Helicity and \( \alpha \)-effect by current-driven instabilities of helical magnetic fields

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Accepted 2011 February 22. Received 2011 January 3

ABSTRACT
Helical magnetic background fields with an adjustable pitch angle are imposed on a conducting fluid in a differentially rotating cylindrical container. The small-scale kinetic and current helicities are calculated for various field geometries, and shown to have the opposite sign to the helicity of the large-scale field. These helicities and also the corresponding \( \alpha \)-effect scale with the current helicity of the background field. The \( \alpha \)-tensor is highly anisotropic as the components \( \alpha_{\phi \phi} \) and \( \alpha_{z z} \) have opposite signs. The amplitudes of the azimuthal \( \alpha \)-effect computed with the cylindrical 3D magnetohydrodynamics code are so small that the operation of an \( \alpha \Omega \)-dynamo on the basis of the current-driven, kink-type instabilities of toroidal fields is highly questionable. In any case, the low value of the \( \alpha \)-effect would lead to very long growth times of a dynamo in the radiation zone of the Sun and early-type stars of the order of megayears.

Key words: dynamo – instabilities – magnetic fields – stars: magnetic field.

1 INTRODUCTION
No hydromagnetic dynamo can exist driven only by differential rotation (Elsasser 1946), but it is known that such dynamos can exist if the turbulence is helical in the sense that its kinetic helicity

\[
\mathcal{H}_{\text{kin}} = (\mathbf{u} \cdot \text{curl} \mathbf{u}) \tag{1}
\]

and/or its current helicity

\[
\mathcal{H}_{\text{curr}} = \frac{1}{\mu_0 \rho} (\mathbf{b} \cdot \text{curl} \mathbf{b}) \tag{2}
\]

do not vanish. Here \( \mathbf{u} \) and \( \mathbf{b} \) are the fluctuating parts of the flow \( \mathbf{U} \) and magnetic field \( \mathbf{B} \). This condition of non-vanishing helicity is clearly fulfilled if the turbulence is rotating and stratified. In such turbulence, a pseudo-scalar exists which allows the pseudo-scalars (1) and (2) to take finite values. The same is true for linear shear flows where the stratified turbulence in the presence of the shear also can form a kinetic helicity (see Rüdiger & Kitchatinov 2006). The simplest pseudo-scalar is the scalar product \( \mathbf{g} \cdot \mathbf{\Omega} \) with \( \mathbf{g} \) as the gradient vector of the turbulence (or the fluid density) and \( \mathbf{\Omega} \) the rotation vector. In spheres, the gradient vector \( \mathbf{g} \) is mainly radial, so the pseudo-scalar \( \mathbf{g} \cdot \mathbf{\Omega} \) has opposite signs in the two hemispheres, and vanishes at the equator. Because of the close relationship of the helicity to the \( \alpha \)-effect in the mean-field electrodynamics of turbulent media,

\[
\langle \mathbf{u} \times \mathbf{b} \rangle = \alpha \mathbf{B}_0 + \ldots
\]

(3)

(the dots represent higher derivatives of \( \mathbf{B}_0 \)); the above-mentioned sign rules are also the sign rules of the \( \alpha \)-effect, that is, \( \alpha \propto \mathbf{g} \cdot \mathbf{\Omega} \). This sort of \( \alpha \)-effect only exists for inhomogeneous turbulence. In planetary cores, however, and also in laboratory experiments, the only inhomogeneities result from boundary conditions, as the density gradients are negligible. One can show that under the presence of magnetic background fields other inhomogeneities also form pseudo-scalars and, as a consequence, lead to new mechanisms for an \( \alpha \)-effect (e.g. Gellert, Rüdiger & Elstner 2008). In this study, we demonstrate that instabilities due to inhomogeneous magnetic fields also lead to finite values of the helicities (1) and (2), and in accordance with equation (3) also to finite values of \( \alpha \). Indeed, in the presence of electric currents, the simplest existing pseudo-scalar is \( \mathbf{B}_0 \cdot \text{curl} \mathbf{B}_0 \) which does not vanish for helical field geometries. We show that for such background fields the small-scale helicities obtain final values with the opposite sign to the helicity of the background field.

