Linear magnetoconvection in a rotating spherical shell, incorporating a finitely conducting inner core

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LINEAR MAGNETOCONVECTION IN A ROTATING SPHERICAL SHELL, INCORPORATING A FINITELY CONDUCTING INNER CORE

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The problem of the onset of convection in a rotating spherical shell with an imposed magnetic field is studied. This problem is relevant to understanding the dynamics of the Earth's outer core. The finite conductivity of the inner core is taken into account and no-slip boundary conditions are assumed at the inner-core and core-mantle boundaries. The problem is investigated numerically, using values of the Ekman number down to $10^{-6}$.

Models in which the toroidal magnetic field vanishes near the core-mantle boundary, as expected in the Earth, are considered. We find the preferred non-axisymmetric wavenumber, $m$, of modes proportional to $\exp(im\varphi)$ as a function of Elsasser number $\Lambda$. We also find that toroidal fields with $\Lambda \geq 10$ are unstable due to magnetic instability even when there is no thermal driving, i.e. at zero Rayleigh number. In the range of Elsasser number appropriate to the geodynamo, convective motions in the interior of the outer core in our model have azimuthal velocities which are only weakly dependent on the coordinate parallel to the rotation axis.

We have also compared the fields and fluid velocities arising from our model with those deduced from geomagnetic data, to the extent possible in our very simplified models. We find that solutions with the $m = 2$ mode best resemble published maps of the geomagnetic field at the core surface. Our calculations generally support the hypothesis that large scale convection is occurring in the Earth's outer core.

KEY WORDS: Earth's core, magnetoconvection.

1. INTRODUCTION

The Earth's interior is divided into three regions; the inner core, the outer core, and the mantle. The solid inner core extends to a radius $r_i \approx 1220$ km. Next comes the liquid outer core, extending to a radius $r_o \approx 3480$ km. Finally, the silicate mantle, which extends almost to the Earth's surface at $r \approx 6370$ km, provides the upper bound to the metallic core. It is generally accepted that the Earth's magnetic field is created in the liquid outer core by so-called 'dynamo action' in which the mechanical energy of convective fluid motions is converted into magnetic energy.

The driving force for the dynamo is thought to be the fluid motion generated by compositional convection (Braginsky, 1963; Gubbins et al., 1979; Fearn and Loper,
1981; Loper, 1989). The outer core consists of liquid iron, with a slight admixture of lighter elements. Pure iron then crystallizes out onto the inner core surface, and the lighter elements are released. The buoyancy created thereby drives the compositional convection. The details of the mechanism are not well understood, particularly as regards the boundary condition at the inner core boundary, and so it is usual to consider the related case of thermal convection in the hope that the large scale convection patterns will not differ significantly.

By their very nature the governing equations, being nonlinear and three dimensional, form a complex and challenging problem. Due to this complexity, studies to date have concentrated on two separate aspects of the subject: (1) the mean-field problem, in which the small-scale structure is assumed given in parameterized form, and one is interested in the dynamics of the large-scale magnetic field and fluid flow, and (2) the magnetoconvection problem, in which the large-scale structure is assumed given, and one is interested in the small-scale instabilities that may result. Reviews of mean-field dynamos and magnetoconvection may be found in Fearn (1994) and Proctor (1994). In this paper we consider linear magnetoconvection. The ultimate objective is of course to combine these two separate aspects to obtain a single, self-consistent dynamo model.

The first linear, incompressible, magnetoconvection models (Eltayeb, 1972, 1975) considered planar Cartesian geometry. For these models the direction in which the gravitational force acts is either taken to be aligned with the axis of rotation (modelling regions local to the pole) or perpendicular to the axis of rotation (modelling regions local to the equator). The extension to the compressible case has been studied by Jones et al. (1990). Nonlinear theory has been examined in planar geometry by Roberts and Stewartson (1974, 1975), and in cylindrical geometry by Skinner and Soward (1988, 1990). Annulus models have been considered by Busse (1976, 1983) and Busse and Finocchi (1993). Fearn and co-workers (Fearn 1993, and references contained therein) have considered the related problem of magnetic instability in various geometries.

In spherical geometry two main methods may be followed in the solution of the momentum equation. The first is to neglect viscosity and inertial terms, which are small, and solve the resulting equations—the so-called magnetostrophic approximation. Although this can cause problems in a spherical shell (Hollerbach and Proctor, 1993), it is straightforward in a full sphere. This method has been followed by Fearn (1979a, b) and Fearn and Proctor (1983) for linear theory. For the nonlinear theory considered by Fearn et al. (1994), the mechanism that equilibrates the solutions is as follows: the quadratic interaction of the non-axisymmetric field with itself produces an axisymmetric force, which in general does not satisfy Taylor's (1963) constraint. The resulting (axisymmetric) geostrophic flow then reacts back on the non-axisymmetric field to equilibrate it. See Jones (1991) or Fearn (1994) for reviews of Taylor's constraint and this type of equilibration through the geostrophic flow.

