The Realm of Ordinal Analysis

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Denn die Pioniere der Mathematik hatten sich von gewissen Grundlagen brauchbare Vorstellungen gemacht, aus denen sich Schlüsse, Rechnungsarten, Resultate ergaben, deren bemächtigten sich die Physiker, um neue Ergebnisse zu erhalten, und endlich kamen die Techniker, nahmen oft bloß die Resultate, setzten neue Rechnungen darauf und es entstanden Maschinen. Und plötzlich, nachdem alles in schönste Existenz gebracht war, kamen die Mathematiker - jene, die ganz innen herumgrübeln, - darauf, daß etwas in den Grundlagen der ganzen Sache absolut nicht in Ordnung zu bringen sei; tatsächlich, sie sahen zuunterst nach und fanden, daß das ganze Gebäude in der Luft stehe. Aber die Maschinen liefen! Man muß daraufhin annehmen, daß unser Dasein bleicher Spuk ist; wir leben es, aber eigentlich nur auf Grund eines Irrtums, ohne den es nicht entstanden wäre.

ROBERT MUSIL: Der mathematische Mensch (1913)

1 Introduction

A central theme running through all the main areas of Mathematical Logic is the classification of sets, functions or theories, by means of transfinite hierarchies whose ordinal levels measure their 'rank' or 'complexity' in some sense appropriate to the underlying context. In Proof Theory this is manifest in the assignment of 'proof theoretic ordinals' to theories, gauging their 'consistency strength' and 'computational power'. Ordinal-theoretic proof theory came into existence in 1936, springing forth from Gentzen's head in the course of his consistency proof of arithmetic. To put it roughly, ordinal analyses attach ordinals in a given representation system to formal theories. Though this area of mathematical logic has is roots in Hilbert's "Beweistheorie" - the aim of which was to lay to rest all worries[1] about the foundations of mathematics once and for all by securing mathematics via an absolute proof of consistency - technical results in proof theory are not different from those in any other branch of mathematics, inasmuch as they can be understood in a way that does not at all refer to any kind of (modified) Hilbert programme. In actuality, most proof theorists do not consider themselves pursuing consistency proofs.

The present paper is based on the lectures that I gave at the LC '97. The lectures were an attempt to give an overview of results that have been achieved by means of ordinal analyses, and to explain the current rationale and goals of ordinally informative

[1]As, in a rather amusing way, described by Musil in the above quote.

proof theory as well as its salient technical tools of analysis. They were aimed at a general logic audience, assuming very little knowledge of proof theory, basically cut elimination for Gentzen's sequent calculus.

The paper is divided into three parts. In Section 1 I try to explain the nature of the connection between ordinal representation systems and theories established in ordinal analyses. Furthermore, I gather together some general conclusions that can be drawn from an ordinal analysis. In the literature, the result of an ordinal analysis of a given theory T is often stated in a rather terse way by saying that the supremum of the provable recursive well-orderings of T (hereafter called $|T|_{\text{sup}}$) is a certain ordinal α. This is at best a shorthand for a much more informative statement. From questions that I've been asked over the years, I know that sloppy talk about proof-theoretic ordinals has led to misconceptions about ordinal-theoretic proof theory. One of the recurring questions is whether it is always possible, given a decent theory T, to cook up a well-ordering $\prec$ such that the order-type of $\prec$ amounts to $|T|_{\text{sup}}$ and T proves all initial segments of $\prec$ to be well-founded, thereby making a mockery of the task of performing an ordinal analysis of T. This time I decided, I'd better take such questions seriously. Section 1 will scrutinize the norm $|.|_{\text{sup}}$, compare it with other scales of strengths and also attend to the above and related questions.

Section 2 is devoted to results that have been achieved through ordinal analyses. They fall into four groups: (1) Consistency of subsystems of classical second order arithmetic and set theory relative to constructive theories, (2) reductions of theories formulated as conservation theorems, (3) combinatorial independence results, and (4) classifications of provable functions and ordinals.

As an introduction to the techniques used in ordinal analysis and in order to illustrate its more subtle features, Section 3 provides sketches of ordinal analyses for two theories. These theories are Kripke-Platek set theory and an extension of the latter, called **KPM**, which formalizes a recursively Mahlo universe of sets. **KPM** is considerably stronger than the fragment of second order arithmetic with Δ^1_2 comprehension. It is distinguished by the fact that it is essentially the 'strongest' classical theory for which a consistency proof in Martin-Löf type theory can be carried out.

2 Measures in proof theory

2.1 Gentzen's result

Gentzen showed that transfinite induction up to the ordinal

$$\varepsilon_0 \;=\; \sup\{\omega, \omega^{\omega}, \omega^{\omega^{\omega}}, \dots\} \;=\; \text{least } \alpha.\, \omega^{\alpha} = \alpha$$

suffices to prove the consistency of Peano Arithmetic, **PA**. To appreciate Gentzen's result it is pivotal to note that he applied transfinite induction up to ε_0 solely to primitive recursive predicates and besides that his proof used only finitistically justified means. Hence, a more precise rendering of Gentzen's result is

$$\mathbf{F} + \text{PR-TI}(\varepsilon_0) \vdash \mathsf{Con}(\mathbf{PA}), \tag{1}$$

where $\mathbf{F}$ signifies a theory that is acceptable in finitism (e.g. $\mathbf{F} = \mathbf{PRA} =$ *Primitive Recursive Arithmetic*) and PR-TI(ε_0) stands for transfinite induction up to ε_0 for primitive recursive predicates. Gentzen also showed that his result is best possible in that **PA** proves transfinite induction up to α for arithmetic predicates for any $\alpha < \varepsilon_0$. The compelling picture conjured up by the above is that the non-finitist part of **PA** is encapsulated in PR-TI(ε_0) and therefore "measured" by ε_0, thereby tempting one to adopt the following definition of *proof-theoretic ordinal* of a theory T:

$$|T|_{Con} \quad = \quad \text{least } \alpha.\, \mathbf{PRA} + \text{PR-TI}(\alpha) \vdash \mathsf{Con}(T). \tag{2}$$

The foregoing definition of $|T|_{Con}$ is, however, inherently vague because the following issues have not been addressed:

- How are ordinals to be represented in **PRA**?
- (2) is definitive only with regard to a prior choice of *ordinal representation system.*
- Different ordinal representation systems may provide different answers to (2).

Notwithstanding that, for "natural" theories T and with regard to a "natural" ordinal representation system, the ordinal $|T|_{Con}$ encapsulates important information about the proof strength of T. To demonstrate the serious deficiencies of the above concept it might be illuminating to exhibit a clearly pathological ordinal representation system for the ordinal ω which underscores the dependence of $|T|_{Con}$ on the choice of the ordinal representation system. This example is due to Kreisel [51]. For a given theory T, it shows how to cook up an ordinal representation system which trivializes the determination of $|T|_{Con}$ by coding the proof predicate for T, Proof_T, into the ordinal representation system. Suppose T is a consistent (primitive recursively axiomatized) extension of **PRA**. Define

$$n <_T m \Leftrightarrow \begin{cases} n < m & \text{if } \forall i < n \, \neg \mathsf{Proof}_T(i, \ulcorner \bot \urcorner) \\ m < n & \text{if } \exists i < n \, \mathsf{Proof}_T(i, \ulcorner \bot \urcorner) \end{cases}$$

where $\bot$ is $\bar{0} = \bar{1}$. If T were inconsistent, then there would exists a least natural number k_0 such that $\mathsf{Proof}_T(k_0, \ulcorner \bot \urcorner)$ and the ordering $<_T$ would look like

$$k_0 \,{}_T\!> k_0 - 1 \,{}_T\!> \cdots \,{}_T\!> 0 \,{}_T\!> k_0 + 1 \,{}_T\!> k_0 + 2 \,{}_T\!> k_0 + 3 \,{}_T\!> \cdots .$$

Otherwise, $<_T$ is just the standard ordering on the natural numbers. At any rate, $<_T$ is a linear ordering (provably so in **PRA**). However, by assumption, T is consistent and thus $n <_T m \Leftrightarrow n < m$. Consequently, the order-type of $<_T$ is ω. In view of its definition, $<_T$ is primitive recursive. Furthermore,

$$\mathbf{PRA} + \text{PR} - \text{TI}(<_T) \vdash \mathsf{Con}(T). \tag{3}$$

Let $A(x) := \forall u \leq x \, \neg \mathsf{Proof}_T(u, \ulcorner \bot \urcorner)$. To see that (3) holds, it suffices to prove $\mathbf{PRA} \vdash \forall x \, [\forall y <_T x \, A(y) \rightarrow A(x)]$. So assume $\forall y <_T a \, A(y)$. We have to show $A(a)$. But $\neg A(a)$ would imply $a + 1 <_T a$ and thus yield $A(a+1)$ which implies $A(a)$. Therefore, $A(a)$ must hold.

2.2 How natural ordinal representation systems arise

Natural ordinal representation systems are frequently derived from structures of the form

$$\mathfrak{A} \;=\; \langle \alpha, f_1, \ldots, f_n, <_\alpha \rangle \tag{4}$$

where α is an ordinal, $<_\alpha$ is the ordering of ordinals restricted to elements of α and the f_i are functions

$$f_i : \underbrace{\alpha \times \cdots \times \alpha}_{k_i \text{ times}} \longrightarrow \alpha$$

for some natural number k_i.

$$\mathbb{A} \;=\; \langle A, g_1, \ldots, g_n, \prec \rangle \tag{5}$$

is a *recursive representation of* $\mathfrak{A}$ if the following conditions hold:

1. $A \subseteq \mathbb{N}$
2. A is a recursive set.
3. $\prec$ is a recursive total ordering on A.
4. The functions g_i are recursive.
5. $\mathfrak{A} \cong \mathbb{A}$, i.e. the two structures are isomorphic.

Gentzen's ordinal representation system for ε_0 is based on the *Cantor normal form*, i.e. for any ordinal $0 < \alpha < \varepsilon_0$ there exist uniquely determined ordinals $\alpha_1, \ldots, \alpha_n < \alpha$ such that

- $\alpha_1 \geq \cdots \geq \alpha_n$
- $\alpha = \omega^{\alpha_1} + \cdots \omega^{\alpha_n}$.

To indicate the Cantor normal form we write $\alpha =_{CNF} \omega^{\alpha_1} + \cdots \omega^{\alpha_n}$. Now define a function

$$\ulcorner . \urcorner : \varepsilon_0 \longrightarrow \mathbb{N}$$

by

$$\ulcorner \alpha \urcorner = \begin{cases} 0 & \text{if } \alpha = 0 \\ \langle \ulcorner \alpha_1 \urcorner, \ldots, \ulcorner \alpha_n \urcorner \rangle & \text{if } \alpha =_{CNF} \omega^{\alpha_1} + \cdots \omega^{\alpha_n} \end{cases}$$

where $\langle k_1, \cdots, k_n \rangle := 2^{k_1+1} \cdot \ldots \cdot p_n^{k_n+1}$ with p_i being the ith prime number (or any other coding of tuples). Further define

$$\begin{aligned} A_0 &:= \mathbf{ran}(\ulcorner . \urcorner) \\ \ulcorner \alpha \urcorner \prec \ulcorner \beta \urcorner &:\Leftrightarrow \alpha < \beta \\ \ulcorner \alpha \urcorner \hat{+} \ulcorner \beta \urcorner &:= \ulcorner \alpha + \beta \urcorner \\ \ulcorner \alpha \urcorner \hat{\cdot} \ulcorner \beta \urcorner &:= \ulcorner \alpha \cdot \beta \urcorner \\ \hat{\omega}^{\ulcorner \alpha \urcorner} &:= \ulcorner \omega^\alpha \urcorner. \end{aligned}$$

Then

$$\langle \varepsilon_0, +, \cdot, \delta \mapsto \omega^\delta, < \rangle \;\cong\; \langle A_0, \hat{+}, \hat{\cdot}, x \mapsto \hat{\omega}^x, \prec \rangle.$$

$A_0, \hat{+}, \hat{\cdot}, x \mapsto \hat{\omega}^x, \prec$ are recursive, in point of fact, they are all elementary recursive.

2.3 Elementary ordinal representation systems

The next definition garners some features (following [32]) that ordinal representation systems used in proof theory always have, and collectively calls them *"elementary ordinal representation system"*. One reason for singling out this notion is that it leads to an elegant characterization of the provably recursive functions of theories equipped with transfinite induction principles for such ordinal representation systems (cf. Propositions 3.19, 3.20).

Definition 2.1 *Elementary recursive arithmetic,* **ERA***,* is a weak system of number theory, in a language with $0, 1, +, \times, E$ (exponentiation), $<$, whose axioms are:

1. the usual recursion axioms for $+, \times, E, <$.
2. induction on Δ_0-formulae with free variables.

ERA is referred to as elementary recursive arithmetic since its provably recursive functions are exactly the Kalmar *elementary functions,* i.e. the class of functions which contains the successor, projection, zero, addition, multiplication, and modified subtraction functions and is closed under composition and bounded sums and products (cf. [89]).

Definition 2.2 For a set X and and a binary relation $\prec$ on X, let $\mathrm{LO}(X, \prec)$ abbreviate that $\prec$ linearly orders the elements of X and that for all u, v, whenever $u \prec v$, then $u, v \in X$.

A *linear ordering* is a pair $\langle X, \prec\rangle$ satisfying $\mathrm{LO}(X, \prec)$.

Definition 2.3 An *elementary ordinal representation system* (EORS) for a limit ordinal λ is a structure $\langle A, \lhd, n \mapsto \lambda_n, +, \times, x \mapsto \omega^x\rangle$ such that:

(i) A is an elementary subset of $\mathbb{N}$.

(ii) $\lhd$ is an elementary well-ordering of A.

(iii) $|\lhd| = \lambda$.

(iv) Provably in **ERA**, $\lhd \restriction \lambda_n$ is a proper initial segment of $\lhd$ for each n, and $\bigcup_n \lhd \restriction \lambda_n = \lhd$. In particular, $\mathbf{ERA} \vdash \forall y\, \lambda_y \in A \wedge \forall x \in A \exists y\, [x \lhd \lambda_y]$.

(v) $\mathbf{ERA} \vdash \mathrm{LO}(A, \lhd)$

(vi) $+, \times$ are binary and $x \mapsto \omega^x$ is unary. They are elementary functions on elementary initial segments of A. They correspond to ordinal addition, multiplication and exponentiation to base ω, respectively. The initial segments of A on which they are defined are maximal.

$n \mapsto \lambda_n$ is an elementary function.

(vii) $\langle A, \lhd, +, \times, \omega^x\rangle$ satisfies "all the usual algebraic properties" of an initial segment of ordinals. In addition, these properties of $\langle A, \lhd, +, \times, \omega^x\rangle$ can be proved in **ERA**.

(viii) Let $\tilde{n}$ denote the n^{th} element in the ordering of A. Then the correspondence $n \leftrightarrow \tilde{n}$ is elementary.

(ix) Let $\alpha = \omega^{\beta_1} + \cdots + \omega^{\beta_k}, \beta_1 \geq \cdots \geq \beta_k$ (Cantor normal form). Then the correspondence $\alpha \leftrightarrow \langle \beta_1, \ldots, \beta_k \rangle$ is elementary.

Elements of A will often be referred to as *ordinals*, and denoted $\alpha, \beta, \ldots$.

2.4 The conceptual problem of characterizing natural ordinal representation systems

It is an empirical fact that ordinal representation systems emerging in proof theory are always elementary recursive and their basic properties are provable in weak fragments of arithmetic like **ERA**. Sommer has investigated the question of complexity of ordinal representation systems at great length in [104, 105]. His case studies revealed that with regard to complexity measures considered in complexity theory the complexity of ordinal representation systems involved in ordinal analyses is rather low. It appears that they are always Δ_0-representable (cf. [104]) and that computations on ordinals in actual proof-theoretic ordinal analyses can be handled in the theory $I\Delta_0 + \Omega_1$, where Ω_1 is the assertion that the function $x \mapsto x^{\log_2(x)}$ is total.

Sommer's findings clearly underpin the fact that the naturalness of ordinal representation systems involved in proof-theoretic ordinal analyses cannot be described in terms of the computational complexity of the representations of these ordinals. Intuitively, computational complexity is inadequate because it says nothing about how ordinals are built up. It has been suggested (cf. [50], [25]) that it is important to address the broader question *"What is a natural well-ordering?"* A criterion for naturalness put forward in [50] is uniqueness up to recursive isomorphism. Furthermore, in [50], Kreisel seems to seek naturalness in algebraic characterizations of ordered structures. Feferman, in [22], discerns the properties of completeness, repleteness, relative categoricity and preservation of these under iteration of the critical process as significant features of systems of natural representation. Girard [35] appears to propose dilators to capture the abstract notion of a notation system for ordinals. However, in my opinion, similar attempts in the Philosophy of Science of defining *'natural properties'* and the complete failure of these attempts show that it is futile to look for a formal definition of *'natural well-ordering'* that will exclude every pathological example. Moreover, it is rather unlikely that such a definition would be able to discern and explain an important feature of EORSs found in proof theory, namely their versatility in establishing equivalences between classical non-constructive theories and intuitionistic constructive theories based on radically different ontologies. To obtain the reductions of classical (non-constructive) theories to constructive ones (as related, for instance, in [26], [83],§2) it appears to be pivotal to work with very special and well-structured ordinal representation systems.

"Natural" well-orderings have arisen using several sources of inspiration:

Set-theoretical (*Cantor, Veblen, Gentzen, Bachmann, Schütte, Feferman, Pfeiffer, Isles, Bridge, Buchholz, Pohlers, Jäger, Rathjen*)

- Define hierarchies of functions on the ordinals.

- Build up terms from function symbols for those functions.
- The ordering on the values of terms induces an ordering on the terms.

Reductions in proof figures (*Takeuti, Yasugi, Kino, Arai*)

- Ordinal diagrams; formal terms endowed with an inductively defined ordering on them.

Patterns of partial elementary substructurehood (*Carlson*, cf. [16])

- Finite structures with Σ_n-elementary substructure relations .

Category-theoretical (*Aczel, Girard, Vauzeilles*)

- Functors on the category of ordinals (with strictly increasing functions) respecting direct limits and pull-backs.

Examples for the set-theoretical approach to ordinal representation systems, in particular the use and role of large cardinals therein, will be presented in section 4.

2.5 Proof-theoretical reductions

Ordinal analyses of theories allow one to compare the strength of theories. This subsection defines the notions of *proof-theoretic reducibility* and *proof-theoretic strength* that will be used henceforth.

All theories T considered in the following are assumed to contain a modicum of arithmetic. For definiteness let this mean that the system **PRA** of Primitive Recursive Arithmetic is contained in T, either directly or by translation.

Definition 2.4 Let T_1, T_2 be a pair of theories with languages $\mathcal{L}_1$ and $\mathcal{L}_2$, respectively, and let Φ be a (primitive recursive) collection of formulae common to both languages. Furthermore, Φ should contain the closed equations of the language of **PRA**.

We then say that T_1 is *proof-theoretically Φ-reducible to* T_2, written $T_1 \leq_\Phi T_2$, if there exists a primitive recursive function f such that

$$\mathbf{PRA} \vdash \forall\phi \in \Phi\, \forall x\, [\mathsf{Proof}_{T_1}(x, \phi) \;\rightarrow\; \mathsf{Proof}_{T_2}(f(x), \phi)]. \tag{6}$$

T_1 and T_2 are said to be *proof-theoretically Φ-equivalent*, written $T_1 \equiv_\Phi T_2$, if $T_1 \leq_\Phi T_2$ and $T_2 \leq_\Phi T_1$.

The appropriate class Φ is revealed in the process of reduction itself, so that in the statement of theorems we simply say that T_1 is *proof-theoretically reducible* to T_2 (written $T_1 \leq T_2$) and T_1 and T_2 are *proof-theoretically equivalent* (written $T_1 \equiv T_2$), respectively. Alternatively, we shall say that T_1 and T_2 have the *same proof-theoretic strength* when $T_1 \equiv T_2$.

Remark 2.5 Feferman's notion of proof-theoretic reducibility in [26] is more relaxed in that he allows the reduction to be given by a T_2-recursive function f, i.e.

$$T_2 \vdash \forall\phi \in \Phi\, \forall x\, [\mathsf{Proof}_{T_1}(x, \phi) \;\rightarrow\; \mathsf{Proof}_{T_2}(f(x), \phi)]. \tag{7}$$

The disadvantage of (7) is that one forfeits the transitivity of the relation $\leq_\Phi$. Furthermore, in practice, proof-theoretic reductions always come with a primitive recursive reduction, so nothing seems to be lost by using the stronger notion of reducibility.

2.6 The general form of ordinal analysis

In this section I attempt to say something general about all ordinal analyses that have been carried out thus far. One has to bear in mind that these concern "natural" theories. Also, to circumvent countless and rather boring counter examples, I will only address theories that have at least the strength of **PA** and and always assume the pertinent ordinal representation systems are closed under $\alpha \mapsto \omega^\alpha$.

Before delineating the general form of an ordinal analysis, we need several definitions.

Definition 2.6 Let T be a framework for formalizing a certain part of mathematics. T should be a true theory which contains a modicum of arithmetic.

Let A be a subset of $\mathbb{N}$ ordered by $\prec$ such that A and $\prec$ are both definable in the language of T. If the language of T allows for quantification over subsets of $\mathbb{N}$, like that of second order arithmetic or set theory, *well-foundedness* of $\langle A, \prec\rangle$ will be formally expressed by

$$\mathrm{WF}(A,\prec) \;\; := \;\; \forall X \subseteq \mathbb{N}\,[\forall u{\in}A(\forall v \prec u\, v{\in}X \rightarrow u{\in}X) \;\rightarrow\; \forall u{\in}A\, u{\in}X.] \tag{8}$$

If, however, the language of T does not provide for quantification over arbitrary subsets of $\mathbb{N}$, like that of Peano arithmetic, we shall assume that it contains a new unary predicate $\mathbf{U}$. $\mathbf{U}$ acts like a free set variable, in that no special properties of it will ever be assumed. We will then resort to the following formalization of well-foundedness:

$$\mathrm{WF}(A,\prec) \;\; := \;\; \forall u{\in}A(\forall v \prec u\, \mathbf{U}(v) \rightarrow \mathbf{U}(u)) \;\rightarrow\; \forall u{\in}A\, \mathbf{U}(u), \tag{9}$$

where $\forall v \prec u \ldots$ is short for $\forall v(v \prec u \rightarrow \ldots)$.

We also set

$$\mathrm{WO}(A,\prec) \;\; := \;\; \mathrm{LO}(A,\prec) \wedge \mathrm{WF}(A,\prec). \tag{10}$$

If $\langle A, \prec\rangle$ is well-founded, we use $|\prec|$ to signify its set-theoretic order-type. For $a{\in}A$, the ordering $\prec\restriction a$ is the restriction of $\prec$ to $\{x{\in}A : x \prec a\}$.

