrm{i}s_j)}, \tag{4.4}$$

with $G(z)$ the hyperbolic gamma function. At first we specialize to

$$\mathbf{c} \in \mathbb{R}^4, \quad \mathrm{Re}\ v, \mathrm{Re}\ \hat{v} > 0, \quad s_1, s_2, s_3 \in (-a, a). \tag{4.5}$$

Then the R-function is defined by the contour integral

$$R(\mathbf{c}; v, \hat{v}) = \frac{1}{(a_+ a_-)^{1/2}} \int_{\mathscr{C}} F(c_0; v, z) K(\mathbf{c}; z) F(\hat{c}_0; \hat{v}, z)\, \mathrm{d}z. \tag{4.6}$$

The contour $\mathscr{C}$ depends on the location of the poles in the eight z-dependent G-functions in the integrand, cf. (3.6) and (3.7). Specifically, the function $K(\mathbf{c}; z)$ gives rise to four upward pole sequences on the imaginary axis, beginning at $z = 0, \mathrm{i}(a - s_j)$, $j = 1, 2, 3$, whereas $F(b; y, z)$ yields two downward sequences, beginning at $z = \pm y - \mathrm{i}b$. The contour is given by a horizontal line $\mathrm{Im}\ z = h$, indented (if need be) so that it passes above the points $-v - \mathrm{i}c_0$, $-\hat{v} - \mathrm{i}\hat{c}_0$ in the left half-plane and $v - \mathrm{i}c_0$, $\hat{v} - \mathrm{i}\hat{c}_0$ in the right half-plane, and so that it passes below 0. Thus the four upward pole sequences of the integrand are above $\mathscr{C}$ and the four downward ones are below $\mathscr{C}$. In view of (3.13), the integrand has exponential decay as $|\mathrm{Re}\ z| \to \infty$, so that the integral does not depend on h.

Starting from the integral representation (4.6) with (4.5) in force, the analyticity properties of the R-function can be established in great detail. They are most easily explained from the representation (cf. I, Theorem 2.2)

$$R(a_+, a_-, \mathbf{c}; v, \hat{v}) = \frac{H(a_+, a_-, \mathbf{c}; v, \hat{v}) \prod_{j=1}^{3} G(a_+, a_-; \mathrm{i}s_j)}{p(a_+, a_-, \mathbf{c}; v)\, \hat{p}(a_+, a_-, \mathbf{c}; \hat{v})}. \tag{4.7}$$

The functions H, p and $\hat{p}$ are holomorphic for $\mathrm{Re}\, a_+, \mathrm{Re}\, a_- > 0$ and $(\mathbf{c}, v, \hat{v}) \in \mathbb{C}^6$. The functions p and $\hat{p}$ are factorized as a product of eight holomorphic functions whose zero loci consist of a union of countably many explicitly known hyperplanes. (More specifically, the denominator on the rhs of (4.7) is given by I (2.33), cf. also I (2.23)–(2.24).) Since the analyticity features of $G(\mathrm{i}s_j)$ are also known, (4.7) entails that the R-function is meromorphic in all of its eight arguments (provided $\mathrm{Re}\, a_+$, $\mathrm{Re}\ a_- > 0$), with explicitly known pole hyperplanes.

As a consequence, it now follows that for fixed $a_+, a_- > 0$ and (generic) $\mathbf{c} \in \mathbb{R}^4$ (to which we restrict attention in this survey), the R-function extends to a meromorphic function of v and $\hat{v}$, with poles that can (but need not) occur solely for certain points on the imaginary axis. These points are given by collisions of v- and $\hat{v}$-dependent z-poles in the integrand with z-poles in the three upward s_j-pole sequences, and by poles of the factors $1/G(\pm v + \mathrm{i}c_0 - \mathrm{i}a)$ and $1/G(\pm\hat{v} + \mathrm{i}\hat{c}_0 - \mathrm{i}a)$ in the integrand.

We continue to list some symmetries that are readily established from (4.6) and features of the G-function mentioned in Section 3. These include evenness,

$$R(a_+, a_-, \mathbf{c}; v, \hat{v}) = R(a_+, a_-, \mathbf{c}; \delta v, \delta'\hat{v}), \quad \delta, \delta' = +, - \tag{4.8}$$

scale invariance,

$$R(a_+, a_-, \mathbf{c}; v, \hat{v}) = R(\lambda a_+, \lambda a_-, \lambda\mathbf{c}; \lambda v, \lambda\hat{v}), \quad \lambda > 0 \tag{4.9}$$

and ‘modular invariance’,

$$R(a_+, a_-, \mathbf{c}; v, \hat{v}) = R(a_-, a_+, I\mathbf{c}; v, \hat{v}), \tag{4.10}$$

where I denotes the transposition of c_1 and c_2. (Observe that s_j is invariant under the interchange $(a_+, c_1) \leftrightarrow (a_-, c_2)$, cf. (4.1).) Defining next dual couplings

$$\hat{\mathbf{c}} \equiv J\mathbf{c}, \quad J \equiv \frac{1}{2}\begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & 1 & -1 \\ 1 & -1 & -1 & 1 \end{pmatrix} \tag{4.11}$$

one readily verifies

$$c_0 + c_j = \hat{c}_0 + \hat{c}_j, \quad j = 1, 2, 3. \tag{4.12}$$

Recalling (4.1), it is now not hard to deduce the self-duality property

$$R(a_+, a_-, \mathbf{c}; v, \hat{v}) = R(a_+, a_-, \hat{\mathbf{c}}; \hat{v}, v). \tag{4.13}$$

5. The hyperbolic Askey–Wilson AΔOs

We proceed to expound the eigenfunction properties of the R-function. To this end we introduce the notation

$$s_\delta(y) \equiv \sinh(\pi y/a_\delta), \quad c_\delta(y) \equiv \cosh(\pi y/a_\delta). \tag{5.1}$$

Now we define coefficient functions

$$C_\delta(\mathbf{c}; y) \equiv \frac{s_\delta(y - \mathrm{i}c_0)}{s_\delta(y)}\frac{c_\delta(y - \mathrm{i}c_1)}{c_\delta(y)}\frac{s_\delta(y - \mathrm{i}c_2 - \mathrm{i}a_{-\delta}/2)}{s_\delta(y - \mathrm{i}a_{-\delta}/2)}\frac{c_\delta(y - \mathrm{i}c_3 - \mathrm{i}a_{-\delta}/2)}{c_\delta(y - \mathrm{i}a_{-\delta}/2)} \tag{5.2}$$

and AΔOs

$$A_\delta(\mathbf{c}; y) \equiv C_\delta(\mathbf{c}; y)(T^y_{\mathrm{i}a_{-\delta}} - 1) + C_\delta(\mathbf{c}; -y)(T^y_{-\mathrm{i}a_{-\delta}} - 1) + 2c_\delta(\mathrm{i}(c_0 + c_1 + c_2 + c_3)), \tag{5.3}$$

where $\delta = +, -$ and the translations are defined by (2.7).

Focusing on $A_+(\mathbf{c}; v)$, we begin by pointing out that it is a hyperbolic analog of the trigonometric Askey–Wilson AΔO. Indeed, the latter arises via the analytic continuation $a_+ \to -2\mathrm{i}\pi$. It follows from the scale invariance (3.9) and the analyticity properties summarized above that the R-function allows the same analytic continuation, but in the process several symmetries are destroyed and the integral representation becomes awkward to handle. Moreover, from the viewpoint of relativistic quantum mechanics there is no need for the full R-function in the trigonometric regime: One only needs the Askey–Wilson polynomials to diagonalize the trigonometric Hamiltonian, and these arise via suitable discretizations of the R-function, cf. Section 6.

In any case, we keep our convention (3.1) and continue to sketch why the eigenfunction property

$$A_+(\mathbf{c}; v)R(\mathbf{c}; v, \hat{v}) = 2c_+(2\hat{v})R(\mathbf{c}; v, \hat{v}) \tag{5.4}$$

holds true. Basically, the verification of this second order AΔE can be reduced to one of the first-order AΔEs satisfied by the G-functions in the integrand, cf. (3.4)–(3.5). Indeed, (3.5) entails that $F(b; y, z)$ (4.3) and $K(\mathbf{c}; z)$ (4.4) satisfy a first-order AΔE with shift $\mathrm{i}a_-$ both in y and in z. Choosing first suitable parameters and variables in the R-function, so that the action of the AΔO $A_+(\mathbf{c}; v)$ can be transferred to the integrand, it is now possible to exploit these first order AΔEs to demonstrate (5.4), cf., I, Theorem 3.1.

Taking (5.4) for granted, it is clear from symmetries (4.10) and (4.13) that R solves three more eigenvalue problems, viz.:

$$A_-(I\mathbf{c}; v)R(\mathbf{c}; v, \hat{v}) = 2c_-(2\hat{v})R(\mathbf{c}; v, \hat{v}), \tag{5.5}$$

$$A_+(\hat{\mathbf{c}}; \hat{v})R(\mathbf{c}; v, \hat{v}) = 2c_+(2v)R(\mathbf{c}; v, \hat{v}), \tag{5.6}$$

$$A_-(I\hat{\mathbf{c}}; \hat{v})R(\mathbf{c}; v, \hat{v}) = 2c_-(2v)R(\mathbf{c}; v, \hat{v}). \tag{5.7}$$

In words, the R-function is a joint eigenfunction of four independent hyperbolic AΔOs of Askey–Wilson type.

In this connection we would like to point out that even though these four AΔOs manifestly commute (as operators on meromorphic functions of v and $\hat{v}$), there are no general results ensuring that a joint eigenfunction exists. Stronger yet, restricting attention to two AΔOs $A_\pm(y)$ of form (5.3) with $\mathrm{i}a_\pm$-periodic coefficients $C_\pm(y)$ (so that A_+ and A_- commute), there is no guarantee that any meromorphic $M(v)$ exists that is a joint eigenfunction.