According to the Rayleigh criterion, in the absence of magnetohydrodynamic (MHD) effects, an ideal flow is stable against axisymmetrical perturbations whenever the specific angular momentum increases outwards:

\[
\frac{d}{dR} (R^2 \Omega)^2 > 0, \tag{4}
\]

where \( \Omega \) is the angular velocity and \((R, \phi, z)\) are cylindrical coordinates in a right-handed system. In the presence of an azimuthal
magnetic field \( B_b \), this criterion is modified to
\[
\frac{1}{R^2} \frac{d}{dR} \left( R^2 \Omega \right)^2 - \frac{R}{\mu_0 \rho} \frac{d}{dR} \left( \frac{B_{0,b}}{R} \right)^2 > 0,
\]
where \( \mu_0 \) is the permeability and \( \rho \) is the density (Michael 1954). Note also that this criterion is both necessary and sufficient for (axisymmetric) stability. In particular, all ideal flows can thus be destabilized by azimuthal magnetic fields with the right-handed profiles (steeply increasing outwards) and amplitudes.

On the other hand, for non-axisymmetric modes, one has
\[
\frac{d}{dR} \left( R B_{0,b} \right) < 0
\]
as the necessary and sufficient condition for the stability of an ideal fluid at rest (Tayler 1973). Again, outwardly increasing fields are thus unstable. If equation (6) is violated, the most unstable mode has azimuthal wave numbers of \( m = \pm 1 \).

### 2 EQUATIONS

We are interested in the stability of the background field \( B_b = (0, B_z(R), B_b) \), with \( B_z = \) constant, and the flow \( \mathbf{u}_0 = (0, R \Omega, 0) \). The governing equations for the flow \( \mathbf{U} \) and the field \( \mathbf{B} \) are
\[
\frac{\partial \mathbf{U}}{\partial t} + (\mathbf{U} \cdot \nabla) \mathbf{U} = -\frac{1}{\rho} \nabla p + \nu \Delta \mathbf{U} + \frac{1}{\mu_0 \rho} \nabla \times \mathbf{B} \times \mathbf{B},
\]
\[
\frac{\partial \mathbf{B}}{\partial t} = \nabla \times (\mathbf{U} \times \mathbf{B}) + \eta \Delta \mathbf{B},
\]
and \( \nabla \cdot \mathbf{U} = \nabla \cdot \mathbf{B} = 0 \), where \( p \) is the pressure, \( \nu \) is the kinematic viscosity and \( \eta \) is the magnetic diffusivity. Their ratio is the magnetic Prandtl number
\[
Pm = \frac{\nu}{\eta}.
\]
From now on we drop the subscripts from the large-scale values so that the total flow is \( \mathbf{U} + \mathbf{u} \) and the total field is \( \mathbf{B} + \mathbf{b} \). The stationary background solution is
\[
\Omega = \alpha_\Omega + \frac{b_z}{R^2}, \quad B_z = a_b R + \frac{b_B}{R},
\]
where \( \alpha_\Omega, b_z, a_b \) and \( b_B \) are constants defined by
\[
\alpha_\Omega = \frac{\Omega_0 \mu_0 - \hat{\eta}^2}{1 - \hat{\eta}^2}, \quad b_z = \frac{\Omega_0 R_a^2 - 1 - \mu_0}{1 - \hat{\eta}^2},
\]
\[
a_b = \frac{B_{0,b} \hat{\eta} (\mu_b - \hat{\eta})}{R_a} \quad \text{and} \quad b_B = \frac{B_{0,b} R_a}{1 - \hat{\eta}^2} - \frac{1 - \mu_b \hat{\eta}}{1 - \hat{\eta}^2},
\]
respectively, with
\[
\hat{\eta} = \frac{R_a}{R_{out}}, \quad \mu_0 = \frac{\Omega_{out}}{\Omega_0}, \quad \mu_b = \frac{B_{out}}{B_{in}}.
\]

The inner value \( B_{in} \) is normalized by the uniform vertical field, that is,
\[
\beta = \frac{B_{in}}{B_0}.
\]

For our standard profile \( \mu_b = 1 \), we have \( B_0 \cdot \nabla \cdot B = 2 \beta B_{in}^2 / \rho R_{in} \) for the helicity of the background field. For a fixed toroidal field amplitude, this quantity scales as \( \beta^{-1} \):
\[
B_0 \cdot \nabla \cdot B = \frac{2 B_{in}^2}{3 \beta R_{in}}.
\]

The sign of \( \beta \) determines the sign of the current helicity. If the toroidal field is due to the interaction of a poloidal field with a differential rotation with negative shear, then \( \beta \) is negative and vice versa. Interchanging \( \pm \beta \) simply interchanges left- and right-handed spirals, \( m \rightarrow -m \).