The ordering $\langle A, \prec\rangle$ is said to be *provably well-founded in* T if

$$T \vdash \mathrm{WO}(A,\prec). \tag{11}$$

The supremum of the provable well-orderings of T, $|T|_{\sup}$, is defined as follows:

$$|T|_{\sup} \;\; := \;\; \sup\big\{\alpha : \; \alpha \text{ provably recursive in } T\big\} \tag{12}$$

where an ordinal α is said to be provably recursive in T if there is a recursive well–ordering $\langle A, \prec\rangle$ with order–type α such that

$$T \vdash \mathrm{WO}(A,\prec)$$

with A and $\prec$ being provably recursive in T. Note that, by definition, $|T|_{\sup} \leq \omega_1^{CK}$, where ω_1^{CK} is the supremum of the order-types of all recursive well-orderings on $\mathbb{N}$. Another characterization of ω_1^{CK} is that it is the least admissible ordinal $> \omega$.

Definition 2.7 Suppose $\mathrm{LO}(A, \lhd)$ and $F(u)$ is a formula. Then $\mathrm{TI}_{\langle A,\lhd\rangle}(F)$ is the formula

$$\forall n{\in}A\,[\forall x \lhd n F(x) \to F(n)] \;\to\; \forall n{\in}A\, F(n). \tag{13}$$

$\mathrm{TI}(A, \lhd)$ is the schema consisting of $\mathrm{TI}_{\langle A,\lhd\rangle}(F)$ for all F.

Given a linear ordering $\langle A, \lhd\rangle$ and $\alpha{\in}A$ let $A_\alpha = \{\beta{\in}A : \beta \lhd \alpha\}$ and $\lhd_\alpha$ be the restriction of $\lhd$ to A_α.

In what follows, quantifiers and variables are supposed to range over the natural numbers. When n denotes a natural number, $\bar{n}$ is the canonical name in the language under consideration which denotes that number.

Observation 2.8 *Every ordinal analysis of a classical or intuitionistic theory* $\mathbf{T}$ *that has ever appeared in the literature provides an EORS* $\langle A, \lhd, \dots\rangle$ *such that* $\mathbf{T}$ *is proof-theoretically reducible to* $\mathbf{PA} + \bigcup_{\alpha\in A} \mathrm{TI}(A_{\bar{\alpha}}, \lhd_{\bar{\alpha}})$.

Moreover, if T is a classical theory, then T and $\mathbf{PA} + \bigcup_{\alpha\in A} \mathrm{TI}(A_{\bar{\alpha}}, \lhd_{\bar{\alpha}})$ *prove the same arithmetic sentences, whereas if T is based on intuitionististic, then T and* $\mathbf{HA} + \bigcup_{\alpha\in A} \mathrm{TI}(A_{\bar{\alpha}}, \lhd_{\bar{\alpha}})$ *prove the same arithmetic sentences.*

Furthermore, $|T|_{\sup} = |\lhd|$.

Remark 2.9 There is a lot of leeway in stating the latter observation. For instance, instead of **PA** one could take **PRA** or **ERA** as the base theory, and the scheme of transfinite induction could be restricted to Σ^0_1 formulae as it follows from Proposition 3.20 that $\mathbf{PA} + \bigcup_{\alpha\in A} \mathrm{TI}(A_{\bar{\alpha}}, \lhd_{\bar{\alpha}})$ and $\mathbf{ERA} + \bigcup_{\alpha\in A} \Sigma^0_1\text{-}\mathrm{TI}(A_{\bar{\alpha}}, \lhd_{\bar{\alpha}})$ have the same proof-theoretic strength, providing that A is closed under exponentiation $\alpha \mapsto \omega^\alpha$.

Observation 2.8 lends itself to a formal definition of the notion of *proof-theoretic ordinal* of a theory T. Of course, before one can go about determining the proof-theoretic ordinal of T, one needs to be furnished with representations of ordinals. Not surprisingly, a great deal of ordinally informative proof theory has been concerned with developing and comparing particular ordinal representation systems. Assuming that a sufficiently strong EORS $\langle A, \lhd, \dots\rangle$ has been provided, we define

$$|T|_{\langle A,\lhd,\dots\rangle} \;:=\; \text{least } \rho \in A.\; T \equiv \mathbf{PA} + \bigcup_{\alpha\lhd\rho} \mathrm{TI}(A_{\bar{\alpha}}, \lhd_{\bar{\alpha}}) \tag{14}$$

and call $|T|_{\langle A,\lhd,\dots\rangle}$, providing this ordinal exists, the *proof-theoretic ordinal* of T with respect to $\langle A, \lhd, \dots\rangle$.

Since, in practice, the ordinal representation systems used in proof theory are comparable, we shall frequently drop mentioning of $\langle A, \lhd, \dots\rangle$ and just write $|T|$ for $|T|_{\langle A,\lhd,\dots\rangle}$.

Note, however, that $|T|_{\langle A,\lhd,\dots\rangle}$ might not exist even if the order-type of $\lhd$ is bigger than $|T|_{\sup}$. A simple example is provided by the theory $\mathbf{PA} + \mathsf{Con}(\mathbf{PA})$ (where $\mathsf{Con}(\mathbf{PA})$ expresses the consistency of **PA**) when we take $\langle A, \lhd, \dots\rangle$ to be a standard EORS for ordinals $> \varepsilon_0$; the reason being that $\mathbf{PA} + \mathsf{Con}(\mathbf{PA})$ is proof-theoretically strictly stronger than $\mathbf{PA} + \bigcup_{\alpha\lhd\varepsilon_0} \mathrm{TI}(A_{\bar{\alpha}}, \lhd_{\bar{\alpha}})$ but also strictly weaker than $\mathbf{PA} + \bigcup_{\alpha\lhd\varepsilon_0+1} \mathrm{TI}(A_{\bar{\alpha}}, \lhd_{\bar{\alpha}})$. Therefore, as opposed to $|\cdot|_{\sup}$, the norm $|\cdot|_{\langle A,\lhd,\dots\rangle}$ is only partially defined and does not induce a prewellordering on theories T with $|T|_{\sup} < |\lhd|$.

The remainder of this subsection expounds on important consequences of ordinal analyses that follow from Observation 2.8.

Proposition 2.10 **PA** + $\bigcup_{\alpha\in A} \mathrm{TI}(A_{\bar\alpha}, \lhd_{\bar\alpha})$ *and* **HA** + $\bigcup_{\alpha\in A} \mathrm{TI}(A_{\bar\alpha}, \lhd_{\bar\alpha})$ *prove the same sentences in the negative fragment.*

Proof: **PA** + $\bigcup_{\alpha\in A} \mathrm{TI}(A_{\bar\alpha}, \lhd_{\bar\alpha})$ can be interpreted in **HA** + $\bigcup_{\alpha\in A} \mathrm{TI}(A_{\bar\alpha}, \lhd_{\bar\alpha})$ via the Gödel–Gentzen $\neg\neg$-translation. Observe that for an instance of the schema of transfinite induction we have

$$\big(\forall u\,\big[\forall x\,(\forall y\,[y \prec x \to \phi(y)] \to \phi(x)) \;\to\; \phi(u)\big]\big)^{\neg\neg} \;\equiv$$
$$\big(\forall u\,\big[\forall x\,(\forall y\,[\neg\neg y \prec x \to \neg\neg\phi(y)] \to \neg\neg\phi(x)) \;\to\; \neg\neg\phi(u)\big]\big).$$

Thus for primitive recursive $\prec$ the $\neg\neg$-translation is **HA** equivalent to an instance of the same schema. □

Corollary 2.11 **PA** + $\bigcup_{\alpha\in A} \mathrm{TI}(A_{\bar\alpha}, \lhd_{\bar\alpha})$ *and* **HA** + $\bigcup_{\alpha\in A} \mathrm{TI}(A_{\bar\alpha}, \lhd_{\bar\alpha})$ *prove the same* Π^0_1 *sentences.*

Since many well-known and important theorems as well as conjectures from number theory are expressible in Π^0_1 form (examples: the quadratic reciprocity law, Wiles' theorem, also known as Fermat's conjecture, Goldbach's conjecture, the Riemann hypothesis), Π^0_1 conservativity ensures that many mathematically important theorems which turn out to be provable in S will be provable in T, too.

However, Π^0_1 conservativity is not always a satisfactory conservation result. Some important number-theoretic statements are Π^0_2 (examples are: the twin prime conjecture, miniaturized versions of Kruskal's theorem, totality of the van der Waerden function), and in particular, formulas that express the convergence of a recursive function for all arguments. Consider a formula $\forall n\,\exists m\,P(n,m)$, where $P(n,m)$ is a primitive recursive formula expressing that *"m codes a complete computation of algorithm A on input n."* The $\neg\neg$-translation of this formula is $\forall n\,\neg\forall m\,\neg P(n,m)$, conveying the convergence of the algorithm A for all inputs only in a weak sense. Fortunately, Proposition 2.11 can be improved to hold for sentences of Π^0_2 form.

Proposition 2.12 **PA** + $\bigcup_{\alpha\in A} \mathrm{TI}(A_{\bar\alpha}, \lhd_{\bar\alpha})$ *and* **HA** + $\bigcup_{\alpha\in A} \mathrm{TI}(A_{\bar\alpha}, \lhd_{\bar\alpha})$ *prove the same* Π^0_2 *sentences.*

The missing link to get from Proposition 2.10 to Proposition 2.12 is usually provided by *Markov's Rule* for primitive recursive predicates, $\mathbf{MR}_{PR}$: if $\neg\forall n \neg Q(n)$ (or, equivalently, $\neg\neg\,\exists n\,Q(n)$) is a theorem, where Q is a primitive recursive relation, then $\exists n\,Q(n)$ is a theorem. Kreisel [48] showed that $\mathbf{MR}_{PR}$ holds for **HA**. A variety of intuitionistic systems have since been shown to be closed under $\mathbf{MR}_{PR}$, using a variety of complicated methods, notably Gödel's dialectica interpretation and normalizability. A particularly elegant and short proof for closure under $\mathbf{MR}_{PR}$ is due to Friedman [28] and, independently, to Dragalin [18]. However, though the Friedman–Dragalin argument works for a host of systems, it doesn't seem to work in the case of **HA** + $\bigcup_{\alpha\in A} \mathrm{TI}(A_{\bar\alpha}, \lhd_{\bar\alpha})$.

Proof of Proposition 2.12: We will give a direct proof, i.e. without using Proposition 2.10. So suppose

$$\mathbf{PA} + \bigcup_{\alpha\in A} \mathrm{TI}(A_{\bar\alpha}, \lhd_{\bar\alpha}) \vdash \forall x\,\exists y\,\phi(x,y),$$

where ϕ is Δ_0. Then there already exists a $\delta \in A$ such that

$$\mathbf{PA} + \mathrm{TI}(A_{\bar{\delta}}, \lhd_{\bar{\delta}}) \vdash \forall x\, \exists y\, \phi(x,y). \tag{15}$$

We now use the coding of infinitary $\mathbf{PA}_\infty$ derivations presented in [99], section 4.2.2. Let $d \,\vert\!\frac{\beta}{\rho}\, \ulcorner\psi\urcorner$ signify that d is the code of a $\mathbf{PA}_\infty$ derivation with length $\leq \beta$, cut-rank ρ and end formula ψ. (15) implies that there is a d_0 and $n < \omega$ such that

$$\mathbf{HA} + \bigcup_{\alpha \in A} \mathrm{TI}(A_{\bar{\alpha}}, \lhd_{\bar{\alpha}}) \vdash d_0 \,\vert\!\frac{\delta\cdot\omega}{n}\, \ulcorner\forall x\, \exists y\, \phi(x,y)\urcorner. \tag{16}$$

To obtain a cut-free proof of $\forall x\, \exists y\, \phi(x,y)$ in $\mathbf{PA}_\infty$ one needs transfinite induction up to the ordinal $\omega_n^{\delta\cdot\omega}$, where $\omega_0^\gamma := \gamma$ and $\omega_{m+1}^\gamma := \omega^{\omega_m^\gamma}$. This amount of transfinite induction is available in our background theory $\mathbf{HA} + \bigcup_{\alpha\in A} \mathrm{TI}(A_{\bar{\alpha}}, \lhd_{\bar{\alpha}})$ as A is closed under $\xi \mapsto \omega^\xi$. Also note that the cut-elimination procedure is completely effective. Thus from (16) we obtain, for some d^*,

$$\mathbf{HA} + \bigcup_{\alpha \in A} \mathrm{TI}(A_{\bar{\alpha}}, \lhd_{\bar{\alpha}}) \vdash d^* \,\vert\!\frac{\omega_n^{\delta\cdot\omega}}{0}\, \ulcorner\forall x\, \exists y\, \phi(x,y)\urcorner, \tag{17}$$

and further

$$\mathbf{HA} + \bigcup_{\alpha \in A} \mathrm{TI}(A_{\bar{\alpha}}, \lhd_{\bar{\alpha}}) \vdash \forall x \exists d\, d \,\vert\!\frac{\omega_n^{\delta\cdot\omega}}{0}\, \ulcorner\exists y\, \phi(\dot{x},y)\urcorner \tag{18}$$

(where Feferman's dot convention has been used here). Let Tr_{Σ_1} be a truth predicate for Gödel numbers of disjunctions of Σ_1 formulae (cf. [109], section 1.5, in particular 1.5.7). We claim that

$$\mathbf{HA} + \bigcup_{\alpha \in A} \mathrm{TI}(A_{\bar{\alpha}}, \lhd_{\bar{\alpha}}) \vdash \forall d \forall \beta \leq \omega_n^{\delta\cdot\omega}\, \forall \Gamma \subseteq \Sigma_1 \left[d \,\vert\!\frac{\beta}{0}\, \Gamma \;\rightarrow\; \mathsf{Tr}_{\Sigma_1}(\bigvee \Gamma) \right], \tag{19}$$

where $\forall \Gamma \subseteq \Sigma_1$ is a quantifier ranging over Gödel numbers of finite sets of Σ_1 formulae and $\bigvee \Gamma$ stands for the Gödel number corresponding to the disjunction of all formulae of Γ. (19) is proved by induction on β by observing that all formulae occurring in a cut-free $\mathbf{PA}_\infty$ proof of a set of Σ_1 formulae are Σ_1 themselves and the only inferences therein are either axioms or instances of the $(\exists)$ rule or improper instances of the ω rule. Combining (18) and (19) we obtain

$$\mathbf{HA} + \bigcup_{\alpha \in A} \mathrm{TI}(A_{\bar{\alpha}}, \lhd_{\bar{\alpha}}) \vdash \forall x\, \mathsf{Tr}_{\Sigma_1}(\ulcorner\exists y\, \phi(\dot{x},y)\urcorner). \tag{20}$$

As

$$\mathbf{HA} \vdash \forall x \left[\mathsf{Tr}_{\Sigma_1}(\ulcorner\exists y\, \phi(\dot{x},y)\urcorner) \leftrightarrow \exists y\, \phi(x,y) \right]$$

(cf. [109], Theorem 1.5.6), we finally obtain

$$\mathbf{HA} + \bigcup_{\alpha \in A} \mathrm{TI}(A_{\bar{\alpha}}, \lhd_{\bar{\alpha}}) \vdash \forall x\, \exists y\, \phi(x,y).$$

□

In section 2 we considered the ordinal $|T|_{Con}$. What is the relation between $|T|_{Con}$ and $|T|_{\langle A,\lhd,\ldots\rangle}$? First we have to delineate the meaning of $|T|_{Con}$, though. The latter is only determined with respect to a given ordinal representation system $\langle B, \prec, \ldots \rangle$. Thus let

$$|T|_{Con} \quad = \quad \text{least } \alpha \in B.\ \mathbf{PRA} + \text{PR-TI}(\alpha) \vdash \mathsf{Con}(T).$$

It turns out that the two ordinals are the same when T is proof-theoretically reducible to $\mathbf{PA} + \bigcup_{\alpha\in A} \text{TI}(A_{\bar\alpha}, \lhd_{\bar\alpha})$, A is closed under $\alpha \mapsto \omega^\alpha$ and $\langle B, \prec, \ldots\rangle$ is a proper end extension of $\langle A, \lhd, \ldots\rangle$. The reasons are as follows:

Proposition 2.13 *The consistency of* $\mathbf{PA} + \bigcup_{\alpha\in A} \text{TI}(A_{\bar\alpha}, \lhd_{\bar\alpha})$ *can be proved in the theory* $\mathbf{PRA}+\text{PR-TI}(A, \lhd)$*, where* $\text{PR-TI}(A, \lhd)$ *stands for transfinite induction along* $\lhd$ *for primitive recursive predicates.*

Hint of proof. First note that $\mathbf{PRA} + \text{PR-TI}(A, \lhd) \vdash \Pi^0_1\text{-TI}(A, \lhd)$. The key to showing this is that for each $\alpha \in A$ and each $x \in \omega$ we can code α and x by the ordinal $\omega \cdot \alpha + x$ which is less than $\omega \cdot (\alpha + 1)$ and therefore in A.

Secondly, one has to show that an ordinal analysis of $\mathbf{PA} + \bigcup_{\alpha\in A} \text{TI}(A_{\bar\alpha}, \lhd_{\bar\alpha})$ can be carried out in $\mathbf{PRA} + \Pi^0_1\text{-TI}(A, \lhd)$. The main tool to achieve this is to embed $\mathbf{PA} + \bigcup_{\alpha\in A} \text{TI}(A_{\bar\alpha}, \lhd_{\bar\alpha})$ into a system of Peano arithmetic with an infinitary rule, the so-called ω-rule, and a *repetition rule*, `Rep`, which simply repeats the premise as the conclusion. The ω-rule allows one to infer $\forall x\phi(x)$ from the infinitely many premises $\phi(\bar 0), \phi(\bar 1), \phi(\bar 2), \ldots$ (where $\bar n$ denotes the nth numeral); its addition accounts for the fact that the infinitary system enjoys cut-elimination. The addition of the `Rep` rule enables one to carry out a *continuous cut elimination*, due to Mints [59], which is a continuous operation in the usual tree topology on prooftrees. A further pivotal step consists in making the ω-rule more constructive by assigning codes to proofs, where codes for applications of finitary rules contain codes for the proofs of the premises, and codes for applications of the ω-rule contain Gödel numbers for primitive recursive functions enumerating codes of the premises. Details can be found in [99]. The main idea here is that we can do everything with primitive recursive proof–trees instead of arbitrary derivations. A proof–tree is a tree, with each node labelled by: A sequent, a rule of inference or the designation "Axiom", two sets of formulas specifying the set of principal and minor formulas,respectively, of that inference, and two ordinals (length and cut–rank) such that the sequent is obtained from those immediately above it through application of the specified rule of inference. The well-foundedness of a proof–tree is then witnessed by the (first) ordinal "tags" which are in reverse order of the tree order. As a result, the notion of being a (code of a) proof tree is Π^0_1. The cut elimination for infinitary proofs with finite cut rank (as presented in [99]) can be formalized in $\mathbf{PRA} + \Pi^0_1\text{-TI}(A, \lhd)$. The last step consists in recognizing that every endformula of Π^0_1 form of a cut free infinitary proof is true. The latter employs $\Pi^0_1\text{-TI}(A, \lhd)$. For details see [99]. □

2.7 The orderings of consistency-strength and Π^0_1 conservativity

Two orderings figure prominently among orderings that have been suggested for comparing the strength of theories. These are the orderings of consistency-strength

($\leq_{\mathsf{Con}}$) and Π^0_1 conservativity ($\subseteq_{\Pi^0_1}$) (cf. [19, 113]). If one has ordinal analyses for two theories S, T such that $|S| \leq |T|$, then $S \leq_{\mathsf{Con}} T$ and $S \subseteq_{\Pi^0_1} T$. The latter, however, need not obtain if one merely knows that $|S|_{\sup} \leq |T|_{\sup}$ as will be shown in the subsequent subsection.

I consider the results of this section folklore, though I have no references.

Definition 2.14 T_2 is *conservative over* T_1 *for* Φ, written $T_1 \subseteq_\Phi T_2$ if

$$\forall \phi\, [\phi \in \Phi \wedge T_1 \vdash \phi \;\rightarrow\; T_2 \vdash \phi].$$

Definition 2.15 Let $\mathsf{Proof}_T(x, y)$ express that x is the code of a proof in T such that y is the code of its endformula. We use $\Pr_T(y)$ for $\exists x \mathsf{Proof}_T(x, y)$. The sentence $\mathsf{Con}(T)$ expressing the consistency of T can be taken as $\neg \Pr_T(\ulcorner 0 = 1 \urcorner)$.

The ordering of *consistency strength* between theories is defined by

$$S \leq_{\mathsf{Con}} T \quad :\Leftrightarrow \quad \text{the consistency of } T \text{ implies the consistency of } S. \tag{21}$$

One point needs to be attended to: Where should relative consistency be proven? If one is actually interested in the consistency of S relative to T it would suffice to prove the relative consistency result in T:

$$T \vdash \mathsf{Con}(T) \rightarrow \mathsf{Con}(S). \tag{22}$$

However, the provability within T of such an implication might be rather meaningless, as is the case for $T := \mathbf{PA} + \neg\mathsf{Con}(\mathbf{PA})$, and the relation $\leq_{\mathsf{Con}}$ wouldn't even be transitive.[2]

In practice, one shows the relative consistency result in a sound base theory like **PRA**. Moreover, as Kreisel noted, if the proof of $\mathsf{Con}(T) \rightarrow \mathsf{Con}(S)$ in T provides an effective transformation, i.e.

$$T \vdash \forall n\, \big[\mathsf{Proof}_S(n, \ulcorner 0 = 1 \urcorner) \;\rightarrow \mathsf{Proof}_T(f(n), \ulcorner 0 = 1 \urcorner)\big], \tag{23}$$

where f is primitive recursive, then

$$\mathbf{PRA} + \mathsf{Con}(T) \vdash \forall n\, \big[\mathsf{Proof}_S(n, \ulcorner 0 = 1 \urcorner) \;\rightarrow \mathsf{Proof}_T(f(n), \ulcorner 0 = 1 \urcorner)\big] \tag{24}$$

(cf. [102], Theorem 5.2.1), and therefore

$$\mathbf{PRA} \vdash \mathsf{Con}(T) \rightarrow \mathsf{Con}(S). \tag{25}$$

Moreover, owing to a well-known metamathematical property of **PRA**, (25) yields the existence of a primitive recursive function g such that

$$\mathbf{PRA} \vdash \forall n\, \big[\mathsf{Proof}_S(n, \ulcorner 0 = 1 \urcorner) \;\rightarrow \mathsf{Proof}_T(g(n), \ulcorner 0 = 1 \urcorner)\big]. \tag{26}$$

The upshot of the above is that if the proof of $\mathsf{Con}(T) \rightarrow \mathsf{Con}(S)$ is done at all nicely, i.e. in the sense of (23), it automatically follows that the statement $\mathsf{Con}(T) \rightarrow \mathsf{Con}(S)$ can be proven in the weaker theory **PRA**. On the strength of the latter we adopt (25) as our official definition of $S \leq_{\mathsf{Con}} T$.

[2] Let $T_0 := \mathbf{ACA}$, $T_1 := \mathbf{PA} + \neg\mathsf{Con}(\mathbf{PA})$, and $T_2 := \mathbf{PA} + \mathsf{Con}(\mathbf{PA})$. Then $T_1 \vdash \mathsf{Con}(T_1) \rightarrow \mathsf{Con}(T_0)$ simply because $T_1 \vdash \neg\mathsf{Con}(T_1)$. Also (cf. [102], Corollary 2.2.4) $T_2 \vdash \mathsf{Con}(T_2) \rightarrow \mathsf{Con}(T_1)$. But surely we don't have $T_2 \vdash \mathsf{Con}(T_2) \rightarrow \mathsf{Con}(T_0)$ since T_0 is proof-theoretically stronger than $T_2 + \mathsf{Con}(T_2)$.

