An Ordinal Analysis of parameter free Π^1_2-Comprehension

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Es wurden nun des längeren die einfachsten Modellherleitungen gewertet, mit der Ausweitungsidee. Es ergaben sich transfinite Stufen, verschieden je nach Geschick. Unerfreuliche technische Mühsal. Die Stufen gingen sehr rasch in die Höhe. [...] *So etwas ist durchaus denkbar: Steigerung der Vielfältigkeit der erforderlichen mitzuschleppenden Funktionen mit dem Sequenzgrade. Es hängt sachlich alles vielfältig miteinander zusammen, und der Konstruktivist hat die Aufgabe, alle diese Zusammenhänge auseinander zu klauben und zum Ausdruck zu bringen.* [...] *Man muß den mühsamen Weg der Einzeluntersuchung gehen, des schrittweisen Aufstieges, einen Begriff nach dem anderen hinzunehmend.*

GERHARD GENTZEN (1945) ([9])

Abstract

This paper is the second in a series of three culminating in an ordinal analysis of Π^1_2-comprehension. Its objective is to present an ordinal analysis for the subsystem of second order arithmetic with $\boldsymbol{\Delta}^1_2$-comprehension, bar induction and Π^1_2-comprehension for formulae without set parameters. Couched in terms of Kripke-Platek set theory, **KP**, the latter system corresponds to **KPi** augmented by the assertion that there exists a stable ordinal, where **KPi** is **KP** with an additional axiom stating that every set is contained in an admissible set.

1 Introduction

Ordinal-theoretic proof theory came into existence in 1936, springing forth from Gentzen's head in the course of his consistency proof of arithmetic. Gentzen fostered hopes that with sufficiently large constructive ordinals one could establish the consistency of analysis, i.e., the formal system of second order arithmetic, $\mathbf{Z_2}$. Considerable progress has been made in proof theory since Gentzen's tragic death on August 4th, 1945. Π^1_2 comprehension has been a major stumbling block[1] on the road to understanding full comprehension. The introductory quote taken from Gentzen's notes of 1945 reflect quite vividly the enormous technical difficulties and intricacies which one faces when one embarks on such an enterprise.

*In the main this research was carried out in 1995. The particular presentation, however, is a more recent development.

[1]For more background information see [26],p.259, [8],p.362, [13],p.374.

This paper is the second in a series of three culminating in an ordinal analysis of Π^1_2-comprehension. On the set-theoretic side Π^1_2-comprehension corresponds to Kripke-Platek set theory, **KP**, plus Σ_1-separation. The strength of the latter theory is encapsulated in the fact that it proves the existence of unboundedly many stable ordinals, that is ordinals π such that $\mathbf{L}_\pi$ is a Σ_1-elementary substructure of $\mathbf{L}$.

The objective of this paper is to present an ordinal analysis for the subsystem of second order arithmetic with $\mathbf{\Delta}^1_2$-comprehension, bar induction and Π^1_2-comprehension for formulae without set parameters. Couched in terms of Kripke-Platek set theory, the latter system corresponds to **KPi** plus the assertion that there exists a stable ordinal, where **KPi** is **KP** with an additional axiom stating that every set is contained in an admissible set.

The rational for singling out this theory is that experience has shown that the understanding of the ordinal analysis of Π^1_2-comprehension is greatly facilitated by explicating certain simpler cases first. To put it roughly, the difference between the three papers can be explained in terms of the maximal degree of 'stability' that can be proven to exist in the analyzed theory. It is mirrored by the so-called collapsing or projection functions that figure in the pertaining ordinal representation systems. The theory of the first paper [20] entails that for every ordinal α there exists an ordinal β such that β is $\beta+\alpha$-stable, but it does not prove the existence of an ordinal ρ which is stable all the way up, i.e., γ-stable for all $\gamma > \rho$. While the reflection inferences of the infinitary system of [20] are already very complicated, the projection functions still project down single ordinals. What is novel in this paper is that the projection functions will have to project down intervals $[\pi, \beta]$ of ordinals below π. What distinguishes it from the last paper [18] in the series is that such intervals are not too complicated, in that they do not contain stable ordinals $> \pi$.

An important part of ordinal analysis is the development of ordinal representation systems. Such systems are usually generated from collapsing functions. However, we prefer to call them *projection functions* in the present paper as they no longer bear any resemblance to the Mostowski's collapsing function. The ones needed for the ordinal analysis in this (and also the final paper of the series) will be construed as inverses of certain partial elementary embeddings. As their existence is rather difficult to prove, we will develop a model for them on the basis of large cardinals.

2 Large Cardinals

Extensive ordinal representation systems are difficult to understand from a purely syntactical point of view, often to such an extent that it makes no sense to present an ordinal representation system without giving some kind of semantic interpretation. For ordinal representation systems in impredicative proof theory it is essential to understand the projection functions (also called collapsing functions) which they encapsulate. In this section we will indicate a model for the projection functions, employing rather sweeping large cardinal axioms, in that we shall presume the existence of certain cardinals, featuring a strong form of indescribability.

Large cardinals have been used quite frequently in the definition procedure of strong ordinal representation systems, and large cardinal notions have been an important source of inspiration. In the end, they can be dispensed with, but they add an intriguing twist to the relation between set theory and proof theory. The advantage of working in a strong set–theoretic context is that we can build models without getting buried under complexity considerations.

Another objective of this section is to "find" a large cardinal analogue of a stable ordinal.

Two such candidates will be suggested; both of them can be couched in terms of elementary embeddings. However, regarding consistency strength there will be a considerable difference in that the existence of the first kind is compatible with **L** but not the existence of the second kind.

Definition 2.1 Let

$$V = \bigcup_{\alpha \in ON} V_\alpha$$

be the cumulative hierarchy of sets, i.e.

$$V_0 = \emptyset, \quad V_{\alpha+1} = \{X : X \subseteq V_\alpha\}, \quad V_\lambda = \bigcup_{\xi<\lambda} V_\xi \text{ for limit ordinals } \lambda.$$

Let $\eta > 0$. A cardinal κ is *η-shrewd* if for all $P \subseteq V_\kappa$ and every set-theoretic formula $\phi(v_0, v_1)$, whenever

$$V_{\kappa+\eta} \models \phi[P, \kappa],$$

then there exist $0 < \kappa_0, \eta_0 < \kappa$ such that

$$V_{\kappa_0+\eta_0} \models \phi[P \cap V_{\kappa_0}, \kappa_0].$$

κ is *shrewd* if κ is η-shrewd for every $\eta > 0$.

Let $\mathcal{F}$ be a collection of formulae. A cardinal κ is *η-$\mathcal{F}$-shrewd* if for all $P \subseteq V_\kappa$ and every $\mathcal{F}$-formula $\chi(v_0, v_1)$, whenever

$$V_{\kappa+\eta} \models \chi[P, \kappa],$$

then there exist $0 < \kappa_0, \eta_0 < \kappa$ such that

$$V_{\kappa_0+\eta_0} \models \chi[P \cap V_{\kappa_0}, \kappa_0].$$

We will also consider a notion of shrewdness with regard to a given class.

Let $\mathbf{P}$ be a fresh unary predicate symbol. Given a language $\mathcal{L}$ let $\mathcal{L}(\mathbf{P})$ denote its extension by $\mathbf{P}$. If $\mathcal{A}$ is a class we denote by $\langle V_\alpha; \mathcal{A}\rangle$ the structure $\langle V_\alpha; \in; \mathcal{A} \cap V_\alpha\rangle$. $\mathcal{L}_{set}$ denotes the language of set theory. For an $\mathcal{L}_{set}(\mathbf{P})$-sentence ϕ, let the meaning of "$\langle V_\alpha; \mathcal{A}\rangle \models \phi$" be determined by interpreting $\mathbf{P}(t)$ as $t \in \mathcal{A} \cap V_\alpha$.

Definition 2.2 Assume that $\mathcal{A}$ is a class. Let $\eta > 0$. A cardinal κ is *$\mathcal{A}$-η-shrewd* if for all $P \subseteq V_\kappa$ and every formula $\phi(v_0, v_1)$ of $\mathcal{L}_{set}(\mathbf{P})$, whenever

$$\langle V_{\kappa+\eta}; \mathcal{A}\rangle \models \phi[P, \kappa],$$

then there exist $0 < \kappa_0, \eta_0 < \kappa$ such that

$$\langle V_{\kappa_0+\eta_0}; \mathcal{A}\rangle \models \phi[P \cap V_{\kappa_0}, \kappa_0].$$

κ is *$\mathcal{A}$-shrewd* if κ is $\mathcal{A}$-η-shrewd for every $\eta > 0$.