Returning to the Askey–Wilson case at issue, it may well be that when one of the eigenvalues $2c_\pm(2\hat{v})$ of the AΔOs $A_\pm$ is altered, no solution to the joint eigenfunction problem exists. These open questions exemplify various other ones in the area of linear AΔEs, which is quite underdeveloped at present.

6. The relation to the Askey–Wilson polynomials

The locations of eventual poles in the R-function are known exactly. In particular, provided $\hat{c}_0$ is chosen rationally independent of $a_+, a_-, \hat{c}_1, \hat{c}_2$ and $\hat{c}_3$, no pole occurs at the points

$$\hat{v}_n = \mathrm{i}\hat{c}_0 + \mathrm{i}na_-, \qquad n \in \mathbb{Z}. \tag{6.1}$$

Thus we may define the functions

$$R_n(v) \equiv R(\mathbf{c}; v, \hat{v}_n), \quad n \in \mathbb{Z}. \tag{6.2}$$

We now explain the special character of these functions for $n \in \mathbb{N}$. Note first that for $\hat{v} = \hat{v}_0$ the eigenvalues of $A_+(\mathbf{c}; v)$ and $A_-(I\mathbf{c}; v)$ on $R(\mathbf{c}; v, \hat{v})$ are given by $2c_+(2\mathrm{i}\hat{c}_0)$ and $2c_-(2\mathrm{i}\hat{c}_0)$, cf. (5.4) and

(5.5). These are just the eigenvalues of the AΔOs on the constant functions, as is clear from (5.3). Thus it should not come as a surprise that one has

$$R_0(v) = 1. \tag{6.3}$$

This identity can be shown by shifting the contour in (4.6) across the (simple) pole at $z = 0$, picking up residue 1 due to the normalization factor up front. Now one can let $\hat{v}$ converge to $\hat{v}_0$ without poles colliding with the contour, so that the vanishing factor $1/G(-\hat{v} + \mathrm{i}\hat{c}_0 - \mathrm{i}a)$ implies (6.3).

Next, we write out the eigenvalue AΔE (5.6) for the points $\hat{v} = \hat{v}_n$. It reads

$$\begin{aligned} &C_+(\hat{\mathbf{c}}; \hat{v}_n)[R_{n-1}(v) - R_n(v)] + C_+(\hat{\mathbf{c}}; -\hat{v}_n)[R_{n+1}(v) - R_n(v)] \\ &\quad + 2c_+(2\mathrm{i}c_0)R_n(v) = 2c_+(2v)R_n(v). \end{aligned} \tag{6.4}$$

Due to the rational independence assumption, the coefficients are pole-free and $C_+(\hat{\mathbf{c}}; -\hat{v}_n)$ does not vanish for $n \in \mathbb{N}$, cf. (5.2). But we have $C_+(\hat{\mathbf{c}}; \hat{v}_0) = 0$, so that it follows recursively from (6.4) and (6.3) that one has

$$R_n(v) = P_n(c_+(2v)), \quad n \in \mathbb{N} \tag{6.5}$$

with $P_n(x)$ a polynomial of degree n in x with real coefficients. After an analytic continuation $a_+ \to -2\pi\mathrm{i}$, these polynomials become the Askey–Wilson polynomials $P_n(\cos v)$ and (6.4) becomes their three-term recurrence relation.

7. The $\mathscr{E}$-function: D_4 symmetry and asymptotics

From (5.2) it can be seen why the parameters $c_0, \ldots, c_3$ are couplings, physically speaking. Indeed, when they vanish, the coefficients $C_\pm(\mathbf{c}; y)$ reduce to 1, so there is no interaction and the AΔOs $A_\delta(\mathbf{c}; y)$ (5.3) reduce to the 'free' AΔOs

$$A_\delta^{(0)}(y) \equiv T^y_{\mathrm{i}a_{-\delta}} + T^y_{-\mathrm{i}a_{-\delta}}, \qquad \delta = +, -. \tag{7.1}$$

To obtain a new symmetry property, however, it is crucial to work instead with shifted parameters $\gamma_0, \ldots, \gamma_3$, defined by (inversion of)

$$\mathbf{c}(\gamma) \equiv (\gamma_0 + a, \gamma_1 + a_-/2, \gamma_2 + a_+/2, \gamma_3). \tag{7.2}$$

Then we have

$$\begin{aligned} C_+(\mathbf{c}(\gamma); y) &= -\frac{4\prod_{\mu=0}^3 c_+(y - \mathrm{i}\gamma_\mu - \mathrm{i}a_-/2)}{s_+(2y)s_+(2y - \mathrm{i}a_-)}, \\ C_-(I\mathbf{c}(\gamma); y) &= -\frac{4\prod_{\mu=0}^3 c_-(y - \mathrm{i}\gamma_\mu - \mathrm{i}a_+/2)}{s_-(2y)s_-(2y - \mathrm{i}a_+)}. \end{aligned} \tag{7.3}$$

Hence the AΔOs $A_+(\mathbf{c}(\gamma); y)$ and $A_-(I\mathbf{c}(\gamma); y)$ are invariant under arbitrary permutations of $\gamma_0, \ldots, \gamma_3$.

The shift vector in (7.2) is invariant under J (cf. (4.11)), so when we set

$$\hat{\gamma} \equiv J\gamma \tag{7.4}$$

we obtain

$$s_j = \gamma_0 + \gamma_j + a = \hat{\gamma}_0 + \hat{\gamma}_j + a, \qquad j = 1, 2, 3 \tag{7.5}$$

cf. (4.1). We now introduce a renormalized R-function

$$R_{\mathrm{r}}(a_+, a_-, \gamma; v, \hat{v}) \equiv R(a_+, a_-, \mathbf{c}(\gamma); v, \hat{v}) \Big/ \prod_{j=1}^{3} G(a_+, a_-; \mathrm{i}(\gamma_0 + \gamma_j + a)). \tag{7.6}$$

(This function amounts to the function $R_{\mathrm{ren}}(a_+, a_-, \mathbf{c}; v, \hat{v})$ II (1.13), reparametrized by γ instead of $\mathbf{c}$.) Recalling (4.1)–(4.6), we see this entails

$$R_{\mathrm{r}}(\gamma; v, \hat{v}) = \frac{1}{(a_+a_-)^{1/2}} \int_{\mathscr{C}} \frac{F(\gamma_0 + a; v, z) F(\hat{\gamma}_0 + a; \hat{v}, z)}{G(z + \mathrm{i}a) \prod_{j=1}^{3} G(z + \mathrm{i}(\gamma_0 + \gamma_j + a))} \, \mathrm{d}z, \tag{7.7}$$

whereas properties (4.8)–(4.10) and (4.13) yield

$$R_{\mathrm{r}}(a_+, a_-, \gamma; v, \hat{v}) = R_{\mathrm{r}}(a_+, a_-, \gamma; \delta v, \delta' \hat{v}), \quad \delta, \delta' = +, -, \tag{7.8}$$

$$R_{\mathrm{r}}(a_+, a_-, \gamma; v, \hat{v}) = R_{\mathrm{r}}(\lambda a_+, \lambda a_-, \lambda \gamma; \lambda v, \lambda \hat{v}), \quad \lambda > 0, \tag{7.9}$$

$$R_{\mathrm{r}}(a_+, a_-, \gamma; v, \hat{v}) = R_{\mathrm{r}}(a_-, a_+, \gamma; v, \hat{v}), \tag{7.10}$$

$$R_{\mathrm{r}}(a_+, a_-, \gamma; v, \hat{v}) = R_{\mathrm{r}}(a_+, a_-, \hat{\gamma}; \hat{v}, v). \tag{7.11}$$

From (7.7) one reads off that R_{r} is invariant under permutations of $\gamma_1, \gamma_2, \gamma_3$, whereas the γ_0-dependence is quite different from the γ_j-dependence.

We will presently see that R_{r} is indeed not invariant under permutations involving γ_0. But this is most easily established by similarity transforming to a function $\mathscr{E}(\gamma; v, \hat{v})$ that is not only invariant under any permutation of $\gamma_0, \ldots, \gamma_3$, but also under sign flips involving an even number of γ_μ. These transformations generate the Weyl group W of the Lie algebra D_4, and it is crucial in the sequel that J satisfies

$$JWJ = J. \tag{7.12}$$

(This is easily checked from the definitions. Note that when w is the transposition of γ_0 and γ_j, the transformation JwJ equals the product of a permutation and a double sign flip.)

The similarity transformation involves the c-function

$$c(p; y) \equiv \frac{1}{G(2y + \mathrm{i}a)} \prod_{\mu=0}^{3} G(y - \mathrm{i}p_\mu). \tag{7.13}$$

Specifically, the $\mathscr{E}$-function is defined by

$$\mathscr{E}(\gamma; v, \hat{v}) \equiv \frac{\chi(\gamma)}{c(\gamma; v)} R_{\mathrm{r}}(\gamma; v, \hat{v}) \frac{1}{c(\hat{\gamma}; \hat{v})}. \tag{7.14}$$

Here, χ is the phase factor

$$\chi(\gamma) \equiv \exp(\mathrm{i}\alpha[\gamma \cdot \gamma/4 - (a_+^2 + a_-^2 + a_+a_-)/8]), \quad \alpha = 2\pi/a_+a_-. \tag{7.15}$$

The phase occurs for normalization purposes and is clearly W invariant. The crux is now that $\mathscr{E}$ is W invariant:

$$\mathscr{E}(\gamma; v, \hat{v}) = \mathscr{E}(w(\gamma); v, \hat{v}), \quad \forall w \in W. \tag{7.16}$$

Accepting this, it follows that R_{r} satisfies

$$\frac{R_{\mathrm{r}}(\gamma^{(1)}; v, \hat{v})}{R_{\mathrm{r}}(\gamma^{(2)}; v, \hat{v})} = \frac{c(\gamma^{(1)}; v)c(J\gamma^{(1)}; \hat{v})}{c(\gamma^{(2)}; v)c(J\gamma^{(2)}; \hat{v})}, \ \gamma^{(j)} \equiv w_j(\gamma), \quad w_j \in W, \ \ j = 1, 2. \tag{7.17}$$

(In particular, taking w_1 the identity map and w_2 the transposition of γ_0 and γ_j, the rhs is a nontrivial function of $\hat{v}$.)