Remark 2.16 By definition, $T_1 \leq_\Phi T_2$ implies $T_1 \leq_{\mathsf{Con}} T_2$ and $T_1 \subseteq_\Phi T_2$. The converses are by no means true. (For a trivial counterexample, take $T_1 := \mathbf{ZF}$, $T_2 := \mathbf{PRA}$, and let Φ be the closed equations $\mathcal{L}(\mathbf{PRA})$. Then $T_1 \subseteq_\Phi T_2$ but not $T_1 \leq_\Phi T_2$, assuming T_1 is consistent.)

It is a striking empirical fact that many "natural" theories, i.e. theories which have something like an "idea" to them, are comparable with regard to consistency strength. This has actually been proved in many cases, for theories whose ideas and motivations have nothing at all to do with one another. A plethora of results in proof theory and set theory seems to provide compelling evidence that $\leq_{\mathsf{Con}}$ is a linear ordering on "natural" theories. To illustrate this by way of examples from set theory, with a few exceptions, large cardinal axioms have been shown to form a well-ordered hierarchy when ordered as follows:

$$\phi \leq_{\mathsf{Con}} \psi \quad := \quad \mathbf{ZFC} + \phi \leq_{\mathsf{Con}} \mathbf{ZFC} + \psi,$$

where ϕ and ψ are large cardinal axioms. This has not been established for all of the large cardinal axioms which have been proposed to date; but there is strong conviction among set theorists that this will eventually be accomplished (cf. [19, 113]).

The mere fact of linearity of $\leq_{\mathsf{Con}}$ is remarkable. But one must emphasize "natural" here, because one can construct a pair of self-referential sentences which yield incomparable theories. We first give an example of true theories which are not ordered by $\subseteq_{\Pi^0_1}$.

Proposition 2.17 *There are true arithmetic statements ψ_0 and ψ_1 such that the theories $\mathbf{PA} + \psi_0$ and $\mathbf{PA} + \psi_1$ are not comparable with regard to $\subseteq_{\Pi^0_1}$.*

Proof: For existential, arithmetical formulae $\phi \equiv \exists x \phi_0(x)$ and $\psi \equiv \exists u \psi_0(u)$ define

$$\begin{aligned} \phi \preceq \psi \quad &:\Leftrightarrow \quad \exists x \left[\phi_0(x) \wedge \forall u < x\, \neg\psi_0(u)\right] \\ \phi \prec \psi \quad &:\Leftrightarrow \quad \exists x \left[\phi_0(x) \wedge \forall u \leq x\, \neg\psi_0(u)\right]. \end{aligned} \tag{27}$$

For T an extension of $\mathbf{PRA}$, let $\Box_T(\phi) := \exists x\, \mathsf{Proof}_T(x, \ulcorner\phi\urcorner)$. By the Diagonalization Lemma (cf. [102], Theorem 2.2.1) we find a sentence θ so that

$$\mathbf{PA} \vdash \theta \;\leftrightarrow\; \Box_{\mathbf{PA}}(\neg\theta) \preceq \Box_{\mathbf{PA}}(\theta). \tag{28}$$

Let

$$\begin{aligned} \psi_0 \quad &:= \quad \neg\theta \\ \psi_1 \quad &:= \quad \neg(\Box_{\mathbf{PA}}(\theta) \prec \Box_{\mathbf{PA}}(\neg\theta)). \end{aligned} \tag{29}$$

We claim that ψ_0 and ψ_1 are both true. To see that ψ_0 is true, note that because of (28), θ implies $\mathbf{PA} \vdash \mathrm{Pr}_{\mathbf{PA}}(\ulcorner\neg\theta\urcorner)$, yielding $\neg\theta$, so ψ_0 is true. As a result we get $\theta \to \neg\theta$, which implies $\neg\theta$ and hence ψ_0.

To see that ψ_1 is true, note that $\Box_{\mathbf{PA}}(\theta) \prec \Box_{\mathbf{PA}}(\neg\theta)$ implies $\mathrm{Pr}_{\mathbf{PA}}(\ulcorner\theta\urcorner)$, which yields θ, and by the foregoing arguments also $\neg\theta$; thus $\neg(\Box_{\mathbf{PA}}(\theta) \prec \Box_{\mathbf{PA}}(\neg\theta))$ must hold.

Let $T_0 := \mathbf{PA} + \psi_0$ and $T_1 := \mathbf{PA} + \psi_1$. We claim that

$$T_0 \nvdash \psi_1, \tag{30}$$
$$T_1 \nvdash \psi_0. \tag{31}$$

For a contradiction, assume $T_0 \vdash \psi_1$. Then $\mathbf{PA} \vdash \neg\psi_0 \vee \psi_1$, whence, using (28),

$$\begin{aligned}\mathbf{PA} \ \vdash \ & \exists x[\mathsf{Proof}_{\mathbf{PA}}(x, \ulcorner\neg\theta\urcorner) \wedge \forall u < x \neg\mathsf{Proof}_{\mathbf{PA}}(u, \theta)] \\ & \vee \forall x[\mathsf{Proof}_{\mathbf{PA}}(x, \ulcorner\theta\urcorner) \rightarrow \exists u < x \mathsf{Proof}_{\mathbf{PA}}(u, \neg\theta)].\end{aligned}$$

The latter yields

$$\mathbf{PA} \vdash \forall x \, [\mathsf{Proof}_{\mathbf{PA}}(x, \ulcorner\theta\urcorner) \rightarrow \exists u < x \mathsf{Proof}_{\mathbf{PA}}(u, \neg\theta)]$$

and hence $\mathbf{PA} \vdash \psi_1$. Further, $\mathbf{PA} \vdash \psi_1$ implies $\mathbf{PA} \vdash \mathrm{Pr}_{\mathbf{PA}}(\ulcorner\theta\urcorner) \rightarrow \theta$, and thus $\mathbf{PA} \vdash \theta$ by Löb's theorem (cf. [102], 4.1.1). The latter yields $\mathbf{PA} \vdash \exists x \, \mathsf{Proof}_{\mathbf{PA}}(x, \ulcorner\neg\theta\urcorner)$ and hence $\mathbf{PA} \vdash \neg\theta$. But then we have $\mathbf{PA} \vdash \theta$ as well as $\mathbf{PA} \vdash \neg\theta$ and $\mathbf{PA}$ would be inconsistent.

To show $T_1 \nvdash \psi_0$, we assume $T_1 \vdash \psi_0$, working towards a contradiction. We then get $\mathbf{PA} \vdash \neg\psi_1 \vee \neg\theta$ which yields $\mathbf{PA} \vdash \neg(\Box_{\mathbf{PA}}(\neg\theta) \prec \Box_{\mathbf{PA}}(\theta))$ and hence $\mathbf{PA} \vdash \neg\theta$ by (28). Thus we have $\mathbf{PA} \vdash \mathrm{Pr}_{\mathbf{PA}}(\ulcorner\neg\theta\urcorner)$. $\mathbf{PA} \vdash \neg\theta$ and $\mathbf{PA} \vdash \mathrm{Pr}_{\mathbf{PA}}(\ulcorner\neg\theta\urcorner)$ together imply $\mathbf{PA} \vdash \mathrm{Pr}_{\mathbf{PA}}(\ulcorner\theta\urcorner)$, and hence the contradiction $\mathbf{PA} \vdash \neg\mathsf{Con}(\mathbf{PA})$. □

Next, we give an example of a pair of true theories S_0, S_1 which cannot be compared with regard to $\leq_{\mathsf{Con}}$.

Proposition 2.18 *There is a pair of sound theories S_0, S_1 which are extensions of* **PRA** *such that*

$$\mathbf{PA} \nvdash \mathsf{Con}(S_0) \rightarrow \mathsf{Con}(S_1), \tag{32}$$
$$\mathbf{PA} \nvdash \mathsf{Con}(S_1) \rightarrow \mathsf{Con}(S_0).$$

Proof: By [103], chap. 7, Corollary 2.6, one can construct Π^0_1 sentences χ, η satisfying

$$\mathbf{PRA} + \mathsf{Con}(\mathbf{PRA}) \vdash \psi_0 \leftrightarrow \mathsf{Con}(\mathbf{PRA} + \chi), \tag{33}$$
$$\mathbf{PRA} + \mathsf{Con}(\mathbf{PRA}) \vdash \psi_1 \leftrightarrow \mathsf{Con}(\mathbf{PRA} + \eta),$$

where ψ_0, ψ_1 are from Proposition 2.17 (29). Since ψ_0 and ψ_1 are true, $\mathsf{Con}(\mathbf{PRA} + \chi)$ and $\mathsf{Con}(\mathbf{PRA} + \eta)$ must be true, too. Therefore, as χ is Π^0_1 it must be true as well, for otherwise **PRA** would prove $\neg\chi$, yielding that $\mathbf{PRA} + \chi$ is inconsistent, colliding with $\mathsf{Con}(\mathbf{PRA} + \chi)$ being true. By the same token, η is true.

Now set $S_0 := \mathbf{PRA} + \chi$ and $S_1 := \mathbf{PRA} + \eta$. Note that $\mathbf{PA} \vdash \mathsf{Con}(\mathbf{PRA})$. Thus, using (33), $\mathbf{PA} \vdash \mathsf{Con}(S_0) \rightarrow \mathsf{Con}(S_1)$ would imply $\mathbf{PA} \vdash \psi_0 \rightarrow \psi_1$ and $\mathbf{PA} \vdash \mathsf{Con}(S_1) \rightarrow \mathsf{Con}(S_0)$ would imply $\mathbf{PA} \vdash \psi_1 \rightarrow \psi_0$, both contradicting the results of Proposition 2.17. □

2.8 The proof-theoretic ordinal of a theory and the supremum of its provable recursive well-orderings

In several papers and books the calibration of $|T|_{\text{sup}}$ has been called *ordinal analysis of* T. The definition of $|T|_{\text{sup}}$ has the advantage that it is not notation-sensitive. But as to the activity named "ordinal analysis" it is left completely open what constitutes such an analysis. One often encounters this kind of sloppy talk of ordinals in proof theory, though it is mostly a shorthand for conveying a far more interesting result.

In this subsection the norm $|\cdot|_{\text{sup}}$ will be compared with the other previously introduced norms. It will also become clear that, in general, the mere knowledge of $|T|_{\text{sup}}$ is not the goal of an ordinal analysis of T.

First, it should be mentioned that, in general, $|T|_{\text{sup}}$ has several equivalent characterizations; though some of these hinge upon the mathematical strength of T. As the next the result below will show, the concept $|T|_{\text{sup}}$ is very robust.

Proposition 2.19 *(i) Suppose that for every elementary well-ordering* $\langle A, \prec\rangle$, *whenever* $T \vdash \mathrm{WO}(A,\prec)$, *then*

$$T \vdash \forall u\,[\,A(u) \to (\forall v \prec u P(v)) \to P(u)] \to \forall u\,[A(u) \to P(u)]$$

holds for all provably recursive predicates P *of* T. *Then*

$$\begin{aligned} |T|_{\text{sup}} &= \sup\{\alpha : \alpha \textit{ is provably elementary in } T\} \qquad (34)\\ &= \sup\{\alpha : \alpha \textit{ is provably } \Sigma^0_1 \textit{ in } T\}. \end{aligned}$$

Moreover, if $T \vdash \mathrm{WO}(A,\prec)$ and $A,\prec$ *are provably recursive in* T, *then one can find an elementary well-ordering* $\langle B, \lessdot\rangle$ *and a recursive function* f *such that* $T \vdash \mathrm{WO}(B,\lessdot)$, f *is provably recursive in* T, *and* T *proves that* f *supplies an order isomorphism between* $\langle B,\lessdot\rangle$ *and* $\langle A,\prec\rangle$.

Examples for (i) are the theories $\mathbf{I\Sigma_1}$, $\mathbf{WKL_0}$ *and* $\mathbf{PA}$.

(ii) If T *comprises* $\mathbf{ACA_0}$, *then*

$$|T|_{\text{sup}} = \sup\{\alpha : \alpha \textit{ is provably arithmetic in } T\}. \qquad (35)$$

(iii) If T *comprises* $\mathbf{\Sigma^1_1} - \mathbf{AC_0}$, *then*

$$|T|_{\text{sup}} = \sup\{\alpha : \alpha \textit{ is provably analytic in } T\}, \qquad (36)$$

where a relation on $\mathbb{N}$ *is called* analytic *if it is lightface* Σ^1_1.

Proof: **(i)**: Suppose $T \vdash \mathrm{WO}(A,\lhd)$, where A and $\lhd$ are defined by Σ^0_1 arithmetic formulae. We shall reason informally in T. We may assume that A contains at least two elements since there are elementary well-orderings for any finite order-type. Without loss of generality we may also assume $0 \notin A$ as $\langle A,\lhd\rangle$ could be replaced by $\langle\{n+1 : n\in A\}, \{(n+1,m+1) : n \lhd m\}\rangle$. A crucial observation is now that there are elementary R and f such that $x \lhd y \;\leftrightarrow\; \exists z R(x,y,z)$ and f enumerates A, i.e. $A = \{f(n) : n\in\mathbb{N}\}$. It is wellknown that such A and f can be chosen among the primitive recursive ones; the usual proof actually furnishes this stronger result (cf. [89], p. 30).

Next, define a function h by $h(0) = 0$, and $h(v+1) = f(i)$ if i is the smallest integer $\leq v+1$ such that $f(i) \neq h(0), \ldots, f(i) \neq h(v-1), f(i) \neq h(v)$ and

$$\forall u \leq v\, \exists w \leq v\, [h(u) \neq 0 \;\rightarrow\; R(f(i), h(u), w) \vee R(h(u), f(i), w)];$$

let $h(v+1) = 0$ if there is no such $i \leq v+1$. Clearly, $h(v) \leq \Pi_{u \leq v} f(u)$. Thus h is a primitive recursive function bounded by an elementary function. As the auxiliary functions entering the definition of h are elementary, h is elementary too (cf. [89], Theorem 3.1). Obviously, h enumerates $\{0\} \cup A$, moreover, for each $a \in A$ there is exactly one v such that $h(v) = a$.

Define the elementary relation $\lessdot$ via

$$x \lessdot y \quad \text{iff} \quad \exists w \leq \max(x,y)\, R(h(x), h(y), w). \tag{37}$$

We want to show that $\lessdot$ linearly orders the elementary set $B := \{n : h(n) \neq 0\}$. If x is in the field of $\lessdot$, i.e. $\exists y\, (x \lessdot y \;\vee\; y \lessdot x)$, then clearly $x \in B$ by definition of $\lessdot$ and h. Conversely, if $h(x) \neq 0$, then $h(x) \in A$, and thus $h(x) \lhd a \;\vee\; a \lhd h(x)$ for some a since A has at least two elements. Pick y such that $a = h(y)$. By definition of h, $\exists w \leq \max(x,y)\, [R(h(x), h(y), w) \;\vee\; R(h(y), h(x), w)]$. Hence $x \lessdot y \;\vee\; y \lessdot x$.

As $\lessdot$ is clearly irreflexive, to verify $\mathrm{LO}(B, \lessdot)$ it remains to be shown that $\lessdot$ is transitive. Assume $x \lessdot y \wedge y \lessdot z$. Then $h(x) \lhd h(z)$, and, by definition of h, if $a < z$ then $\exists w \leq z\, R(h(x), h(z), w)$, whereas $z < x$ implies $\exists w \leq x\, R(h(x), h(z), w)$; thus $x \lessdot z$.

To prove $\mathrm{WF}(B, \lessdot)$, assume

$$\forall x \in B\, [\forall y \lessdot x U(y) \;\rightarrow\; U(x)]. \tag{38}$$

We want to show $\forall x \in B\, U(x)$. Define

$$g(v) \quad = \quad \begin{cases} \text{least } x.\, h(x) = v & \text{if } v \in A \\ 0 & \text{otherwise.} \end{cases}$$

Notice that g is provably recursive in T. Let $G(u)$ be the formula $U(g(u))$, and assume $v \in A$ and $\forall u \lhd v\, G(u)$. Then $\forall y \lessdot g(v)\, U(y)$ as $y \lessdot g(v)$ yields $h(y) \lhd h(g(v)) = v$. So (38) yields $U(g(v))$; thus $G(v)$. We then get $\forall v \in A\, G(v)$ employing $\mathrm{WO}(A, \lhd)$. Hence $\forall x \in B\, U(x)$. The upshot of the foregoing is that

$$T \vdash \mathrm{WO}(B, \lessdot). \tag{39}$$

The desired result now follows by noticing that h furnishes an order preserving mapping from $\langle B, \lessdot \rangle$ onto $\langle A, \lhd \rangle$ (provably in T), thereby yielding $|\lessdot| = |\lhd|$.

(ii): We are going to draw on the notations of Lemma 2.20 below. Let $\lhd$ be a binary Σ^1_1 relation on $\mathbb{N}$ such that $T \vdash \mathrm{WO}(\lhd)$. Let

$$\mathfrak{S} \quad := \quad \{e \in \mathit{Rec} : \; \exists f\, \mathsf{Emb}(f, \prec_e, \lhd) \;\wedge\; \mathrm{LO}(\prec_e)\},$$

where $\mathsf{Emb}(f, \prec_e, \lhd)$ stands for $\forall n \forall m\, [n \prec_e m \;\rightarrow\; f(n) \lhd f(m)]$.

The formula "$\exists f \, \mathsf{Emb}(f, \prec_e, \lhd)$" is Σ^1_1. Note that $T \vdash \mathfrak{S} \subseteq \mathfrak{W}_{Rec}$. By Lemma 2.20 below (a formalized, effective version of the Σ^1_1 Bounding Principle) we can find an integer a such that

$$T \vdash \bar{a} \in \mathfrak{W}_{Rec} \setminus \mathfrak{S},$$

in particular, $T \vdash \neg\exists f \, \mathsf{Emb}(f, \prec_{\bar{a}}, \lhd)$, and hence the Π^1_1 faithfulness of T yields $|\lhd| \leq |\prec_a|$, thus $|\lhd| < |T|_{\text{sup}}$.

(iii): This time let $\lhd$ be a binary Σ^1_1 relation on $\mathbb{N}$. The proof is basically the same as for (ii) though the formula "$\exists f \, \mathsf{Emb}(f, \prec_e, \lhd)$" may not be strictly Σ^1_1, but it is equivalent to a Σ^1_1 formula provably in T, by using the Σ^1_1 axiom of choice.

Lemma 2.20 *Let $Rec := \{e \in \mathbb{N} : e$ is an index of a total recursive function$\}$. With each $e \in Rec$ there is associated a relation $\prec_d$ via $n \prec_d m :\Leftrightarrow \{d\}(\langle n, m\rangle) = 0$, where $\langle ., . \rangle$ is a primitive recursive pairing function. Let*

$$\mathfrak{W}_{Rec} \;\; := \;\; \{e \in \mathbb{N} : \; e \in Rec \; \wedge \; \mathrm{WO}(\prec_e)\}.$$

Suppose $H(x)$ is a Σ^1_1 formula such that

$$T \vdash \forall n[H(n) \rightarrow x \in \mathfrak{W}_{Rec}].$$

Then there exists $e \in Rec$ such that

$$T \vdash \bar{e} \in \mathfrak{W}_{Rec} \; \wedge \; \neg H(\bar{e}).$$

Proof: See [75], Lemma 1.1. □

Theorem 2.21 *Let T be a Σ^1_1 axiomatizable theory.*

(i) If T is Π^1_1-faithful, then $|T|_{\text{sup}} < \omega_1^{CK}$.

(ii) If $\mathbf{ACA_0} \subseteq T$ and $|T|_{\text{sup}} < \omega_1^{CK}$, then T is Π^1_1-faithful.

(iii) There are consistent primitive recursive theories T such that $|T|_{\text{sup}} = \omega_1^{CK}$.

Proof: **(i)**: The set $X := \{e \in Rec : \; T \vdash \mathrm{WO}(\prec_{\bar{e}})\}$ is Σ^1_1. Π^1_1-faithfulness ensures that $X \subseteq \mathfrak{W}_{Rec}$. So by Σ^1_1 bounding there is a recursive well-ordering that has a bigger order-type than all the orderings $\prec_e$ with $e \in$ Rec. Consequently, $|T|_{\text{sup}} < \omega_1^{CK}$.

(ii): For a contradiction, suppose T is not Π^1_1-faithful. Then there is a false Π^1_1-sentence B such that $T \vdash B$. Rendering B in Π^1_1 normal form, one obtains a primitive recursive well-ordering $\prec$ such that $\mathbf{ACA_0} \vdash B \leftrightarrow \mathrm{WF}(\prec)$. As a result, $T \vdash \mathrm{WF}(\prec)$, but $\prec$ is not well-founded. Now let $\lhd$ be an arbitrary recursive well-ordering. Put

$$\mathbb{T} := \{\langle\rangle\} \; \cup \; \{\langle\langle x_0, y_0\rangle, \ldots, \langle x_i, y_i\rangle\rangle : \; x_i \lhd \cdots \lhd x_0; \; y_i \prec \cdots \prec y_0; \; i \in \mathbb{N}\}$$

and let $<_{\mathbb{T}}$ be the Kleene-Brouwer linearization of $\mathbb{T}$. Since $T \vdash \mathrm{WF}(\prec)$ it follows $T \vdash \mathrm{WO}(<_{\mathbb{T}})$. Since $\lhd$ is well-founded, $<_{\mathbb{T}}$ is a well-ordering in the "real world". We claim that $<_{\mathbb{T}}$ has at least the order-type of $\lhd$. To this end, let $(f(n))_{n\in\mathbb{N}}$ be an infinite descending $\prec$ sequence, i.e. $f(n+1) \prec f(n)$ for all n. Put

$$\mathbb{S} := \{\langle\rangle\} \; \cup \; \{\sigma \in \mathbb{T} : \; \sigma = \langle\langle x_0, f(0)\rangle, \ldots, \langle x_i, f(i)\rangle\rangle\}.$$

Being a subtree of $\mathbb{T}$, the Kleene-Brouwer ordering on $\mathbb{S}$, $<_{\mathbb{S}}$, is also well-founded. Define $g(x)$ be the $<_{\mathbb{S}}$-least $\sigma \in \mathbb{S}$ of the form $\sigma * \langle\langle x, f(i)\rangle\rangle$. Now, if $y \lhd x$, then

$$g(y) \leq_{\mathbb{S}} g(x) * \langle\langle y, f(i+1)\rangle\rangle <_{\mathbb{S}} g(x).$$

This shows that $\mathsf{Emb}(g, \lhd, <_{\mathbb{S}})$ and a fortiori $\mathsf{Emb}(g, \lhd, <_{\mathbb{T}})$, verifying the claim. As $\lhd$ was an arbitrary recursive well-ordering, it follows $|T|_{\sup} = \omega_1^{CK}$, contradicting $|T|_{\sup} < \omega_1^{CK}$.