Likewise, for $\mathcal{F}$ a collection of formulae in a language $\mathcal{L}(\mathbf{P})$, we say that a cardinal κ is *$\mathcal{A}$-η-$\mathcal{F}$-shrewd* if for all $P \subseteq V_\kappa$ and every $\mathcal{F}$-formula $\chi(v_0, v_1)$, whenever

$$\langle V_{\kappa+\eta}; \mathcal{A}\rangle \models \chi[P, \kappa],$$

then there exist $0 < \kappa_0, \eta_0 < \kappa$ such that

$$\langle V_{\kappa_0+\eta_0}; \mathcal{A}\rangle \models \chi[P \cap V_{\kappa_0}, \kappa_0].$$

A simple lemma about the uniform definability of V_α from α in any V_β with $\alpha < \beta$ will be used frequently and mostly tacitly.

Lemma 2.3 *There is a formula $\phi(v_0, v_1)$ of the language of set theory such that for any $\alpha < \beta$,*

$$V_\beta \models \phi[\alpha, x] \text{ iff } x = V_\alpha.$$

Proof: See [11], 23.2. □

Corollary 2.4 *If κ is $\mathcal{A}$-δ-shrewd and $0 < \eta < \delta$, then κ is $\mathcal{A}$-η-shrewd.*

Proof: Suppose $P \subseteq V_\kappa$ and $\langle V_{\kappa+\eta}; \mathcal{A}\rangle \models \phi[P, \kappa]$. Let

$$\chi(u, v) := \exists \xi > 0 \text{ ``}\langle V_{v+\xi}; \mathbf{P}\rangle \models \phi(u, v)\text{''}.$$

Then $\langle V_{\kappa+\delta}; \mathcal{A}\rangle \models \chi[P, \kappa]$ by lemma 2.3. Thus there exist $\kappa_0, \delta_0 < \kappa$ such that $\langle V_{\kappa_0+\delta_0}; \mathcal{A}\rangle \models \chi[P \cap V_{\kappa_0}, \kappa_0]$. This implies $\langle V_{\kappa_0+\eta_0}; \mathcal{A}\rangle \models \phi[P \cap V_{\kappa_0}, \kappa_0]$ for some $0 < \eta_0 < \kappa$ by lemma 2.3. □

To my knowledge the notion of shrewdness has not been considered in the set-theoretic literature. There are similarities between the notions of η-shrewdness and η-indescribability (see [4], Ch.9, §4). However, it should be noted that if κ is η-indescribable and $\rho < \eta$, it does not necessarily follow that κ is also ρ-indescribable (see [4], 9.4.6). Furthermore, shrewdness can be construed as a weak pullback version of *extendibility*, an analogy we shall ponder on in the next subsection.

A reason for calling the above cardinals *shrewd* is that if there is a shrewd cardinal κ in the universe, then, loosely speaking, for any notion of large cardinal N which does not make reference to the totality of all ordinals, if there exists an N-cardinal then the least such cardinal is below κ. So for instance, if there are measurable and shrewd cardinals in the universe, then the least measurable is smaller than the least shrewd cardinal.

To situate the notion of shrewdness with regard to consistency strength in the usual hierarchy of large cardinals, we recall the notion of a subtle cardinal.

Definition 2.5 A cardinal κ is said to be *subtle* if for any sequence $\langle S_\alpha : \alpha < \kappa\rangle$ such that $S_\alpha \subseteq \alpha$ and C closed and unbounded in κ, there are $\beta < \delta$ both in C satisfying

$$S_\delta \cap \beta = S_\beta.$$

Since subtle cardinals are not covered in many of the standard texts dealing with large cardinals, we mention the following facts (see [12], §20):

Remark 2.6 *Let $\kappa(\omega)$ denote the first ω-Erdös cardinal.*

(i) $\{\pi < \kappa(\omega) : \pi \text{ is subtle}\}$ is stationary in $\kappa(\omega)$.

(ii) 'Subtlety' relativises to $\mathbf{L}$, i.e. if π is subtle, then $\mathbf{L} \models$ "π is subtle".

Lemma 2.7 *Assume that π is a subtle cardinal and that $\mathcal{A} \subseteq V_\pi$. Then for every $B \subseteq \pi$ closed and unbounded in π there exists $\kappa \in B$ such that*

$$\langle V_\pi; \mathcal{A}\rangle \models \text{``}\kappa \text{ is } \mathcal{A}\text{-shrewd''}.$$

Proof: Assume that π is subtle. Since π is inaccessible, we may select a bijective mapping

$$F : V_\pi \longrightarrow \pi$$

such that

$$C_F \;=\; \{\kappa < \pi :\; F \restriction V_\kappa \text{ maps } V_\kappa \text{ bijectively into } \kappa\} \tag{1}$$

is closed and unbounded in π.

Now let B be closed and unbounded in π. By the preceding we may assume $B \subseteq C_F$. In addition, we may assume that B consists only of cardinals. For a contradiction assume that there is no cardinal $\kappa \in B$ satisfying $\langle V_\pi; \mathcal{A}\rangle \models$ "*κ is $\mathcal{A}$-shrewd*". Since B is unbounded in π, for any $\kappa \in B$, we can choose $\sigma_\kappa \in B$ such that $\kappa < \sigma_\kappa$ and κ fails to be $\mathcal{A}$-σ_κ-shrewd. For $\rho \notin B$ put $\sigma_\rho = \rho$. Let

$$E \;=\; \{\rho \in B : \rho \text{ is closed under } \nu \mapsto \sigma_\nu\}.$$

Then E is also closed and unbounded in π. Notice that for $\kappa_0 < \kappa_1$ both in E, using Corollary 2.4, κ_0 is not $\mathcal{A}$-κ_1-shrewd.

For $\kappa \in E$, let κ^s be the successor of κ in E. Since κ is not $\mathcal{A}$-κ^s-shrewd, we can find an $\mathcal{L}_{set}(\mathbf{P})$-formula ϕ_κ and a subset $P_\kappa \subseteq V_\kappa$ so that (note that $\kappa + \kappa^s = \kappa^s$)

$$\langle V_{\kappa^s}; \mathcal{A}\rangle \models \phi_\kappa(P_\kappa, \kappa) \tag{2}$$

and

$$\forall \nu < \kappa\, \forall \delta \in \kappa \backslash \{0\}\; \langle V_{\nu+\delta}; \mathcal{A}\rangle \models \neg\phi_\kappa(P_\kappa \cap V_\nu, \nu). \tag{3}$$

Put

$$\theta_\kappa(u, v) \;:=\; \text{"}\exists \xi > v\, \langle V_\xi; \mathbf{P}\rangle \models \phi_\kappa(u, v)\text{"}.$$

If now $\kappa^s < \rho$, then $\langle V_\rho; \mathcal{A}\rangle \models \theta_\kappa(P_\kappa, \kappa)$. Further, for all $0 < \mu < \kappa$, $\langle V_\kappa; \mathcal{A}\rangle \models \neg\theta_\kappa(P_\kappa \cap V_\mu, \mu)$. Let E_∞ be the set of all limit points of E below π. The upshot of the foregoing is that for $\kappa < \rho$ both in E_∞,

$$\langle V_\rho; \mathcal{A}\rangle \models \theta_\kappa(P_\kappa, \kappa), \tag{4}$$

however,

$$\forall \mu \in \kappa \backslash \{0\}\; \langle V_\kappa; \mathcal{A}\rangle \models \neg\theta_\kappa(P_\kappa \cap V_\mu, \mu). \tag{5}$$

Define

$$\begin{aligned} P^*_\kappa \;=\; & F''P_\kappa \cap (\kappa \setminus \omega) \;\cup\; \{3n :\, n \in F''P_\kappa \cap \omega\} \\ & \cup \{3n+1 :\, \langle V_{\kappa^d}; \mathcal{A}\rangle \models \psi_n(P_\kappa, \kappa)\} \;\cup\; \{3n+2 :\, \langle V_{\kappa^d}; \mathcal{A}\rangle \models \neg\psi_n(P_\kappa, \kappa)\} \end{aligned} \tag{6}$$

where $\langle \psi_n : n \in \omega\rangle$ is an enumeration of the $\mathcal{L}_{set}(\mathbf{P})$-formulas with two free variables, and κ^d denotes the successor of κ in E_∞.