As usual, the toroidal field amplitude is measured by the Hartmann number and the global rotation by the Reynolds number, that is,
\[
Ha = \frac{B_{in} D}{\sqrt{\mu_0 \rho \nu}}, \quad Re = \frac{\Omega_{in} D^2}{\nu}.
\]

\( D = R_{out} - R_{in} \) is used as the unit of length, \( \nu D \) as the unit of velocity and \( B_{in} \) as the unit of the azimuthal fields. Frequencies, including the rotation \( \Omega \), are normalized with the inner rotation rate \( \Omega_{in} \). The Lundquist number \( S \) is defined by \( S = \sqrt{Pm \, Ha} \). The magnetic-diffusion frequency is \( \omega_\eta = \eta D^2 \) and then the Alfvén frequency \( \Omega_\Lambda = \omega_\eta \) is
\[
\Omega_\Lambda = \frac{B_{in}^2}{\mu_0 \rho D}.
\]

Throughout this paper, numerical values of helicities are given in units of \( \Omega_\Lambda^2 D \). In this notation, the helicity (14) of the background field can be written as
\[
\frac{1}{\mu_0 \rho} B_0 \cdot \nabla \times B = \frac{2}{3 \beta} \Omega_\Lambda^2 D \frac{D}{R_{in}} \simeq \frac{\Omega_\Lambda^2 D}{\beta}.
\]

The boundary conditions associated with the perturbation equations are no-slip for \( \mathbf{u} \) and perfectly conducting for \( \mathbf{b} \), at both \( R = R_a \) and \( R = R_{out} \), where we fix \( R_{out} = 2 R_{in} \), that is, \( \hat{\eta} = 0.5 \). The computational domain is periodic in \( \phi \). The non-linear MHD code used for the solution of equations (7) and (8) has been described in detail by Gellert, Rüdiger & Fournier (2007) (see also Fournier et al. 2004, 2005).

### 3 RESULTS

Fig. 1 shows the growth rates for a purely toroidal field (\( \beta = \infty \)), and no rotation. We see that beyond the critical Lundquist number, the growth rate is essentially linear, that is,
\[
\omega_{gr} \propto \Omega_\Lambda,
\]
where \( Pm = 1 \) has the steepest slope, and is thus more unstable than both \( Pm < 1 \) and \( Pm > 1 \).

The azimuthal wavenumber of the modes shown in Fig. 1 is \( m = \pm 1 \). For a purely toroidal field, \( \pm m \), corresponding to left- and right-handed spirals, are degenerate and necessarily have exactly the same growth rate curves. See also Hollerbach et al. (2009) who obtained the same effect in magnetorotational instabilities and Rüdiger, Kitchatinov & Elstner (in preparation) who consider instabilities of toroidal fields in spheres.

We next consider the non-linear equilibration of these modes. As Fig. 2 shows, even though \( m = \pm 1 \) are degenerate, the equilibrated
solutions do not consist of equal mixtures of both modes. Instead, either the left- or the right-handed mode wins out and completely suppresses the other. Which mode one obtains depends on the precise initial conditions. If these already favour one mode, then (not surprisingly) that one wins, but if the initial condition is evenly balanced between the two modes, it is ultimately just numerical noise that determines which mode wins. Eventually though one mode always wins; the solution consisting of an equal mixture of both is unstable.

Spontaneous parity-breaking bifurcations of this type are well known in classical, non-magnetic Taylor–Couette flow (e.g. Hoffmann et al. 2009, Almeier et al. 2010, and reference therein), but are also unknown in MHD problems. To the best of our knowledge, the only other example is in the very recent work by Chatterjee et al. (2010). Given the importance of the helicity in mean-field dynamics, any effect that generates helicity from an underlying basic state without helicity could be significant.