(iii): Due to (ii), an example is provided by $\mathbf{ACA_0} + \neg\mathsf{Con}(\mathbf{ACA_0})$. □

Remark 2.22 If one considers it worthwhile looking at theories which are not Π^1_1 faithful, though consistent, one can amuse oneself by producing theories S, T such that S and T are equiconsistent but differ with respect to their $|.|_{\sup}$ norms. Just let $T := \mathbf{ACA_0}$ and $S := \mathbf{ACA_0} + \neg\mathsf{Con}(\mathbf{ACA_0})$. Then $T \equiv_{\mathsf{Con}} S$ by [102], Corollary 2.2.4 and $|T|_{\sup} < |S|_{\sup}$ by Theorem 2.21, (ii).

Sloppy talk about what constitutes an ordinal analysis of a theory T is prone to trivialization. Given a faithful theory T, one easily concocts a definition of a well-ordering whose order-type is $|T|_{\sup}$ by simply amalgamating the provable well-orderings of T into one big ordering.

Theorem 2.23 *Let T be a primitive recursive Π^1_1-faithful theory which comprises $\mathbf{RCA_0}$. Then there exists a primitive recursive well-ordering $\lhd$ such that*

$$|T|_{\sup} = |\lhd|, \tag{40}$$

$$\mathbf{RCA_0} \vdash \mathrm{WF}(\lhd) \rightarrow \mathsf{Con}(T); \tag{41}$$

$$\textit{For each proper initial segment } \lhd' \textit{ of } \lhd,\ T \vdash \mathrm{WO}(\lhd'). \tag{42}$$

The third assertion probably requires some clarification. For definiteness, by a proper initial segment of $\lhd$ we mean any ordering of the form $\{(n,m) : n \lhd m \wedge m \lhd n_0\}$ such that $n_0 \lhd k$ for some k.

Define

$$\begin{aligned}
\phi(n) &\;:\Leftrightarrow\; \exists e\, \exists m\, \big[n = \langle e, m\rangle \wedge \mathsf{Proof}_T(m, \ulcorner \mathrm{WO}(\prec_{\bar{e}})\urcorner)\big];\\
x \prec_n y &\;:\Leftrightarrow\; \phi(n) \wedge x \prec_{(n)_0} y;\\
\langle n, x\rangle \lhd \langle n', y\rangle &\;:\Leftrightarrow\; \phi(n) \wedge \phi(n') \wedge \big[n < n' \vee (n = n' \wedge x \prec_{(n)_0} y)\big].
\end{aligned}$$

In view of its definition, $\lhd$ is primitive recursive and $|T|_{\sup} = |\lhd|$.

To verify (41), we reason in $\mathbf{RCA_0}$. Assume $\neg\mathsf{Con}(T)$. Then T is inconsistent and thus proves every statement. In particular, T proves then that the ordering $0 \gtrdot 1 \gtrdot 2 \gtrdot 3 \gtrdot \cdots$ is a well-ordering. But $\lessdot$ is embeddable into $\lhd$; thus $\lhd$ cannot be a well-ordering.

As to (42), let $\lhd\!\restriction\! r := \{(n,m) : n \lhd m \wedge m \lhd r\}$ be an initial segment of $\lhd$. Then there exists s such that $r \lhd s$. In particular, $\phi(r)$ and $T \vdash \phi(\bar{r})$. Let $\langle e_0, p_0\rangle, \ldots, \langle e_t, p_t\rangle$ be the list of all pairs $< s$ such that $\mathsf{Proof}_T(e_i, \ulcorner \mathrm{WO}(\prec_{\overline{p_i}})\urcorner)$. Then

$$\begin{aligned}
&T \vdash \forall u \lhd \bar{r}\, [(u)_0 = \langle \overline{e_0}, \overline{p_0}\rangle \vee \cdots \vee (u)_0 = \langle \overline{e_t}, \overline{p_t}\rangle];\\
&T \vdash \mathrm{WO}(\prec_{\overline{e_0}}) \wedge \cdots \wedge \mathrm{WO}(\prec_{\overline{e_t}}).
\end{aligned}$$

The latter implies $T \vdash \mathrm{WO}(\lhd\restriction\bar{r})$. □

Another reason why the ordinal $|T|_{\sup}$, even when presented in the shape of a natural ordinal representation system, does not convey all the information obtained by an ordinal analysis of T is that one can find theories T_1, T_2 of different proof-theoretic strength which satisfy $|T_1|_{\sup} = |T_2|_{\sup}$. More precisely, the ordinal $|T|_{\sup}$ usually doesn't change when one augments T by true Σ^1_1 statements.

In the main, the next result is due to Kreisel. But I couldn't find a reference and don't know how Kreisel proved it.

Proposition 2.24 *Let T be a primitive recursive, Π^1_1-faithful theory of second order arithmetic such that $\mathbf{PA} \subseteq T$. Let $\lhd$ be a primitive recursive well-ordering such that $|T|_{\sup} = |\lhd|$ and*

$$\mathbf{PA} + \mathrm{TI}(\lhd) \vdash \mathsf{Proof}_{\mathbf{T}}(\ulcorner \mathbf{F} \urcorner) \rightarrow \mathbf{F} \tag{43}$$

holds for all arithmetic formulae F which may contain free second order set variables but no free number variables. Then, for any true Σ^1_1 statement B,

$$|T|_{\sup} = |T + B|_{\sup}.$$

Proof: Let $B := \exists X\, C(X)$ be a true Σ^1_1 sentence with $C(X)$ being arithmetic. Let $S := T + B$. Note that S is also Π^1_1 faithful.

We want to show that $|T|_{\sup} = |S|_{\sup}$. So suppose

$$S \vdash \mathrm{WO}(\prec)$$

for some arithmetic well-ordering $\prec$. Then let $E(U)$ be the statement that U is the graph of a function on $\mathbb{N}$ which maps the field of $\lhd$ order-preservingly onto an initial segment (not necessarily proper) of the field of $\prec$. Then

$$S + \exists X\, E(X) \vdash \mathrm{TI}(\lhd).$$

Thus, in view of (43), one gets

$$S \vdash \exists X\, E(X) \;\rightarrow\; \mathsf{Proof}_S(\ulcorner \forall X \neg E(X) \urcorner) \;\rightarrow\; \forall X \neg E(X). \tag{44}$$

The latter yields (using predicate logic)

$$S \vdash \mathsf{Proof}_S(\ulcorner \forall X \neg E(X) \urcorner) \;\rightarrow\; \forall X \neg E(X), \tag{45}$$

and thus, by Löb's Theorem (cf. [102], Theorem 4.1.1),

$$S \vdash \forall X \neg E(X). \tag{46}$$

As a result, since S proves only true statements, $\forall X \neg E(X)$ must be true and therefore the order-type of $\prec$ must be less than the order-type of $\lhd$. In conclusion,

$$|T|_{\sup} = |S|_{\sup}.$$

□

Remark 2.25 In all the examples I know, if T is a subsystem of classical second order arithmetic for which an ordinal analysis has been carried out via an ordinal representation system $(A, \lhd)$, (43) is satisfied.

If one takes, e.g. $B := \mathsf{Con}(T)$, then S is of greater proof-theoretic strength than T.

3 Rewards of ordinal analyses and ordinal representation systems

This section is devoted to results that have been achieved through ordinal analyses. They fall into four groups: (1) Consistency of subsystems of classical second order arithmetic and set theory relative to constructive theories, (2) reductions of theories formulated as conservation theorems, (3) combinatorial independence results, and (4) classifications of provable functions and ordinals.

3.1 Hilbert's programme extended: Constructive consistency proofs

A natural modification of Hilbert's programme consists in loosening the requirement of reduction to finitary methods by allowing reduction to constructive methods more generally.[3]

The point of an extended Hilbert programme (H.P.) is that one wants a constructive conception for which there is an absolute guarantee that, whatever one proves in a sufficiently strong classical theory T, say, a fragment of second order arithmetic or set theory, there would be an interpretation of the proof according to which the theorem is contructively true. Moreover, one wants the theory T to be such as to make the process of formalization of mathematics in T almost trivial, in particular T should be sufficiently strong for all practical purposes. This is a very Hilbertian attitude: show once and for all that non-constructive methods do not lead to false constructive conclusions and then proceed happily on with non-constructive methods.

There are several aspects of an extended H.P. that require clarification. Let's first dispense with the question of how to delineate a sufficiently strong classical theory T as this is an easy one. It was already observed by Hilbert-Bernays [37] that classical analysis can be formalized within second order arithmetic. Further scrutiny revealed that a small fragment is sufficient. Even without knowledge of that program carried out under the rubric of "reverse mathematics", it is easily seen that most of ordinary mathematics can be formalized in $\Delta^1_2 - \mathbf{CA} + \mathbf{BI}$ without effort (**BI** stands for the principle of bar induction, i.e. the assertion that transfinite induction along a well-founded set relation holds for arbitrary classes; this is the pendant of the foundation axiom in set theory). A more convenient framework for formalizing mathematics is set theory. A set theory which proves the same theorems of second order arithmetic is the set theory **KPi** which is an extension of Kripke-Platek set theory via an axiom that asserts the existence of many admissible sets, namely every set is contained in an admissible set.

It may not be clear how ordinal analysis can contribute to an extended H.P. The

[3] Such a shift from the original programme is implicit in Hilbert-Bernays' [37] apparent acceptance of Gentzen's consistency proof for **PA** under the heading "Überschreitung des bisherigen methodischen Standpunktes der Beweistheorie". The need for a modified Hilbert programme has clearly been recognized by Gentzen (cf. [34]) and Bernays [6]: *It thus became apparent that the "finite Standpunkt" is not the only alternative to classical ways of reasoning and is not necessarily implied by the idea of proof theory. An enlarging of the methods of proof theory was therefore suggested: instead of reduction to finitist methods of reasoning it was required only that the arguments be of a constructive character, allowing us to deal with more general forms of inferences.*

system of ordinal representations used in consistency proofs of stronger and stronger theories becomes more and more complicated. To say that the consistency proof has been carried out by transfinite induction on a certain complicated ordering tells us nothing about what constructive principles are involved in the proof of its well-ordering. Are we to take transfinite induction with respect to these ordinal representation as a fundamental constructive principle? The answer could hardly be "yes" lest only specialists on ordinal representations should be convinced. Therefore it becomes necessary to give a detailed account of what constructive principles are allowed in any well-ordering proof and to carry out well-ordering proofs for ordinal representations using only these principles. The problem is thus to find some basic constructive principles upon which a coherent system of constructive reasoning may be built. Several frameworks for constructivism that relate to Bishop's constructive mathematics as theories like **ZFC** relate to Cantorian set theory have been proposed by Myhill, Martin–Löf, Feferman and Aczel. Among those are Feferman's "Explicit Mathematics", a constructive theory of operations and classes ([23, 24]), and Martin-Löf's intuitionistic type theory of [58] (the latter does not contain Russell's infamous *reducibility axiom*). Type theory is a logic free theory of constructions within which the logical notions can be defined whereas systems of Explicit mathematics leave the logical notions unanalysed. For this reason we consider type theory to be more fundamental.

By employing an ordinal analysis for **KPi** it has been shown that **KPi** and consequently $\Delta^1_2 - \mathbf{CA}+\mathbf{BI}$ can be reduced to both these theories.

Theorem 3.1 (Feferman [23], Jäger [42], Jäger and Pohlers [44]) $\Delta^1_2 - \mathbf{CA}+\mathbf{BI}$ *(or* **KPi***) and* $\mathbf{T_0}$ *are proof-theoretically equivalent. In particular, these theories prove the same theorems in the negative arithmetic fragment.*

Theorem 3.2 (Rathjen [80]; Setzer [101]) *The consistency of* $\Delta^1_2 - \mathbf{CA}+\mathbf{BI}$ *and* **KPi** *is provable in Martin-Löf's 1984 type theory.*

On the part of the intuitionists/constructivists, the following objection could be raised against the significance of consistency proofs: even if it had been constructively demonstrated that the classical theory T cannot lead to mutually contradictory results, the theorems of T would nevertheless be propositions without sense and their investigation therefore an idle pastime. Well, it turns out that the constructive well-ordering proof of the representation system used in the analysis of $\Delta^1_2 - \mathbf{CA}+\mathbf{BI}$ yields more than the mere consistency of the latter system. For the important class of Π^0_2 statements one obtains a conservativity result.

Theorem 3.3 (Rathjen [80]; Setzer [101])

- *The soundness of the negative arithmetic fragment of* $\Delta^1_2 - \mathbf{CA}+\mathbf{BI}$ *(or* **KPi***) is provable in Martin-Löf's 1984 type theory.*
- *Every* Π^0_2 *statement provable in* $\Delta^1_2 - \mathbf{CA}+\mathbf{BI}$ *(or* **KPi***) has a proof in Martin-Löf's 1984 type theory.*

3.2 Reductions of theories formulated as conservation theorems

The motivation for an extended Hilbert programme depends on the conviction that constructive methods are, in some sense, superior. Another way to conceive of the results mentioned in the previous subsection is to simply view them as proof-theoretic reductions and to formulate them as conservation theorems. Ordinal analyses have been used many times to prove that a foundationally interesting theory is in some sense reducible to or equiconsistent with another foundationally interesting theory. Here I'm going to list just a few examples, their selection being a very biased choice. A plethora of further reductions can be found in [69, 43, 15, 72, 71].

1. The proofs that the theories $\mathbf{ATR_0}$ and $\mathbf{KPi_0}$ are reducible to Feferman's system of predicative analysis, **IR**, in the sense that they are conservative over **IR** for Π^1_1 sentences involves the ordinal Γ_0. The foundational significance of these systems is as follows. $\mathbf{ATR_0}$ is a subsystem of second order arithmetic that frequently arises in reverse mathematics and is equivalent to many mathematical statements, e.g. the open Ramsey theorem, the perfect set theorem, Ulm's theorem, the König duality theorem for countable bipartite graphs, etc. $\mathbf{KPi_0}$ is, on the one hand, an extension of Kripke-Platek set theory via an axiom that asserts that any set is contained in an admissible set, but, on the other hand, a weakening of Kripke-Platek set theory in that the foundation axiom is completely missing.

   The reductions can be described as contributing to a foundational program of predicative reductionism. The reduction of $\mathbf{ATR_0}$ to **IR** was obtained in [29]. The reduction of $\mathbf{KPi_0}$ to **IR** is due to [43].

2. The study of formal theories featuring inductive definitions in both single and iterated form was initiated by Kreisel [49]. The immediate stimulus was the question of constructive justification of Spector's 1961 consistency proof for analysis via his interpretation in the so-called bar-recursive functionals of finite type. Let ν denote a fixed ordinal in a given ordinal representation system. Ordinal analysis has shown that the classical theory $\mathbf{ID}^{\mathbf{c}}_{\nu}$ of ν-times iterated arithmetical inductions is reducible to the intuitionistic theory $\mathbf{ID}^{i}_{\nu}(\mathfrak{O})$ of ν-times iterated constructive number classes $\mathfrak{O}$. The history of these results is described in the monograph [14].[4]

3. T_0 is Feferman's system of Explicit mathematics. The results of Feferman [23], Jäger [42], Jäger and Pohlers [44]) yield that $\Delta^1_2-\mathbf{CA}+\mathbf{BI}$, **KPi**, and $\mathbf{T_0}$ are proof-theoretically equivalent. In particular, these theories prove the same theorems in the negative arithmetic fragment.

   No proof of the above result has been found that doesn't use ordinal representations.

4. Inspired by work of Myhill [61] on constructive set theories, Aczel (cf. [1, 2, 3]) proposed an intuitionistic set theory, termed *Constructive Zermelo-Fraenkel set*

[4]For limit ordinals ν, Sieg (cf. [14]) obtained the reduction $\bigcup_{\alpha<\nu}\mathbf{ID}^{\mathbf{c}}_{\alpha}\equiv\bigcup_{\alpha<\nu}\mathbf{ID}^{\mathbf{i}}_{\alpha}(\mathfrak{O})$ without the use of ordinal analysis, but his approach is still proof-theoretic as it employs cut-elimination for infinitary derivations.

theory (**CZF**), that bears a close relation to Martin-Löf type theory. The novel ideas were to replace Powerset by the (classically equivalent) Subset Collection Axiom and to discard full Comprehension while strengthening Collection to Strong Collection. Aczel corroborated the constructiveness of **CZF** by interpreting it in Martin-Löf's intuitionistic type theory. A very nice aspect of **CZF** is the fact that one can develop a good theory of large sets with the right consistency strength. Since in intuitionistic set theory $\in$ is not a linear ordering on ordinals the notion of a cardinal does not play a central role. Consequently, one talks about *"large set properties"* instead of *"large cardinal properties"*. Classically though, the large cardinal axioms and the pertinent large set axiom are of the same strength.

Up to now, the notions of inaccessible, Mahlo and 2-strong sets that classically correspond to inaccessible, Mahlo and weakly compact cardinals, respectively, have been investigated (cf. [85, 86, 87]). As to consistency strength and conservativity the following theories have the same consistency strength and actually prove the same Π^0_2-sentences:

(i) $\mathbf{CZF} + \forall x \exists I\, [x \in I \,\wedge$ "I inaccessible"$]$ and
$\mathbf{KP} + \forall \alpha \exists \kappa\, [\alpha \in \kappa \,\wedge$ "κ recursively inaccessible"$]$.

(ii) $\mathbf{CZF} + \forall x \exists M\, [x \in M \,\wedge$ "M Mahlo"$]$ and
$\mathbf{KP} + \forall \alpha \exists \kappa\, [\alpha \in \kappa \,\wedge$ "κ recursively Mahlo ordinal"$]$.

(iii) $\mathbf{CZF} + \forall x \exists K\, [x \in K \,\wedge$ "K 2-strong"$]$ and
$\mathbf{KP} + \forall \alpha \exists \kappa\, [\alpha \in \kappa \,\wedge$ "κ Π_3-reflecting"$]$.

The proof that the intuitionistic theory has at least the strength of the classical one requires an ordinal analysis of the classical theories as given in [73, 74, 82] and a proof of the well-foundedness of the pertinent ordinal representation system in the intuitionistic theory.

3.3 Combinatorial independence results and new combinatorial principles

Since 1931, the year Gödel's Incompleteness Theorems were published, logicians have been looking for a strictly mathematical example of an incompleteness in first-order Peano arithmetic, one which is mathematically simple and interesting and does not require the numerical coding of notions from logic. The first such examples were found early in 1977. The most elegant of these is a strengthening of the Finite Ramsey Theorem due to Paris and Harrington (cf. [64]). The original proofs of the independence of combinatorial statements from **PA** all used techniques from non-standard models of arithmetic. Only later on alternative proofs using proof-theoretic techniques were found. However, results from ordinal-theoretic proof theory turned out to be pivotal in providing independence results for stronger theories than **PA**, and even led to a new combinatorial statement. The stronger theories referred to are Friedman's system $\mathbf{ATR}_0$ of *arithmetical transfinite recursion* and the system $\Pi^1_1 - CA$ based on Π^1_1-comprehension. The independent combinatorial statements have their origin in certain embeddability questions in the theory of finite graphs. The first is a famous theorem of Kruskal asserting that every set of finite trees has only finitely many minimal elements.

Definition 3.4 A *finite tree* is a finite partially ordered set $\mathbb{B} = (B, \leq)$ such that:

(i) B has a smallest element (called the *root* of $\mathbb{B}$);

(ii) for each $s \in B$ the set $\{t \in B : t \leq s\}$ is a totally ordered subset of B.

Definition 3.5 For finite trees $\mathbb{B}_1$ and $\mathbb{B}_2$, an *embedding* of $\mathbb{B}_1$ into $\mathbb{B}_2$ is a one-to-one mapping $f : \mathbb{B}_1 \to \mathbb{B}_2$ such that $f(a \wedge b) = f(a) \wedge f(b)$ for all $a, b \in \mathbb{B}_1$, where $a \wedge b$ denotes the infimum of a and b.

We write $\mathbb{B}_1 \leq \mathbb{B}_2$ to mean that there exists an embedding $f : \mathbb{B}_1 \to \mathbb{B}_2$.

Theorem 3.6 (Kruskal's theorem) *For every infinite sequence of trees* $\big(\mathbb{B}_k : k < \omega\big)$, *there exist indices* i *and* j *such that* $i < j < \omega$ *and* $\mathbb{B}_i \leq \mathbb{B}_j$. *(In particular, there is no infinite set of pairwise nonembeddable trees.)*

Theorem 3.7 *Kruskal's Theorem is not provable in* $\mathbf{ATR}_0$ *(*cf. [100]*).*

The proof of the above independence result exploits a connection between finite trees and ordinal representations for ordinals $< \Gamma_0$ and the fact that Γ_0 is the proof-theoretic ordinal of $\mathbf{ATR}_0$. Each ordinal representation $\mathfrak{a}$ is assigned a finite tree $\mathbb{B}_{\mathfrak{a}}$ to the effect that for two representations $\mathfrak{a}$ and $\mathfrak{b}$, $\mathbb{B}_{\mathfrak{a}} \leq \mathbb{B}_{\mathfrak{b}}$ implies $\mathfrak{a} \leq \mathfrak{b}$. Hence Kruskal's theorem implies the well-foundedness of Γ_0 and is therefore not provable in $\mathbf{ATR}_0$. The connection between finite trees and ordinal representations for ordinals $< \Gamma_0$ was noticed by Friedman (cf. [100]) and independently by Diana Schmidt (cf. [93]).

A hope in connection with ordinal analyses is that they lead to new combinatorial principles which encapsulate considerable proof-theoretic strength. Examples are still scarce. One case where ordinal notations led to a new combinatorial result was Friedman's extension of Kruskal's Theorem, EKT, which asserts that finite trees are well-quasi-ordered under gap embeddability (see [100]). The gap condition imposed on the embeddings is directly related to an ordinal notation system that was used for the analysis of Π^1_1 comprehension. The principle EKT played a crucial role in the proof of the graph minor theorem of Robertson and Seymour (see [30]).

Definition 3.8 For $n < \omega$, let $\mathcal{B}_n$ be the set of all finite trees with labels from n, i.e. $(\mathbb{B}, \ell) \in \mathcal{B}_n$ if $\mathbb{B}$ is a finite tree and $\ell : B \to \{0, \ldots, n-1\}$. The set $\mathcal{B}_n$ is quasiordered by putting $(\mathbb{B}_1, \ell_1) \leq (\mathbb{B}_2, \ell_2)$ if there exists an embedding $f : \mathbb{B}_1 \to \mathbb{B}_2$ with the following properties:

1. for each $b \in B_1$ we have $\ell_1(b) = \ell_2(f(b))$;

2. if b is an immediate successor of $a \in \mathbb{B}_1$, then for each $c \in \mathbb{B}_2$ in the interval $f(a) < c < f(b)$ we have $\ell_2(c) \geq \ell_2(f(b))$.