By subtlety of π, we find $\kappa_0 < \kappa_1$ both in E_∞, so that

$$P^*_{\kappa_0} \;=\; P^*_{\kappa_1} \cap \kappa_0. \tag{7}$$

(7) yields

$$P_{\kappa_0} \;=\; P_{\kappa_1} \cap V_{\kappa_0}. \tag{8}$$

Now, $\langle V_{\kappa_1^d}; \mathcal{A}\rangle \models \theta_{\kappa_1}(P_{\kappa_1}, \kappa_1)$ holds by (4). Therefore (7) viewed together with (6) implies

$$\langle V_{\kappa_0^d}; \mathcal{A}\rangle \models \theta_{\kappa_1}(P_{\kappa_0}, \kappa_0). \tag{9}$$

Hence, using (8), $\langle V_{\kappa_0^d}; \mathcal{A}\rangle \models \theta_{\kappa_1}(P_{\kappa_1} \cap V_{\kappa_0}, \kappa_0)$. The latter implies

$$\langle V_{\kappa_1}; \mathcal{A}\rangle \models \theta_{\kappa_1}(P_{\kappa_1} \cap V_{\kappa_0}, \kappa_0),$$

contradicting (5). □

Corollary 2.8 *Assume that π is a subtle cardinal. Then there exists a cardinal $\kappa < \pi$ such that κ is η-shrewd for all $\eta < \pi$.*

Next we shall derive a cardinal notion from shrewdness which is couched in terms of elementary equivalence of structures and gives rise to partial (definable) elementary embeddings.

Notation. When addressing a structure of the form $\langle V_\zeta; \in; \mathfrak{V}; \dots\rangle$, $\mathfrak{V}$ is meant to stand for the function $(\alpha \mapsto V_\alpha)_{\alpha<\zeta}$.

We want the relation "$x = V_\alpha$" (between x and α) to be preserved under Σ_1-elementary equivalence. But the formula ϕ of lemma 2.3 is not Σ_1. To remedy this, we will consider $\mathfrak{V}$ as a basic function in our structures. This accounts for the insertion of $\mathfrak{V}$ in the next definition.

Definition 2.9 Let $\mathcal{A}$ be a class and $\eta > \kappa$. κ is said to be *$\mathcal{A}$-η-reducible* if for every $P \subseteq V_\eta$ there exist $0 < \kappa_0 < \eta_0 < \kappa$ and $Q \subseteq V_{\eta_0}$ such that

$$\langle V_{\eta_0}; \in; \mathfrak{V}; \kappa_0; \mathcal{A}; Q; x\rangle_{x\in V_{\kappa_0}} \;\equiv\; \langle V_\eta; \in; \mathfrak{V}; \kappa; \mathcal{A}; P; x\rangle_{x\in V_{\kappa_0}}, \tag{10}$$

where "$\equiv$" denotes elementary equivalence of the structures. It is also useful to have a notion of κ being *$\mathcal{A}$-κ-reducible.* κ is *$\mathcal{A}$-κ-reducible* if for every $P \subseteq V_\kappa$ there exist $0 < \kappa_0 < \kappa$ and $Q \subseteq V_{\kappa_0}$ such that

$$\langle V_{\kappa_0}; \in; \mathfrak{V}; \mathcal{A}; Q; x\rangle_{x\in V_{\kappa_0}} \;\equiv\; \langle V_\kappa; \in; \mathfrak{V}; \mathcal{A}; P; x\rangle_{x\in V_{\kappa_0}}. \tag{11}$$

κ is *$\mathcal{A}$-reducible* if κ is $\mathcal{A}$-ξ-reducible for every $\xi \geq \kappa$. For $\xi \geq \kappa$, κ is ξ-reducible if κ is V-ξ-reducible. κ is *reducible* if κ is ξ-reducible for every $\xi \geq \kappa$.

Observe that $Q \cap V_{\kappa_0} \;=\; P \cap V_{\kappa_0}$ is a consequence of (10).

Note that in the situation of (10) there exists a *partial* embedding p from V_{η_0} into V_η satisfying $p \restriction V_{\kappa_0} \;=\; \mathrm{id} \restriction V_{\kappa_0}$ and $p(\kappa_0) \;=\; \kappa$. Moreover, p can be canonically extended so as to being defined on all elements of V_{η_0} which are definable in the structure $\langle V_{\eta_0}; \in; \mathfrak{V}; \kappa_0; \mathcal{A}; Q; x\rangle_{x\in V_{\kappa_0}}$.

We will use

$$p : \langle V_{\kappa_0,\eta_0}; \in; \mathcal{A}; Q\rangle \underset{\equiv}{\longrightarrow} \langle V_{\kappa,\eta}; \in; \mathcal{A}; P\rangle$$

as a shorthand for conveying the foregoing situation.

As to be expected from [20], we shall also have use for refined hierarchies of reducibility.

Definition 2.10 Let $\mathcal{F}$ be a collection of formulae. If $\eta > \kappa$, κ is *$\mathcal{A}$-η-$\mathcal{F}$-reducible* if for every $P \subseteq V_\eta$ there exist $0 < \kappa_0 < \eta_0 < \kappa$ and $Q \subseteq V_{\eta_0}$ such that

$$\langle V_{\eta_0}; \in; \mathfrak{V}; \kappa_0; \mathcal{A}; Q; x\rangle_{x \in V_{\kappa_0}} \quad \equiv_{\mathcal{F}} \quad \langle V_\eta; \in; \mathfrak{V}; \kappa; \mathcal{A}; P; x\rangle_{x\in V_{\kappa_0}}, \tag{12}$$

where $\equiv_{\mathcal{F}}$ signifies that in both structures the same sentences of $\mathcal{F}$ hold true.

κ is *$\mathcal{A}$-κ-$\mathcal{F}$-reducible* if for every $P \subseteq V_\kappa$ there exist $0 < \kappa_0 < \kappa$ and $Q \subseteq V_{\kappa_0}$ such that

$$\langle V_{\kappa_0}; \in; \mathfrak{V}; \mathcal{A}; Q; x\rangle_{x\in V_{\kappa_0}} \quad \equiv_{\mathcal{F}} \quad \langle V_\kappa; \in; \mathfrak{V}; \mathcal{A}; P; x\rangle_{x \in V_{\kappa_0}}. \tag{13}$$

To facilitate notation, (12) and (13) will be shortened into

$$\langle V_{\kappa_0,\eta_0}; \in; \mathcal{A}; Q\rangle \ \underset{\equiv}{\overset{\mathcal{F}}{\longrightarrow}}\ \langle V_{\kappa,\eta}; \in; \mathcal{A}; P\rangle.$$

Similarly, we shall use

$$p : \langle V_{\kappa_0,\eta_0}; \in; \mathcal{A}; Q\rangle \ \underset{\equiv}{\overset{\mathcal{F}}{\longrightarrow}}\ \langle V_{\kappa,\eta}; \in; \mathcal{A}; P\rangle$$

to indicate the above situation together with p being the induced map defined on the $\mathcal{F}$-definable elements of $\langle V_{\kappa_0,\eta_0}; \in; \mathfrak{V}; \kappa_0; Q\rangle$.

Definition 2.11 We shall also have use for a notion of reducibility with regard to a prescribed set of pairs: Suppose $C \subseteq \kappa \times \kappa$. If $\eta > \kappa$, κ is said to be *$\mathcal{A}$-η-$\mathcal{F}$-reducible in C* if for every $P \subseteq V_\eta$ there exist $\langle \kappa_0, \eta_0\rangle \in C$ with $0 < \kappa_0 < \eta_0$ and a $Q \subseteq V_{\eta_0}$ such that

$$\langle V_{\eta_0}; \in; \mathfrak{V}; \kappa_0; \mathcal{A}; Q; x\rangle_{x \in V_{\kappa_0}} \quad \equiv_{\mathcal{F}} \quad \langle V_\eta; \in; \mathfrak{V}; \kappa; \mathcal{A}; P; x\rangle_{x\in V_{\kappa_0}}. \tag{14}$$

κ is *$\mathcal{A}$-κ-$\mathcal{F}$-reducible in C* if for every $P \subseteq V_\kappa$ there exist $\langle \kappa_0, \kappa_0\rangle \in C$ and $Q \subseteq V_{\kappa_0}$ such that

$$\langle V_{\kappa_0}; \in; \mathfrak{V}; \mathcal{A}; Q; x\rangle_{x\in V_{\kappa_0}} \quad \equiv_{\mathcal{F}} \quad \langle V_\kappa; \in; \mathfrak{V}; \mathcal{A}; P; x\rangle_{x \in V_{\kappa_0}}. \tag{15}$$

Next we show that shrewd cardinals are reducible.