To appreciate why (7.16) holds true, it is important to examine the similarity transformed AΔOs. We begin by noting that (7.13) and the G-AΔEs (3.4)–(3.5) entail

$$c(\gamma; y)/c(\gamma; y - \mathrm{i}a_-) = C_+(\mathbf{c}(\gamma); y), \quad c(\gamma; y)/c(\gamma; y - \mathrm{i}a_+) = C_-(I\mathbf{c}(\gamma); y) \tag{7.18}$$

cf. (7.3). From this we deduce that the AΔOs

$$\mathscr{A}_+(\gamma; y) \equiv c(\gamma; y)^{-1}A_+(\mathbf{c}(\gamma); y)c(\gamma; y), \quad \mathscr{A}_-(\gamma; y) \equiv c(\gamma; y)^{-1}A_-(I\mathbf{c}(\gamma); y)c(\gamma; y) \tag{7.19}$$

can be written as

$$\mathscr{A}_\delta(\gamma; y) = T_{\mathrm{i}a_{-\delta}} + V_{a,\delta}(\gamma; y)T_{-\mathrm{i}a_{-\delta}} + V_{b,\delta}(\gamma; y), \quad \delta = +, - \tag{7.20}$$

with

$$V_{a,\delta}(\gamma; y) = \frac{16\prod_{\mu=0}^{3} c_\delta(y + \mathrm{i}\gamma_\mu + \mathrm{i}a_{-\delta}/2)c_\delta(y - \mathrm{i}\gamma_\mu + \mathrm{i}a_{-\delta}/2)}{s_\delta(2y)s_\delta(2y + \mathrm{i}a_{-\delta})^2 s_\delta(2y + 2\mathrm{i}a_{-\delta})}, \tag{7.21}$$

$$V_{b,\delta}(\gamma; y) \equiv -C_\delta(\mathbf{c}(\gamma); y) - C_\delta(\mathbf{c}(\gamma); -y) - 2c_\delta\left(\mathrm{i}\sum_{\mu=0}^{3}\gamma_\mu + \mathrm{i}a_{-\delta}\right). \tag{7.22}$$

Obviously, $V_{a,\delta}(\gamma; y)$ is not only S_4 invariant, but also invariant under arbitrary sign flips. At face value, $V_{b,\delta}(\gamma; y)$ is only S_4 invariant. In fact, however, $V_{b,\delta}(\gamma; y)$ is D_4 invariant. This follows from an alternative representation, namely,

$$V_{b,\delta}(\gamma; y) = \frac{(p_{c,\delta} - p_{s,\delta})c_\delta(2y) + (p_{c,\delta} + p_{s,\delta})c_\delta(\mathrm{i}a_{-\delta})}{s_\delta(y - \mathrm{i}a_{-\delta}/2)s_\delta(y + \mathrm{i}a_{-\delta}/2)}, \tag{7.23}$$

where

$$p_{c,\delta} \equiv 4\prod_{\mu=0}^{3} c_\delta(\mathrm{i}\gamma_\mu), \quad p_{s,\delta} \equiv 4\prod_{\mu=0}^{3} s_\delta(\mathrm{i}\gamma_\mu). \tag{7.24}$$

(Equality to (7.22) can be readily checked by comparing periodicity, residues and asymptotics.) As a consequence, we obtain

$$\mathscr{A}_\delta(w(\gamma); y) = \mathscr{A}_\delta(\gamma; y), \quad \delta = +, -, \ \ w \in W. \tag{7.25}$$

The upshot of these developments is that $\mathscr{E}(\gamma; v, \hat{v})$ is a joint eigenfunction of the four D_4 invariant AΔOs $\mathscr{A}_\delta(\gamma; v)$, $\mathscr{A}_\delta(\hat{\gamma}; \hat{v})$ with eigenvalues $2c_\delta(2\hat{v})$, $2c_\delta(2v)$, $\delta = +, -$. Although this 'explains' why $\mathscr{E}(\gamma; v, \hat{v})$ is itself D_4 invariant (cf. (7.16)), we are not aware of any general result from which this conclusion rigorously follows.

Even so, in the special context at issue a complete proof of (7.16) can be constructed by exploiting a quite different feature of the $\mathscr{E}$-function, namely its $\operatorname{Re} v \to \infty$ asymptotic behavior. Introducing the '*S*-matrix'

$$u(p; y) = -c(p; y)/c(p; -y) \tag{7.26}$$

and leading asymptotics function

$$\mathscr{E}_{\mathrm{as}}(\gamma; v, \hat{v}) \equiv \exp(\mathrm{i}\alpha v\hat{v}) - u(\hat{\gamma}; -\hat{v}) \exp(-\mathrm{i}\alpha v\hat{v}) \tag{7.27}$$

this asymptotics reads, roughly speaking,

$$\mathscr{E}(\gamma; v, \hat{v}) - \mathscr{E}_{\mathrm{as}}(\gamma; v, \hat{v}) = \mathrm{O}(\exp(-\rho \operatorname{Re} v)), \quad \operatorname{Re} v \to \infty, \tag{7.28}$$

where the rate $\rho > 0$ depends only on the parameters (a_+, a_-, γ). (The precise result is rather technical, and involves in particular a proviso for the special case $a_+ = a_-$, $\gamma = 0$. We refer to Theorem 1.2 in II for the details.)

The relevance of asymptotics (7.28) for the problem of proving D_4 symmetry is due to $u(p; y)$ being manifestly D_4 symmetric. Indeed, using the reflection equation (3.7) we obtain the representation

$$u(p; y) = -\frac{\prod_{\mu=0}^{3} G(y - \mathrm{i}p_\mu) G(y + \mathrm{i}p_\mu)}{G(2y + \mathrm{i}a) G(2y - \mathrm{i}a)} \tag{7.29}$$

which reveals that the *u*-function is even invariant under arbitrary sign flips of the parameters $p_0, \ldots, p_3$.

Our proof of D_4 symmetry, as encoded in (7.16), and (a strong form of) the asymptotics (7.28) in II is quite involved. It is beyond our scope to even sketch it, but we do add that it involves an entanglement of the two distinct features that we are unable to avoid.

To conclude this section, we note that the duplication formula (3.10) entails

$$c(\gamma_{\mathrm{f}}; y) = 1, \tag{7.30}$$

where γ_{f} corresponds to the 'free' case $\mathbf{c} = 0$, cf. (7.2):

$$\gamma_{\mathrm{f}} \equiv (-a, -a_-/2, -a_+/2, 0). \tag{7.31}$$

Thus we get (recall (7.1))

$$A_\delta(0; y) = \mathscr{A}_\delta(\gamma_{\mathrm{f}}; y) = A_\delta^{(0)}(y), \quad \delta = +, -. \tag{7.32}$$

Since γ_{f} is also self-dual, it should not come as a surprise that for zero coupling the $\mathscr{E}$-function coincides with its asymptotics. Specifically, we have

$$\mathscr{E}(\gamma_{\mathrm{f}}; v, \hat{v}) = 2\cos(\alpha v\hat{v}). \tag{7.33}$$

Since $\chi(\gamma_{\mathrm{f}}) = 1$ (cf. (7.15) and (7.31)), this identity amounts to (recall (7.14))

$$R_{\mathrm{r}}(\gamma_{\mathrm{f}}; v, \hat{v}) = 2\cos(\alpha v\hat{v}). \tag{7.34}$$

Yet another equivalent formula reads

$$R(0; v, \hat{v}) = \cos(\alpha v \hat{v}). \tag{7.35}$$

Indeed, taking γ equal to γ_{f} in (7.6), the G-product reduces to $\frac{1}{2}$, cf. (3.14). In Section 11 we sketch the proof of (7.34).

8. The nonrelativistic limit

In this section, we specify the limiting transitions leading from the functions R, $\mathscr{E}$ and AΔOs $A_{\pm}$, $\mathscr{A}_{\pm}$ to their counterparts in Section 2. To start with, we define

$$\psi_{\mathrm{rel}}(\lambda, \mathbf{c}; v, \hat{v}) \equiv R(\pi, \lambda, \lambda\mathbf{c}; v, \hat{v}/2). \tag{8.1}$$

Then we have

$$\lim_{\lambda\downarrow 0} \psi_{\mathrm{rel}}(\lambda, \mathbf{c}; v, \lambda\hat{v}) = \psi_{\mathrm{nr}}(c_0 + c_2, c_1 + c_3; v, \hat{v}) \tag{8.2}$$

with ψ_{nr} given by (2.2). Thus this amounts to a limit $R \to {}_2F_1$.

To date, this limit is a formal one. We conjecture that (8.2) holds true uniformly on compact subsets of the v-region

$$\mathscr{R} \equiv \{v \in \mathbb{C} \mid \mathrm{Re}\, v > 0, |\mathrm{Im}\, v| < \pi/2\} \tag{8.3}$$

and compact subsets of the $\hat{v}$-plane. (Note that the boundary of $\mathscr{R}$ corresponds to the ${}_2F_1$-cut, cf. (2.1)–(2.2).) Not even pointwise convergence has been rigorously proved, though. We now explain the most important reason why the conjecture is plausible.