The condition (ii) above is called a *gap condition.*

Theorem 3.9 *For each* $n < \omega$, $\mathcal{B}_n$ *is a well quasi ordering (abbreviated* $\mathrm{WQO}(\mathcal{B}_n)$*), i.e. there is no infinite set of pairwise nonembeddable trees.*

Theorem 3.10 *$\forall n < \omega$ WQO($\mathcal{B}_n$) is not provable in $\Pi^1_1 - \mathbf{CA}_0$.*

The proof of Theorem 3.10 employs an ordinal representation system for the proof-theoretic ordinal of $\Pi^1_1 - \mathbf{CA}_0$. The ordinal is $\psi_0(\Omega_\omega)$ in the ordinal representation system of [8] or $\theta\Omega_\omega 0$ in that of [97]. Let $\mathcal{T}(\psi_0(\Omega_\omega))$ denote the ordinal representation system. The connection between $< \omega$ labelled trees and $\mathcal{T}(\psi_0(\Omega_\omega))$ is that $\forall n < \omega$ WQO($\mathcal{B}_n$) implies the wellfoundedness of $\mathcal{T}(\psi_0(\Omega_\omega))$ on the basis of $\mathbf{ACA}_0$. The connection is even closer in that the gap condition imposed on the embeddings between trees is actually gleaned from the ordering of the terms in $\mathcal{T}(\psi_0(\Omega_\omega))$. If one views these terms as labelled trees, then the gap condition is exactly what one needs to ensure that an embedding of two such trees implies that the ordinal corresponding to the first tree is less than the ordinal corresponding to the second tree.

It is also for that reason that criticism had been levelled against the principle EKT for being too contrived or too metamathematical. But this was superseded by the crucial role that EKT played in the proof of the graph minor theorem of Robertson and Seymour (see [30]).

As to the importance attributed to the graph minor theorem, I quote from a book on Graph Theory [17], p. 249.

> *Our goal [...] is a single theorem, one which dwarfs any other result in graph theory and may doubtless be counted among the deepest theorems that mathematics has to offer:* in every infinite set of graphs there are two such that one is a minor of the other. *This* minor theorem*, inconspicuous though it may look at first glance, has made a fundamental impact both outside graph theory and within. Its proof, due to Neil Robertson and Paul Seymour, takes well over 500 pages.*

Definition 3.11 Let $e = xy$ be an edge of a graph $G = (V, E)$, where V and E denote its vertex and edge set, respectively. By G/e we denote the graph obtained from G by *contracting* the edge e into a new vertex v_e, which becomes adjacent to all the former neighbours of x and of y. Formally, G/e is a graph (V', E') with vertex set $V' := (V \setminus \{x, y\}) \cup \{v_e\}$ (where v_e is the "new" vertex, i.e. $v \notin V \cup E$) and edge set

$$\begin{aligned} E' \;&:=\; \big\{vw \in E | \{v, w\} \cap \{x, y\} = \emptyset\big\} \\ &\quad\; \cup \big\{v_e w | xw \in E \setminus \{e\} \;\; \vee \;\; yw \in E \setminus \{e\}\big\}. \end{aligned}$$

If X is obtained from Y by first deleting some vertices and edges, and then contracting some further edges, X is said to be a *minor* of Y. In point of fact, the order in which deletions and contractions are applied is immaterial as any graph obtained from another by repeated deletions and contractions in any order is its minor.

Theorem 3.12 (Robertson and Seymour 1986-1997) *If $G_0, G_1, G_2, \ldots$ is an infinite sequence of finite graphs, then there exist $i < j$ so that G_i is isomorphic to a minor of G_j.*

Corollary 3.13 *(i)* (Vázsonyi's conjecture) *If all the G_k are trivalent, then there exist $i < j$ so that G_i is embeddable into G_j.*

(ii) (Wagner's conjecture) *For any 2-manifold M there are only finitely many graphs which are not embeddable in M and are minimal with this property.*

Theorem 3.14 (Friedman, Robertson, Seymour [30])

(i) GMT *implies* EKT *within, say,* $\mathbf{RCA}_0$.

(ii) GMT *is not provable in* $\Pi^1_1 - \mathbf{CA}_0$.

A further independence result that ensues from ordinal analysis is due to Buchholz [9]. It concerns an extension of the hydra game of Kirby and Paris. It is shown in [9] that the assertion that Hercules has a winning strategy in this game is not provable in the theory $\Pi^1_1 - \mathbf{CA} + \mathbf{BI}$.

3.4 Classifications of provable functions and ordinals

An apt leitmotif for this subsection is provided by Kreisel's question (cf. [48]): *"What more do we know if we have proved a theorem by restricted means than if we merely know that it is true?"*

3.4.1 Provable recursive functions

In the case of **PA** an answer to the foregoing question was provided by Kreisel in [47], where he characterized the provably recursive functions of **PA** as those which are α-recursive for some $\alpha < \varepsilon_0$. However, there is nothing special about **PA** when it comes to extracting the latter kind of information. Indeed, it is a general fact that an ordinal analysis of a theory T yields, as a by-product, a characterization of the provably recursive functions of T. As stated in section 2, an ordinal analysis of T via an ordinal representation system $\langle A, \lhd, \dots \rangle$ provides a reduction of T to $\mathbf{PA} + \bigcup_{\alpha \in A} \mathrm{TI}(A_{\bar{\alpha}}, \lhd_{\bar{\alpha}})$ and further ensures Π^0_2-conservativity. On the strength of the latter, it suffices to characterize the provably recursive functions of

$$\mathbf{S} := \mathbf{PA} + \bigcup_{\alpha \in A} \mathrm{TI}(A_{\bar{\alpha}}, \lhd_{\bar{\alpha}})$$

for EORSs $\langle A, \lhd, \dots \rangle$.

Definition 3.15 Let $\alpha \in A$ such that $0 \lhd \alpha$. A number-theoretic function f is called α-recursive if it can be generated by the usual schemes for generating primitive recursive functions plus the following scheme:

$$f(m, \vec{n}) = \begin{cases} h(m, \vec{n}, f(\theta(m, \vec{n}), \vec{n})) & \text{if } 0 \lhd m \lhd \alpha \\ g(m, \vec{n}) & \text{otherwise,} \end{cases}$$

where g, h, θ are α-recursive and θ satisfies $\theta(\beta, \vec{x}) \lhd \beta$ whenever $0 \lhd \beta \lhd \alpha$.

Theorem 3.16 *The provably recursive functions of* $\mathbf{PA} + \bigcup_{\alpha \in A} \mathrm{TI}(A_{\bar{\alpha}}, \lhd_{\bar{\alpha}})$ *are excactly the recursive functions which are* α*-recursive for some* $\alpha \in A$.

The technical tool for achieving this characterization is to embed $\mathbf{PA} + \bigcup_{\alpha \in A} \mathrm{TI}(A_{\bar{\alpha}}, \lhd_{\bar{\alpha}})$ into a system of Peano arithmetic with an infinitary rule, the so-called ω-rule, and a *repetition rule*, `Rep`, which simply repeats the premise as the conclusion. The ω-rule allows one to infer $\forall x \phi(x)$ from the infinitely many premises

$\phi(\bar{0}), \phi(\bar{1}), \phi(\bar{2}), \ldots$ (where $\bar{n}$ denotes the nth numeral); its addition accounts for the fact that the infinitary system enjoys cut-elimination. The addition of the `Rep` rule enables one to carry out a *continuous cut elimination*, due to Mints [59], which is a continous operation in the usual tree topology on prooftrees. A further pivotal step consists in making the ω-rule more constructive by assigning codes to proofs, where codes for applications of finitary rules contain codes for the proofs of the premises, and codes for applications of the ω-rule contain Gödel numbers for partial functions enumerating codes of the premises. The aforementioned enumerating functions can be required to be partial recursive, making the proof trees recursive, or even primitive recursive in the presence of the rule `Rep` which enables one to stretch recursive trees into primitive recursive trees. Theorem 3.16 can be extracted from Kreisel-Mints-Simpson [52], Lopez-Escobar [53], or Schwichtenberg [99] and was certainly known to these authors. A variant of the characterization of Theorem 3.16 is given in Friedman-Sheard [32], where the provable functions of $\mathbf{PA} + \bigcup_{\alpha \in A} \mathrm{TI}(A_{\bar{\alpha}}, \lhd_{\bar{\alpha}})$ are classified as the *descent recursive functions over A.* But before discussing this and related results, I'd like to draw attention to a more recent approach which has the great advantage over the previous one that one need not bother with codes for infinite derivations. In this approach one adds an extra feature to infinite derivations by which one can exert a greater control on derivations so as to be able to directly read off numerical bounds from cut free proofs of Σ^0_1 statements. This has been carried out by Buchholz-Wainer [11] for the special case of **PA**. In much greater generality and flexibility this approach has been developed by Weiermann [112].

The remainder of this section presents further results about theories of the shape $\mathbf{PA} + \bigcup_{\alpha \in A} \mathrm{TI}(A_{\bar{\alpha}}, \lhd_{\bar{\alpha}})$, thereby providing more information that can be extracted from ordinal analyses. Propositions 3.19, 3.20, and 3.23 are due to Friedman-Sheard [32].

Definition 3.17 For each $\alpha \in A$, $\mathrm{ERWF}(\lhd, \bar{\alpha})$ is the schema

$$\forall \vec{x} \exists y [f(\vec{x}, y) \unlhd f(\vec{x}, y+1) \ \vee \ f(\vec{x}, y) \notin A \ \vee \ \bar{\alpha} \unlhd f(\vec{x}, y)]$$

for each (definition of an) elementary function f.

$\mathrm{ERWF}(\lhd)$ is the schema

$$\forall \vec{x} \exists y [f(\vec{x}, y) \unlhd f(\vec{x}, y+1) \ \vee \ f(\vec{x}, y) \notin A]$$

for each elementary function f.

The schemata $\mathrm{PRWF}(\lhd, \bar{\alpha})$ and $\mathrm{PRWF}(\lhd)$ are defined identically, except that f ranges over the primitive recursive functions.

Definition 3.18 $\mathbf{DRA}_{\langle \mathbf{A}, \lhd \rangle}$ (*Descent Recursive Arithmetic*) is the theory whose axioms are $\mathbf{ERA} + \bigcup_{\alpha \in A} \mathrm{ERWF}(\lhd, \bar{\alpha})$.

$\mathbf{DRA}(\lhd^{+})$ is the theory whose axioms are $\mathbf{ERA} + \mathrm{ERWF}(\lhd)$.

The difference is that $\mathbf{DRA}(\lhd)$ asserts only the non-existence of elementary infinitely descending sequences below each $\alpha \in A$, where α is given at the meta-level.

Combined with 2.8 the latter result leads to a neat characterization of the provably recursive functions of $\mathbf{T}$ due to the following observation:

Proposition 3.19 ([32]) *The provably recursive functions of* $\mathbf{DRA}_{\langle\mathbf{A},\lhd\rangle}$ *are all functions* f *of the form*

$$f(\vec{m}) \quad = \quad g(\vec{m}, \textit{least}\, n.h(\vec{m}, n) \unlhd h(\vec{m}, n+1)) \tag{47}$$

where g *and* h *are elementary functions and* $\mathbf{ERA} \vdash \forall\vec{x}y\, h(\vec{x}, y){\in}A_{\bar{\alpha}}$ *for some* $\alpha{\in}A$.

The above class of recursive functions will be referred to as the *descent recursive functions over* A.

Proposition 3.20 ([32, 4.4]) $\mathbf{DRA}_{\langle\mathbf{A},\lhd\rangle}$ *and* $\mathbf{PA} + \bigcup_{\alpha\in A} \mathrm{TI}(A_{\bar{\alpha}}, \lhd_{\bar{\alpha}})$ *prove the same* Π^0_2 *sentences.*

From 2.8 and 3.20 we get:

Observation 3.21 *Suppose an ordinal analysis of the formal system* T *has been attained using an EORS* $\langle A, \lhd, \dots\rangle$. *Then the provably recursive functions of* T *are the descent recursive functions over* A.

We shall list some complementary results.

Definition 3.22 If T is a theory, the 1-*consistency of* T is the schema

$$\forall u[Pr_T(\ulcorner F(\dot{u})\urcorner) \to F(u)]$$

for Σ^0_1 formulae $F(u)$ with one free variable u.

Proposition 3.23 ([32, 4.5]) *The following are equivalent over* **PRA**:

(i) 1-consistency of $\mathbf{PA} + \bigcup_{\alpha\in A} \mathrm{TI}(A_{\bar{\alpha}}, \lhd_{\bar{\alpha}})$

(ii) $\mathrm{PRWF}(\lhd^+)$

(ii) $\mathrm{ERWF}(\lhd^+)$.

Observation 3.24 *Again, let* T *be a theory for which an ordinal analysis has been carried out via* $\langle A, \lhd\rangle$. *Then the following are equivalent over* **PRA**:

(i) 1-consistency of T

(ii) $\mathrm{PRWF}(\lhd^+)$

(ii) $\mathrm{ERWF}(\lhd^+)$.

A characterization of the provably recursive functions of a formal system T as the α-recursive functions for $\alpha \in A$ or the descent recursive functions over A is notation-sensitive. None the less, it is sometimes possible to extract further notation-free information, in particular independence results that are not couched in terms of a given ordinal representation system $\langle A, \lhd, \dots\rangle$. Usually, though, to obtain such results one needs to be furnished with a specific well-structured hierarchy $(F_\alpha)_{\alpha\in A}$ of functions such that every provable function of T is majorized by a function in the hierarchy. An example is Kirby and Paris' result on the unprovability of the termination of all Goodstein sequences in **PA**. A proof-theoretic proof of this result (cf. [11, 108])

employs the fact that for every provable recursive functions of **PA** there is a function H_α for some $\alpha < \varepsilon_0$ that majorizes it. Here H_α is the αth function in the so-called Hardy hierarchy. What complicates matters is that the definition of the functions H_α hinges upon a particular assignment of fundamental sequences to limit ordinals. It appears that only "natural" assignments of fundamental sequences, which take into account their algebraic properties rather than their codes, lead to function hierarchies that can be used for combinatorial independence results. A general discussion about different hierarchies $(F_\alpha)_{\alpha \in A}$ and their relations can be found in [13].

3.4.2 Provable set functions and ordinals

The extraction of classifications of provable functions from ordinal analyses is not confined to recursive functions on natural numbers. In the case of fragments of second order arithmetic, one may also classify the provable hyperarithmetical as well as the provable Δ^1_2 functions on $\mathbb{N}$. In the case of set theories one may classify several kinds of provable set functions.

In the following we will be concerned with norms that can be assigned to set theories. In general, they can also be extracted from an ordinal analysis of a set theory T. Among other results, they lead to a classification of the provable set functions of T.

The first of these norms will be denoted $|T|^E$, where the superscript E signifies E*-recursion*, also termed *set recursion*. E-recursion theory extends the notion of computation from the natural numbers to arbitrary sets. For details see [90].

Definition 3.25 The intent is to assign meaning to $\{e\}(x)$ for every set x via an appropriate notion of computation. E-recursion is defined by the following schemes:

1. $e = \langle 1, n, i \rangle$,
   $\{e\}(x_1, \ldots, x_n) = x_i$.
2. $e = \langle 2, n, i, j \rangle$,
   $\{e\}(x_1, \ldots, x_n) = x_i \backslash x_j$.
3. $e = \langle 3, n, i, j \rangle$,
   $\{e\}(x_1, \ldots, x_n) = \{x_i, x_j\}$.
4. $e = \langle 4, n, c \rangle$,
   $\{e\}(x_1, \ldots, x_n) = \bigcup \{\{c\}(y, x_2, \ldots, x_n) : \ y \in x_1\}$.
   The left side is not defined unless $\{c\}(y, x_2, \ldots, x_n)$ is defined for all $y \in x_1$.
5. $e = \langle 5, n, m, e', e_1, \ldots, e_n \rangle$,
   $\{e\}(x_1, \ldots, x_n) \simeq \{e'\}(\{e_1\}(x_1, \ldots, x_n), \ldots, \{e_m\}(x_1, \ldots, x_n))$.
6. $e = \langle 6, n, m \rangle$,
   $\{e\}(e_1, x_1, \ldots, x_n, y_1, \ldots, y_m) \simeq \{e_1\}(x_1, \ldots, x_n)$.

$\simeq$ is Kleene's symbol for strong equality. If g and f are partial functions, then $f(x) \simeq g(x)$ iff neither $f(x)$ nor $g(x)$ is defined, or $f(x)$ and $g(x)$ are defined and equal.

Recall that L_α, the αth level of Gödel's constructible hierarchy L, is defined by $L_0 = \emptyset$, $L_{\beta+1} = \big\{X : X \subseteq L_\beta;\ X \text{ definable over } \langle L_\beta, \in\rangle\big\}$ and $L_\lambda = \bigcup\{L_\beta : \beta < \lambda\}$ for limits λ. So any element of L of level α is definable from elements of L with levels $< \alpha$ and L_α.

Definition 3.26 For a collection of formulae (in the language of set theory), $\mathcal{F}$, we say that L_α is an *$\mathcal{F}$-model of T* if for all $B \in \mathcal{F}$, whenever $T \vdash B$, then $L_\alpha \models B$. Let

$$|T|_{\mathcal{F}} \ := \ \min\{\alpha :\ L_\alpha \text{ is an } \mathcal{F}\text{-model of } T\}.$$

Definition 3.27 The next notions are due to A. Schlüter [91].

$$|T|^E_{\Sigma_1} \ := \ \min\{\alpha :\ \text{for all } e \in \omega,\ T \vdash \{e\}(\omega)\downarrow \text{ implies } \{e\}(\omega) \in L_\alpha\}.$$

$$|T|_{\Pi^E_2} := \\ \min\{\alpha > \omega :\ \text{for all } e \in \omega,\ T \vdash \forall x\, \{e\}(x)\downarrow \text{ implies } \forall x{\in}L_\alpha \{e\}(x) \in L_\alpha\}.$$

Definition 3.28 Let $\mathcal{F}$ be a collection of sentences. A set theory T is said to be *$\mathcal{F}$-sound* if for every $\mathcal{F}$ theorem ϕ of T, $L \models \phi$ holds.

For a collection of formulae $\mathcal{F}$, let $\mathcal{F}(L_\alpha)$ consist of all formulae A^{L_α} with $A \in \mathcal{F}$.

The system **PRST** (for *Primitive Recursive Set Theory*) is formulated in the language of set theory augmented by symbols for all primitive recursive set functions. The *axioms of* **PRST** are Extensionality, Pair, Union, Infinity, Δ_0-Separation, the Foundation Axiom (i.e. $x \neq \emptyset \to (\exists y{\in}x)(\forall z{\in}y)\, z{\notin}x$) and the defining equations for the primitive recursive set functions.

In the following we shall assume that all set theories contain **PRST** either directly or via interpretation.

Proposition 3.29 *Suppose T is Π_2 sound and comprises Δ_0-collection. Furthermore, suppose that $T \vdash B$ implies $T \vdash \exists\alpha\exists x(x{=}L_\alpha \wedge B^x)$ for all Σ_1-sentences B. If T has a Σ_1-model then T has a Π_2-model and*

$$|T|_{\Sigma_1} \ = \ |T|_{\Pi_2}. \tag{48}$$

Proof: [76]. Theorem 2.1. □

Proposition 3.30 *If T is a Π_2 sound theory, then*

$$|T|^E_{\Sigma_1} \ = \ |T|^E_{\Pi_2}. \tag{49}$$

Recall that ω_1^{CK} stands for the least admissible ordinal $> \omega$. If, in addition, T proves the existence of ω_1^{CK}, then

$$|T|_{\Sigma_1(L(\omega_1^{CK}))} \ = \ |T|_{\Pi_2(L(\omega_1^{CK}))}. \tag{50}$$

Proof: (50) is an immediate consequence of the proof of [76], Theorem 2.1 and a slight modification of the latter proof yields (49). (49) is stated and proved in [91], 6.14. □

Theorem 3.31 *If T is Π_2-sound and $T \vdash \forall x\, \exists y\, [x \in y \ \wedge$ "y is an admissible set"], then*

$$|T|_{\sup} = |T|^E_{\Sigma_1} = |T|^E_{\Pi_2} = |T|_{\Sigma_1(L(\omega_1^{CK}))} = |T|_{\Pi_2(L(\omega_1^{CK}))}.$$

Proof: A detailed proof of $|T|^E_{\Sigma_1} = |T|_{\Sigma_1(L(\omega_1^{CK}))}$ can be found in [91], Satz 6.15. The equality $|T|_{\sup} = |T|_{\Sigma_1(L(\omega_1^{CK}))}$ also follows from the proof of [91], Satz 6.15, but has been observed previously (cf. [74], Theorem 7.14). □

Definition 3.32 Another notion that is closely related to the the norm $|T|_{\Sigma_1}$ is the notion of *good* Σ_1*-definition* from admissible set theory (see [5], II.5.13). Given a set theory T, we say that an ordinal α has a *good* Σ_1*-definition in* T if there is a Σ_1-formula $\phi(u)$ such that

$$L \models \phi[\alpha] \text{ and } T \vdash \exists! x \phi(x).$$

Let

$$\mathbf{sp}_{\Sigma_1}(T) := \{\alpha : \ \alpha \text{ has a good } \Sigma_1 \text{ definition in } T\}.$$

One obviously has $\sup(\mathbf{sp}_{\Sigma_1}(T)) = |T|_{\Sigma_1}$. In many cases the set $\mathbf{sp}_{\Sigma_1}(T)$ bears interesting connections to the ordinals of the representation system that has been used to analyze T. Ordinal representation systems that have been developed via a detour through large cardinals allow for an alternative interpretation wherein the large cardinals are replaced by their recursively large counterparts. The latter interpretation gives rise to a canonical interpretation of the ordinal terms of the representation system in $\mathbf{sp}_{\Sigma_1}(T)$. In general, however, the ordinals of $\mathbf{sp}_{\Sigma_1}(T)$ stemming from the ordinal representation form a proper subset of $\mathbf{sp}_{\Sigma_1}(T)$ with many 'holes'. It would be very desirable to find a 'natural' property which could distinguish the ordinals of the representation system within $\mathbf{sp}_{\Sigma_1}(T)$ so as to illuminate their naturalness. I consider this to be one of the most important problems in the area of strong ordinal representation systems. A more thorough discussion will follow in section 4.

4 Examples of ordinal analyses

In this last section, I'm going to sketch the ordinal analyses of two systems of set theory which are intended to illustrate the main ideas and techniques used in ordinal analysis. Some attempts will be made to explain the role of large cardinals that appear in the definition procedures of so-called *collapsing functions* which then give rise to strong ordinal representation systems.

4.1 A brief history of ordinal analyses

To set the stage for the following, a very brief history of ordinal-theoretic proof theory since Gentzen reads as follows: In the 1950's proof theory flourished in the hands of Schütte: in [94] he introduced an infinitary system for first order number theory with the so-called ω-rule, which had already been proposed by Hilbert [36]. Ordinals were assigned as lengths to derivations and via cut-elimination he re-obtained Gentzen's ordinal analysis for number theory in a particularly transparent way. Further, Schütte extended his approach to systems of ramified analysis and brought this technique to perfection in his monograph "Beweistheorie" [95]. Independently, in 1964 Feferman [21] and Schütte [96], [97] determined the ordinal bound Γ_0 for theories of autonomous ramified progressions.

A major breakthrough was made by Takeuti in 1967, who for the first time obtained an ordinal analysis of a strong fragment of second order arithmetic. In [106] he gave an ordinal analysis of Π^1_1 comprehension, extended in 1973 to Δ^1_2 comprehension in [107] jointly with Yasugi. For this Takeuti returned to Gentzen's method of assigning ordinals (ordinal diagrams, to be precise) to purported derivations of the empty sequent (inconsistency).