Lemma 2.12 *If κ is $\mathcal{A}$-ρ-shrewd and $0 < \mu < \rho$, then κ is $\mathcal{A}$–$\kappa + \mu$–reducible.*

Proof: Suppose $P \subseteq V_{\kappa+\mu}$. Set

$$\mathfrak{A} \ = \ \langle V_{\kappa+\mu}; \in; \mathfrak{V}; \kappa; \mathcal{A}; P; x\rangle_{x \in V_\kappa}$$

and let D be the elementary diagram of $\mathfrak{A}$. To be more precise, put

$$D \ = \ \{\langle \vec{x}, \ulcorner \phi \urcorner\rangle : \vec{x} \in V_\kappa;\ \phi \text{ is an } \mathcal{L}_{set}(\mathbf{P})\text{-formula; } \mathfrak{A} \models \phi[\vec{x}, P, \kappa, \mathfrak{V}]\}.$$

Observe that $D \subseteq V_\kappa$. Further, set

$$\theta(U, W, v, w) := \forall \ulcorner\phi\urcorner \forall \vec{x} \in V_v \left(\text{“}\langle V_{v+w}; \in; \mathfrak{V}; v; \mathbf{P}; U\rangle \models \phi[\vec{x}, U, v, \mathfrak{V}]\text{”} \ \leftrightarrow\ \langle \ulcorner\phi\urcorner, \vec{x}\rangle \in W\right).$$

Then $\langle V_{\kappa+\rho}; \in; \mathcal{A}\rangle \models \theta(P, D, \kappa, \mu)$, hence

$$\langle V_{\kappa+\rho}; \in; \mathcal{A}\rangle \models \exists \zeta > 0\, \exists Z \subseteq V_{\kappa+\zeta} \text{ “}\langle V_{\kappa+\zeta}; \in; \mathbf{P}\rangle \models \theta(Z, D, \kappa, \zeta)\text{”}.$$

Employing the $\mathcal{A}$-ρ-shrewdness of κ, there exist $0 < \kappa_0, \rho_0 < \kappa$ satisfying

$$\langle V_{\kappa_0+\rho_0}; \in; \mathcal{A}\rangle \models \exists \zeta > 0\, \exists Z \subseteq V_{\kappa_0+\zeta} \text{ “}\langle V_{\kappa_0+\zeta}; \in; \mathbf{P}\rangle \models \theta(Z, D \cap V_{\kappa_0}, \kappa_0, \zeta)\text{”}.$$

Thus there exist $0 < \mu_0 < \rho_0$ and $Q \subseteq V_{\kappa_0+\mu_0}$ such that

$$\forall \ulcorner\phi\urcorner \forall \vec{x} \in V_{\kappa_0} \left(\text{“ }\langle V_{\kappa_0+\mu_0}; \in; \mathfrak{V}; \kappa_0; \mathcal{A}; Q\rangle \models \phi[\vec{x}, Q, \kappa_0, \mathfrak{V}]\text{”} \ \leftrightarrow\ \langle \ulcorner\phi\urcorner, \vec{x}\rangle \in D \cap V_{\kappa_0}\right).$$

By the very definition of D, the latter yields

$$\langle V_{\kappa_0+\mu_0}; \in; \mathfrak{V}; \kappa_0; \mathcal{A}; Q; x\rangle_{x\in V_{\kappa_0}} \ \equiv \ \langle V_{\kappa+\mu}; \in; \mathfrak{V}; \kappa; \mathcal{A}; P; x\rangle_{x \in V_{\kappa_0}}. \tag{16}$$

□

2.1 Relating reducibility to strong large cardinal notions

In one approach to devising projection functions with specific features, the author used an elementary embedding analogue of shrewdness and drew on *extendible cardinals* (cf. [25]). This subsection retraces these developments and points at similarities between the notion of reducible cardinal and well-known strong large cardinal notion expressed in terms of elementary embeddings. The material is not needed for the rest of the paper.

Definition 2.13 If $\eta > 0$, κ is *η-extendible* if there is a ζ and a $j : V_{\kappa+\eta} \longrightarrow V_\zeta$ with critical point κ, where $\kappa + \eta < j(\kappa) < \zeta$.

κ is *extendible* if κ is η-extendible for every $\eta > 0$.

Corollary 2.14 *Let κ be η-extendible.*

(i) *When $\eta < \kappa$, then $\zeta = j(\kappa) + \eta$ and $j(\kappa + \delta) = j(\kappa) + \delta$ holds for all $\delta < \eta$.*

(ii) *If $0 < \delta < \eta$, then κ is δ-extendible.*

Proof: Exercise, or see [25], §5. □

Extendible cardinals enjoy a pullback property which can be seen as an elementary embedding analogue of shrewdness.

Lemma 2.15 *If κ is ρ-extendible and $0 < \eta < \rho$, then for every $P \subseteq V_{\kappa+\eta}$ there exist $\kappa_0, \eta_0 < \kappa$, $Q \subseteq V_{\kappa_0+\eta_0}$ and an i such that*

$$Q \cap V_{\kappa_0} \; = \; P \cap V_{\kappa_0}$$

and

$$i : \langle V_{\kappa_0+\eta_0}; \in, Q\rangle \longrightarrow \langle V_{\kappa+\eta}; \in, P\rangle$$

with critical point κ_0 and $i(\kappa_0) = \kappa$.

Proof: Pick a ζ and a $j : V_{\kappa+\rho} \longrightarrow V_\zeta$ with critical point κ such that $\kappa + \rho < j(\kappa) < \zeta$. Notice that $P \cap V_\kappa \; = \; j(P) \cap V_\kappa$. With $i \; = \; j \restriction V_{\kappa+\eta}$ we then get

$$i : \langle V_{\kappa+\eta}; \in, P\rangle \longrightarrow \langle V_{j(\kappa+\eta)}; \in, j(P)\rangle \;\wedge\; P \cap V_\kappa \; = \; j(P) \cap V_\kappa. \tag{17}$$

Therefore, letting $\kappa_0 = \kappa$, $\eta_0 = \eta$, and $Q = P$,

$$\begin{aligned} V_\zeta \models\; & \exists i\, \exists \kappa_0 < j(\kappa)\, \exists \eta_0 < j(\kappa)\, \exists Q \subseteq V_{\kappa_0+\eta_0} \\ & \big(i : \langle V_{\kappa_0+\eta_0}; \in, Q\rangle \longrightarrow \langle V_{j(\kappa+\eta)}; \in, j(P)\rangle \\ & \wedge \mathrm{crit}(i) = \kappa_0 \,\wedge\, i(\kappa_0) = j(\kappa) \,\wedge\, Q \cap V_{\kappa_0} \; = \; j(P) \cap V_{\kappa_0} \big). \end{aligned} \tag{18}$$

Using the elementarity of j, we get

$$\begin{aligned} V_{\kappa+\rho} \models\; & \exists i\, \exists \kappa_0 < \kappa\, \exists \eta_0 < \kappa\, \exists Q \subseteq V_{\kappa_0+\eta_0} \\ & \big(i : \langle V_{\kappa_0+\eta_0}; \in, Q\rangle \longrightarrow \langle V_{\kappa+\eta}; \in, P\rangle \\ & \wedge \mathrm{crit}(i) = \kappa_0 \,\wedge\, i(\kappa_0) = \kappa \,\wedge\, Q \cap V_{\kappa_0} \; = \; P \cap V_{\kappa_0} \big) \end{aligned} \tag{19}$$

from which the desired assertion follows. □

Definition 2.16 Let $\mathcal{A}$ be a class. As above, $\langle V_\alpha; \in; \mathcal{A}; \ldots \rangle$ is short for $\langle V_\alpha; \in; \mathcal{A} \cap V_\alpha; \ldots \rangle$.