First, we substitute $z \to \lambda z$ in the integral representation of ψ_{rel} (given by (8.1) and (4.6)) and factorize it into two 'side' functions and a 'middle' function, given by

$$S_L(\lambda, c_0; v, z) \equiv \exp(2\mathrm{i}z \ln 2) F(\pi, \lambda, \lambda c_0; v, \lambda z), \tag{8.4}$$

$$M(\lambda, \mathbf{c}; z) \equiv \left(\frac{\lambda}{\pi}\right)^{1/2} \exp(-2\mathrm{i}z \ln(4\lambda)) K(\pi, \lambda, \lambda\mathbf{c}; \lambda z), \tag{8.5}$$

$$S_R(\lambda, \hat{c}_0; \hat{v}, z) \equiv \exp(2\mathrm{i}z \ln(2\lambda)) F(\pi, \lambda, \lambda\hat{c}_0; \lambda\hat{v}/2, \lambda z). \tag{8.6}$$

Using (3.11)–(3.12), we now deduce

$$\lim_{\lambda\downarrow 0} S_L(\lambda, c_0; v, z) = \exp(-\mathrm{i}z \ln(\sinh^2 v)), \quad \mathrm{Re}\, v > 0, \tag{8.7}$$

$$\lim_{\lambda\downarrow 0} M(\lambda, \mathbf{c}; z) = \frac{\Gamma(\mathrm{i}z)\Gamma(c_0 + c_2 + 1/2)}{2\pi\Gamma(c_0 + c_2 + 1/2 - \mathrm{i}z)}, \tag{8.8}$$

$$\lim_{\lambda\downarrow 0} S_R(\lambda, \hat{c}_0; \hat{v}, z) = \frac{\Gamma(\hat{c}_0 + \mathrm{i}\hat{v}/2 - \mathrm{i}z)}{\Gamma(\hat{c}_0 + \mathrm{i}\hat{v}/2)} \frac{\Gamma(\hat{c}_0 - \mathrm{i}\hat{v}/2 - \mathrm{i}z)}{\Gamma(\hat{c}_0 - \mathrm{i}\hat{v}/2)}. \tag{8.9}$$

Thus the integrand corresponding to ψ_{rel} converges to that of ψ_{nr} for $\lambda \downarrow 0$, cf. (2.1) and (2.2). This holds true uniformly on sufficiently small discs around any point on the contour. To control the limit, however, one would need a suitable uniform bound on the tails so as to invoke the dominated convergence theorem, and no such bound has been proved yet.

Next, we consider the limiting behavior of the four AΔOs $A_+(\mathbf{c}; v)$, $A_-(I\mathbf{c}; v)$, $A_+(\hat{\mathbf{c}}; \hat{v})$ and $A_-(I\hat{\mathbf{c}}; \hat{v})$ when the substitutions going with (8.1) and (8.2) are made, and λ is taken to 0. Thus we should substitute

$$a_+, a_-, \mathbf{c}, v, \hat{v} \to \pi, \lambda, \lambda\mathbf{c}, v, \lambda\hat{v}/2 \tag{8.10}$$

and study the behavior of the coefficients, translations and eigenvalues as $\lambda \downarrow 0$. As regards the translations, we note that (2.7) entails

$$T_\eta^y \simeq \exp\left(-\eta\frac{\mathrm{d}}{\mathrm{d}y}\right) = 1 - \eta\frac{\mathrm{d}}{\mathrm{d}y} + \frac{\eta^2}{2}\frac{\mathrm{d}^2}{\mathrm{d}y^2} + \mathrm{O}(\eta^3), \quad \eta \to 0, \tag{8.11}$$

whereas T_η^y has no reasonable behavior for $\eta \to \infty$. Since substitution of (8.10) in $A_-(I\hat{\mathbf{c}}; \hat{v})$ yields an AΔO with diverging translations

$$T_{\pm\mathrm{i}\pi}^{\lambda\hat{v}/2} \simeq \exp\left(\mp\frac{2\mathrm{i}\pi}{\lambda}\frac{\mathrm{d}}{\mathrm{d}\hat{v}}\right) \tag{8.12}$$

as $\lambda \downarrow 0$, it becomes useless. (Note that the reparametrized eigenvalue $2\cosh(2\pi v/\lambda)$ diverges, too.)

Setting

$$d = c_0 + c_2, \quad \tilde{d} = c_1 + c_3 \tag{8.13}$$

it is readily verified that the remaining three AΔOs satisfy

$$A_+(\lambda\mathbf{c}; v) = 2 + \lambda^2 H_v + \mathrm{O}(\lambda^4), \tag{8.14}$$

$$A_-(\lambda I\mathbf{c}; v) = [\exp(-\mathrm{i}\pi(d+\tilde{d})) + \mathrm{O}(\mathrm{e}^{-2\pi v/\lambda})]T_{\mathrm{i}\pi}^v + (\mathrm{i} \to -\mathrm{i}) + \mathrm{O}(\mathrm{e}^{-2\pi v/\lambda}), \quad \mathrm{Re}\ v > 0, \tag{8.15}$$

$$A_+(\lambda\hat{\mathbf{c}}; \lambda\hat{v}/2) = A_{\hat{v}} + \mathrm{O}(\lambda^2), \tag{8.16}$$

where H_v and $A_{\hat{v}}$ are given by (2.4) and (2.6), resp. The eigenvalue of $A_+(\lambda\mathbf{c}; v)$ is given by

$$2\cosh(\lambda\hat{v}) = 2 + \lambda^2\hat{v}^2 + \mathrm{O}(\lambda^4), \tag{8.17}$$

whereas the eigenvalues of the two AΔOs on the lhs of (8.15) and (8.16) are λ-independent, namely, $2\cosh(\pi\hat{v})$ and $2\cosh(2v)$, resp.

We now turn to $\mathscr{E}(\gamma; v, \hat{v})$ and the AΔOs $\mathscr{A}_\pm(\gamma; v)$, $\mathscr{A}_\pm(\hat{\gamma}; \hat{v})$. The substitutions for γ and $\hat{\gamma}$ associated with (8.10) are given by (cf. (7.2))

$$\gamma \to \gamma(\lambda) \equiv \lambda\mathbf{c} - \sigma(\lambda), \quad \hat{\gamma} \to \hat{\gamma}(\lambda) \equiv \lambda\hat{\mathbf{c}} - \sigma(\lambda), \quad \sigma(\lambda) \equiv ((\pi+\lambda)/2, \lambda/2, \pi/2, 0). \tag{8.18}$$

From this and (7.20)–(7.22) we obtain

$$\mathscr{A}_+(\gamma(\lambda); v) = 2 + \lambda^2\mathscr{H}_v + \mathrm{O}(\lambda^4), \tag{8.19}$$

$$\mathscr{A}_-(\gamma(\lambda); v) = T_{\mathrm{i}\pi}^v + [\exp(2\mathrm{i}\pi(d+\tilde{d})) + \mathrm{O}(\mathrm{e}^{-2\pi v/\lambda})]T_{-\mathrm{i}\pi}^v + \mathrm{O}(\mathrm{e}^{-2\pi v/\lambda}), \quad \mathrm{Re}\ v > 0, \tag{8.20}$$

$$\mathscr{A}_+(\hat{\gamma}(\lambda); \lambda\hat{v}/2) = \mathscr{A}_{\hat{v}} + \mathrm{O}(\lambda^2) \tag{8.21}$$

with $\mathscr{H}_v$ and $\mathscr{A}_{\hat{v}}$ given by (2.12) and (2.13)–(2.15), resp.

Next, we use the duplication formula (3.10) and limit (3.11) to obtain (recall (2.9))

$$\lim_{\lambda\downarrow 0}\chi(\gamma(\lambda))/c(\gamma(\lambda);v)=w_{\mathrm{nr}}(d,\tilde{d};v)^{1/2},\quad \mathrm{Re}\, v>0. \tag{8.22}$$

Likewise, using also (3.12) we get (recall (2.10))

$$\lim_{\lambda\downarrow 0}c(\hat{\gamma}(\lambda);\lambda\hat{v}/2)\prod_{j=1}^{3}G(\mathrm{i}s_j(\lambda))=\frac{1}{2}\hat{c}_{\mathrm{nr}}(d,\tilde{d};\hat{v}). \tag{8.23}$$

Combining all this, we finally obtain

$$\lim_{\lambda\downarrow 0}\mathscr{E}(\pi,\lambda,\gamma(\lambda);v,\lambda\hat{v}/2)=\mathscr{E}_{\mathrm{nr}}(d,\tilde{d};v,\hat{v}),\quad d=c_0+c_2,\ \tilde{d}=c_1+c_3 \tag{8.24}$$

with $\mathscr{E}_{\mathrm{nr}}$ given by (2.8)–(2.10).

We point out that under the nonrelativistic limit almost all of the symmetries of the R- and $\mathscr{E}$-functions disappear. The D_4 symmetry leaves one footprint, however. Indeed, when we rewrite the sign flip $(\gamma_1,\gamma_3)\to(-\gamma_1,-\gamma_3)$ in terms of $\gamma(\lambda)$ (given by (8.18)), then it amounts to

$$c_1+c_3\to 1-c_1-c_3. \tag{8.25}$$

The resulting $\tilde{d}\to 1-\tilde{d}$ invariance of $\mathscr{E}_{\mathrm{nr}}(d,\tilde{d};v,\hat{v})$ amounts to the well-known identity

$${}_2F_1(\alpha,\beta,\gamma;x)=(1-x)^{\gamma-\alpha-\beta}\,{}_2F_1(\gamma-\beta,\gamma-\alpha,\gamma;x). \tag{8.26}$$

Next, we note that $\mathscr{H}_v$ (2.12) is not only invariant under $\tilde{d}\to 1-\tilde{d}$, but also under $d\to 1-d$. But the latter symmetry cannot be viewed as the remnant of a D_4 transformation, and indeed $\mathscr{E}_{\mathrm{nr}}(d,\tilde{d};v,\hat{v})$ is not invariant under $d\to 1-d$. (Of course, $\mathscr{E}_{\mathrm{nr}}(1-d,\tilde{d};v,\hat{v})$ does yield a second $\mathscr{H}_v$-eigenfunction.)