The next wave of results, which concerned theories of iterated inductive definitions, were obtained by Buchholz, Pohlers, and Sieg in the late 1970's (see [14]). Takeuti's methods of reducing derivations of the empty sequent ("the inconsistency") were extremely difficult to follow, and therefore a more perspicuous treatment was to be hoped for. Since the use of the infinitary ω-rule had greatly facilitated the ordinal analysis of number theory, new infinitary rules were sought. In 1977 (see [7]) Buchholz introduced such rules, dubbed Ω-rules to stress the analogy. They led to a proof-theoretic treatment of a wide variety of systems, as exemplified in the monograph [15] by Buchholz and Schütte. Yet simpler infinitary rules were put forward a few years later by Pohlers, leading to the *method of local predicativity*, which proved to be a very versatile tool (see [66, 67, 68]). With the work of Jäger and Pohlers (see [40, 41, 44]) the forum of ordinal analysis then switched from the realm of second-order arithmetic to set theory, shaping what is now called *admissible proof theory*, after the models of *Kripke-Platek set theory*, **KP**. Their work culminated in the analysis of the system with Δ^1_2 comprehension plus **BI** [44]. In essence, admissible proof theory is a gathering of cut-elimination techniques for infinitary calculi of ramified set theory with Σ and/or Π_2 reflection rules[5] that lend itself to ordinal analyses of theories of the form **KP**+ *"there are x many admissibles"* or **KP**+ *"there are many admissibles"*. By way of illustration, the subsystem of analysis with Δ^1_2 comprehension and bar induction can be couched in such terms, for it is naturally interpretable in the set theory $\mathbf{KPi} := \mathbf{KP} + \forall y \exists z (y \in z \wedge z \textit{ is admissible})$ (cf. [44]).

After an intermediate step [74], which dealt with a set theory **KPM** that formalizes a recursively Mahlo universe, a major step beyond admissible proof theory was taken in [82]. That paper featured ordinal analyses of extensions of **KP** by Π_n reflection. A generalization of the methods of [82] underlies the treatment of $\Pi^1_2 - \mathbf{CA}$ sketched in [83].

4.2 An ordinal analysis of Kripke-Platek set theory

Until the late 70s the systems treated by ordinal analysis were either fragments of second order arithmetic or theories of iterated inductive definitions. A direct proof-theoretic treatment of systems of set theory was pioneered by Jäger (cf. [40, 41]). A first impression of ordinal analysis will be given by way of the example of Kripke-Platek set theory.

[5]Recall that the salient feature of admissible sets is that they are models of Δ_0 collection and that Δ_0 collection is equivalent to Σ reflection on the basis of the other axioms of **KP** (see [5]). Furthermore, admissible sets of the form L_α also satisfy Π_2 reflection.

4.2.1 The system KP

Though considerably weaker than **ZF**, a great deal of set theory requires only the axioms of **KP**. The axioms of **KP** are:[6]

Extensionality:	$a = b \rightarrow [F(a) \leftrightarrow F(b)]$ for all formulas F.
Foundation:	$\exists x G(x) \rightarrow \exists x[G(x) \wedge (\forall y \in x)\neg G(y)]$
Pair:	$\exists x\ (x = \{a, b\})$.
Union:	$\exists x\ (x = \bigcup a)$.
Infinity:	$\exists x\ \big[x \neq \emptyset\ \wedge\ (\forall y \in x)(\exists z \in x)(y \in z)\big]$.[7]
Δ_0 *Separation:*	$\exists x\ \big(x = \{y \in a : F(y)\}\big)$[8] for all Δ_0–formulas F in which x does not occur free.
Δ_0 *Collection:*	$(\forall x \in a)\exists y G(x, y) \rightarrow \exists z(\forall x \in a)(\exists y \in z)G(x, y)$ for all Δ_0–formulas G.

By a Δ_0 formula we mean a formula of set theory in which all the quantifiers appear restricted, that is have one of the forms $(\forall x \in b)$ or $(\exists x \in b)$.

KP arises from **ZF** by completely omitting the power set axiom and restricting separation and collection to absolute predicates (cf. Barwise [1975]), i.e. Δ_0 formulas. These alterations are suggested by the informal notion of 'predicative'.

4.2.2 An ordinal representation system for the Bachmann-Howard ordinal

This section introduces an ordinal representation system which encapsulates the strength of **KP**.

Definition 4.1 The *Veblen-function* φ figures prominently in elementary proof theory (cf. [22, 70, 98]). It is defined by transfinite recursion on α by letting $\varphi_0(\xi) := \omega^\xi$ and, for $\alpha > 0$, φ_α be the function that enumerates the class of ordinals

$$\{\gamma : \forall \xi < \alpha\, [\varphi_\xi(\gamma) = \gamma]\}.$$

We shall write $\varphi\alpha\beta$ instead of $\varphi_\alpha(\beta)$. Let Γ_α be the α^{th} ordinal $\rho > 0$ such that for all $\beta, \gamma < \rho$, $\varphi\beta\gamma < \rho$

Corollary 4.2 *(i)* $\varphi 0\beta = \omega^\beta$.

(ii) $\xi, \eta < \varphi\alpha\beta \Longrightarrow \xi + \eta < \varphi\alpha\beta$.

(iii) $\xi < \zeta \Longrightarrow \varphi\alpha\xi < \varphi\alpha\zeta$.

[6] For technical convenience, $\in$ will be taken to be the only predicate symbol of the language of set theory. This does no harm, since equality can be defined by $a = b :\Leftrightarrow (\forall x \in a)(x \in b)\ \wedge\ (\forall x \in b)(x \in a)$, provided that we state extensionality in a slightly different form than usually.

[7] This contrasts with Barwise [1975] where Infinity is not included in **KP**.

[8] $x = \{y \in a : F(y)\}$ stands for the Δ_0–formula $(\forall y \in x)[y \in a \wedge F(y)] \wedge (\forall y \in a)[F(y) \rightarrow y \in x]$.

(iv) $\alpha < \beta \Longrightarrow \varphi\alpha(\varphi\beta\xi) = \varphi\beta\xi$.

The least ordinal (> 0) closed under the function φ is called Γ_0. The proof-theoretic ordinal of **KP**, however, is bigger than Γ_0 and we need another function to obtain a sufficiently large ordinal representation system.

Definition 4.3 Let Ω be a "big" ordinal. By recursion on α we define sets $C^\Omega(\alpha, \beta)$ and the ordinal $\psi_\Omega(\alpha)$ as follows:

$$C^\Omega(\alpha,\beta) = \begin{cases} \textit{closure of } \beta \cup \{0,\Omega\} \\ \textit{under:} \\ +, (\xi \mapsto \omega^\xi) \\ (\xi \longmapsto \psi_\Omega(\xi))_{\xi<\alpha} \end{cases} \tag{51}$$

$$\psi_\Omega(\alpha) \simeq \min\{\rho < \Omega : \ C^\Omega(\alpha,\rho) \cap \Omega = \rho\}. \tag{52}$$

Note that if $\rho = \psi_\Omega(\alpha)$, then $\psi_\Omega(\alpha) < \Omega$ and $[\rho,\Omega) \cap C^\Omega(\alpha,\rho) = \emptyset$, thus the order-type of the ordinals below Ω which belong to the Skolem hull $C^\Omega(\alpha,\rho)$ is ρ . In more pictorial terms, ρ is the α^{th} *collapse* of Ω.

Lemma 4.4 *$\psi_\Omega(\alpha)$ is always defined; in particular $\psi_\Omega(\alpha) < \Omega$.*

Proof: The claim is actually not a definitive statement as I haven't yet said what largeness properties Ω has to satisfy. In the proof below, we assume $\Omega := \aleph_1$, i.e. Ω is the first uncountable cardinal.

Observe first that for a limit ordinal λ,

$$C^\Omega(\alpha,\lambda) = \bigcup_{\xi<\lambda} C^\Omega(\alpha,\xi)$$

since the right hand side is easily shown to be closed under the clauses that define $C^\Omega(\alpha,\lambda)$. Now define

$$\begin{aligned} \eta_0 &= \sup C^\Omega(\alpha,0) \cap \Omega \\ \eta_{n+1} &= \sup C^\Omega(\alpha,\eta_n) \cap \Omega \\ \eta^* &= \sup_{n<\omega} \eta_n. \end{aligned} \tag{53}$$

Since for $\eta < \Omega$ the cardinality of $C^\Omega(\alpha,\eta)$ is the same as that of $\max(\eta,\omega)$ and therefore less than Ω, the regularity of Ω implies that $\eta_0 < \Omega$. By repetition of this argument one obtains $\eta_n < \Omega$, and consequently $\eta^* < \Omega$. The definition of η^* then ensures

$$C^\Omega(\alpha,\eta^*) \cap \Omega = \bigcup_n C^\Omega(\alpha,\eta_n) \cap \Omega = \eta^* < \Omega.$$

Therefore, $\psi_\Omega(\alpha) < \Omega$. □

Let $\varepsilon_{\Omega+1}$ be the least ordinal $\alpha > \Omega$ such that $\omega^\alpha = \alpha$. The next definition singles out a subset $\mathcal{T}(\Omega)$ of $C^\Omega(\varepsilon_{\Omega+1}, 0)$ which gives rise to an ordinal representation system, i.e., there is an elementary ordinal representation system $\langle \mathcal{OR}, \lhd, \hat{\Re}, \hat{\psi}, \dots \rangle$, so that

$$\langle \mathcal{T}(\Omega), <, \Re, \psi, \dots \rangle \cong \langle \mathcal{OR}, \lhd, \hat{\Re}, \hat{\psi}, \dots \rangle. \tag{54}$$

"..." is supposed to indicate that more structure carries over to the ordinal representation system.

Definition 4.5 $\mathcal{T}(\Omega)$ is defined inductively as follows:

1. $0, \Omega \in \mathcal{T}(\Omega)$.
2. If $\alpha_1, \ldots, \alpha_n \in \mathcal{T}(\Omega)$ and $\omega^{\alpha_1} + \cdots + \omega^{\alpha_n} > \alpha_1 \geq \ldots \geq \alpha_n$, then $\omega^{\alpha_1} + \cdots + \omega^{\alpha_n} \in \mathcal{T}(\Omega)$.
3. If $\alpha \in \mathcal{T}(\Omega)$ and $\alpha \in C^{\Omega}(\alpha, \psi_\Omega(\alpha))$, then $\psi_\Omega(\alpha) \in \mathcal{T}(\Omega)$.

The side condition in 4.5.2 is easily explained by the desire to have unique representations in $\mathcal{T}(\Omega)$. The requirement $\alpha \in C^{\Omega}(\alpha, \psi_\Omega(\alpha))$ in 4.5.3 also serves the purpose of unique representations (and more) but is probably a bit harder to explain. The idea here is that from $\psi_\Omega(\alpha)$ one should be able to retrieve the stage (namely α) where it was generated. This is reflected by $\alpha \in C^{\Omega}(\alpha, \psi_\Omega(\alpha))$.

It can be shown that the foregoing definition of $\mathcal{T}(\Omega)$ is deterministic, that is to say every ordinal in $\mathcal{T}(\Omega)$ is generated by the inductive clauses of 4.5 in exactly one way. As a result, every $\gamma \in \mathcal{T}(\Omega)$ has a unique representation in terms of symbols for $0, \Omega$ and function symbols for $+, (\alpha \mapsto \omega^{\alpha}), (\alpha \mapsto \psi_\Omega(\alpha))$. Thus, by taking some primitive recursive (injective) coding function $\ulcorner \cdots \urcorner$ on finite sequences of natural numbers, we can code $\mathcal{T}(\Omega)$ as a set of natural numbers as follows:

$$\ell(\alpha) = \begin{cases} \ulcorner 0, 0 \urcorner & \text{if } \alpha = 0 \\ \ulcorner 1, 0 \urcorner & \text{if } \alpha = \Omega \\ \ulcorner 2, \ell(\alpha_1), \cdots, \ell(\alpha_n) \urcorner & \text{if } \alpha = \omega^{\alpha_1} + \cdots + \omega^{\alpha_n} \\ \ulcorner 3, \ell(\beta), \ell(\Omega) \urcorner & \text{if } \alpha = \psi_\Omega(\beta), \end{cases}$$

where the distinction by cases refers to the unique representation of 4.5. With the aid of ℓ, the ordinal representation system of (54) can be defined by letting $\mathcal{OR}$ be the image of ℓ and setting $\lhd \ := \{(\ell(\gamma), \ell(\delta)) : \gamma < \delta \wedge \delta, \gamma \in \mathcal{T}(\Omega)\}$ etc. However, for a proof that this definition of $\langle \mathcal{OR}, \lhd, \hat{\Re}, \hat{\psi}, \ldots \rangle$ in point of fact furnishes an elementary ordinal representation system, we have to refer to the literature (cf. [8, 12, 82]).

4.2.3 A reminder: Ordinal analysis of PA à la Schütte

It is well known that the axioms of Peano Arithmetic, **PA**, can be derived in a sequent calculus, $\mathbf{PA}_\omega$, augmented by an infinitary rule, the so–called *ω–rule*[9]

$$\frac{\Gamma, A(\bar{n}) \text{ for all } n}{\Gamma, \forall x A(x)}.$$

An ordinal analysis for **PA** is then attained as follows:

- Each **PA**–proof can be "unfolded" into a $\mathbf{PA}_\omega$–proof of the same sequent.
- Each such $\mathbf{PA}_\omega$–proof can be transformed into a cut–free $\mathbf{PA}_\omega$–proof of the same sequent of length $< \varepsilon_0$.

In order to obtain a similar result for set theories like **KP**, we have to work a bit harder. Guided by the ordinal analysis of **PA**, we would like to invent an infinitary rule which, when added to **KP**, enables us to eliminate cuts. As opposed to the

[9] $\bar{n}$ stands for the n^{th} numeral

natural numbers, it is not clear how to bestow a canonical name to each element of the set–theoretic universe. However, within the confines of the constructible universe, which is made from the ordinals, it is pretty obvious how to "name" sets once we have names for ordinals at our disposal.

4.2.4 The language of RS_Ω

The problem of "naming" sets will be solved by erecting a formal constructible hierarchy using the ordinals from $\mathcal{T}(\Omega)$. Henceforth, we shall restrict ourselves to ordinals from $\mathcal{T}(\Omega)$.

Definition 4.6 We adopt a language of set theory, $\mathcal{L}$, which has only the predicate symbol $\in$. The *atomic formulae* of $\mathcal{L}$ are those of either form $(a{\in}b)$ or $\neg(a{\in}b)$. The $\mathcal{L}$–*formulae* are obtained from atomic ones by closing off under $\wedge, \vee, (\exists x{\in}a), (\forall x{\in}a), \exists x$, and $\forall x$.

Definition 4.7 The RS_Ω–*terms* and their *levels* are generated as follows.

1. For each $\alpha < \Omega$, $\mathbb{L}_\alpha$ is an RS_Ω–term of level α.
2. The formal expression $[x{\in}\mathbb{L}_\alpha : F(x,\vec{s})^{\mathbb{L}_\alpha}]$ is an RS_Ω–term of level α if $F(a,\vec{b})$ is an $\mathcal{L}$–formula (whose free variables are among the indicated) and $\vec{s} \equiv s_1, \cdots, s_n$ are RS_Ω–terms with levels $< \alpha$. $F(x,\vec{s})^{\mathbb{L}_\alpha}$ results from $F(x,\vec{s})$ by restricting all unbounded quantifiers to $\mathbb{L}_\alpha$.

We shall denote the level of an RS_Ω–term t by $|\,t\,|$; $t{\in}\mathcal{T}(\alpha)$ stands for $|\,t\,| < \alpha$ and $t{\in}\mathcal{T}$ for $t{\in}\mathcal{T}(\Omega)$.

The RS_Ω–*formulae* are the expressions of the form $F(\vec{s})$, where $F(\vec{a})$ is an $\mathcal{L}$–formula and $\vec{s} \equiv s_1, \ldots, s_n{\in}\mathcal{T}$.

For technical convenience, we let $\neg A$ be the formula which arises from A by (i) putting $\neg$ in front of each atomic formula, (ii) replacing $\wedge, \vee, (\forall x \in a), (\exists x \in a)$ by $\vee, \wedge, (\exists x \in a), (\forall x \in a)$, respectively, and (iii) dropping double negations.

Definition 4.8 We use the relation $\equiv$ to mean syntactical identity. For terms s, t with $|\,s\,| < |\,t\,|$ we set

$$s\mathring{\in}t \equiv \begin{cases} B(s) & \text{if } \; t \equiv [x{\in}\mathbb{L}_\beta : B(x)] \\ \mathsf{True}_s & \text{if } \; t \equiv \mathbb{L}_\beta \end{cases}$$

where True_s is a true formula, say $s{\notin}\mathbb{L}_0$.

Observe that $s{\in}t$ and $s\mathring{\in}t$ have the same truth value under the standard interpretation in the constructible hierarchy.

4.2.5 The rules of $\mathcal{L}_{RS}$

Having created names for a segment of the constructible universe, we can introduce infinitary rules analogous to the the ω-rule.

Let $A, B, C, \ldots, F(t), G(t), \ldots$ range over RS_Ω–formulae. We denote by upper case Greek letters $\Gamma, \Delta, \Lambda, \ldots$ finite sets of RS_Ω–formulae. The intended meaning of $\Gamma = \{A_1, \cdots, A_n\}$ is the disjunction $A_1 \vee \cdots \vee A_n$. Γ, A stands for $\Gamma \cup \{A\}$ etc.. We also use the shorthands $r \neq s := \neg(r = s)$ and $r{\notin}t := \neg(r{\in}t)$.

Definition 4.9 The *rules* of RS_Ω are:

$$(\wedge) \qquad \frac{\Gamma, A \quad \Gamma, A'}{\Gamma, A \wedge A'}$$

$$(\vee) \qquad \frac{\Gamma, A_i}{\Gamma, A_0 \vee A_1} \quad \text{if } i = 0 \text{ or } i = 1$$

$$(b\forall) \qquad \frac{\cdots \Gamma, s \mathring{\in} t \rightarrow F(s) \cdots (s \in \mathcal{T}(|t|))}{\Gamma, (\forall x \in t) F(x)}$$

$$(b\exists) \qquad \frac{\Gamma, s \mathring{\in} t \wedge F(s)}{\Gamma, (\exists x \in t) F(x)} \quad \text{if } s \in \mathcal{T}(|t|)$$

$$(\forall) \qquad \frac{\cdots \Gamma, F(s) \cdots (s \in \mathcal{T})}{\Gamma, \forall x F(x)}$$

$$(\exists) \qquad \frac{\Gamma, F(s)}{\Gamma, \exists x F(x)} \quad \text{if } s \in \mathcal{T}$$

$$(\notin) \qquad \frac{\cdots \Gamma, s \mathring{\in} t \rightarrow r \neq s \cdots \cdots (s \in \mathcal{T}(|t|))}{\Gamma, r \notin t}$$

$$(\in) \qquad \frac{\Gamma, s \mathring{\in} t \ \wedge \ r = s}{\Gamma, r \in t} \quad \text{if } s \in \mathcal{T}(|t|)$$

$$(\mathrm{Cut}) \qquad \frac{\Gamma, A \qquad \Gamma, \neg A}{\Gamma}$$

$$(\mathrm{Ref}_{\Sigma(\Omega)}) \qquad \frac{\Gamma, A}{\Gamma, \exists z A^z} \quad \text{if } A \text{ is a } \Sigma\text{-formula,}$$

where a formula is said to be Σ if all unbounded quantifiers are existential. A^z results from A by restricting all unbounded quantifiers to z.

4.2.6 $\mathcal{H}$–controlled derivations

If we dropped the rule $(\mathrm{Ref}_{\Sigma(\Omega)})$ from RS_Ω, the remaining calculus would enjoy full cut elimination owing to the symmetry of the pairs of rules $\langle (\wedge), (\vee) \rangle$, $\langle (\forall), (\exists) \rangle$, $\langle (\notin), (\in) \rangle$. However, partial cut elimination for RS_Ω can be attained by delimiting a collection of derivations of a very uniform kind. Fortunately, Buchholz has provided us with a very elegant and flexible setting for describing uniformity in infinitary proofs, called *operator controlled derivations* (see [10]).

Definition 4.10 Let $P(ON) = \{X : X \text{ is a set of ordinals}\}$. A class function $\mathcal{H} : P(ON) \rightarrow P(ON)$ will be called *operator* if $\mathcal{H}$ is a *closure operator*, i.e monotone, inclusive and idempotent, and satisfies the following conditions for all $X \in P(ON)$: $0 \in \mathcal{H}(X)$, and, if α has Cantor normal form $\omega^{\alpha_1} + \cdots + \omega^{\alpha_n}$, then $\alpha \in \mathcal{H}(X) \iff \alpha_1, ..., \alpha_n \in \mathcal{H}(X)$. The latter ensures that $\mathcal{H}(X)$ will be closed under $+$ and $\sigma \mapsto \omega^\sigma$, and decomposition of its members into additive and multiplicative components. For $Z \in P(ON)$, the operator $\mathcal{H}[Z]$ is defined by $\mathcal{H}[Z](X) := \mathcal{H}(Z \cup X)$.

If $\mathfrak{X}$ consists of "syntactic material", i.e. terms, formulae, and possibly elements from $\{0,1\}$, then let $\mathcal{H}[\mathfrak{X}](X) := \mathcal{H}(k(\mathfrak{X}) \cup X)$, where $k(\mathfrak{X})$ is the set of ordinals needed to build this "material". Finally, if s is a term, then define $\mathcal{H}[\int]$ by $\mathcal{H}[\{s\}]$.

To facilitate the definition of $\mathcal{H}$–controlled derivations, we assign to each RS_Ω–formula A, either a (possibly infinite) disjunction $\bigvee(A_\iota)_{\iota\in I}$ or a conjunction $\bigwedge(A_\iota)_{\iota\in I}$ of RS_Ω–formulae. This assignment will be indicated by $A \cong \bigvee(A_\iota)_{\iota\in I}$ and $A \cong \bigwedge(A_\iota)_{\iota\in I}$, respectively. Define: $r{\in}t \cong \bigvee(s\mathring{\in}t \wedge r = s)_{s\in\mathcal{T}_{|t|}}$; $\exists x\, F(x) \cong \bigvee(F(s))_{s\in\mathcal{T}}$; $(\exists x{\in}t)F(x) \cong \bigvee(s\mathring{\in}t \wedge F(s))_{s\in\mathcal{T}_{|t|}}$; $A_0 \vee A_1 \cong \bigvee(A_\iota)_{\iota\in\{0,1\}}$; $\neg A \cong \bigwedge(\neg A_\iota)_{\iota\in I}$, if $A \cong \bigvee(A_\iota)_{\iota\in I}$. Using this representation of formulae, we can define the *subformulae* of a formula as follows. When $A \cong \bigwedge(A_\iota)_{\iota\in I}$ or $A \cong \bigvee(A_\iota)_{\iota\in I}$, then B is a subformula of A if $B \equiv A$ or, for some $\iota{\in}I$, B is a subformula of A_ι.

Since one also wants to keep track of the complexity of cuts appearing in derivations, each formula F gets assigned an ordinal rank $rk(F)$ which is roughly the sup of the level of terms in F plus a finite number.

Using the formula representation, in spite of the many rules of RS_Ω, the notion of $\mathcal{H}$–controlled derivability can be defined concisely. We shall use $I{\restriction}\alpha$ to denote the set $\{\iota{\in}I : |\,\iota\,| < \alpha\}$.