If the viscosity of the fluid is included (but inertial terms neglected) the resulting equations may be solved directly. However, when the viscosity is realistically small and the correct no-slip boundary conditions are applied, boundary layers may be generated. For low viscosity these boundary layers require excessive computation to resolve and thus, if this method is to be successful, it must be shown that the prescribed value of viscosity is sufficiently small to be in the small viscosity limit. Another related method
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(see for example Zhang, 1994) retains both the inertial and viscous terms. Using stress-free boundary conditions, thereby eliminating the Ekman boundary layers, the problem is posed as the determination of eigenvalues resulting from a spectral representation.

In this paper the second method will be used, ignoring inertial terms but including a finite viscosity, with any boundary layers that may develop being explicitly resolved. In addition, we include a finitely conducting inner core. This has not been considered before, but in the context of the above-mentioned mean-field problem Hollerbach and Jones (1993a, b, 1995) have demonstrated that it may be significant, and so we implement it here as well. In Section 2 the basic equations governing linear magnetoconvection are outlined. Section 3 describes the numerical method. The results are given in Section 4 and are discussed in Section 5.

2. EQUATIONS GOVERNING THE LINEAR THEORY

We consider motions in which the inertial terms are small compared to the Coriolis terms, so that the Rossby number is negligibly small. In the Boussinesq approximation the equations governing the evolution of the magnetic field \( B \), the fluid velocity \( U \) and the temperature \( T \), in a reference frame rotating with angular velocity \( \Omega = \Omega \hat{z} \), are then given by (Fearn, 1994)

\[
\hat{t} \times U = -\nabla P + (\nabla \times B) \times B + E \nabla^2 U + q Ra Tr, \tag{2.1}
\]

\[
\frac{\partial B}{\partial t} = \nabla^2 B + \nabla \times (U \times B), \tag{2.2}
\]

\[
\frac{\partial T}{\partial t} = q \nabla^2 T - U \cdot \nabla T + s, \tag{2.3}
\]

\[
\nabla \cdot B = \nabla \cdot U = 0, \tag{2.4}
\]

where \( s \) is a uniformly distributed heat source and \( P \) is the fluid pressure. The above equations are in dimensionless form, the unit of length being \( l_0(= r_0 - r_i) \), where \( r_0 \) and \( r_i \) are the outer core and inner core radii respectively. The unit of time is then \( l_0^2/\eta \), and the unit of magnetic field is \((2\Omega \mu \rho_0 \eta)^{1/2}\).

Note that it is more common in magnetoconvection studies to use the imposed magnetic field as the unit of field strength. However, as this work is a preliminary to a study of self-consistent dynamos, for which there is no imposed field, we prefer to work in terms of \((2\Omega \mu \rho_0 \eta)^{1/2}\). The temperature is non-dimensionalised so that in the absence of convection the temperature distribution is \( T = -r^2/2 \). The assumption of uniform heating in the outer core has been made in a number of previous papers that do not incorporate an inner core. It is made here so that the results may be compared with previous work, although with the addition of an inner core other prescriptions are possible.
The dimensionless parameters in the equations are

\[ E = \frac{v}{2\Omega l_0^2}, \quad q = \frac{\kappa}{\eta}, \quad Ra = \frac{g_0 \beta l_0^2}{2 \Omega \kappa}, \quad (2.5) \]

the Ekman number, the Roberts number, and the modified Rayleigh number respectively. Here \( \eta \) is the magnetic diffusivity, \( v \) is the coefficient of kinematic viscosity, \( \kappa \) is the thermal diffusivity and \( -g_0 r \) is the acceleration due to gravity (taken to act radially inwards). The coefficient of expansion is \( \alpha \) and \( \beta l_0 \) is the unit of temperature.

It is helpful to decompose the magnetic field, fluid flow and temperature into axisymmetric and non-axisymmetric components. In the following the barred variables represent the unperturbed, axisymmetric parts of the physical state and the unbarred variables the corresponding non-axisymmetric perturbations. Thus the magnetic field \( B = B + b \), the flow \( U = \bar{U} + u \), the temperature \( T = \bar{T} + T' \) and the pressure \( P = \bar{P} + p \). Then the perturbation equations we must solve are

\[ \dot{z} \times u = -\nabla p + [(\nabla \times \bar{B}) \times b + (\nabla \times b) \times \bar{B}] + EV^2 u + qRa T' r, \quad (2.6) \]

\[ \frac{\partial b}{\partial t} = \nabla^2 b + \nabla \times [\bar{U} \times b + u \times \bar{B}], \quad (2.7) \]

\[ \frac{\partial T'}{\partial t} = q \nabla^2 T' - \bar{U} \cdot \nabla T' - u \cdot \nabla \bar{T}, \quad (2.8) \]

\[ \nabla \cdot b = \nabla \cdot u = 0. \quad (2.9) \]

At the CMB and the inner-outner core boundary vanishing fluid flow is imposed (i.e. no-slip conditions) and perfect thermal conduction is assumed. The inner and outer core magnetic fields are explicitly calculated from the induction equation (2.7) and are matched at the inner-outner core boundary. The field at the CMB is matched to an external potential solution. This last boundary condition assumes that the mantle is a perfect electrical insulator.