Let $\eta > \kappa$. κ is *strongly $\mathcal{A}$-η-reducible* if for every $P \subseteq V_\eta$ there exist $0 < \kappa_0 < \eta_0 < \kappa$ and $Q \subseteq V_{\eta_0}$ and an elementary embedding i such that $Q \cap V_{\kappa_0} = P \cap V_{\kappa_0}$ and

$$i : \langle V_{\eta_0}; \in; \mathcal{A}; Q \rangle \longrightarrow \langle V_\eta; \in; \mathcal{A}; P \rangle$$

with critical point κ_0 and $i(\kappa_0) = \kappa$.

κ is *strongly $\mathcal{A}$-reducible* if κ is strongly $\mathcal{A}$-η-reducible for all $\eta > \kappa$.

κ is *strongly η-reducible* if κ is strongly V-η-reducible. κ is *strongly reducible* if κ is strongly η-reducible for all $\eta > 0$.

From the proof of Theorem 5.7 [25], it follows that the notion of strong reducibility is actually equivalent to *supercompactness.*

Definition 2.17 κ is *δ-supercompact* if there is a transitive class M and a $j : V \longrightarrow M$ such that $\mathrm{crit}(j) = \kappa$ and $\delta < j(\kappa)$, and ${}^\delta M \subseteq M$.

κ is *supercompact* if κ is δ-supercompact for every $\delta \geq \kappa$.

Proposition 2.18 *κ is strongly reducible iff κ is supercompact.*

Proof: The direction "$\Rightarrow$"follows from [11], 22.10.

For the backward direction, suppose that $\delta > \kappa$ and $P \subseteq V_{\kappa+\delta}$. Let $j : V \longrightarrow M$ witness the $|V_{\kappa+\delta+1}|$-supercompactness of κ. Set $\bar{j} = j \restriction V_{\kappa+\delta}$; it is simple to see that

$$\bar{j} : \langle V_{\kappa+\delta}; \in; P \rangle \longrightarrow \langle (V_{j(\kappa+\delta)})^M; \in; j(P) \rangle.$$

By the closure of M under $|V_{\kappa+\delta}|$-sequences, $V_\zeta = (V_\zeta)^M \in M$ for $\zeta \leq \kappa + \delta$ by induction, and so also $\bar{j} \in M$. Hence,

$$M \models \bar{j} : \langle V_{\kappa+\delta}; \in; P \rangle \longrightarrow \langle (V_{j(\kappa+\delta)})^M; \in; j(P) \rangle.$$

Noting that $P \cap V_\kappa = j(P) \cap V_\kappa$, it follows

$$M \models \exists i, \kappa_0, \delta_0 \, \exists Q \subseteq V_{\kappa_0+\delta_0} \big(i : \langle V_{\kappa_0+\delta_0}; \in; Q \rangle \longrightarrow \langle V_{j(\kappa_0+\delta_0)}; \in; j(P) \rangle \\ \wedge \mathrm{crit}(i) = \kappa_0 \wedge i(\kappa_0) = j(\kappa) \wedge Q \cap V_{\kappa_0} = j(P) \cap V_{\kappa_0} \big).$$

The desired result now follows from the elementarity of j. □

A similar equivalence can be shown for $\mathcal{A}$-supercompact cardinals (cf. [25], 6.7).

Proposition 2.19 *κ is strongly $\mathcal{A}$-reducible iff κ is $\mathcal{A}$-supercompact.*

3 Skolem Hulls and Projection Functions

In this section we use the large cardinal notions of the previous section to develop projection functions. The large cardinal assumptions that will be in force for the rest of this paper are the following: There exists an inaccessible cardinal $\mathbf{I}$ and a cardinal $\Xi < \mathbf{I}$ such that Ξ is ρ-reducible for all $\Xi \leq \rho < \mathbf{I}$. We will also assume that $\mathbf{I}$ is the least such cardinal. The existence of such cardinals is a consequence of the existence of a subtle cardinal owing to Corollary 2.7 and Lemma 2.12.

Definition 3.1 Let β^+ denote the least cardinal $> \beta$. Note that β^+ is always a regular cardinal (on the basis of **ZFC**). Let $\beta^{+(0)} := \beta$ and $\beta^{+(n+1)} := (\beta^{+(n)})^+$.

The *Veblen-function* (cf. [6], [24]) $\varphi\alpha\beta := \varphi_\alpha(\beta)$ is defined by transfinite recursion on α by letting φ_α be the function that enumerates the class of ordinals

$$\{\omega^\gamma : \ (\forall \xi < \alpha)[\varphi_\xi(\omega^\gamma) = \omega^\gamma]\}.$$

Let $\mathbf{Veb}_{Cl}$ be the class of ordinals $\alpha > 0$ such $\forall \beta < \alpha \, \forall \gamma < \alpha \; \varphi\beta\gamma < \alpha$.

Definition 3.2 By recursion on α we shall define sets of ordinals $C(\alpha, \zeta)$ (the α^{th} *Skolem Hull generated from ordinals* $< \zeta$), *projection instances* and *reflection instances* of level α, and triples $(\alpha, \mathbb{X}, f)$ and $(\alpha, \mathbb{X}, \mathfrak{D})$, respectively, where $\mathbb{X}$ is a projection instance of level $\leq \alpha$, f is a partial function and $\mathfrak{D}$ is a set. The triples $(\alpha, \mathbb{X}, f)$ and $(\alpha, \mathbb{X}, \mathfrak{D})$ will be called *projection functions* and *projection structures*, respectively.

To every projection instance and reflection instance we also assign an interval of ordinals $[\pi, \rho]$ with $\rho \geq \pi$ (where $[\pi, \rho] = \{\nu : \pi \leq \nu \leq \rho\}$.)

The set of projection instances and reflection instances of levels $\leq \alpha$ will be denoted by $\Re^\alpha$. Likewise, the sets of projection functions and projection structures of levels $\leq \alpha$ will be denoted by $\mathfrak{F}^\alpha$ and $\mathfrak{S}^\alpha$, respectively.

Let $\Re = \bigcup_\alpha \Re^\alpha$, $\mathfrak{F} = \bigcup_\alpha \mathfrak{F}^\alpha$ and $\mathfrak{S} = \bigcup_\alpha \mathfrak{S}^\alpha$. The definition of $\mathfrak{F}$ and $\mathfrak{S}$ will actually guarantee that for any α and $\mathbb{X} \in \Re$ there is at most one f and $\mathfrak{D}$ satisfying $(\alpha, \mathbb{X}, f) \in \mathfrak{F}$ and $(\alpha, \mathbb{X}, \mathfrak{D}) \in \mathfrak{S}$, respectively. This justifies the following definitions:

$$\Psi^\alpha_{\mathbb{X}} = \begin{cases} f & \text{if } (\alpha, \mathbb{X}, f) \in \mathfrak{F} \text{ for some } f \\ \emptyset & \text{otherwise.} \end{cases}$$

$$\mathfrak{D}^\alpha_{\mathbb{X}} = \begin{cases} \mathfrak{D} & \text{if } (\alpha, \mathbb{X}, \mathfrak{D}) \in \mathfrak{S} \text{ for some } \mathfrak{D} \\ \emptyset & \text{otherwise.} \end{cases}$$

Let's stipulate some conventions for ease of presentation. Instead of $(\alpha, \mathbb{X}, f) \in \mathfrak{F}$ we shall write $\Psi^\alpha_{\mathbb{X}} \in \mathfrak{F}$. A projection instance $\mathbb{X} \in \Re^\alpha$ is derived from a reflection pattern and determines the projection function $\Psi^\alpha_{\mathbb{X}}$. Formally $\mathbb{X}$ is an expression built up from ordinals, symbols $\mathsf{RLC}, \mathsf{RSC}, \mathsf{M}, \mathsf{P}_n$ and parentheses. By $\mathbb{X} \in C(\alpha, \zeta)$ we mean that $\mathbb{X} \in \Re^{<\alpha} := \bigcup_{\xi<\alpha} \Re^\xi$ and any ordinal occurring in $\mathbb{X}$ belongs to $C(\alpha, \zeta)$.

It appears to be useful for the technical organization and illuminating for the understanding to group related projection instances together into *reflection patterns*. Formally reflection patterns are projection instance valued functions which arise from a fixed reflection instance by varying certain ordinal parameters (we also allow for the zero arity case, where a reflection pattern is reflection instance, too).