Finally, we mention that in II we did not study the nonrelativistic limit of the $\mathscr{E}$-function and associated AΔOs. In our recent lecture notes [21], however, we briefly looked at this question, cf. [21, (6.19)–(6.22)]. We would like to point out that the right-hand sides of (6.20) and (6.22) have an incorrect dependence on the couplings. This is rectified in (8.23); also, the above definition (2.10) differs from [21, (6.21)] by the three factors up front.

9. The Hilbert space transform associated to $\mathscr{E}$

For parameters (a_+,a_-,γ) in

$$\Pi\equiv\{(a_+,a_-,p)\in\mathbb{R}^6\mid a_+,a_->0\} \tag{9.1}$$

the function $\mathscr{E}(a_+,a_-,\gamma;v,\hat{v})$ is meromorphic in v and $\hat{v}$, with eventual poles that are located solely on the imaginary axis. These locations are known as linear functions of the parameters. In particular, in the polytope

$$P\equiv\{(a_+,a_-,p)\in\Pi\mid |p_\mu|<a,\ \mu=0,\ldots,3\} \tag{9.2}$$

the $\mathscr{E}$-function has no poles at the origin. More generally, no such poles occur for generic parameters in Π, but it is likely that there do exist parameters in Π for which $\mathscr{E}$ has a pole at the origin.

Restricting (a_+, a_-, γ) to P from now on, we can define a linear operator (generalized Fourier transform)

$$\mathscr{F} : \mathscr{C} \equiv C_0^\infty((0,\infty)) \subset \hat{\mathscr{H}} \equiv L^2((0,\infty), \mathrm{d}\hat{v}) \to \mathscr{H} \equiv L^2((0,\infty), \mathrm{d}v) \tag{9.3}$$

by using $\mathscr{E}$ as a kernel:

$$(\mathscr{F}\phi)(v) \equiv \left(\frac{\alpha}{2\pi}\right)^{1/2} \int_0^\infty \mathscr{E}(v, \hat{v})\phi(\hat{v})\,\mathrm{d}\hat{v}, \qquad \phi \in \mathscr{C}. \tag{9.4}$$

Due to the regularity of $\mathscr{E}$ for real v and its plane wave asymptotics for $v \to \infty$ (cf. (7.28)), the function $(\mathscr{F}\phi)(v)$ is indeed in $\mathscr{H}$. Moreover, it is the restriction of a meromorphic function (denoted by $(\mathscr{F}\phi)(v)$ as well) to $(0,\infty)$, so that the AΔOs $\mathscr{A}_\pm(\gamma; v)$ have a well-defined action on it. Using the known meromorphy properties of $\mathscr{E}$ and the eigenvalue equations

$$\mathscr{A}_\delta(\gamma; v)\mathscr{E}(v, \hat{v}) = 2c_\delta(2\hat{v})\mathscr{E}(v, \hat{v}) \quad \delta = +, - \tag{9.5}$$

it is not hard to see that this action is given by

$$\mathscr{A}_\delta(\gamma; v)(\mathscr{F}\phi)(v) = \left(\frac{\alpha}{2\pi}\right)^{1/2} \int_0^\infty \mathscr{E}(\gamma; v, \hat{v})2c_\delta(2\hat{v})\phi(\hat{v})\,\mathrm{d}\hat{v}. \tag{9.6}$$

This implies in particular that the meromorphic function $\mathscr{A}_\delta(\gamma; v)(\mathscr{F}\phi)(v)$ has a restriction to $(0,\infty)$ that belongs to $\mathscr{H}$. Thus we obtain well-defined Hilbert space operators

$$\mathscr{A}_\delta : \mathscr{F}\mathscr{C} \subset \mathscr{H} \to \mathscr{H} \tag{9.7}$$

satisfying

$$\mathscr{A}_\delta \mathscr{F}\phi = \mathscr{F}\mathscr{M}_\delta\phi, \qquad \delta = +, -, \ \ \phi \in \mathscr{C}, \tag{9.8}$$

where $\mathscr{M}_\delta$ denotes multiplication by $2c_\delta(2\hat{v})$ on $\hat{\mathscr{H}}$.

With due effort, it can now be shown that the operators $\mathscr{A}_\pm$ are essentially self-adjoint on $\mathscr{F}\mathscr{C}$ and that $\mathscr{F}$ is isometric. We proceed to sketch a few key steps in the proof of these properties. To this end it is convenient to work with parameters

$$a_\mathrm{s} \equiv \min(a_+, a_-), \qquad a_\mathrm{l} \equiv \max(a_+, a_-). \tag{9.9}$$

First, symmetry of the operator $\mathscr{A}_\mathrm{s}$ with the smallest step size a_s is shown via contour shifts and Cauchy's theorem. Second, essential self-adjointness of $\mathscr{A}_\mathrm{s}$ is derived from Nelson's analytic vector theorem. Hence the 'interacting evolution' $\exp(-\mathrm{i}t\mathscr{A}_\mathrm{s})$ is diagonalized by $\mathscr{F}$, in the sense that

$$\exp(-\mathrm{i}t\mathscr{A}_\mathrm{s})\mathscr{F}\phi = \mathscr{F}\exp(-\mathrm{i}t\mathscr{M}_\mathrm{s})\phi, \qquad \phi \in \mathscr{C}. \tag{9.10}$$

At this stage, however, it is neither clear whether $\mathscr{F}$ is a bounded operator, nor whether it is invertible on $\mathscr{F}\mathscr{C}$.

Third, this interacting evolution is compared to a free evolution defined by

$$\exp(-\mathrm{i}t\mathscr{A}_\mathrm{s}^{(0)}) \equiv \mathscr{F}_0 \exp(-\mathrm{i}t\mathscr{M}_\mathrm{s})\mathscr{F}_0^*, \tag{9.11}$$

where $\mathscr{F}_0$ is essentially the sine transform, namely,

$$(\mathscr{F}_0\phi)(v) \equiv \left(\frac{\alpha}{2\pi}\right)^{1/2} \int_0^\infty [\exp(\mathrm{i}\alpha v\hat{v}) - \exp(-\mathrm{i}\alpha v\hat{v})]\phi(\hat{v})\,\mathrm{d}\hat{v}. \tag{9.12}$$

(Note this implies that the action of the generator $\mathscr{A}_s^{(0)}$ on the core $\mathscr{F}_0\mathscr{C}$ amounts to that of the AΔO $A_s^{(0)}$, cf. (7.1).) The result is that one obtains

$$W_+ = \mathscr{F}\mathscr{F}_0^*, \tag{9.13}$$

where W_+ denotes the $t \to \infty$ wave operator. Since $\mathscr{F}_0$ is unitary and W_+ isometric, it now follows not only that $\mathscr{F}$ is isometric, but also that $u(\hat{\gamma}; \hat{v})$ encodes the scattering.

Fourth, symmetry of the second AΔO $\mathscr{A}_l$ with the largest step size a_l follows from isometry of $\mathscr{F}$, and its essential self-adjointness from the analytic vector theorem. Then another application of time-dependent scattering theory shows that $u(\hat{\gamma}; \hat{v})$ is also the S-matrix for the pair of evolutions $\exp(-\mathrm{i}t\mathscr{A}_l)$, $\exp(-\mathrm{i}t\mathscr{A}_l^{(0)})$.

We mention in passing that at face value $\mathscr{A}_l$ does not appear to be symmetric, in as much as for $a_l > a_s$ the contour shifts involved give rise to nonzero residues. But since $\mathscr{A}_l$ *is* symmetric (as follows from isometry of $\mathscr{F}$), the residue *sum* must vanish. This exemplifies that the issue whether a Hilbert space operator associated to a formally self-adjoint AΔO is symmetric is quite delicate.

Next, using the self-duality property of the kernel $\mathscr{E}$ of $\mathscr{F}$ (which can be derived from (7.11) and (7.14)), it is not hard to see that $\mathscr{F}^*$ is also isometric for parameters (a_+, a_-, γ) in $P \cap \hat{P}$, where

$$\hat{P} \equiv \{(a_+, a_-, p) \mid (a_+, a_-, \hat{p}) \in P\}. \tag{9.14}$$

Thus the scattering states are complete in $\mathscr{H}$ for parameters in $P \cap \hat{P}$.

The results sketched thus far extend to a parameter set P_e that is slightly larger than P. It is defined by allowing one p_μ to become equal to a or $-a$. In particular, $\mathscr{F}$ is unitary for (a_+, a_-, γ) in $P_e \cap \hat{P}_e$, with $\hat{P}_e$ defined by (9.14) with $P \to P_e$. Note that the self-dual parameters (a_+, a_-, γ_f) belong to $P_e \backslash P$, and that the associated transform amounts to the cosine transform, cf. (7.33).

For parameters in P_e that do not belong to $\hat{P}_e$, unitarity of $\mathscr{F}$ breaks down. It is not hard to see that parameters (a_+, a_-, γ) belonging to P_e do not belong to $\hat{P}_e$ if and only if

$$\max(|\hat{\gamma}_0|, |\hat{\gamma}_1|, |\hat{\gamma}_2|, |\hat{\gamma}_3|) > a. \tag{9.15}$$

By D_4 invariance we may and will assume (in addition to our standing assumption $(a_+, a_-, \gamma) \in P$)

$$\hat{\gamma}_0 < -a \tag{9.16}$$

from now on. The key point is that since the $\mathscr{A}_\pm(\gamma; v)$-eigenfunction $1/c(\gamma; v)$ satisfies

$$c(\gamma; v)^{-1} \sim \chi(\gamma)^{-1} \exp(\alpha(\hat{\gamma}_0 + a)v), \quad v \to \infty \tag{9.17}$$

(as follows from (7.13) and (3.13)), assumption (9.16) entails that it is in $\mathscr{H}$. More generally, the eigenfunctions

$$\psi_n(v) \equiv P_n(\cosh(\alpha a_s v))/c(v), \quad n \in \mathbb{N} \tag{9.18}$$

(where $P_n(x)$ are the polynomials from Section 6) are in $\mathscr{H}$ whenever

$$\hat{\gamma}_0 + a + na_s < 0, \quad n = 0, \ldots, N-1 \tag{9.19}$$

as is clear from (9.17). (Here, $N \geqslant 1$ is the largest integer so that the inequality holds true.)