3. THE NUMERICAL METHOD

We make the usual poloidal/toroidal decomposition of the flow and field,

\[ \bar{U} = \nabla \times [\psi \hat{e}_\theta] + v \hat{e}_\phi, \quad (3.1) \]

\[ \bar{B} = \nabla \times [A \hat{e}_\theta] + B \hat{e}_\phi, \quad (3.2) \]

\[ u = \nabla \times [e \hat{e}_\theta] + \nabla \times \nabla \times [f \hat{e}_r], \quad (3.3) \]

\[ b = \nabla \times [g \hat{e}_\theta] + \nabla \times \nabla \times [h \hat{e}_r]. \quad (3.4) \]
Again, the axisymmetric variables $A, B, \psi, \text{ and } v$ are given, prescribed quantities. The non-axisymmetric variables $g, h, e, f$, as well as the temperature $T'$, are expanded in the outer core as

$$
g = \sum_{n=1}^{N_1} \sum_{l=1}^{M_1+2} g_{nl}^{(0)} T_{l-1}^{-1} (x_0) P_{2n+l-2}^{(m)} (\cos \theta) e^{im\phi},$$

$$
h = \sum_{n=1}^{N_1} \sum_{l=1}^{M_1+2} h_{nl}^{(0)} T_{l-1}^{-1} (x_0) P_{2n+l-1}^{(m)} (\cos \theta) e^{im\phi},$$

$$
e = \sum_{n=1}^{N_2} \sum_{l=1}^{M_2+2} e_{nl} T_{l-1}^{-1} (x_0) P_{2n+l-1}^{(m)} (\cos \theta) e^{im\phi},$$

$$
f = \sum_{n=1}^{N_2} \sum_{l=1}^{M_2+4} f_{nl} T_{l-1}^{-1} (x_0) P_{2n+l-2}^{(m)} (\cos \theta) e^{im\phi},$$

$$
T' = \sum_{n=1}^{N_1} \sum_{l=1}^{M_1+2} T'^{n}_{l-1} (x_0) P_{2n+l-2}^{(m)} (\cos \theta) e^{im\phi}.
$$

where $P_{n}^{(m)} (\cos \theta)$ are associated Legendre functions, $T_l$ are Chebyshev polynomials, with $x_0$ being the radial coordinate normalized to $(-1, 1)$ across the spherical shell.

In addition, since we are including a finitely conducting inner core, we must expand $g$ and $h$ in the inner core as

$$
g = \sum_{n=1}^{N_1} \sum_{l=1}^{M_1/2+1} g_{nl}^{(i)} x_i T_{2l-1} (x_i) P_{2n+l-2}^{(m)} (\cos \theta) e^{im\phi},$$

$$
h = \sum_{n=1}^{N_1} \sum_{l=1}^{M_1/2+1} h_{nl}^{(i)} x_i T_{2l-1} (x_i) P_{2n+l-1}^{(m)} (\cos \theta) e^{im\phi},$$

where $x_i = r/r_i$, and $k_1 = 2, k_2 = 1$ for even $m$, $k_1 = 1, k_2 = 2$ for odd $m$ ensure the proper symmetry in $r$. Remember that $m$ is the azimuthal wavenumber of the mode under consideration, and that, because we are only considering the linear problem, we can consider each $m$ in isolation.

Note that in the above expansions we have imposed a particular symmetry about the equator. The prescribed axisymmetric quantities $A, B, \psi, \text{ and } T$ are chosen to satisfy a particular symmetry, with $A, \text{ and } T$ symmetric, and $B$ and $\psi$ antisymmetric. With this choice of axisymmetric quantities, the non-axisymmetric variables then take one of two parities:

(i) $u_r, u_\phi, b_r, T'$ symmetric
$u_\phi, b_\phi$ antisymmetric

(ii) $u_\phi, b_r, b_\phi$ symmetric
$u_r, u_\phi, b_\phi, T'$ antisymmetric.
Surface observations can be used to determine the magnetic field at the CMB (see for example Gubbins and Bloxham, 1987). The observed data suggest that modes with the parity \((i)\) predominate, and that is the parity we have implemented above.