Variables $\mathbb{A}, \mathbb{B}, \mathbb{C}, \mathbb{D}, \mathbb{E}, \mathbb{F}, \mathbb{G}, \mathbb{H}$ are supposed to range over reflection patterns. Variables $\mathbb{X}, \mathbb{Y}, \mathbb{U}, \mathbb{V}, \mathbb{W}, \mathbb{Z}$ are reserved for projection instances. Variables $\mathbb{I}, \mathbb{J}, \mathbb{K}$ are reserved for reflection instances.

In referring to an n-tuple of ordinals as *the least tuple* having a certain property P, we mean the least such n-tuple with respect to the lexicographic order on tuples of ordinals.

$C_n(\alpha,\zeta)$ is defined by recursion on n as follows:

$$\begin{aligned}
C_0(\alpha,\zeta) &= \zeta \,\cup\, \{0,\Xi,\mathbf{I}\} \\
C_{n+1}(\alpha,\zeta) &= C_n(\alpha,\zeta) \\
&\cup \left\{\beta : \beta =_{NF} \omega^{\xi}+\eta;\ \xi,\eta\in C_n(\alpha,\zeta)\right\} \\
&\cup \left\{\beta : \beta =_{NF} \varphi\xi\eta;\ \xi,\eta\in C_n(\alpha,\zeta);\ \xi>0; \xi,\eta<\mathbf{I}\right\} \\
&\cup \left\{\rho^{+(k)} :\ 0<k<\omega,\ \rho\in C_n(\alpha,\zeta);\ \rho=\omega \text{ or } \rho \text{ is a limit cardinal} < \mathbf{I}\right\} \\
&\cup \left\{\Psi^{\beta}_{\mathbb{X}}(\gamma) :\ \mathbb{X},\beta,\gamma\in C_n(\alpha,\zeta);\ \beta<\alpha;\ \gamma\in \mathbf{dom}(\Psi^{\beta}_{\mathbb{X}})\right\} \\
C(\alpha,\zeta) &= \bigcup_n C_n(\alpha,\zeta).
\end{aligned}$$

The third clause in the definition of $C_{n+1}(\alpha,\zeta)$ guarantees that the sets $C(\alpha,\zeta)$ will be closed under the function $(\beta\mapsto\beta^+)_{\beta<\mathbf{I}}$. The reason for not having a clause which simply expresses closure under this function is that we want to ensure uniqueness of representations.

Every projection instance $\mathbb{X}$ is equipped with a *thinning hierarchy*

$$(\mathfrak{M}^{\xi}_{\mathbb{X}})_{\xi\geq o(\mathbb{X})}$$

where $o(\mathbb{X})$ denotes the least ordinal η with $\mathbb{X}\in\mathfrak{R}^{\eta}$. Let $[\pi,\rho]$ be the interval assigned to $\mathbb{X}$. $\mathfrak{M}^{\xi}_{\mathbb{X}}$ consists of pairs of ordinals $\langle\kappa,\delta\rangle$ with $\kappa\leq\delta<\pi$. Moreover, to each $\langle\kappa,\delta\rangle\in\mathfrak{M}^{\xi}_{\mathbb{X}}$ we assign a function $\mathsf{f}^{\xi,\mathbb{X}}_{\kappa,\delta}$. In addition, to each $\mathsf{f}^{\xi,\mathbb{X}}_{\kappa,\delta}$ we also assign an interval of ordinals.

Before we can define the projection instances, reflection instances, projection functions and projection structures of level α, we have to fix a coding of $C(\alpha,\gamma)$ and its pertaining structure as a subset of $V_{\max(\gamma,\omega)}$ for ordinals γ.

Description of $\mathfrak{C}(\alpha,\gamma)$: Each $\beta\in C(\alpha,\gamma)$ is generated in finitely many steps from ordinals $<\gamma$ and the ordinals $\{0,\mathbf{I},\Xi\}\cup\{\omega^{+(n)} :\ 0<n<\omega\}$. Thus, using a set primitive recursive coding of finite sequences of sets, one assigns a Gödel number $\ulcorner\beta\urcorner$ to $\beta\in C(\alpha,\gamma)$ that codes the build-up of β from ordinals in $\gamma\cup\{0,\mathbf{I},\Xi\}\cup\{\omega^{+(n)} :\ 0<n<\omega\}$. If we put $\ulcorner 0\urcorner := \langle 0,0\rangle$, $\ulcorner\mathbf{I}\urcorner := \langle 0,1\rangle$, $\ulcorner\Xi\urcorner := \langle 0,2\rangle$ and code the elements of $\{\omega^{+(n)} :\ 0<n<\omega\}$ via $\ulcorner\omega^{+(n)}\urcorner := \langle 0,2+n\rangle$, all codes will land in $V_{\max(\gamma,\omega)}$. Note however (at least at this point) that we cannot exclude that β enters $C(\alpha,\gamma)$ in different ways. So we have to allow for multiple codes.

In the same vein we assign Gödel numbers $\ulcorner\mathbb{X}\urcorner$ to projection instances $\mathbb{X}\in\mathfrak{R}^{<\alpha}$. Also it will be assumed that the generation process of an ordinal in $C(\alpha,\gamma)$ can be read off set primitive recursively from its code.

Let $\mathrm{Code}^{\alpha,\gamma}(t,\beta)$ signify that t is a code for $\beta\in C(\alpha,\gamma)$. The latter will also be abbreviated by $t =_{\alpha,\gamma} \ulcorner\beta\urcorner$.

Furthermore, we define the following relations on $V_{\max(\gamma,\omega)}$:

$$
\begin{array}{rll}
R_1^{\alpha,\gamma}(t) & \text{if} & t =_{\alpha,\gamma} \ulcorner\beta\urcorner \text{ for some } \beta \in C(\alpha,\gamma) \\
R_2^{\alpha,\gamma}(s) & \text{if} & s =_{\alpha,\gamma} \ulcorner\mathbb{X}\urcorner \text{ for some } \mathbb{X} \in \mathfrak{R}^{<\alpha} \text{ with } \mathbb{X} \in C(\alpha,\gamma) \\
R_3^{\alpha,\gamma}(t,\beta) & \text{if} & t =_{\alpha,\gamma} \ulcorner\beta\urcorner \text{ and } \beta \in C(\alpha,\gamma) \cap \gamma \\
R_4^{\alpha,\gamma}(t,s) & \text{if} & t =_{\alpha,\gamma} \ulcorner\beta\urcorner,\ s =_{\alpha,\gamma} \ulcorner\eta\urcorner \text{ for some } \beta,\eta \in C(\alpha,\gamma) \text{ such that } \beta < \eta. \\
R_5^{\alpha,\gamma}(t,s,r) & \text{if} & t =_{\alpha,\gamma} \ulcorner\beta\urcorner,\ s =_{\alpha,\gamma} \ulcorner\eta\urcorner,\ r =_{\alpha,\gamma} \ulcorner\xi\urcorner \\
 & & \text{for some } \beta,\xi,\zeta \in C(\alpha,\gamma) \text{ such that } \beta \le \alpha \text{ and } \xi \in C(\beta,\zeta). \\
R_6^{\alpha,\gamma}(t,s,r,q) & \text{if} & t =_{\alpha,\gamma} \ulcorner\beta\urcorner,\ s =_{\alpha,\gamma} \ulcorner\mathbb{X}\urcorner,\ r =_{\alpha,\gamma} \ulcorner\xi\urcorner, \text{ and } q =_{\alpha,\gamma} \ulcorner\zeta\urcorner \text{ for some} \\
 & & \mathbb{X},\beta,\xi,\zeta \in C(\alpha,\gamma) \text{ with } \mathbb{X} \in \mathfrak{R}^{<\alpha},\ \Psi^\beta_{\mathbb{X}} \in \mathfrak{F}^{<\alpha} \text{ and } \Psi^\beta_{\mathbb{X}}(\xi) = \zeta. \\
R_7^{\alpha,\gamma}(t,s,x,j) & \text{if} & t =_{\alpha,\gamma} \ulcorner\beta\urcorner,\ s =_{\alpha,\gamma} \ulcorner\mathbb{X}\urcorner,\ x = \langle a_1,\dots,a_k\rangle \text{ for some} \\
 & & \mathbb{X},\beta,a_1,\dots,a_n \in C(\alpha,\gamma) \text{ with } \mathbb{X} \in \mathfrak{R}^{<\alpha},\ \mathfrak{D}^\beta_{\mathbb{X}} \in \mathfrak{S}^{<\alpha}, \text{ and} \\
 & & j \text{ is the Gödel number of a formula } \mathcal{F} \text{ with at most } k \text{ free} \\
 & & \text{variables such that } \mathfrak{D}^\beta_{\mathbb{X}} \models \mathcal{F}[a_1,\dots,a_k]. \\
R_8^{\alpha,\gamma}(t,s,r,q) & \text{if} & t =_{\alpha,\gamma} \ulcorner\beta\urcorner,\ s =_{\alpha,\gamma} \ulcorner\mathbb{X}\urcorner,\ r =_{\alpha,\gamma} \ulcorner\mu\urcorner, \text{ and } q =_{\alpha,\gamma} \ulcorner\nu\urcorner \text{ for some} \\
 & & \mathbb{X},\beta,\mu,\nu \in C(\alpha,\gamma) \text{ such that } \mathbb{X} \in \mathfrak{R}^{<\alpha} \text{ and } \langle\mu,\nu\rangle \in \mathfrak{M}^\beta_{\mathbb{X}}. \\
R_9^{\alpha,\gamma}(t,s,r,q,p,u) & \text{if} & t =_{\alpha,\gamma} \ulcorner\beta\urcorner,\ s =_{\alpha,\gamma} \ulcorner\mathbb{X}\urcorner,\ r =_{\alpha,\gamma} \ulcorner\mu\urcorner,\ q =_{\alpha,\gamma} \ulcorner\nu\urcorner,\ p =_{\alpha,\gamma} \ulcorner\tau\urcorner \text{ and} \\
 & & u =_{\alpha,\gamma} \ulcorner\zeta\urcorner \text{ for some } \mathbb{X},\beta,\mu,\nu,\tau,\zeta \in C(\alpha,\gamma) \text{ such that } \mathbb{X} \in \mathfrak{R}^{<\alpha}, \\
 & & \langle\mu,\nu\rangle \in \mathfrak{M}^\beta_{\mathbb{X}}, \tau \in \mathbf{dom}(\mathsf{f}^{\beta,\mathbb{X}}_{\mu,\nu}), \text{ and } \mathsf{f}^{\beta,\mathbb{X}}_{\mu,\nu}(\tau) = \zeta.
\end{array}
\tag{20}
$$