Definition 4.11 Let $\mathcal{H}$ be an operator and let Γ be a finite set of RS_Ω–formulae. $\mathcal{H} \,\frac{\alpha}{\rho}\, \Gamma$ is defined by recursion on α. It is always demanded that $\{\alpha\} \cup k(\Gamma) \subseteq \mathcal{H}(\emptyset)$. The inductive clauses are:

$$(\bigvee) \qquad \frac{\mathcal{H} \,\frac{\alpha_0}{\rho}\, \Lambda, A_{\iota_0}}{\mathcal{H} \,\frac{\alpha}{\rho}\, \Lambda, \bigvee(A_\iota)_{\iota\in I}} \qquad \begin{array}{c}\alpha_0 < \alpha \\ \iota_0{\in}I{\restriction}\alpha\end{array}$$

$$(\bigwedge) \qquad \frac{\mathcal{H}[\iota] \,\frac{\alpha_\iota}{\rho}\, \Lambda, A_\iota \textit{ for all } \iota{\in}I}{\mathcal{H} \,\frac{\alpha}{\rho}\, \Lambda, \bigwedge(A_\iota)_{\iota\in I}} \qquad |\,\iota\,| \le \alpha_\iota < \alpha$$

$$\textit{(Cut)} \qquad \frac{\mathcal{H} \,\frac{\alpha_0}{\rho}\, \Lambda, B \qquad \mathcal{H} \,\frac{\alpha_0}{\rho}\, \Lambda, \neg B}{\mathcal{H} \,\frac{\alpha}{\rho}\, \Lambda} \qquad \begin{array}{c}\alpha_0 < \alpha \\ rk(B) < \rho\end{array}$$

$$(Ref_{\Sigma(\Omega)}) \qquad \frac{\mathcal{H} \,\frac{\alpha_0}{\rho}\, \Lambda, A}{\mathcal{H} \,\frac{\alpha}{\rho}\, \Lambda, \exists z\, A^z} \qquad \begin{array}{c}\alpha_0, \Omega < \alpha \\ A{\in}\Sigma\end{array}$$

The specification of the operators needed for an ordinal analysis will, of course, hinge upon the particular theory and ordinal representation system.

To connect **KP** with the infinitary system RS_Ω one has to show that **KP** can be embedded into RS_Ω. Indeed, the finite **KP**-derivations give rise to very uniform infinitary derivations.

Theorem 4.12 *If* **KP** $\vdash B(a_1, \ldots, a_r)$*, then* $\mathcal{H} \,\frac{\Omega\cdot m}{\Omega+n}\, B(s_1, \ldots, s_r)$ *holds for some* m, n *and all set terms* $s_1, \ldots, s_r$ *and operators* $\mathcal{H}$ *satisfying*

$$\{\xi \,:\, \xi \textit{ occurs in } B(\vec{s})\} \,\cup\, \{\Omega\} \,\subseteq\, \mathcal{H}(\emptyset).$$

m and n depend only on the **KP***-derivation of* $B(\vec{a})$*.*

The usual cut elimination procedure works as long as the cut formulae have not been introduced by an inference $(Ref_{\Sigma(\Omega)})$. As the main formula of an inference $(Ref_{\Sigma(\Omega)})$ has rank Ω one gets the following result.

Theorem 4.13 (Cut elimination I)

$$\mathcal{H} \mid\frac{\alpha}{\Omega+n+1} \Gamma \quad \Rightarrow \quad \mathcal{H} \mid\frac{\omega_n(\alpha)}{\Omega+1} \Gamma$$

where $\omega_0(\beta) := \beta$ *and* $\omega_{k+1}(\beta) := \omega^{\omega_k(\beta)}$.

The reason why the usual cut-elimination method fails for cuts with rank Ω is that it is too limited to treat a cut in the following scenario:

$$\dfrac{\dfrac{\mathcal{H} \mid\frac{\delta}{\Omega} \Gamma, A}{\mathcal{H} \mid\frac{\xi}{\Omega} \Gamma, \exists z\, A^z}(\Sigma\text{–}Ref_\Omega) \qquad \dfrac{\cdots \mathcal{H}[s] \mid\frac{\xi_s}{\Omega} \Gamma, \neg A^s \cdots (s \in \mathcal{T})}{\mathcal{H} \mid\frac{\xi}{\Omega} \Gamma, \forall z\, \neg A^z}(\forall)}{\mathcal{H} \mid\frac{\alpha}{\Omega+1} \Gamma}(Cut)$$

Fortunately, it is possible to eliminate cuts in the above situation provided that the side formulae Γ are of complexity Σ. The technique is known as "collapsing" of derivations.

In the course of "collapsing" one makes use of a simple bounding principle.

Lemma 4.14 (Boundedness) *Let A be a Σ-formula, $\alpha \leq \beta < \Omega$, and $\beta \in \mathcal{H}(\emptyset)$. If $\mathcal{H} \mid\frac{\alpha}{\rho} \Gamma, A$, then $\mathcal{H} \mid\frac{\alpha}{\rho} \Gamma, A^{\mathbb{L}_\beta}$.*

If the length of the derivation is already $\geq \Omega$, then "collapsing" results in a shorter derivation, however, at the cost of a much more complicated controlling operator.

Theorem 4.15 (Collapsing Theorem) *Let Γ be a set of Σ-formulae. Then we have*

$$\mathcal{H}_\eta \mid\frac{\alpha}{\Omega+1} \Gamma \quad \Rightarrow \quad \mathcal{H}_{f(\eta,\alpha)} \mid\frac{\psi_\Omega(f(\eta,\alpha))}{\psi_\Omega(f(\eta,\alpha))} \Gamma\,,$$

where $\left(\mathcal{H}_\xi\right)_{\xi \in \mathcal{T}(\Omega)}$ is a uniform sequence of ever stronger operators.

From Lemma 4.14 it follows that all instances of $(Ref_{\Sigma(\Omega)})$ can be removed from derivations of length $< \Omega$. For the latter kind of derivations there is a well-known cut-elimination procedure, the so-called *predicative cut-elimination*. Below this is stated in precise terms. It should also be mentioned that the φ function can be defined in terms of the functions of $\mathcal{T}(\Omega)$ and that $\varphi\alpha\beta < \Omega$ holds whenever $\alpha, \beta < \Omega$.

Theorem 4.16 (Predicative cut elimination)

$$\mathcal{H} \mid\frac{\delta}{\rho} \Gamma \;\text{ and } \delta, \rho < \Omega \quad \Rightarrow \quad \mathcal{H} \mid\frac{\varphi\rho\delta}{0} \Gamma\,.$$

The ordinal $\psi_\Omega(\varepsilon_{\Omega+1})$ is known as the *Bachmann-Howard ordinal*. Combining the previous results of this section, one obtains:

Corollary 4.17 *If A is a Σ-formula and* **KP** $\vdash A$, *then* $L_{\psi_\Omega(\varepsilon_{\Omega+1})} \models A$.

The bound of Corollary 4.17 is sharp, that is, $\psi_\Omega(\varepsilon_{\Omega+1})$ is the first ordinal with that property. Below we list further results that follow from the ordinal analysis of **KP**.

Corollary 4.18 *(i)* $|\mathbf{KP}| = |\mathbf{KP}|_{\sup} = |\mathbf{KP}|_{\Pi_2} = |\mathbf{KP}|^E_{\Pi_2} = \psi_\Omega(\varepsilon_{\Omega+1})$.

(ii) $\mathbf{sp}_{\Sigma_1}(\mathbf{KP}) = \psi_\Omega(\varepsilon_{\Omega+1})$.

4.3 Ordinal analysis of KPM

In many respects, **KP** is a very special case. Several fascinating aspects of ordinal analysis do not yet exhibit themselves at the level of **KP**. An example for the latter is that, in general, $\mathbf{sp}_{\Sigma_1}(T)$ is not contained in the ordinal representation system; the connection between them only emerges at the level of stronger theories. Furthermore, up to now the approach of "using" large cardinals to devise strong ordinal representation systems is only exemplified in a very weak sense namely in the shape of an uncountable cardinal. For these reasons, I shall outline the ordinal analysis of the stronger theory **KPM**. **KPM** formalizes a recursively Mahlo universe of sets and is considerably stronger than $\Delta^1_2 - \mathbf{CA}+\mathbf{BI}$. It is distinguished by the fact that it is essentially the 'strongest' classical theory for which a consistency proof in Martin-Löf type theory can be carried out. The particular formal system of Martin-Löf type theory that suffices for such a consistency proof is based on P. Dybjer's schema of simultaneous inductive-recursive definition (cf. [20]) or E. Palmgren's higher order universes (cf. [63]) and proceeds by showing the well-foundedness of the representation system $\mathcal{T}(\mathbf{M})$ that was used in the ordinal analysis of **KPM** (cf.[74]) in type theory. However, I should be a little cautious here as a full proof has not yet been written down, mainly because it taxes the limits of human tolerance. Though, for a strong fragment of **KPM** (wherein the foundation scheme is restricted to set-theoretic Π_2 formulas) there is a full proof, using techniques of [86, 76].

4.3.1 The theory KPM

KPM is an extension of **KP** by a schema stating that for every Σ_1-definable (class) function there exists an admissible set closed under this function. Its canonical models are the sets $\mathbf{L}_\mu$ with μ recursively Mahlo. To be more precise, the *language of* **KPM**, denoted by $\mathcal{L}_{Ad}$, is an extension of the language of **KP** by a unary predicate **Ad** which is used to express that a set is an admissible set. In addition to the axioms of **KP**, **KPM** has the following axioms:

Ad-*Limit:* $\forall x\,\exists y\,(x \in y \wedge \mathbf{Ad}(y))$.

Ad-*Linearity:* $\forall u\,\forall v\,[\mathbf{Ad}(u) \wedge \mathbf{Ad}(v) \rightarrow u \in v \vee u = v \vee v \in u]$.

(**Ad**1)*:* $\mathbf{Ad}(a) \rightarrow \omega \in a \wedge \forall x \in a \forall z \in x\, z \in a$.

(**Ad**2)*:* $\mathbf{Ad}(a) \rightarrow A^a$,
where the sentence A is a universal closure of one of the following axioms:

Pairing: $\exists x\ (x = \{a, b\})$.

Union: $\exists x\ (x = \bigcup a)$.

Δ_0–*Sep:* $\exists x\ \big(x = \{y \in a : F(y)\}\big)$ for all Δ_0–formulae $F(b)$

Δ_0–*Coll:* $(\forall x \in a)\exists y G(x, y) \rightarrow \exists z(\forall x \in a)(\exists y \in z)G(x, y)$
for all Δ_0–formulae $G(b)$.

(**M**): $\forall x\,\exists y\, G(x, y) \rightarrow \exists z\,[\mathbf{Ad}(z) \wedge (\forall x \in z)\,(\exists y \in z) G(x, y)]$
for all Δ_0–formulae $G(a, b)$.

4.3.2 Ordinal functions based on a weakly Mahlo cardinal

To develop a sufficiently strong ordinal representation system we first develop certain collapsing under the assumption that a weakly Mahlo cardinal exists (cf. [73]).

In a paper from 1911 Mahlo [55] investigated two hierarchies of regular cardinals. Mahlo called the cardinals considered in the first hierarchy π_α-*numbers*. In modern terminology they are spelled out as follows:

$$\begin{array}{rcl} \kappa \text{ is } 0\textit{-weakly inaccessible} & \text{iff} & \kappa \text{ is regular;} \\ \kappa \text{ is } (\alpha+1)\textit{-weakly inaccessible} & \text{iff} & \kappa \text{ is a regular limit of } \alpha\text{-weakly inaccessibles} \\ \kappa \text{ is } \lambda\textit{-weakly inaccessible} & \text{iff} & \kappa \text{ is } \alpha\text{-weakly inaccessible for every } \alpha < \lambda \end{array}$$

for limit ordinals λ. This hierarchy could be extended through diagonalization, by taking next the cardinals κ such that κ is κ-weakly inaccessible and after that choosing regular limits of the previous kind etc.

Mahlo also discerned a second hierarchy which is generated by a principle superior to taking regular fixed-points. Its starting point is the class of ρ_0-numbers which later came to be called *weakly Mahlo cardinals*. Weakly Mahlo cardinals are larger than any of those that can be obtained by the above processes from below. Here we shall define an extension of Mahlo's π-hierarchy by using ordinals above a weakly Mahlo to keep track of diagonalization.

Definition 4.19 Let

$$\mathbf{M} \;:=\; \text{first weakly Mahlo cardinal} \tag{55}$$

and set

$$\Re^{\mathbf{M}} \;:=\; \{\pi < \mathbf{M} : \pi \text{ regular}, \pi > \omega\}. \tag{56}$$

Variables κ, π will range over $\Re^{\mathbf{M}}$.

An ordinal representation system for the analysis of **KPM** can be derived from the following functions and Skolem hulls of ordinals, defined by recursion on α:

$$C^{\mathbf{M}}(\alpha,\beta) = \begin{cases} \textit{closure of } \beta \cup \{0,\mathbf{M}\} \\ \textit{under:} \\ +, (\xi \longmapsto \omega^\xi) \\ (\xi\delta \mapsto \chi^\xi(\delta))_{\xi<\alpha} \\ (\xi\pi \longmapsto \psi^\xi(\pi))_{\xi<\alpha} \end{cases} \tag{57}$$

$$\chi^\alpha(\delta) \;\simeq\; \delta^{th} \text{ regular } \pi < \mathbf{M} \text{ s.t. } C^{\mathbf{M}}(\alpha,\pi) \cap \mathbf{M} = \pi \tag{58}$$

$$\psi^\alpha(\pi) \;\simeq\; \min\{\rho < \pi :\; C^{\mathbf{M}}(\alpha,\rho) \cap \pi = \rho \;\wedge\; \pi \in C^{\mathbf{M}}(\alpha,\rho)\}. \tag{59}$$

Lemma 4.20 *For all α,*

$$\chi^\alpha : \mathbf{M} \to \mathbf{M}$$

i.e. χ^α is a total function on $\mathbf{M}$.

Proof: Set

$$X_\alpha \;:=\; \{\rho < \mathbf{M} : C^{\mathbf{M}}(\alpha,\rho) \cap \mathbf{M} = \rho\}.$$

We want to show that X_α is closed and unbounded in $\mathbf{M}$. As $\mathbf{M}$ is weakly Mahlo the latter will imply that X_α contains $\mathbf{M}$-many regular cardinals, ensuring that χ^α is total on $\mathbf{M}$.

Unboundedness: Given $\eta < \mathbf{M}$, define

$$\begin{aligned} \eta_0 &= \sup(C^{\mathbf{M}}(\alpha, \eta+1) \cap \mathbf{M}) \\ \eta_{n+1} &= \sup(C^{\mathbf{M}}(\alpha, \eta_n) \cap \mathbf{M}) \\ \eta^* &= \sup_n \eta_n. \end{aligned}$$

One easily verifies $C^{\mathbf{M}}(\alpha, \eta^*) \cap \mathbf{M} = \eta^*$. Hence, $\eta < \eta^*$ and $\eta^* \in X_\alpha$.

Closedness: If $X_\alpha \cap \lambda$ is unbounded in a limit $\lambda < \mathbf{M}$, then

$$C^{\mathbf{M}}(\alpha, \lambda) = \bigcup_{\xi \in X_\alpha \cap \lambda} C^{\mathbf{M}}(\alpha, \xi),$$

whence

$$C^{\mathbf{M}}(\alpha, \lambda) \cap \mathbf{M} = \sup\{\xi : \xi \in X_\alpha \cap \lambda\} = \lambda,$$

verifying $\lambda \in X_\alpha$. □

For a comparison with Mahlo's π_α numbers let $\mathbf{I}_\alpha$ be the function that enumerates, monotonically, the α-weakly inaccessibles. Neglecting finitely many exceptions, the function $\mathbf{I}_\alpha$ enumerates Mahlo's π_α numbers.

Proposition 4.21 *For $\alpha < \mathbf{M}$ let*

$$\Delta(\alpha) \;:=\; \textit{the } \alpha^{th}\ \kappa < \mathbf{M} \textit{ such that } \kappa \textit{ is } \kappa\textit{-weakly inaccessible.}$$

(i) $\forall \alpha < \Delta(0)\, \forall \xi < \mathbf{M}\ \mathbf{I}_\alpha(\xi) = \chi^\alpha(\xi)$.

(ii) $\Delta(\alpha) = \chi^{\mathbf{M}}(\alpha)$.

(iii) If $\chi^{\mathbf{M}}(\alpha) \le \beta < \chi^{\mathbf{M}}(\alpha+1)$, *then* $\forall \xi \le \alpha\ \chi^\beta(\xi) = \chi^{\mathbf{M}}(\xi)$.

(iv) If $\beta = \chi^{\mathbf{M}}(\alpha)$, *then* $\forall \xi \le \mathbf{M}\ \chi^\beta(\alpha+\xi) = \mathbf{I}_\beta(\xi)$.

(v) If $\chi^{\mathbf{M}}(\alpha) < \beta < \chi^{\mathbf{M}}(\alpha+1)$, *then* $\forall \xi < \mathbf{M}\ \chi^\beta(\alpha+1+\xi) = \mathbf{I}_\beta(\xi)$.

Ever higher levels of diagonalizations are obtained by the functions χ^{M^M}, $\chi^{M^{M^M}}$, etc.

The preceding gives rise to an EORS $\mathcal{T}(\mathbf{M})$ (similarly as sketched for $\mathcal{T}(\Omega)$) which is essentially order isomorphic to $C^{\mathbf{M}}(\varepsilon_{\mathbf{M}+1}, 0)$. This EORS exactly captures the strength of **KPM**.

4.3.3 The rules of $RS_{\mathbf{M}}$

The next step consists in utilizing $\mathcal{T}(\mathbf{M})$ for an ordinal analysis of **KPM**. Here we restrict ourselves to ordinals from $\mathcal{T}(\mathbf{M})$. The $RS_{\mathbf{M}}$*–terms* and their *levels* are generated as the RS_Ω-terms, except that in the starting case, for each $\alpha < \mathbf{M}$, $\mathbb{L}_\alpha$ is an $RS_{\mathbf{M}}$–term of level α. We will use $s \in RS_{\mathbf{M}}$ to convey that s is an $RS_{\mathbf{M}}$-term. The *atomic formulae* of $RS_{\mathbf{M}}$ are those of either form $(s \in t)$, $\neg(s \in t)$, $\mathbf{Ad}(s)$, or $\neg\mathbf{Ad}(s)$.

Definition 4.22 The *rules* of $RS_{\mathbf{M}}$ comprise $(\wedge), (\vee), (b\forall), (b\exists), \forall, \exists, (\notin)$, $(\in)$, (Cut) as for RS_Ω. The additional rules are:

$$(\neg\mathbf{Ad}) \qquad \frac{\cdots \Gamma, \mathbb{L}_\kappa \neq t \cdots (\kappa \leq |\, t\,|)}{\Gamma, \neg\mathbf{Ad}(t)}$$

$$(\mathbf{Ad}) \qquad \frac{\Gamma, \mathbb{L}_\kappa = t}{\Gamma, \mathbf{Ad}(t)} \quad \text{if } \kappa \leq |\, t\,|$$

$$(\mathrm{Ref}_{\Sigma(\pi)}) \qquad \frac{\Gamma, A^{\mathbb{L}_\pi}}{\Gamma, (\exists z \in \mathbb{L}_\pi)\, A^z} \quad \text{if } A \text{ is a } \Sigma\text{-formula whose terms have levels} < \pi$$

$$(\mathbf{M}) \qquad \frac{\cdots \Gamma,\, \exists y\, F(s,y) \cdots (s \in RS_{\mathbf{M}})}{\Gamma, \exists z\, [\mathbf{Ad}(z) \;\wedge\; (\forall x \in z)(\exists y \in z) F(x,y)]} \quad \text{if } F \text{ is } \Delta_0.$$

Extending Definition 4.10, we assign to the $RS_{\mathbf{M}}$–formula $\mathbf{Ad}(t)$ the disjunction $\mathbf{Ad}(t) \cong \bigvee(\mathbb{L}_\pi = t)_{\mathbb{L}_\pi \in I}$, where $I := \{\mathbb{L}_\kappa : \kappa \in \Re^{\mathbf{M}}; \kappa \leq |\, t\,|\}$.

Definition 4.23 Let $\mathcal{H}$ be an operator and let Γ be a finite set of $RS_{\mathbf{M}}$–formulae. $\mathcal{H} \overset{\alpha}{\underset{\rho}{\vert}} \Gamma$ is defined by recursion on α. It is always demanded that $\{\alpha\} \cup k(\Gamma) \subseteq \mathcal{H}(\emptyset)$. The inductive clauses are:

$$(\bigvee) \qquad \frac{\mathcal{H} \overset{\alpha_0}{\underset{\rho}{\vert}} \Lambda, A_{\iota_0}}{\mathcal{H} \overset{\alpha}{\underset{\rho}{\vert}} \Lambda, \bigvee(A_\iota)_{\iota \in I}} \qquad \begin{array}{r} \alpha_0 < \alpha \\ \iota_0 \in I \restriction \alpha \end{array}$$

$$(\bigwedge) \qquad \frac{\mathcal{H}[\iota] \overset{\alpha_\iota}{\underset{\rho}{\vert}} \Lambda, A_\iota \text{ for all } \iota \in I}{\mathcal{H} \overset{\alpha}{\underset{\rho}{\vert}} \Lambda, \bigwedge(A_\iota)_{\iota \in I}} \qquad |\, \iota\,| \leq \alpha_\iota < \alpha$$

$$\textit{(Cut)} \qquad \frac{\mathcal{H} \overset{\alpha_0}{\underset{\rho}{\vert}} \Lambda, B \qquad \mathcal{H} \overset{\alpha_0}{\underset{\rho}{\vert}} \Lambda, \neg B}{\mathcal{H} \overset{\alpha}{\underset{\rho}{\vert}} \Lambda} \qquad \begin{array}{r} \alpha_0 < \alpha \\ rk(B) < \rho \end{array}$$

$$(\mathit{Ref}_{\Sigma(\pi)}) \qquad \frac{\mathcal{H} \overset{\alpha_0}{\underset{\rho}{\vert}} \Lambda, A^{\mathbb{L}_\pi}}{\mathcal{H} \overset{\alpha}{\underset{\rho}{\vert}} \Lambda, (\exists z \in \mathbb{L}_\pi)\, A^z} \qquad \begin{array}{r} \alpha_0, \pi < \alpha \\ \pi \in \Re^{\mathbf{M}} \\ A \in \Sigma \end{array}$$

$$(\mathbf{M}) \qquad \frac{\mathcal{H}[s] \overset{\alpha_s}{\underset{\rho}{\vert}} \Lambda, \exists y\, F(s,y) \text{ for all } s \in RS_{\mathbf{M}}}{\mathcal{H} \overset{\alpha}{\underset{\rho}{\vert}} \Lambda, \exists z\, [\mathbf{Ad}(z) \;\wedge\; (\forall x \in z)(\exists y \in z)\, F(x,y)]} \qquad \begin{array}{r} |\, s\,| \leq \alpha_s < \alpha \\ F \in \Delta_0 \end{array}$$

As in the case of **KP** and RS_Ω, the proof system $RS_{\mathbf{M}}$ is tailored for an embedding of **KPM**.