We next present two series of solutions where $B_0$ includes an axial component ($\beta < \infty$). In contrast to Figs 1 and 2, we also include a differential rotation here. The profiles of the basic-state field and flow are fixed at $\mu_B = 1$ and $\mu_{11} = 0.5$. Their amplitudes are $Ha = 100$ and $Re = 200$ for the first series, and $Ha = 200$ and $Re = 20$ for the second. The first series is thus rotationally dominated ($\Omega > \Omega_A$), whereas the second is magnetically dominated ($\Omega_A > \Omega$).

The astrophysically relevant case is rotationally dominated, which is not the classical realization of the Tayler instability. We have called this instability the Azimuthal Magnetorotational Instability (see Hollerbach et al. 2009).

For both series of runs, Fig. 3 shows the kinetic and magnetic turbulence intensities ($u^2$) and ($b^2$). For sufficiently large $\beta$, its influence is very small; the axial component of $B_0$ is then so weak that it has no further influence. This is not true for small $\beta$, where the axial field starts to dominate. For $\beta < 1$, the kink instability is strongly stabilized (Rüdiger, Schultz & Elstner 2011) and the resulting energies of the perturbations are reduced.

Fig. 4 shows the kinetic (1) and current (2) helicities for the two series of runs. For both series, both helicities have the opposite sign to $\beta$ (see also Käpylä & Brandenburg 2009 for comparison). In stating this result, it is important though to specify carefully the nature of the initial conditions used in each run. For $\beta = O(1)$, the basic state has a sufficiently strong handedness that it forces the instabilities to have a particular parity as well, which as indicated turns out to be opposite to that of the basic state. If one then gradually increases $\beta$, each time using the previous solution as the new initial condition, this parity of the instabilities is preserved all the way to $\beta \to \infty$, where the basic state no longer has a handedness, and both the left- and the right-handed instabilities could exist equally well, as in Fig. 2.

That is, by the time one reaches $\beta = 500$, say, the basic state makes sufficiently little distinction between the left- and right-handed modes that both could exist, but because of the way we have reached $\beta = 500$, we consistently obtain the right-handed mode.
Figure 4. The kinetic and current helicities of the non-axisymmetric perturbations as functions of $\beta$. Top panel: for $\Omega > \Omega_A$ ($Ha = 100$ and $Re = 200$). Bottom panel: for $\Omega_A > \Omega$ ($Ha = 200$ and $Re = 20$). The dash–dotted lines indicate the limits $\pm 6 \times 10^{-4}$ of the kinetic helicity of the left- and right-handed modes in Fig. 2.

However, suppose one does the following experiment now: take the right-handed mode at $\beta = 500$, swap its parity to be left handed and use that as a new initial condition for a series of runs where $\beta$ is now gradually reduced. Eventually, there comes a point where the basic-state handedness is sufficiently great that it no longer allows the instability to have the ‘wrong’ parity and the solution reverts to the right-handed mode. This feature that both the left- and right-handed modes are allowed for sufficiently large $\beta$ (where the degeneracy between the two modes is only weakly broken) but not for smaller $\beta$ (where the degeneracy is strongly broken) is in many ways analogous to an imperfect pitchfork bifurcation.

4 α-EFFECT AND DYNAMO THEORY

We have also calculated the $\alpha$-effect in equation (3) with the same averaging procedure over the azimuth. Because of the complex structure of the background field, it is even possible to determine parts of the tensorial structure of the $\alpha$-tensor. In particular, we are interested in the signs and amplitudes of the $\alpha$-effect in both azimuthal and axial directions. According to the general rule that the azimuthal $\alpha$-effect is anticorrelated with the (kinetic) helicity, we expect the azimuthal $\alpha$-effect to be positive for $\beta > 0$. The expected sign of the axial $\alpha$-effect is not clear. There are theories and simulations leading to $\alpha_{\phi\phi}$ and $\alpha_{zz}$ with opposite signs (see Rädiger & Hollerbach 2004 for an overview). We should not be surprised to find a similar behaviour in the present simulations. It also means that any dynamo with very weak differential rotation cannot be treated with a scalar $\alpha$-effect.