Finally let $\mathfrak{C}(\alpha,\gamma)$ be the disjoint union of all the above relations coded into one subset of $V_{\max(\gamma,\omega)}$. This may actually require some "flat" pairing function which does not raise the rank if δ happens to be a successor (cf. [4], 9.1 and [22], Proposition 2.8, pp. 241). Such a flat pairing function can be defined by

$$
\mathsf{f}(x,y) = \begin{cases} \langle x,y\rangle & \text{if } x,y \in V_\omega \\ \{\mathsf{f}(a,b) : a{\in}x \wedge b{\in}y\} & \text{otherwise} \end{cases}
$$

f is uniformly definable in the structures $\langle V_\delta; \in\rangle$ for $\delta > \omega$.

New projection and reflection instances are actually only generated at level 0 and at successor levels. If $\mathbb{X} \in \mathfrak{R}$, let $o(\mathbb{X})$ denote the level of $\mathbb{X}$.

In the following let us fix an ordinal α and assume that $C(\alpha,\gamma)$ has been defined for all ordinals γ, and that $\mathfrak{R}^\alpha, \mathfrak{F}^{<\alpha}, \mathfrak{S}^{<\alpha}$ have been defined as well. We then define the projection functions and projection structures of level α and the thinning hierarchies up to level α for each projection instance in $\mathfrak{R}^\alpha$. Clauses 1.1 and 1.2 define the projection instances of level 0 while the projection instances and reflection instances defined in later clauses are of level $\alpha+1$. Right next to the definition of a new projection instance we shall indicate by "$\to n$" where the pertaining thinning hierarchy can be found. Unfortunately, there are many clauses to consider.

1.1.

$$(\mathbf{I}; \mathsf{RLC}; \emptyset; 0) \qquad \to 1.6$$

is a projection instance and a reflection instance in $\mathfrak{R}^0$ with interval $[\mathbf{I},\mathbf{I}]$. $(\mathbf{I}; \mathsf{RLC}; \emptyset; 0)$ is also a reflection pattern.

1.2. For every $0 < n < \omega$,

$$(\omega^{+(n)}; \mathsf{RSC}; \emptyset; 0) \qquad \to 1.6$$

and

$$(\Xi^{+(n)}; \mathsf{RSC}; \emptyset; 0) \qquad \to 1.6$$

are projection instances and reflection instances in $\mathfrak{R}^0$ with intervals $[\omega^{+(n)}, \omega^{+(n)}]$ and $[\Xi^{+(n)}, \Xi^{+(n)}]$, respectively. $(\omega^{+(n)}; \mathsf{RSC}; \emptyset; 0)$ and $(\Xi^{+(n)}; \mathsf{RSC}; \emptyset; 0)$ are also reflection patterns.

1.3. If $\mathbb{X} \in \mathfrak{R}^\alpha$ is a projection instance with interval $[\pi, \rho]$, $\rho \in \mathbf{dom}(\Psi^\alpha_\mathbb{X})$, and $\mathbb{X}$ is not of the form $(\pi; \mathsf{RSC}; \dots)$, then for every $0 < n < \omega$,

$$(\Psi^\alpha_\mathbb{X}(\rho)^{+(n)}; \mathsf{RSC}; \mathbb{X}; \alpha + 1) \qquad \to 1.6$$

is a projection instance and a reflection instance in $\mathfrak{R}^{\alpha+1}$ with interval $[\Psi^\alpha_\mathbb{X}(\rho)^{+(n)}, \Psi^\alpha_\mathbb{X}(\rho)^{+(n)}]$. $(\Psi^\alpha_\mathbb{X}(\rho)^{+(n)}; \mathsf{RSC}; \mathbb{X}; \alpha + 1)$ is also a reflection pattern.

1.4. If $\mathbb{X} \in \mathfrak{R}^\alpha$ is a projection instance with interval $[\pi, \rho]$, $\pi < \kappa < \rho$, $\kappa \in \mathbf{dom}(\Psi^\alpha_\mathbb{X})$, and there is a projection instance $\mathbb{Y} \in \mathfrak{R}^{<\alpha}$ of the form $(\kappa; \mathsf{RSC}; \dots)$, then

$$(\Psi^\alpha_\mathbb{X}(\kappa); \mathsf{RSC}; \mathbb{X}; \alpha + 1) \qquad \to 1.6$$

is a projection instance and a reflection instance in $\mathfrak{R}^{\alpha+1}$ with interval $[\Psi^\alpha_\mathbb{X}(\kappa), \Psi^\alpha_\mathbb{X}(\kappa)]$. $(\Psi^\alpha_\mathbb{X}(\kappa); \mathsf{RSC}; \mathbb{X}; \alpha + 1)$ is also a reflection pattern.

1.5. Suppose $\mathbb{X} \in \mathfrak{R}^\alpha$ is a projection instance with interval $[\pi, \kappa]$, $\kappa \in \mathbf{dom}(\Psi^\alpha_\mathbb{X})$, $\pi < \kappa$, and there is a projection instance $\mathbb{Y} \in \mathfrak{R}^{<\alpha}$ of the form $(\kappa; \mathsf{RSC}; \dots)$. If $\mathbb{X}$ is of the form $(\pi; \kappa\text{-}\mathsf{P}_n; \dots)$ or $(\pi; \mathsf{M}^{<\xi}_\mathbb{Z}\text{-}\kappa\text{-}\mathsf{P}_n; \dots)$ with $n \geq 3$, then

$$(\Psi^\alpha_\mathbb{X}(\kappa); \mathsf{RSC}; \mathbb{X}; \alpha + 1) \qquad \to 1.6$$

is a projection instance and a reflection instance in $\mathfrak{R}^{\alpha+1}$ with interval $[\Psi^\alpha_\mathbb{X}(\kappa), \Psi^\alpha_\mathbb{X}(\kappa)]$. $(\Psi^\alpha_\mathbb{X}(\kappa); \mathsf{RSC}; \mathbb{X}; \alpha + 1)$ is also a reflection pattern.