It can now be shown that the vectors $\psi_0, \ldots, \psi_{N-1} \in \mathscr{H}$ are pairwise orthogonal, and orthogonal to $\mathrm{Ran}(\mathscr{F})$ as well, so that $\mathscr{F}^*$ is not isometric. Moreover, these bound states and the scattering states $\mathscr{F}\phi$, $\phi \in \hat{\mathscr{H}}$, are complete:

$$\mathscr{H} = \mathscr{F}(\hat{\mathscr{H}}) \oplus \mathrm{Span}(\psi_0, \ldots, \psi_{N-1}). \tag{9.20}$$

We proceed to sketch the main steps of the proof of orthogonality and completeness in III.

First, in view of the AΔO action

$$\mathscr{A}_{\mathrm{s}}(\gamma; v)\psi_n(v) = 2\cos(2\pi(\hat{\gamma}_0 + a + na_{\mathrm{s}})/a_{\mathrm{l}})\psi_n(v), \quad n = 0, \ldots, N-1 \tag{9.21}$$

the action of the Hilbert space operator $\mathscr{A}_{\mathrm{s}}$ (thus far defined only on $\mathscr{F}\mathscr{C}$) can be extended in an obvious way to $\psi_0, \ldots, \psi_{N-1}$, namely via (9.21). Distinctness of the eigenvalues in (9.21) now yields pairwise orthogonality, and orthogonality to $\mathrm{Ran}(\mathscr{F})$ follows from the eigenvalues being smaller than the spectral values $2\cosh(2\pi\hat{v}/a_{\mathrm{l}}) \geqslant 2$ on $\mathrm{Ran}(\mathscr{F})$.

Second, the isometry violation of $\mathscr{F}^*$ can be explicitly related to the symmetry violation of the operator $\tilde{\mathscr{A}}_{\mathrm{s}}$ on $\mathscr{F}^*\mathscr{C}$ associated to the pertinent dual AΔO. Specifically, this yields the identity

$$\begin{aligned}&(\tilde{\mathscr{A}}_{\mathrm{s}}\mathscr{F}^*\phi_1, \mathscr{F}^*\phi_2) - (\mathscr{F}^*\phi_1, \tilde{\mathscr{A}}_{\mathrm{s}}\mathscr{F}^*\phi_2)\\ &\quad = \mathscr{N}\int_0^\infty \mathrm{d}v_1 \overline{\phi_1(v_1)} \int_0^\infty \mathrm{d}v_2 \phi_2(v_2) u(\gamma; v_2) B(v_1, v_2),\end{aligned} \tag{9.22}$$

where $\mathscr{N}$ is a normalization constant and

$$B(v_1, v_2) \equiv \psi_N(v_1)\psi_{N-1}(v_2) - (v_1 \leftrightarrow v_2). \tag{9.23}$$

The third and last step exploits the Christoffel–Darboux identity

$$B(v_1, v_2) = [\cosh(\alpha a_{\mathrm{s}} v_1) - \cosh(\alpha a_{\mathrm{s}} v_2)] \sum_{n=0}^{N-1} \beta_n \psi_n(v_1)\psi_n(v_2) \tag{9.24}$$

and the relation

$$u(\gamma; v_2)\psi_n(v_2) = -\overline{\psi_n(v_2)}, \quad v_2 > 0 \tag{9.25}$$

(recall (7.26)) to arrive at the formula

$$\mathscr{F}\mathscr{F}^* = \mathbf{1}_{\mathscr{H}} - \sum_{n=0}^{N-1} v_n \psi_n \otimes \overline{\psi_n}, \tag{9.26}$$

where $v_0, \ldots, v_{N-1}$ are positive normalization coefficients. From this we deduce the completeness relation (9.20), concluding the proof.

In III we did not study the transform for parameters outside P_{e}. For two one-parameter subfamilies, however, we previously obtained the operator-theoretic properties of the transform in [13]. There we established breakdown of isometry outside (the analog of) P_{e} in explicit detail. It may be expected that for the full four-parameter case the picture emerging from [13] remains basically the same.

10. A hyperbolic analog of the Askey–Wilson integral

The key identity (9.22) arises from a contour shift on the lhs, where residues at two poles of $\mathscr{E}(v, \hat{v})$ are encountered that give rise to $\psi_{N-1}(v)$ and $\psi_N(v)$. The normalization constant $\mathscr{N}$ follows from this residue calculation. It involves the value of $1/c(\hat{\gamma}; \hat{v})$ at $\hat{v} = -\hat{v}_N$ and the residue of $1/c(\hat{\gamma}; \hat{v})$ at $\hat{v} = \hat{v}_{N-1}$. (Recall $\hat{v}_n$ is defined by (6.1).) Now these quantities can be expressed in terms of the G-function, and the recurrence coefficients of the bound states are explicitly known from (6.4). Therefore the normalization coefficients v_n in (9.26) (yielding $\|\psi_n\|$) can be calculated in closed form.

In particular, we have

$$(\psi_0, \psi_0) = \int_0^{\infty} \frac{\mathrm{d}v}{\overline{c(\gamma; v)}c(\gamma; v)} = \frac{1}{v_0}. \tag{10.1}$$

Using

$$\overline{c(a_+, a_-, \gamma; v)} = c(a_+, a_-, \gamma; -v), \qquad v \in \mathbb{R},\ (a_+, a_-, \gamma) \in \Pi \tag{10.2}$$

(cf. (3.8)), the c-function definition (7.13) and the reflection formula (3.7), formula (10.1) now takes the explicit form

$$\int_0^{\infty} \frac{\prod_{\mu=0}^{3} G(v + \mathrm{i}\gamma_\mu) G(-v + \mathrm{i}\gamma_\mu)}{G(2v - \mathrm{i}a) G(-2v - \mathrm{i}a)} \mathrm{d}v = (a_+ a_-)^{1/2} \frac{\prod_{0 \leqslant \mu < \nu \leqslant 3} G(\mathrm{i}\gamma_\mu + \mathrm{i}\gamma_\nu + \mathrm{i}a)}{G(\mathrm{i}\sum_{\mu=0}^{3} \gamma_\mu + 3\mathrm{i}a)}. \tag{10.3}$$

This identity may be viewed as a hyperbolic counterpart of the 'trigonometric' Askey–Wilson weight function integral [1,4]. Indeed, provided the latter is expressed in terms of the trigonometric gamma function from [9], it has essentially the same appearance as (10.3). To demonstrate this, we reparametrize [4, (6.1.1)–(6.1.2)] by setting

$$q \to \mathrm{e}^{-2a}, \quad a \to \mathrm{e}^{-\alpha_0 - a}, \quad b \to \mathrm{e}^{-\alpha_1 - a}, \quad c \to \mathrm{e}^{-\alpha_3 - a}, \quad d \to \mathrm{e}^{-\alpha_3 - a}. \tag{10.4}$$

Then the Askey–Wilson integral can be written

$$\int_0^{\pi} \frac{\prod_{\mu=0}^{3} G_{\mathrm{t}}(\theta + \mathrm{i}\alpha_\mu) G_{\mathrm{t}}(-\theta + \mathrm{i}\alpha_\mu)}{G_{\mathrm{t}}(2\theta - \mathrm{i}a) G_{\mathrm{t}}(-2\theta - \mathrm{i}a)} \mathrm{d}\theta = 2\pi G_{\mathrm{t}}(\mathrm{i}a) \frac{\prod_{0 \leqslant \mu < \nu \leqslant 3} G_{\mathrm{t}}(\mathrm{i}\alpha_\mu + \mathrm{i}\alpha_\nu + \mathrm{i}a)}{G_{\mathrm{t}}(\mathrm{i}\sum_{\mu=0}^{3} \alpha_\mu + 3\mathrm{i}a)}. \tag{10.5}$$

Here we have

$$G_{\mathrm{t}}(\theta) \equiv G_{\mathrm{trig}}(1/2, 2a; \theta) \tag{10.6}$$

with $G_{\mathrm{trig}}(r, a; z)$ the trigonometric gamma function from [9]. To check that [4, (6.1.1)] can indeed be written as (10.5), the duplication formula for the trigonometric gamma function (cf. [9, (3.148)]) should be used to expand the denominator on the lhs of (10.5).

11. Parameter shifts

The factor $\prod_j G(\mathrm{i}s_j)$ in (4.4) ensures the simple normalization $R(\mathbf{c}; v, \mathrm{i}\hat{c}_0) = 1$, cf. (6.1)–(6.3). Due to its v- and $\hat{v}$-independent zeros and poles, however, this normalization factor is awkward for several other purposes. The renormalized R-function R_{r} (given by (7.6)) does not have this drawback. As will become clear shortly, this is only one of the reasons why we focus on $R_{\mathrm{r}}(a_+, a_-, \gamma; v, \hat{v})$ in the account that follows.

As mentioned at the end of Section 5, to date the general theory of linear AΔEs leaves many natural questions unanswered. In particular, the specific context of independent commuting AΔOs leads to problems concerning joint eigenfunctions about which little appears to be known. Specializing to the commuting Askey–Wilson-type AΔOs $A_+(\mathbf{c}(\gamma); v)$ and $A_-(I\mathbf{c}(\gamma); v)$ (given by (5.1)–(5.3) and (7.2)) we now assume until further notice

$$a_+/a_- \notin \mathbb{Q}. \tag{11.1}$$

The only meromorphic functions with periods $\mathrm{i}a_+$ and $\mathrm{i}a_-$ are then the constants. This leads to the conjecture that the space of meromorphic joint solutions to the AΔEs

$$A_+(\mathbf{c}(\gamma); v)F(v) = 2c_+(2\hat{v})F(v), \tag{11.2}$$

$$A_-(I\mathbf{c}(\gamma); v)F(v) = 2c_-(2\hat{v})F(v) \tag{11.3}$$

is at most two-dimensional. (Since it contains $R_{\mathrm{r}}(a_+, a_-, \gamma; v, \hat{v})$, it is at least one-dimensional.)