Given these expansions, the equations are then time-stepped until the dominant eigenmode emerges. The details of how the temperature and induction equations are time-stepped, and how the momentum equation is solved at each timestep, are essentially as in Hollerbach (1994b). This time-stepping method is not as efficient as Zhang's (1994) eigenvalue method, but it is more versatile. This will become particularly important when this model is combined with the mean-field model of Hollerbach and Jones, since the resulting non-linear system will have to be time-stepped. Finally, for reasonable convergence times a truncation of \(N_1 = 18, N_2 = 24, M_1 = 18\) and \(M_1 = 36\) has been taken, allowing solutions to be down to a value of \(E \approx 10^{-4}\) (although for some runs, at lower \(E\), higher truncations have been used).

4. RESULTS

The equilibrium toroidal field for most runs was taken to be

\[
B = \Lambda^{1/2} 8 (r - r_i)^2 \left[ 1 - (r - r_i)^2 \right] \sin \theta \cos \theta.
\]

The toroidal field (4.1) vanishes at the inner core boundary as well as the CMB. This is in line with the dynamo models of Hollerbach and Jones (1993a, b), where toroidal field is expelled from the inner core. However, in order to explore whether the results were sensitive to having a field in the inner core some runs were done with the toroidal field

\[
B = \Lambda^{1/2} \frac{512}{81} \left[ \frac{1}{8} + (r - r_i)^2 \right] \left[ 1 - (r - r_i)^2 \right] \sin \theta \cos \theta,
\]

which is non-zero at the inner core boundary, but has the same maximum field-strength as 4.1. A sketch of the fields 4.1 and 4.2 is given in Figure 1. Note that it is only the behaviour of the outer core field that can affect the solution.

The poloidal field and the flow are taken as

\[
A = \Lambda^{1/2} B_p (r - r_i) \left[ 1 - \frac{3}{4} (r - r_i) \right] \sin \theta,
\]

\[
\nu = -\frac{25}{16} R_m \sqrt{5} (r - r_i) \left[ 1 - (r - r_i)^2 \right]^2 \sin \theta,
\]

\[
\psi = 0.
\]

The choice (4.1) and (4.3) for the magnetic field has previously been made by a number of authors (see for example Fearn and Proctor, 1983; Fearn et al., 1994) and corresponds to a basic toroidal component confined to the outer core together with a dipole component.
Figure 1 The profiles of the toroidal magnetic field for a given $\theta$ as a function of radius (normalized to give a maximum field strength of 1). The prescriptions are $B = 8(r - r_i)^2 [1 - (r - r_i)^2] \sin \theta \cos \theta$ (solid line) and $B = \frac{8}{\pi^3} \left[ \frac{1}{4} + (r - r_i)^2 \right] (1 - (r - r_i)^2) \sin \theta \cos \theta$ (dashed line).

The prescription for the fluid flow (see Roberts, 1972; Fearn and Proctor, 1983) represents the form of differential rotation that would produce the toroidal field specified by (4.1) from a poloidal field. The strength of the magnetic field is measured by the Elsasser number $\Lambda$, which in terms of dimensional quantities is $B_0^2/(2\Omega \mu \rho \eta)$, $B_0$ being the (dimensional) maximum strength of the imposed magnetic field, $\mu$ being the permeability and $\rho$ being the density. The strength of the differential rotation is governed by the magnetic Reynolds number $R_m$. Note that while the toroidal field and flow, (4.1) and (4.4), vanish at both the inner core boundary ($r = r_i$) and at the CMB ($r = r_o, r - r_i = 1$) the poloidal field (4.3) vanishes only at the inner core boundary.

For magnetoconvection in the earth’s core the appropriate non-dimensional inner and outer core radii are $r_i = 1/2$ and $r_o = 3/2$ respectively. The Ekman number associated with earth’s outer core is $E \approx 10^{-15}$. Such values would require an excessive number of terms in the expansion series to resolve any boundary layers that may form, and so as noted above we usually only go down to $E \approx 10^{-4}$.

A Roberts number of $q \approx 10^{-5}$ is normally taken. However it has been shown that such a low value can cause severe numerical difficulties for non-zero Reynolds number. Indeed turbulent mixing may, anyway, lead to a higher value. In the following the prescription $q = 1$ has been taken, although the effect of reducing $q$ to more ‘realistic’ values is examined when $R_m = 0$.

The effect of changing $q$ and $E$ when $R_m = B_p = 0$ is now summarized. If there is no differential rotation ($R_m = 0$) varying $q$ has little effect on Rayleigh number, as can be seen in Figure 2. Figure 2 also shows the associated critical frequencies ($\omega$); it is clear that $\omega$ scales so that $\omega/q \approx$ constant. Note also that there is rapid convergence of both
critical Rayleigh number and frequency as $q$ is decreased. At larger values of $\Lambda \approx 10$, the $q$ dependence is complicated by the presence of magnetic instability. The effect of changing the Ekman number depends on the value of the Elsasser number being considered. For small values of $\Lambda$ on reducing $E$ the critical Rayleigh number rapidly increases (Zhang, 1994) while for larger values of $\Lambda$ on reducing $E$ the critical Rayleigh number first decreases before slowly rising to a constant value. The convergence of the critical Rayleigh number and frequency at larger $\Lambda$ as the viscosity is decreased is shown for $m = 1, \Lambda = 10$ and $m = 2, \Lambda = 5$ in Figure 3.