Figs 5 and 6 give the results for slow and rapid rotation. On the basis of equation (3), the dimensionless $\alpha$-effect in the form

$$C_{\alpha} = \frac{\alpha D}{\eta}$$

is plotted for the components $\alpha_{\phi\phi}$ and $\alpha_{zz}$. In both cases, these two components have opposite signs, with $\alpha_{\phi\phi} > 0$ and $\alpha_{zz} < 0$ almost everywhere in the meridional plane. This anticorrelation between the two components is also strongest at the centre of the gap and weakest near the boundaries. It is therefore not caused by the boundaries.

That Figs 5 and 6 show such similar results is surprising and is one of the basic results of this paper. The influence of rotation on $\alpha$ is evidently rather weak. The fact that – contrary to previous results for rotating convection – $\alpha_{\phi\phi}$ is actually smaller for rapid rotation than for slow rotation illustrates just how different these magnetic-induced helicities are from some of the previous results. Finally, Fig. 7 shows how the amplitudes of $\alpha_{\phi\phi}$ vary with $\beta$, being roughly inversely proportional in both cases.

To consider some possible astrophysical implications of these results, imagine a disc dynamo with the dominant field components $B_\phi$ and $B_R$. Dynamo waves of $\alpha \Omega$ type require for self-excitation that the product of equation (19) and

$$C_{\alpha} = -\frac{D^1 d\Omega}{\eta DR}$$


We know from Fig. 7 that $C_a \simeq C/\beta$ with $C \ll 1$, so that equation (22) gives, at least as an order-of-magnitude estimate, the condition

$$C \simeq \frac{|B_R|}{|B_\phi|}$$  (23)

for the self-excitation of a dynamo with the differential rotation and current-driven $\alpha$-effect. For disc dynamos, $B_R$ dominates $B_\phi$, and for spherical dynamos, $B_R$ is comparable to $B_\phi$. In both cases, the condition for self-excitation becomes $C > 1$, which cannot be fulfilled according to Fig. 7, which suggests instead that $C \lesssim 0.05$. The $\alpha$-effect due to the current helicity of the background field appears too small to allow for the operation of an $\alpha \Omega$ dynamo.

Another argument concerns the growth rate of such a dynamo (if it exists at all) in relation to the very long magnetic diffusion times in radiative zones. Assume that for self-excitation $C_a C_\Omega > 1$, then the growth rate $\omega_{\delta \psi \phi}$ is given by

$$\omega_{\delta \psi \phi} = \frac{\eta}{D^2} \sqrt{C_a C_\Omega} \simeq \frac{\eta}{D^2} C \frac{|B_\phi|}{|B_R|}.$$  (24)

Hence, only for $C > 1$ the growth time of the dynamo would be shorter than the magnetic diffusion time $D^2/\eta$, which is known to be of the order of gigayears for the radiative interior of stars.

One can also argue as follows. The relation (24) also reads

$$\omega_{\delta \psi \phi} \simeq \sqrt{\alpha/\Omega},$$  (25)

independent of the magnetic diffusivity. On the other hand, for given $C_a$, equation (25) states

$$\omega_{\delta \psi \phi} \simeq \frac{1}{D} \sqrt{C_a \eta/\Omega},$$  (26)

which for the computed value $C_a \simeq 0.01$ taken from Fig. 7 and $\eta \simeq 500$ cm$^2$ s$^{-1}$ for the solar core leads to values of the order of $10^{-15}$ s$^{-1}$, that is, to growth times of the order of 10 Myr. As it is typical for $\alpha \Omega$ dynamos, their growth times are only slightly shorter than the basic magnetic decay time.

5 SUMMARY

We have shown that the current-driven instability of helical large-scale fields does produce small-scale helicity (kinetic plus current helicity) and even $\alpha$-effects, but the resulting numerical values seem to be too small for the operation of large-scale dynamos in radiative zones of early-type stars.

ACKNOWLEDGMENTS

MG would like to acknowledge support from Deutsche Forschungsgemeinschaft (DFG) within SPP1488.

REFERENCES

Elsasser W. M., 1946, Phys. Rev., 69, 106

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Michael D., 1954, Mathematica, 1, 45

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