We are not aware of a proof of this conjecture. Under an additional assumption, however, it can indeed be proved. To be specific, the assumption is that two joint solutions $F^{(\pm)}(v)$ exist satisfying

$$\lim_{\mathrm{Im}\, v \to \infty} F^{(+)}(v)/F^{(-)}(v) = 0, \quad \mathrm{Re}\, v \in I, \tag{11.4}$$

where I is some interval, and the proof can be found in [16, Section 1].

This result plays a pivotal role in the sequel. We first exploit it for the special case $\gamma = \gamma_{\mathrm{f}}$ to deduce (7.34). To begin with, it is evident from the first paragraph of Section 7 that for $\mathrm{Re}\, \hat{v} > 0$ (say), the plane waves

$$F_{\hat{v}}^{(\pm)}(v) \equiv \exp(\pm \mathrm{i}\alpha v \hat{v}) \tag{11.5}$$

are joint solutions to (11.2)–(11.3) satisfying the extra assumption (11.4) for any $I \subset \mathbb{R}$. Thus the joint solution space is two-dimensional, and so we have

$$R_{\mathrm{r}}(\gamma_{\mathrm{f}}; v, \hat{v}) = p_+(\hat{v})F_{\hat{v}}^{(+)}(v) + p_-(\hat{v})F_{\hat{v}}^{(-)}(v) \tag{11.6}$$

for certain prefactors $p_\pm(\hat{v})$. Now R_{r} is even in v, implying $p_+(\hat{v}) = p_-(\hat{v}) = p(\hat{v})$. Hence we obtain

$$R_{\mathrm{r}}(\gamma_{\mathrm{f}}; v, \hat{v}) = 2p(\hat{v}) \cos(\alpha v \hat{v}) \tag{11.7}$$

Finally, R_{r} has leading asymptotics $2\cos(\alpha v \hat{v})$ for $v \to \infty$ (since $R_{\mathrm{r}} = \mathscr{E}$ for $\gamma = \gamma_{\mathrm{f}}$), so $p(\hat{v})$ equals 1 and (7.34) follows for parameters a_+, a_- obeying (11.1). Since such parameters are dense in $(0, \infty)^2$, we deduce (7.34).

Formula (7.34) can be viewed as an explicit evaluation of the integral on the rhs of (7.7) for the special case $\gamma = \gamma_{\mathrm{f}}$. From the perspective of understanding the R_{r}-function, a principal result of [20] is that this integral admits explicit evaluation as an ‘elementary’ function (in a sense defined shortly) for (a_+, a_-, γ) in a subset Π_{el} of Π (9.1) that is *dense* in Π. Dropping assumption (11.1) from now on, there are two equivalent definitions of Π_{el} that are both useful.

Embarking on the first one, we define a subset $\mathscr{Z}$ of $\mathbb{Z}^4 \times \mathbb{Z}^4$ by requiring that for $(M, N) \in \mathscr{Z}$ the four pairs (M_μ, N_μ), $\mu \in \{0, 1, 2, 3\}$, are distinct mod(2); equivalently, the pairs are of the form (even, even), (odd, odd), (even, odd), (odd, even). Then Π_{el} can be defined by

$$\Pi_{\mathrm{el}} \equiv \left\{ (a_+, a_-, p) \in \Pi \mid p = \frac{1}{2} \sum_{v=0}^{3} (M_v a_- + N_v a_+) e_v, (M, N) \in \mathscr{Z} \right\}, \tag{11.8}$$

where $e_0, \ldots, e_3$ are the canonical basis vectors of $\mathbb{R}^4$.

It is clear from this definition that Π_{el} is invariant under the Weyl group W of the Lie algebra D_4. In the sequel the weight lattice $\mathscr{P}$ of the latter is crucial. For our present purposes, it suffices to characterize $\mathscr{P}$ as the lattice generated by $e_0, \ldots, e_3$ and the row vectors

$$r_0 \equiv (1, 1, 1, 1)/2, \quad r_1 \equiv (1, 1, -1, -1)/2, \quad r_2 \equiv (1, -1, 1, -1)/2, \quad r_3 \equiv (1, -1, -1, 1)/2 \tag{11.9}$$

of the matrix J (cf. (4.11)). Note that we have

$$J r_\mu = e_\mu, \quad J e_\mu = r_\mu, \quad \mu = 0, 1, 2, 3 \tag{11.10}$$

so that

$$J\mathscr{P} = \mathscr{P}. \tag{11.11}$$

The second definition now reads

$$\Pi_{\mathrm{el}} \equiv \{ (a_+, a_-, p) \in \Pi \mid p = w(\gamma_{\mathrm{f}}) + a_- \lambda_- + a_+ \lambda_+, w \in W, \lambda_\pm \in \mathscr{P} \} \tag{11.12}$$

(Noting (11.8) entails γ_{f} belongs to Π_{el}, cf. (7.31), the equivalence of the two definitions is readily verified.) In view of (7.12) and (11.11), the second definition (11.12) implies

$$(a_+, a_-, p) \in \Pi_{\mathrm{el}} \quad \Leftrightarrow \quad (a_+, a_-, \hat{p}) \in \Pi_{\mathrm{el}}. \tag{11.13}$$

From now on, we call a function

$$\rho(e_+(v), e_-(v), e_+(\hat{v}), e_-(\hat{v})), \quad e_\delta(y) \equiv \exp(\pi y / a_\delta), \quad \delta = +, - \tag{11.14}$$

that has rational dependence on its four arguments a *hyperbolic* function. Likewise, we reserve the term *elementary* function for functions of the form

$$\sum_{\sigma=+,-} \rho^{(\sigma)}(e_+(v), e_-(v), e_+(\hat{v}), e_-(\hat{v})) \exp(\mathrm{i}\sigma\alpha v \hat{v}), \tag{11.15}$$

where the coefficients $\rho^{(\pm)}$ of the plane waves are hyperbolic. (Observe that the coefficients of an elementary function are uniquely determined.)

To appreciate the special character of parameters in Π_{el}, we fix $(a_+, a_-, \gamma) \in \Pi_{\mathrm{el}}$ and begin by showing that the two c-functions $c(a_+, a_-, \gamma; v)$ and $c(a_+, a_-, \hat{\gamma}; \hat{v})$ are hyperbolic. Thanks to (11.13), we need only consider the first one. Recalling (7.13) and the duplication formula (3.10), we can invoke the first definition (11.8) of Π_{el} to infer that $c(a_+, a_-, \gamma; v)$ is the product of four functions of the form

$$\frac{G(a_+, a_-; w + \mathrm{i}k a_+ + \mathrm{i}l a_-)}{G(a_+, a_-; w)}, \quad k, l \in \mathbb{Z} \tag{11.16}$$

with $w=v+\mathrm{i}a, v+\mathrm{i}a_-/2, v+\mathrm{i}a_+/2, v$. In view of the G-AΔEs (3.4)–(3.5), each of these is hyperbolic, so $c(a_+,a_-,\gamma;v)$ is hyperbolic, as asserted.

Recalling (7.14), we now see that for parameters in Π_{el}, elementarity of R_{r} is equivalent to elementarity of $\mathscr{E}$. From (7.17) and (11.12) we also deduce that to prove elementarity of R_{r} on Π_{el}, we need only show elementarity for parameters of the form

$$(a_+,a_-,\gamma_{\mathrm{f}}+a_-\lambda_-+a_+\lambda_+),\quad \lambda_\pm\in\mathscr{P}. \tag{11.17}$$

This can be achieved via the parameter shifts of [20], starting from the free case $(a_+,a_-,\gamma_{\mathrm{f}})$, where elementarity of R_{r} is plain from (7.34).

In order to detail this, we define 16 AΔOs

$$S_\delta^{(r_0)}(y)\equiv\frac{-\mathrm{i}}{2s_\delta(2y)}(T^y_{\mathrm{i}a_{-\delta}/2}-T^y_{-\mathrm{i}a_{-\delta}/2}), \tag{11.18}$$

$$S_\delta^{(-r_0)}(\gamma;y)\equiv\frac{-\mathrm{i}}{2s_\delta(2y)}\left(\prod_{\mu=0}^{3}2c_\delta(y-\mathrm{i}\gamma_\mu)\cdot T^y_{\mathrm{i}a_{-\delta}/2}-\prod_{\mu=0}^{3}2c_\delta(y+\mathrm{i}\gamma_\mu)\cdot T^y_{-\mathrm{i}a_{-\delta}/2}\right), \tag{11.19}$$

$$S_\delta^{(-r_k)}(\gamma;y)\equiv\frac{-\mathrm{i}}{2s_\delta(2y)}(4c_\delta(y-\mathrm{i}\gamma_0)c_\delta(y-\mathrm{i}\gamma_k)T^y_{\mathrm{i}a_{-\delta}/2}-(\mathrm{i}\to-\mathrm{i})),\quad k=1,2,3, \tag{11.20}$$

$$S_\delta^{(r_k)}(\gamma;y)\equiv\frac{-\mathrm{i}}{2s_\delta(2y)}(4c_\delta(y-\mathrm{i}\gamma_l)c_\delta(y-\mathrm{i}\gamma_m)T^y_{\mathrm{i}a_{-\delta}/2}-(\mathrm{i}\to-\mathrm{i})), \tag{11.21}$$

where $\{k,l,m\}=\{1,2,3\}$. They satisfy 32 shift relations

$$S_\delta^{(\varepsilon r_\mu)}(\gamma;y)A_+(\mathbf{c}(\gamma);y)=A_+(\mathbf{c}(\gamma)+\varepsilon a_{-\delta}r_\mu;y)S_\delta^{(\varepsilon r_\mu)}(\gamma;y), \tag{11.22}$$

$$S_\delta^{(\varepsilon r_\mu)}(\gamma;y)A_-(I\mathbf{c}(\gamma);y)=A_-(I\mathbf{c}(\gamma)+\varepsilon a_{-\delta}Ir_\mu;y)S_\delta^{(\varepsilon r_\mu)}(\gamma;y) \tag{11.23}$$

and 16 identities compatible with their shift features:

$$S_+^{(-\varepsilon r_\mu)}(\gamma+\varepsilon a_-r_\mu;y)S_+^{(\varepsilon r_\mu)}(\gamma;y)=A_+(\mathbf{c}(\gamma);y)+2c_+(2\mathrm{i}\hat{\gamma}_\mu+\mathrm{i}\varepsilon a_-), \tag{11.24}$$

$$S_-^{(-\varepsilon r_\mu)}(\gamma+\varepsilon a_+r_\mu;y)S_-^{(\varepsilon r_\mu)}(\gamma;y)=A_-(I\mathbf{c}(\gamma);y)+2c_-(2\mathrm{i}\hat{\gamma}_\mu+\mathrm{i}\varepsilon a_+). \tag{11.25}$$