Contour plots of the fluid flow ($u_\varphi, u_\theta$ and $u_z$) are shown, in a meridional plane, in Figure 4 and, in the equatorial plane, in Figure 5. Plots in meridional planes depend on which plane is chosen, but nevertheless features common to all azimuthal angles can be seen, so these plots are still informative. For both figures a number of viscosities are considered corresponding to $E = 10^{-3}, 10^{-4}$ and $10^{-5}$. For small values of $E$ a shear layer is observed to form, for each component of the flow, on a cylinder tangent to the inner core and parallel to the axis of rotation. This behaviour has been noted and examined previously by Hollerbach and Proctor (1993). In this study an excessive truncation is needed to resolve these layers and so for most cases the onset of this shear layer determines the lower limit on $E$. It has been pointed out by Hollerbach (1994a, b) that the presence of a poloidal field reduces and, for a strong enough field, eliminates such an internal shear layer. It is worth noting here that for all the parameters examined the azimuthal component of the fluid flow remains almost two dimensional, and the form of the convection is essentially $2m$ convection rolls parallel to and about the axis of
rotation. Thus the Proudman-Taylor constraint (see for example Chandrasekhar, 1961), valid in the limit of vanishing Lorentz force, has a significant influence for all the parameter values considered. This includes values of $\Lambda$ as high as $\approx 10$. The Lorentz force is, however, responsible for reducing the critical azimuthal wavenumber. In Figure 6 the fluid velocity vectors in the equatorial plane are shown for $E = 10^{-3}, 10^{-4}$ and $10^{-5}$. The convection rolls are seen to be confined to a region close to the CMB as the viscosity is decreased.

Contour plots of the field ($b_r, b_\theta$ and $b_\phi$) and temperature ($T'$) are shown, in a meridional plane, in Figure 7 and, in the equatorial plane, in Figure 8.

For a given Elsasser number ($\Lambda$) and azimuthal wave number ($m$) the critical Rayleigh number ($Ra_{\text{crit}}$) for the onset of instability may be calculated. The smallest temperature gradient to drive an instability is defined by determining the smallest critical Rayleigh number over all possible values of $m$. This will be denoted by $Ra$, and the associated critical azimuthal wave number by $m_c$. Previous work (Eltayeb and Kumar, 1977; Fearn, 1979a, 1979b; Fearn and Proctor, 1983) has suggested that as the Elsasser number rises from small values the critical Rayleigh numbers ($Ra_{\text{crit}}$) fall, for all $m$, to a minimum before rising again for large $\Lambda$. $m_c$ also falls with increasing $\Lambda$, and at $\Lambda = O(1)$ $Ra_c$ has a minimum with $m_c = O(1)$. However, recent papers (Zhang and Fearn, 1994; Zhang, 1994) propose that the critical Rayleigh numbers continue to fall with increasing $\Lambda$. The critical Rayleigh numbers associated with the $m = O(1)$ modes becoming negative for $\Lambda \geq 10$. Such negative Rayleigh numbers are characteristic of a magnetically driven instability. This behaviour is seen in Figure 9 which shows the critical Rayleigh numbers for $m = 1, 2$ and $3$ as $\Lambda$ is varied. The solid and dashed curves
Figure 4  Contour plots of fluid flow in a meridional plane. Components of flow are, from left to right, \( u_r \), \( u_\theta \), and \( u_\phi \). The Ekman numbers are, from top to bottom, \( E = 10^{-3}, 10^{-4} \) and \( 10^{-5} \). Other parameters are \( m = 2 \) and \( \Lambda = 5 \)

correspond to \( E = 10^{-3} \) and \( 10^{-4} \) respectively. As noted above if the viscosity is decreased further the Rayleigh numbers for small \( \Lambda \) tend to infinity while for larger \( \Lambda \) they remain relatively constant. The associated critical frequencies for \( m = 1, 2 \) and \( 3 \), with \( E = 10^{-4} \), are shown in Figure 10.

Also shown in Figure 9 is the curve for \( m = 1 \) with \( E = 10^{-4} \) for the toroidal field (4.2), which has non-zero field at the inner core boundary. As can be seen, the general form of the curve is similar to the case (4.1), indicating that it is the vanishing of the field at the CMB and the antisymmetry of the toroidal field about the equator which is essential for the curve to cut the \( Ra = 0 \) axis. Note, however, that a second mode is found at negative Rayleigh number which travels in the opposite direction to the usual mode. Magnetic instability can depend quite sensitively on the field profile, so that care must be exercised in drawing general conclusions from particular profiles.