Theorem 4.24 *If* $\mathbf{KPM} \vdash B(a_1, \ldots, a_r)$, *then* $\mathcal{H} \overset{\Omega \cdot m}{\underset{\Omega + n}{\vert}} B(s_1, \ldots, s_r)$ *holds for some* m, n *and all set terms* $s_1, \ldots, s_r$ *and operators* $\mathcal{H}$ *satisfying*

$$\{\xi \,:\, \xi \text{ occurs in } B(\vec{s})\} \;\cup\; \{\mathbf{M}\} \;\subseteq\; \mathcal{H}(\emptyset).$$

m and n depend only on the **KPM**-*derivation of* $B(\vec{a})$.

The cut-elimination procedure for $RS_{\mathbf{M}}$ is rather intricate (cf. [74]) and involves many more steps than in the case of RS_{Ω}. Omitting further details, we just state the outcome of it and a well-ordering proof for all initial segments of $\mathcal{T}(\mathbf{M})$ in **KPM**.

Corollary 4.25 *Letting $\Omega := \chi^0(0)$, we have:*

$$|\mathbf{KPM}| = |\mathbf{KPM}|_{\sup} = |\mathbf{KPM}|_{\Pi_2(L(\omega_1^{CK}))} = |\mathbf{KPM}|^E_{\Pi_2} = \psi^{\varepsilon_{\mathbf{M}+1}}(\Omega).$$

4.3.4 Recursively large ordinals and ordinal representation systems

The large cardinal hypothesis that **M** is the first weakly Mahlo cardinal is outrageous when compared with the strength of **KPM**. However, it enters the definition procedure of the collapsing function χ, which is then employed in the shape of terms to "name" a countable set of ordinals. As one succeeds in establishing recursion relations for the ordering between those terms, the set of terms gives rise to an ordinal representation system. It has long been suggested (cf. [25], p. 436) that, instead, one should be able to interpret the collapsing functions as operating directly on the recursively large counterparts of those cardinals. For example, taking such an approach in Definition 4.19 would consist in letting

$$\mathbf{M} := \text{first recursively Mahlo ordinal}$$

and setting $\Re^{\mathbf{M}} := \{\pi < \mathbf{M} : \pi \text{ admissible}, \pi > \omega\}$. The difficulties with this approach arise with the proof of Lemma 4.20. One wants to show that, for all α, $\chi^\alpha(\beta) < \mathbf{M}$ whenever $\beta < \mathbf{M}$. However, the arguments of the cardinal setting no longer work here. To get a similar result for a recursively Mahlo ordinal μ one would have to work solely with μ-recursive operations. In addition, the functions ψ^α would have to operate on admissible ordinals π. Here one wants $\psi_\pi(\alpha) < \pi$. In the cardinal setting this comes down to a simple cardinality argument. To get a similar result for an admissible π one would have to work solely with π-recursive operations. How this can be accomplished is far from being clear as the definition of $C^{\mathbf{M}}(\alpha, \rho)$ for $\rho < \pi$ usually refers to higher admissibles than just π. Notwithstanding that, the admissible approach is workable as was shown in [77, 81, 92]. A key idea therein is that the higher admissibles which figure in the definition of $\psi_\pi(\alpha)$ can be mimicked via names within the structure $\mathbf{L}_\pi$ in a π-recursive manner.

The drawback of the admissible approach is that it involves quite horrendous definition procedures and computations, which when taken as the first approach are at the limit of human tolerance.

On the other hand, the admissible approach provides a natural semantics for the terms in the EORSs. Recalling the notion of *good* Σ_1-*definition* from Definition 3.32, it turns out that all the ordinals of $\mathcal{T}(\mathbf{M}) \cap \mathbf{M}$ possess a good Σ_1-definition in **KPM** (cf. [81]) under the interpretation which takes **M** to be the first recursively Mahlo ordinal and lets the functions ψ^α operate on admissible ordinals instead of regular cardinals.

Unlike in the case of **KP**, $\mathcal{T}(\mathbf{M}) \cap \mathbf{M}$ only forms a proper subset of the **KPM**-definable ordinals, having many 'holes'.[10] Therefore, to illuminate the nature of the ordinals in $\mathcal{T}(\mathbf{M})$, it would be desirable to find another property which singles them out from the **KPM**-definable ordinals.

[10] The ordinals of $\mathcal{T}(\mathbf{M}) \cap \mathbf{M}$ are cofinal in $\mathbf{sp}_{\Sigma_1}(\mathbf{KPM})$, though. Letting $\pi_0 := \chi^{\varepsilon_{\mathbf{M}+1}}(0)$, one has $\sup(\mathbf{sp}_{\Sigma_1}(\mathbf{KPM})) = \psi^0(\pi_0)$.

References

[1] P. Aczel: *The Type Theoretic Interpretation of Constructive Set Theory*, in: MacIntyre, A., Pacholski, L., and Paris, J. (eds.), *Logic Colloquium '77*, North–Holland, Amsterdam 1978.

[2] P. Aczel: *The Type Theoretic Interpretation of Constructive Set Theory: Choice Principles*, in: Troelstra, A. S., van Dalen, D. (eds), *The L.E.J. Brouwer Centenary Symposium*, North–Holland, Amsterdam 1982.

[3] P. Aczel: *The Type Theoretic Interpretation of Constructive Set Theory: Inductive Definitions*, in: Marcus, R. B. et al. (eds), *Logic, Methodology, and Philosopy of Science VII*, North–Holland, Amsterdam 1986.

[4] H. Bachmann: *Die Normalfunktionen und das Problem der ausgezeichneten Folgen von Ordinalzahlen*, Vierteljahresschrift Naturforsch. Ges. Zürich 95 (1950) 115–147.

[5] J. Barwise: *Admissible Sets and Structures* (Springer, Berlin 1975).

[6] P. Bernays: *Hilbert, David,* Encyclopedia of philosophy, Vol. 3 (Macmillan and Free Press, New York, 1967) 496–504.

[7] W. Buchholz: *Eine Erweiterung der Schnitteliminationsmethode*, Habilitationsschrift (München 1977).

[8] W. Buchholz: *A new system of proof–theoretic ordinal functions,* Ann. Pure Appl. Logic **32** (1986) 195–207.

[9] W. Buchholz: *An independence result for (Π^1_1-CA+BI)*, Ann. Pure Appl. Logic 33 (1987) 131–155.

[10] W. Buchholz: *A simplified version of local predicativity,* in: Aczel, Simmons, Wainer (eds.): *Leeds Proof Theory 90* (Cambridge University Press, Cambridge, 1993) 115–147.

[11] W. Buchholz, S. Wainer: *Provable computable functions and the fast growing hierarchy,* in: Contemporary Mathematics 65 (American Mathematical Society, Providence,1987) 179–198.

[12] W. Buchholz, K. Schütte: *Proof theory of impredicative subsystems of analysis* (Bibliopolis, Naples, 1988).

[13] W. Buchholz, A. Cichon, A. Weiermann: *A uniform approach to fundamental sequences and hierarchies.* Mathematical Logic Quarterly 40 (1994) 273–286.

[14] W. Buchholz, S. Feferman, W. Pohlers, W. Sieg: *Iterated inductive definitions and subsystems of analysis* (Springer, Berlin, 1981).

[15] W. Buchholz and K. Schütte: *Proof theory of impredicative subsystems of analysis* (Bibliopolis, Naples, 1988).

[16] T. Carlson: *Ordinal arithmetic and Σ_1 elementarity* (1997) 12 pages, to appear in: Archive for Mathematical Logic.

[17] R. Diestel: *Graph Theory* (Springer, New York-Berlin-Heidelberg, 1997).

[18] A.G. Dragalin: *New forms of realizability and Markov's rule* (Russian), Dokl. Acad. Nauk. SSSR 2551 (1980) 543–537; translated in: Sov. Math. Dokl. 10, 1417–1420.

[19] F.R. Drake: *How recent work in mathematical logic relates to the foundations of mathematics*, The Mathematical Intelligencer vol. 7, no. 4 (1985) 27–35.

[20] P. Dybjer: *A general formulation of simultaneous inductive-recursive definitions in type theory* (preprint 1994) 21 pages. To appear in: Journal of Symbolic Logic.

[21] S. Feferman: *Systems of predicative analysis*, Journal of Symbolic Logic 29 (1964) 1–30.

[22] S. Feferman: *Systems of predicative analysis II. Representations of ordinals*, Journal of Symbolic Logic 33 (1968) 193–220.

[23] S. Feferman: *A Language and Axioms for Explicit Mathematics*, Lecture Notes in Math. 450 (Springer, Berlin, 1975), 87–139.

[24] S. Feferman: *Constructive Theories of Functions and classes* in: Boffa, M., van Dalen, D., McAloon, K. (eds.), *Logic Colloquium '78* (North-Holland, Amsterdam, 1979) 159–224.

[25] S. Feferman: *Proof theory: a personal report*, in: G. Takeuti, Proof Theory, 2^{nd} edition (North-Holland, Amsterdam, 1987) 445–485.

[26] S. Feferman: *Hilbert's program relativized: Proof-theoretical and foundational reductions*, J. Symbolic Logic 53 (1988) 364–384.

[27] S. Feferman: *Gödel's program for new axioms: Why, where, how and what?* to appear in: Gödel '96 conference, Brno. 23 pages.

[28] H. Friedman: *Classically and intuitionistically provably recursive functions.* In: G.H. Müller, D.S. Scott: *Higher set theory* (Springer, Berlin, 1978) 21–27.

[29] H. Friedman, K. McAloon, and S. Simpson: *A finite combinatorial principle which is equivalent to the 1-consistency of predicative analysis,* in: G. Metakides (ed.): *Patras Logic Symposium* (North-Holland, Amsterdam, 1982) 197–220.

[30] H. Friedman, N. Robertson, P. Seymour: *The metamathematics of the graph minor theorem*, Contemporary Mathematics 65 (1987) 229–261.

[31] H. Friedman and S. Ščedrov: *Large sets in intuitionistic set theory*, Annals of Pure and Applied Logic 27 (1984) 1–24.

[32] H. Friedman and S. Sheard: *Elementary descent recursion and proof theory*, Annals of Pure and Applied Logic 71 (1995) 1–45.

[33] H. Gaifman: *A generalization of Mahlo's method for obtaining large cardinal numbers,* Israel Journal of Mathematics 5 (1967) 188–200.

[34] G. Gentzen: *Die Widerspruchsfreiheit der reinen Zahlentheorie.* Mathematische Annalen 112 (1936) 493–565.

[35] J.-.Y. Girard: *A survey of* Π^1_2*-logic. Part I: Dilators*, Annals of Mathematical Logic 21 (1981) 75–219.

[36] D. Hilbert: *Die Grundlegung der elementaren Zahlentheorie.* Mathematische Annalen 104 (1931).

[37] D. Hilbert and P. Bernays: *Grundlagen der Mathematik II.* (Springer, Berlin, 1938).

[38] P.G. Hinman: *Recursion-theoretic hierarchies.* (Springer, Berlin, 1978).

[39] D. Isles: *Regular ordinals and normal forms,* in: A. Kino, J. Myhill, R.E. Vesley (eds.): Intuitionism and proof theory (North-Holland, Amsterdam, 1968) 288–300.

[40] G. Jäger: *Beweistheorie von KPN*, Archiv f. Math. Logik 2 (1980) 53–64.

[41] G. Jäger: *Zur Beweistheorie der Kripke–Platek Mengenlehre über den natürlichen Zahlen*, Archiv f. Math. Logik 22 (1982) 121–139.

[42] G. Jäger: *A well-ordering proof for Feferman's theory* T_0, Archiv f. Math. Logik 23 (1983) 65–77.

[43] Jäger, G.: *Theories for admissible sets: a unifying approach to proof theory* (Bibliopolis, Naples, 1986).

[44] G. Jäger and W. Pohlers: *Eine beweistheoretische Untersuchung von* $\mathbf{\Delta^1_2}$– $\mathbf{CA + BI}$ *und verwandter Systeme*, Sitzungsberichte der Bayerischen Akademie der Wissenschaften, Mathematisch–Naturwissenschaftliche Klasse (1982).

[45] A. Kanamori: *The higher infinite.* (Springer, Berlin, 1995).

[46] A. Kanamori, M. Magidor: *The evolution of large cardinal axioms in set theory.* In: G. H. Müller, D.S. Scott (eds.) *Higher Set Theory.* Lecture Notes in Mathematics 669 (Springer, Berlin, 1978) 99-275.

[47] G. Kreisel: *On the interpretation of non-finitist proofs II,* Journal of Symbolic Logic 17 (1952) 43–58.

[48] G. Kreisel: *Mathematical significance of consistency proofs.* Journal of Symbolic Logic 23 (1958) 155–182.

[49] G. Kreisel: *Generalized inductive definitions,* in: Stanford Report on the Foundations of Analysis (Mimeographed, Stanford, 1963) Section III.

[50] G. Kreisel: *A survey of proof theory,* Journal of Symbolic Logic 33 (1968) 321–388.

[51] G. Kreisel: *Notes concerning the elements of proof theory.* Course notes of a course on proof theory at U.C.L.A. 1967 - 1968.

[52] G. Kreisel, G. Mints, S. Simpson: *The use of abstract language in elementary metamathematics: Some pedagogic examples,* in: *Lecture Notes in Mathematics*, vol. 453 (Springer, Berlin, 1975) 38–131.

[53] E. Lopez-Escobar: *An extremely restricted* ω*-rule,* Fundamenta Mathematicae 90 (1976) 159–172.

[54] A. Macintyre, H. Simmons: *Algebraic properties of number theories*, Israel Journal of Mathematics, Vol. 22, No. 1 (1975) 7–27.

[55] P. Mahlo: *Über lineare transfinite Mengen*, Berichte über die Verhandlungen der Königlich Sächsischen Gesellschaft der Wissenschaften zu Leipzig, Mathematisch-Physische Klasse, 63 (1911) 187–225.

[56] P. Mahlo: *Zur Theorie und Anwendung der* ρ_0*-Zahlen*, ibid. 64 (1912) 108–112.

[57] P. Mahlo: *Zur Theorie und Anwendung der* ρ_0*-Zahlen*, ibid. 65 (1913) 268–282.

[58] P. Martin-Löf: *Intuitionistic Type Theory*, (Bibliopolis, Naples 1984).

[59] G.E. Mints: *Finite investigations of infinite derivations,* Journal of Soviet Mathematics 15 (1981) 45–62.

[60] Y.N. Moschovakis: *Recursion in the universe of sets*, mimeographed note, 1976.

[61] J. Myhill: *Constructive Set Theory*, J. Symbolic Logic 40 (1975) 347–382.

[62] D. Normann: *Set recursion*, in: Fenstad et al. (eds.): *Generalized recursion theory II* (North-Holland, Amsterdam, 1978) 303–320.

[63] E. Palmgren: *On universes in type theory.* To appear in: *Proceedings of "Twenty-five years of type theory"*, Venice, 1995 (Oxford University Press).

[64] J. Paris, L. Harrington: *A mathematical incompleteness in Peano arithmetic.* In: J. Barwise (ed.): *Handbook of Mathematical Logic* (North Holland, Amsterdam, 1977) 1133-1142.

[65] H. Pfeiffer: *Ausgezeichnete Folgen für gewisse Abschnitte der zweiten und weiterer Zahlklassen* (Dissertation, Hannover, 1964).

[66] W. Pohlers: *Cut elimination for impredicative infinitary systems, part I: Ordinal analysis of* ID_1, Arch. f. Math. Logik 21 (1981) 69–87.

[67] W. Pohlers: *Proof–theoretical analysis of ID_ν by the method of local predicativity*, in: W. Buchholz, S. Feferman, W. Pohlers, W. Sieg: *Iterated inductive definitions and subsystems of analysis* (Springer, Berlin, 1981) 261–357.

[68] W. Pohlers: *Cut elimination for impredicative infinitary systems, part II: Ordinal analysis for iterated inductive definitions*, Arch. f. Math. Logik 22 (1982) 113–129.

[69] W. Pohlers: *Contributions of the Schütte school in Munich to proof theory*, in: G. Takeuti, *Proof Theory,* 2^{nd} edition (North-Holland, Amsterdam, 1987) 406–431.

[70] W. Pohlers: *A short course in ordinal analysis,* in: P. Aczel, H. Simmons, S. Wainer (eds.): *Proof Theory* (Cambridge University Press, Cambridge, 1992) 27–78 .

[71] W. Pohlers: *Subsystems of set theory and second order number theory,* in: S. Buss (ed.): *Handbook of proof theory* (Elsevier Science B.V.) to appear.

[72] M. Rathjen: *Untersuchungen zu Teilsystemen der Zahlentheorie zweiter Stufe und der Mengenlehre mit einer zwischen $\Delta^1_2 - CA$ und $\Delta^1_2 - CA + BI$ liegenden Beweisstärke* (Publication of the Institute for Mathematical Logic and Foundational Research of the University of Münster, 1989).

[73] M. Rathjen: *Ordinal notations based on a weakly Mahlo cardinal*, Archive for Mathematical Logic 29 (1990) 249–263.

[74] M. Rathjen: *Proof-Theoretic Analysis of KPM*, Arch. Math. Logic 30 (1991) 377–403.

[75] M. Rathjen: *The role of parameters in bar rule and bar induction,* Journal of Symbolic Logic 56 (1991) 715–730.

[76] M. Rathjen: *Fragments of Kripke-Platek set theory with infinity,* in: P. Aczel, H. Simmons, S. Wainer (eds.): *Proof Theory* (Cambridge University Press, Cambridge, 1992) 251–273.

[77] M. Rathjen: *How to develop proof–theoretic ordinal functions on the basis of admissible sets.* Mathematical Quarterly 39 (1993) 47–54.

[78] M. Rathjen and A. Weiermann: *Proof–theoretic investigations on Kruskal's theorem,* Annals of Pure and Applied Logic 60 (1993) 49–88.

[79] M. Rathjen: *Admissible proof theory and beyond,* in: Logic, Methodology and Philosophy of Science IX (D. Prawitz, B. Skyrms and D. Westerstahl, eds.), Elsevier Science B.V. (1994) 123–147.

[80] M. Rathjen: *The strength of some Martin–Löf type theories.* Archive for Mathematical Logic 33 (1994) 347–385.

[81] M. Rathjen: *Collapsing functions based on recursively large ordinals: A well–ordering proof for KPM.* Archive for Mathematical Logic 33 (1994) 35–55.

[82] M. Rathjen: *Proof theory of reflection.* Annals of Pure and Applied Logic 68 (1994) 181–224.

[83] M. Rathjen: *Recent advances in ordinal analysis: Π^1_2-CA and related systems.* Bulletin of Symbolic Logic 1 (1995) 468–485.

[84] M. Rathjen: *An ordinal analysis of Π^1_2 comprehension and related systems*, preprint.

[85] M. Rathjen: *The higher infinite in proof theory,* in: J.A. Makowsky and E.V. Ravve (eds.): *Logic Colloquium '95.* Lecture Notes in Logic, vol. 11 (Springer, New York, Berlin, 1998) 275–304.

[86] M. Rathjen and E. Palmgren: *Inaccessibility in constructive set theory and type theory.* Annals of Pure and Applied Logic 94 (1998) 181–200.

[87] M. Rathjen: *Large set axioms in constructive set theory and their strengths*, in preparation.

[88] Richter, W. and Aczel, P.: *Inductive definitions and reflecting properties of admissible ordinals.* In: J.E. Fenstad, Hinman (eds.) Generalized Recursion Theory (North Holland, Amsterdam, 1973) 301-381.

[89] H.E. Rose: *Subrecursion: functions and hierarchies.* (Clarendon Press, Oxford, 1984).

[90] G.E. Sacks: *Higher recursion theory* (Springer, Berlin, 1990).

[91] A. Schlüter: *Zur Mengenexistenz in formalen Theorien der Mengenlehre.* (Thesis, University of Münster, 1993).

[92] A. Schlüter: *Provability in set theories with reflection*, preprint, 1995.

[93] D. Schmidt: *Well-partial orderings and their maximal order types*, Habilitationsschrift (Heidelberg, 1979) 77 pages.

[94] K. Schütte: *Beweistheoretische Erfassung der unendlichen Induktion in der Zahlentheorie*, Mathematische Annalen 122 (1951) 369–389.

[95] K. Schütte: *Beweistheorie* (Springer, Berlin, 1960).

[96] K. Schütte: *Eine Grenze für die Beweisbarkeit der transfiniten Induktion in der verzweigten Typenlogik*, Archiv für Mathematische Logik und Grundlagenforschung 67 (1964) 45–60.

[97] K. Schütte: *Predicative well-orderings*, in: Crossley, Dummett (eds.), Formal systems and recursive functions (North Holland, 1965) 176–184.

[98] K. Schütte: *Proof Theory* (Springer, Berlin, 1977).

[99] H. Schwichtenberg: *Proof theory: Some applications of cut-elimination.* In: J. Barwise (ed.): *Handbook of Mathematical Logic* (North Holland, Amsterdam, 1977) 867–895.

[100] S. Simpson: *Nichtbeweisbarkeit von gewissen kombinatorischen Eigenschaften endlicher Bäume*, Archiv f. Math. Logik 25 (1985) 45–65.

[101] A. Setzer: *A well-ordering proof for the proof theoretical strength of Martin-Löf type theory*, to appear in: Annals of Pure and Applied Logic.

[102] C. Smorynski: *The incompleteness theorems.* In: J. Barwise (ed.): *Handbook of Mathematical Logic* (North Holland, Amsterdam, 1977) 821–864.

[103] C. Smorynski: *Self-reference and modal logic* (Springer, New York - Berlin, 1985).

[104] R. Sommer: *Ordinal arithmetic in* $I\Delta_0$. In: P. Clote and J. Krajicek (eds.): *Arithmetic, proof theory, and computational complexity* (Clarendon Press, Oxford, 1993) 320–363.

[105] R. Sommer: *Ordinal functions in fragments of arithmetic*, Preprint (1992) 28 pages.

[106] G. Takeuti: *Consistency proofs of subsystems of classical analysis*, Ann. Math. 86, 299–348.

[107] G. Takeuti, M. Yasugi: *The ordinals of the systems of second order arithmetic with the provably* Δ^1_2*–comprehension and the* Δ^1_2*–comprehension axiom respectively*, Japan J. Math. 41 (1973) 1–67.

[108] G. Takeuti: *Proof theory*, second edition (North Holland, Amsterdam, 1987).

[109] A. S. Troelstra: *Metamathematical investigations of intuitionistic arithmetic and analysis*, (Springer, Berlin, 1973).

[110] A. S. Troelstra and D. van Dalen: *Constructivism in Mathematics: An Introduction*, volume I, II, North–Holland, Amsterdam 1988.

[111] O. Veblen: *Continuous increasing functions of finite and transfinite ordinals,* Trans. Amer. Math. Soc. 9 (1908) 280–292.

[112] A. Weiermann: *How to characterize provably total functions by local predicativity,* Journal of Symbolic Logic 61 (1996) 52–69.

[113] W.H. Woodin: *Large cardinal axioms and independence: The continuum problem revisited*, The Mathematical Intelligencer vol. 16, No. 3 (1994) 31–35.