(In these formulas, we have $\mu=0,1,2,3$ and $\varepsilon,\delta=+,-$. We point out that in Eqs. (2.14)–(2.16) of [20] we forgot to include the transposition I in A_-.) Moreover, all of the shift commutators save those following from (11.24)–(11.25) vanish:

$$\begin{aligned}&S_{\sigma'}^{(-\varepsilon' r_{\mu'})}(\gamma+\varepsilon a_{-\sigma}r_\mu;y)S_\sigma^{(\varepsilon r_\mu)}(\gamma;y)-S_\sigma^{(\varepsilon r_\mu)}(\gamma-\varepsilon' a_{-\sigma'}r_{\mu'};y)S_{\sigma'}^{(-\varepsilon' r_{\mu'})}(\gamma;y)\\&\quad=4\varepsilon\delta_{\sigma\sigma'}\delta_{\mu\mu'}\delta_{\varepsilon\varepsilon'}s_\sigma(2\mathrm{i}\hat{\gamma}_\mu)s_\sigma(\mathrm{i}a_{-\sigma})\end{aligned} \tag{11.26}$$

(Here we have $\sigma,\sigma',\varepsilon,\varepsilon'=+,-$ and $\mu,\mu'=0,1,2,3$.)

The proofs of relations (11.22)–(11.26) consist of long, but routine calculations, using symmetries wherever possible, cf. [20] Section 2. To establish the action of the shifts on R_{r}, though, we need a far more conceptual reasoning. Skipping this for the moment, the result reads

$$S_{\delta}^{(-r_{\mu})}(\gamma; v)R_{\mathrm{r}}(\gamma; v, \hat{v}) = R_{\mathrm{r}}(\gamma - a_{-\delta}r_{\mu}; v, \hat{v}), \tag{11.27}$$

$$S_{\delta}^{(r_{\mu})}(\gamma; v)R_{\mathrm{r}}(\gamma; v, \hat{v}) = [2c_{\delta}(2\hat{v}) + 2c_{\delta}(2\mathrm{i}\hat{\gamma}_{\mu} + \mathrm{i}a_{-\delta})]R_{\mathrm{r}}(\gamma + a_{-\delta}r_{\mu}; v, \hat{v}). \tag{11.28}$$

By (11.10) and the self-duality relation (7.11), this implies

$$S_{\delta}^{(-r_{\mu})}(\hat{\gamma}; \hat{v})R_{\mathrm{r}}(\gamma; v, \hat{v}) = R_{\mathrm{r}}(\gamma - a_{-\delta}e_{\mu}; v, \hat{v}), \tag{11.29}$$

$$S_{\delta}^{(r_{\mu})}(\hat{\gamma}; \hat{v})R_{\mathrm{r}}(\gamma; v, \hat{v}) = [2c_{\delta}(2v) + 2c_{\delta}(2\mathrm{i}\gamma_{\mu} + \mathrm{i}a_{-\delta})]R_{\mathrm{r}}(\gamma + a_{-\delta}e_{\mu}; v, \hat{v}). \tag{11.30}$$

Taking (11.27)–(11.30) for granted, it is easy to deduce elementarity of R_{r} for the parameters (11.17). Indeed, it is clear from their definition that the 32 shifts featuring in (11.27)–(11.30) leave the space of elementary functions invariant. Now R_{r} is elementary for $\gamma = \gamma_{\mathrm{f}}$ (as shown above), and the square bracket factors in (11.28) and (11.30) are hyperbolic. Hence it follows recursively that R_{r} is elementary for parameters (11.17). (Recall $\mathscr{P}$ is generated by translations over e_{μ} and r_{μ}.) Therefore, R_{r} and $\mathscr{E}$ are elementary on Π_{el}, as announced.

Obviously, the shift actions (11.27)–(11.30) are compatible with (11.22)–(11.26) and the eigenfunction characteristics of R_{r}. But we have not found a proof of these formulas that involves solely the algebraic relations (11.18)–(11.26) and the eigenfunction features. In this connection we would like to point out that the integral representation (7.7) defining R_{r} appears of no help: acting with the shifts on the integrand yields no clue as to why (11.27)–(11.30) should hold true.

We proceed to sketch the proof of (11.27)–(11.28), cf. [20] Section 3. It involves auxiliary functions and γ-values

$$F_{M,N}^{(\pm)}(v, \hat{v}), \quad \gamma(M, N) \equiv \gamma_{\mathrm{f}} + \sum_{v=0}^{3}(M_{v}a_{-} + N_{v}a_{+})r_{v}, \quad M, N \in \mathbb{Z}^{4}. \tag{11.31}$$

The former are generated from the plane waves

$$F_{0,0}^{(\pm)}(v, \hat{v}) \equiv \exp(\pm \mathrm{i}\alpha v\hat{v}) \tag{11.32}$$

by acting solely with the 16 shifts $S_{\delta}^{(\varepsilon r_{\mu})}(\cdot; v)$ in a stepwise fashion. Therefore their general structure can be determined, cf. [20, pp. 490–491].

Requiring again an irrational quotient a_{+}/a_{-}, the functions $F_{M,N}^{(\pm)}(v, \hat{v})$ now play the same role as the plane waves $F_{\hat{v}}^{(\pm)}(v)$ in the above argument proving (7.34). Indeed, it readily follows from their definition that for sufficiently large Re $\hat{v}$ and I of the form (Λ, ∞) with Λ sufficiently large, they satisfy (11.4). Since they are also joint solutions to (11.2) and (11.3) with $\gamma = \gamma(M, N)$, we deduce as before (exploiting evenness features)

$$R_{\mathrm{r}}(\gamma(M, N); v, \hat{v}) = p_{M,N}(\hat{v}) \sum_{\sigma=+,-} F_{M,N}^{(\sigma)}(v, \hat{v}). \tag{11.33}$$

A suitable use of the shifts, combined with the known $v \to \infty$ asymptotics of the relevant functions and shifts, yields recurrence relations for the prefactors $p_{M,N}(\hat{v})$. (This step requires again substantial calculations.) Using $p_{0,0}(\hat{v})=1$, the shift relations (11.27)–(11.28) now follow for Re $\hat{v}$ sufficiently large, a_+/a_- irrational, and γ of the form $\gamma(M,N)$ (11.31). By analyticity, they are then valid for $(a_+,a_-,\gamma)\in\Pi$ and $v,\hat{v}\in\mathbb{C}$, and so the proof is complete.

We conclude this section with some remarks. First, the presence of the normalization factor $\prod_j G(\mathrm{i}s_j)$ in the R-function renders its shift formulas slightly more involved. On the other hand, provided a_+/a_- is irrational, this factor takes values in $\mathbb{R}^*$ on Π_{el} that can be determined in closed form. Indeed, for $(a_+,a_-,\gamma)\in\Pi_{\mathrm{el}}$ the quantities $s_j=\gamma_0+\gamma_j+a$ are given by

$$s_j=\tfrac{1}{2}(M_j a_- + N_j a_+),\qquad j=1,2,3,\ \ M_j,N_j\in\mathbb{Z} \tag{11.34}$$

with the parity of the three pairs (M_j,N_j) being (odd, even), (even, odd) and (even, even). (This readily follows from the first definition (11.8) of Π_{el}.) Using the G-AΔEs (3.4)–(3.5), we can therefore calculate $G(\mathrm{i}s_j)$ explicitly, using either $G(0)=1$ or one of evaluations (3.14).

Next, we note that the plane wave summands

$$R_{\mathrm{r}}^{(\pm)}(a_+,a_-,\gamma;v,\hat{v}),\qquad (a_+,a_-,\gamma)\in\Pi_{\mathrm{el}} \tag{11.35}$$

inherit the eigenfunction properties and symmetries of R_{r}, except evenness (7.8): the latter formula implies that $R_{\mathrm{r}}^{(+)}$ and $R_{\mathrm{r}}^{(-)}$ are related by

$$R_{\mathrm{r}}^{(+)}(-v,\hat{v})=R_{\mathrm{r}}^{(-)}(v,\hat{v})=R_{\mathrm{r}}^{(+)}(v,-\hat{v}),\qquad (a_+,a_-,\gamma)\in\Pi_{\mathrm{el}}. \tag{11.36}$$

Finally, let us require once more irrationality of a_+/a_-. Then functions (11.35) span the joint solution space. This suggests that the joint solution space remains two-dimensional for all γ in $\mathbb{R}^4$. In point of fact, though, this is not the case. More precisely, only even linear combinations of $R_{\mathrm{r}}^{(+)}$ and $R_{\mathrm{r}}^{(-)}$ (i.e., multiples of R_{r}) admit continuous interpolation to all of $\mathbb{R}^4$. For the pertinent 'no-go' result, see [19, pp. 532–533].

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