Figure 11 shows the effect of including a poloidal field through varying \( B_p \). The curves plotted are for \( m = 1, \Lambda = 10 \) and \( m = 2, \Lambda = 5 \). Although for some parameter
Figure 5  Contour plots of fluid flow in the equatorial plane. Components of flow are $u_x$, top and $u_y$, bottom with, from left to right, $E = 10^{-3}$, $10^{-4}$ and $10^{-5}$. Other parameters are $m = 2$ and $\Lambda = 5$. 
Figure 6  The fluid velocity in the equatorial plane. Ekman numbers are, from left to right, $E = 10^{-3}$, $10^{-4}$ and $10^{-5}$. Other parameters are $m = 2$ and $\Lambda = 5$.

Figure 7  Contour plots of the three components of the magnetic field $b_x$, $b_y$ and $b_z$ (top) and temperature $T^*$ (bottom) in a meridional plane. Parameters are $E = 10^{-5}$, $m = 2$ and $\Lambda = 5$. 
Figure 8  Contour plots of the $b_y$ (left) and $T'$ (right) in the equatorial plane. Parameters are $E = 10^{-4}$, $m = 2$ and $\Lambda = 5$.

Figure 9  The critical Rayleigh numbers for $m = 1, 2, 3$ and $E = 10^{-3}$ (solid curves), $10^{-4}$ (short dashed curves) as $\Lambda$ is varied. Note that for small $\Lambda$, $R_{\text{crit}}$ rapidly increases with decreasing $E$, while for larger $\Lambda$, $R_{\text{crit}}$ remains relatively constant with decreasing $E$. Also shown is the case $m = 1$, $E = 10^{-4}$ (long dashed curve) for the toroidal field (4.2).
Figure 10 The critical frequencies for \( m = 1, 2, 3 \) with \( E = 10^{-4} \) as \( \Lambda \) is varied.

Figure 11 The critical Rayleigh number as the strength of the poloidal field \( B_p \) is increased. Parameters are \( m = 1, \Lambda = 10 \) (solid curve) and \( m = 2, \Lambda = 5 \) (dotted curve) with, in both cases, \( E = 10^{-3} \).
values there is a change in the critical Rayleigh number by up to a factor of five, for reasonable values of $\Lambda$ the variation is by less than a factor of two. The rapid change in the Rayleigh number for $m = 1, \Lambda = 10$ is probably due to the transition between two modes at $B_p \approx 0.3$. This may also be seen in Figure 12 with a point of inflection in the associated frequency at $B_p \approx 0.3$.

The effect of differential rotation may be gauged by including the equilibrium flow (4.4) with a non-zero magnetic Reynolds numbers ($R_m$). As has been previously stated, the inclusion of a shear flow can cause convergence problems for low $q$. Results for $10^{-3} \lesssim q \lesssim 1$ confirm previous findings (Fearn and Proctor, 1983) that for $R_m \lesssim O(q)$ the critical Rayleigh number ($Ra_{\text{crit}}$) is basically constant, while for $R_m > O(q)$ $Ra_{\text{crit}}$ grows linearly with $R_m$. This behaviour is seen in Figure 13 which shows how the Rayleigh number behaves with increasing $R_m$ for $q = 0.01$ and 1.0.

It is also of interest to compare the results of these calculations with observations of the Earth's magnetic field. Data gathered near the Earth's surface from a number of sources can be used to give the radial component of the magnetic field at the CMB. Such data, by for instance Bloxham et al. (1989) or Gubbins and Kelly (1993), may be compared with that predicted by our theoretical model. In our model, the nonaxisymmetric convection pattern drifts on the thermal diffusion timescale. This is long compared to the timescale of the Bloxham et al. data, but may not be long on the timescales of the Gubbins and Kelly work, depending on the effective value of the thermal diffusivity. Gubbins and Kelly argue that convection is locked to the mantle, in which case direct comparison between their work and ours is reasonable. Observations show the flux distribution to be (to a first approximation) symmetric about the equator.

**Figure 12** The critical frequencies as the strength of the poloidal field $B_p$ is increased. Parameters are $m = 1, \Lambda = 10$ (solid curve) and $m = 2, \Lambda = 5$ (dotted curve) with, in both cases, $E = 10^{-3}$. 

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In each of the northern and southern hemispheres two regions of high flux (at a latitude of \( \pm 50^\circ \) and longitudes \( \approx 90^\circ \) W and 90° E) are symmetrically distributed about the pole. A region of low flux lies between each of the highs. The maximum observed field strength is approximately 0.5 mT.

In order to match magnetic field observations the \( m = 2 \) mode must be chosen. We also have to add in the axisymmetric component of the field, which has to be prescribed in our calculation. For this axisymmetric component we take a dipole radial field 0.37 cos \( \theta \) mT at the CMB which corresponds to the dipole component of the Earth’s field at the present time. As only linear theory is considered here, an arbitrary amplitude must be assigned to our computed nonaxisymmetric component of the magnetic field. Clearly, it would be more satisfactory to obtain the amplitude of the field from a self-consistent dynamo calculation, but nevertheless it is still of interest to see whether the form of our computed field bears any resemblance to the observed field. Taking \( \Omega = 7.3 \times 10^{-5} \) rad s \(^{-1} \), \( \rho_0 = 10^4 \) kg m \(^{-3} \), \( \mu = 4\pi 10^{-7} \) H m \(^{-3} \) and \( \eta = 2.7 \) m \(^2\) s \(^{-1} \) then gives \( (2\Omega \mu \rho_0 \eta)^{1/2} \approx 2.22 \) mT. Adding the observed axisymmetric dipole radial field to our computed nonaxisymmetric radial field gives \( B_r = 3.7 \cos \theta + 2.22 \) Ab mT. Here \( A \) is the arbitrary amplitude due to the linear treatment of this paper. We have chosen the amplitude \( A \) by taking that value which gives the best fit of \( B_r \) at the CMB to the magnetic flux data of Gubbins and Kelly (1993), thus allowing field strengths to be calculated. One further parameter must be chosen, the longitudinal phase of the nonaxisymmetric component, and this again is determined by comparison with observations.

**Figure 13** The critical Rayleigh number for \( q = 0.01 \) (x) and \( q = 1 \) (+) as the Reynolds number \( (R_m) \) is varied. Other parameters are \( m = 2, B_r = 0.1, \Lambda = 1 \) and \( E = 10^{-5} \).

\[ q = 0.01 \quad q = 1.0 \]
Figure 14 The radial component of the magnetic field at the CMB, plotted here for the toroidal field of (4.1). The arbitrary amplitude from linear theory and an arbitrary phase shift in longitude have been chosen to best match the observed flux. Contours are at 0.05 mT intervals with positive, negative, and zero contours represented by thin solid, dashed, and thick solid lines respectively. Different toroidal field profiles have little effect on the overall pattern or field strengths.
Figure 14 shows the contour plot of the magnetic field at the CMB (here for the toroidal field profile shown in Figure 15). Each contour represents a 0.05 mT interval. Here a value of $A = 5$ has been taken and no axisymmetric differential rotation was imposed. For all of the cases considered here, choosing the amplitude to match the observed magnetic field at the CMB always gives a similar pattern and leads to reasonable values for the maximum field strength. The picture here is to be compared with Figure 3 of Gubbins and Kelly (1993). That Figure 14 is generally comparable with their figure is not perhaps too surprising as the amplitude and phase have been chosen to give the best fit; however, there are similarities such as the latitude of the field maxima which cannot be explained simply by the adjusting of the parameters. This is a robust feature and probably depends mainly on the radius ratio of the inner and outer cores. The fact that the peak fields occur at latitudes $\approx \pm 50^\circ$ both in the theory and the observations provides some support for the hypothesis that large scale convection is occurring in the outer core.

Once the value of the amplitude $A$ and the longitudinal phase is known the velocity field at different radii may be determined. Taking $\eta/l_o = 37.5 \text{ m yr}^{-1}$ gives a flow velocity of 37.5 Au. For no-slip boundary conditions the flow velocity must vanish at the CMB. The secular variation is thus taken to be due to the flow just below the Ekman layer. It can be noted from the velocity eigenfunctions that once below the Ekman layer the amplitudes of the velocities remain relatively constant. Incidentally this behaviour with depth has been inferred from observations, see for example Bloxham and Jackson (1991). The flow patterns thus calculated may be compared with the observed secular variation, see for example Bloxham et al. (1989) and Bloxham (1992). Although, as pointed out by Bloxham and Jackson (1991), there are many

![Figure 15](diagram.png)

**Figure 15** The profile of the toroidal magnetic field for a given $\theta$ as a function of radius (normalized to give a maximum field strength of 1). The prescription is $B = \frac{4\pi}{120}(r - r_i)[1 - (r - r_i)]^3 \sin \theta \cos \theta$. 


Figure 16. The fluid flow at a radius of 1.4 (in dimensionless units) plotted here for the toroidal field profile of Figure 15. The maximum velocity is \( \approx 23 \text{ km yr}^{-1} \) with an RMS value of \( \approx 11 \text{ km yr}^{-1} \). Different toroidal field profiles have little effect on the overall pattern but, in order to get observed velocities, the field must have a maximum close to the inner-outer core boundary. Note again that there is the same arbitrary phase shift in longitude.
uncertainties associated with this data some consistency in the overall flow pattern and velocities can be determined.

Observations show a number of regions of inflow and outflow in the equatorial region, along with up to four convection rolls situated approximately at latitudes of ±45−55°. The observational evidence for the four rolls is clearer in the northern hemisphere; there is some evidence that the axisymmetric component of the azimuthal flow may be larger in the southern hemisphere, altering the flow pattern. Values for the maximum and RMS velocity seem basically independent of the method used to treat the data, and are typically 25 km yr⁻¹ and 11 km yr⁻¹ respectively.

Since the axisymmetric component of the azimuthal flow is not very accurately determined from observations, we have not added any such component in making the comparison. The basic flow pattern is fairly independent of the specific model considered. However, with toroidal field given by (4.1) the amplitude of the velocities found from our model were considerably smaller than the observed flows underneath the CMB. Varying \( A \) within plausible ranges did not correct this disagreement. There may of course be many reasons for this lack of agreement, but the fact that the flow pattern agreed although the amplitude did not led us to consider one simple hypothesis; instead of using the toroidal field distribution given by 4.1, which is plotted in Figure 1, we tried the field distribution

\[
B = A^{1/2} \frac{3125}{128} (r - r_i) [1 - (r - r_i)]^4 \sin \theta \cos \theta
\]  

which is plotted in Figure 15. This distribution of toroidal field peaks much nearer the ICB than the CMB and would be appropriate if the dynamo action is centred near the ICB. Figure 16 shows the flow at a radius of 1.4 in dimensionless units using profile (4.6). The amplitude of the velocity field is now in fair agreement with observed flow. The essential feature required to get the large flows near the CMB (see the review by Bloxham and Jackson, 1991) is that the toroidal field just below the CMB should be small. Then these large flows are only associated with comparatively small nonaxisymmetric field perturbations of the observed order of magnitude. If the toroidal field rises steeply as we move down from the CMB, the amplitude of the convection is constrained to a low value to get agreement with the magnetic field amplitudes, and the corresponding velocities are then too low.

5. DISCUSSION

In this paper the linear magnetoconvection equations have been solved for the case of a rotating spherical shell, with a conducting inner core and a finite outer core viscosity. When no-slip boundary conditions are imposed, as here, boundary layers (such as Ekman layers) may develop. The method described explicitly resolves any such layers if a sufficiently high truncation is taken.

In general including a conducting inner core and viscosity with no-slip boundary conditions seems to have little effect on the general behaviour of magnetoconvection.
The results for varying the Reynolds number \((R_m)\) essentially follow the same scaling law as given in Fearn and Proctor (1983). The results obtained for varying the Roberts number \((q)\), the Ekman number \((E)\) and the Elsasser number \((A)\) are in agreement with those of Zhang (1994), who considered the case of no-stress boundary conditions.

When applying these results to convection in the Earth, the question of the appropriate value of \(q\), the Roberts number, is of importance. Molecular values of the diffusivity suggest that \(q\) should be very small for thermal convection, \(\approx 10^{-6}\), and the equivalent parameter would be even smaller for compositional convection. However, there are serious difficulties with taking \(q\) small when considering large scale convection. First, the thermal diffusion timescale becomes very long, so that instabilities would take a long time to reach a steady nonlinear state. Second, it becomes very difficult to imagine how a dynamo could operate, since at low \(q\) convection is inhibited at low \(R_m\) (with \(R_m \approx q\)) whereas for dynamo action \(R_m\) must be at least \(\approx 10\). A possible resolution of this difficulty is that convection at small \(q\) is unstable, and generates small scale motion (turbulence) as well as large scale motion. If this is the case, the most realistic effective value of \(q\) to adopt for large scale motions is \(q \approx 1\), in which case the thermal and magnetic diffusion timescales are comparable.

The calculation of the flow and radial component of magnetic field at the CMB offers a direct comparison of the theory with observations. To get any agreement it is clear that the \(m = 2\) mode must be considered. Then the general patterns of field and flow seem fairly robust as the other parameters are varied. However if realistic values of flow velocity are to be attained an equilibrium toroidal field concentrated close to the inner-outer core boundary and with a small radial gradient at the CMB must be prescribed.

Two obvious extensions to this work are, (i) the solution of the mean field dynamo problem for the convection patterns obtained, and (ii) the inclusion of nonlinear terms in the magnetoconvection problem. The mean field geodynamo has been studied, for the case of a finitely conducting inner core and viscous outer core fluid, by Hollerbach and Jones (1993a, b, 1995). Their method of solution is analogous to that considered in this paper, and so it should be relatively easy to incorporate the fluid flow and temperature distribution calculated here. Thus essentially this should determine whether it is possible to generate the equilibrium field prescribed here from the flow and temperature calculated in this paper.

The addition of nonlinear terms to the magnetoconvection equations may pose a harder problem. Their inclusion determines explicitly the arbitrary amplitude used in this paper. Some of the nonlinear terms have been included by Fearn et al. (1994) in a study of the complementary magnetostrophic problem. A number of solutions were obtained but there was no clear evidence of an approach to a Taylor state. The effect of the explicit inclusion of a finite viscosity is of great interest. The final objective would be to combine the two nonlinear methods to give a self-consistent convective dynamo solution.

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