1.6. If $\mathbb{X} \in \mathfrak{R}^\alpha$ is a projection instance of either form $(\pi; \mathsf{RLC}; \dots; \beta)$ or $(\pi; \mathsf{RSC}; \dots; \beta)$, then $\mathfrak{M}^\alpha_\mathbb{X}$ consists of all ordinal pairs $\langle \rho, \rho \rangle$ with $\rho < \pi$ such that

$$C(\alpha, \rho) \cap \pi = \rho,$$
$$\mathbb{X}, \alpha \in C(\alpha, \rho).$$

$X^\alpha_\mathbb{X}$ consists of all triples $\langle \rho, \rho, \emptyset \rangle$, where $\langle \rho, \rho \rangle \in \mathfrak{M}^\alpha_\mathbb{X}$.

For $\langle \rho, \rho \rangle \in \mathfrak{M}^\alpha_\mathbb{X}$ let $\mathsf{f}^{\alpha,\mathbb{X}}_{\rho,\rho}$ be the function with domain $\{\pi\}$ and $\mathsf{f}^{\alpha,\mathbb{X}}_{\rho,\rho}(\pi) := \rho$; the interval assigned to $\mathsf{f}^{\alpha,\mathbb{X}}_{\rho,\rho}$ is $[\pi, \pi]$.

If $\mathfrak{M}^\alpha_\mathbb{X} \neq \emptyset$, then $(\alpha, \mathbb{X}, \mathsf{f}^{\alpha,\mathbb{X}}_{\rho_0,\rho_0}) \in \mathfrak{F}^\alpha$ and $(\alpha, \mathbb{X}, \emptyset) \in \mathfrak{S}^\alpha$, where ρ_0 is the least ordinal such that $\langle \rho_0, \rho_0 \rangle \in \mathfrak{M}^\alpha_\mathbb{X}$.

1.7. If $\rho \in [\Xi, \mathbf{I})$ and α is the least ordinal such that $\rho \in C(\alpha, \Xi)$, then

$$\mathbb{A}(\rho) = (\Xi; \rho\text{-}\mathsf{P}_0; \mathbf{I}\text{-}\mathsf{P}_0; \emptyset; \alpha + 1) \qquad \to 2.$$

is a projection instance and a reflection instance in $\mathfrak{R}^{\alpha+1}$ with interval $[\Xi, \rho]$.

The function $\rho' \mapsto \mathbb{A}(\rho')$ with domain $\{\rho' \in [\Xi, \mathbf{I}) : \exists\xi' \, [\rho' \in C(\xi', \Xi)]\}$ is a reflection pattern. The latter definition, perhaps calls for an explanation. It will be given in the following remark.

A general remark on the definition of reflection patterns. To be more precise, in 1.7 we are defining the graph of a partial class function $\mathbb{A}$ simultaneously with the sets $\mathfrak{R}^\alpha$ by induction on α. We have just added the pair $\langle \rho, \mathbb{A}(\rho) \rangle$ to the graph of $\mathbb{A}$. By saying that $\rho' \mapsto \mathbb{A}(\rho')$ is a function with domain $\{\rho' \in [\Xi, \mathbf{I}) : \exists\xi' \, [\rho' \in C(\xi', \Xi)]\}$ we mean that the graph of $\mathbb{A}$ consists of all pairs $\langle \rho', \mathbb{A}(\rho') \rangle$, where $\mathbb{A}(\rho')$ is a projection instance of the form $(\Xi; \rho'\text{-}\mathsf{P}_0; \mathbf{I}\text{-}\mathsf{P}_0; \emptyset; \alpha' + 1)$ for some α'. Note that by 1.7, $\mathbb{A}(\rho')$ is defined if and only if $\exists\xi' \, [\rho' \in C(\xi', \Xi)]$. Here we consider the parameter ρ' to be the main parameter in $\mathbb{A}(\rho')$. Note that the last parameter $\alpha' + 1$ in $\mathbb{A}(\rho')$ just indicates the stage of $\mathbb{A}(\rho')$ as a projection instance. For the set-theoretic definition of projection instances it is not really necessary to make it an ingredient of their definition, but later on when we concern ourselves with ordinal representation systems it will useful to have an easy way of retrieving a projection instance's level.

The upshot of the above is that all projection instances of the form $(\Xi; \rho'\text{-}\mathsf{P}_0; \mathbf{I}\text{-}\mathsf{P}_0; \emptyset; \alpha' + 1)$ constitute one type of projection instances, whereby ρ' is the main varying parameter, rendering them a reflection pattern $\rho' \mapsto \mathbb{A}(\rho')$ with domain $\{\rho' \in [\Xi, \mathbf{I}) : \exists\xi' \, [\rho' \in C(\xi', \Xi)]\}$.

Similar remarks apply to definitions of reflection patterns to come. But we trust that this remark clarifies matters and therefore shall not repeat it below.

2. Let $\mathbb{A}$ be a reflection pattern and $\rho \in \mathbf{dom}(\mathbb{A})$. Suppose further that $\mathbb{A}(\rho)$ is of the form $\mathbb{A}(\rho) = (\pi; \rho\text{-}\mathsf{P}_0; \lambda\text{-}\mathsf{P}_0; \mathbb{X}; \beta)$ or $\mathbb{A}(\rho) = (\pi; \rho\text{-}\mathsf{P}_0; \lambda\text{-}\mathsf{P}_0; \emptyset; \beta)$, where λ is a limit and $\beta \leq \alpha$. Then $X^\alpha_{\mathbb{A}(\rho)}$ consist of all triples $\langle \kappa, \delta, \vec{\mathbf{D}} \rangle$ such that $\kappa \leq \delta < \pi$, κ is inaccessible, and the following are satisfied:

1. $C(\alpha, \kappa) \cap \pi = \kappa$ and $\mathbb{A}(\rho), \alpha \in C(\alpha, \kappa)$.
2. $\langle V_{\kappa,\delta}; \in; \vec{\mathbf{D}} \rangle \xrightarrow[\equiv]{\Pi^1_0} \langle V_{\pi,\rho}; \in; \mathfrak{C}(\alpha, \rho) \rangle$, where $\vec{\mathbf{D}} \subseteq V_\delta$ is the $<_{\mathbf{L}}$-least such set.
3. If $\xi \in C(\xi, \kappa) \cap [\beta, \alpha)$ and $\vartheta \in C(\xi, \kappa) \cap (\rho, \lambda)$, then κ is δ-Π^1_0-reducible in

$$ {}^o\mathfrak{M}^\xi_{\mathbb{A}(\vartheta)} \;:=\; \{\langle \kappa^*, \rho^* \rangle : \exists\vartheta^* \left(\langle \kappa^*, \vartheta^* \rangle \in \mathfrak{M}^\xi_{\mathbb{A}(\vartheta)} \wedge \kappa^* \leq \rho^* < \vartheta^* \right)\}. $$

Set

$$ \mathfrak{M}^\alpha_{\mathbb{A}(\rho)} := \{\langle \kappa, \delta \rangle : \exists\vec{\mathbf{D}} \, \langle \kappa, \delta, \vec{\mathbf{D}} \rangle \in X^\alpha_{\mathbb{A}(\rho)}\}. $$

Suppose $\langle \pi_0, \rho_0 \rangle \in \mathfrak{M}^\alpha_{\mathbb{A}(\rho)}$. Then $\langle \pi_0, \rho_0, \vec{\mathbf{D}}_0 \rangle \in X^\alpha_{\mathbb{A}(\rho)}$ for some $\vec{\mathbf{D}}_0$ which is uniquely determined by π_0, ρ_0. Let j_0 be determined by

$$ j_0 : \langle V_{\pi_0,\rho_0}; \in; \vec{\mathbf{D}}_0 \rangle \xrightarrow[\equiv]{\Pi^1_0} \langle V_{\pi,\rho}; \in; \mathfrak{C}(\alpha, \rho) \rangle. $$

Define

$$ \bar{j}_0 \;:=\; j_0 \restriction [\pi_0, \rho_0) \;\cup\; \{\langle \rho_0, \rho \rangle\} $$

and set

$$ \mathsf{f}^{\alpha,\mathbb{A}(\rho)}_{\pi_0,\rho_0} \;:=\; \bar{j}_0^{\,-1} \restriction C(\alpha, \pi_0). $$

$[\pi, \rho]$ is the interval assigned to $\mathsf{f}^{\alpha,\mathbb{A}(\rho)}_{\pi_0,\rho_0}$.

If $\mathfrak{M}^{\alpha}_{\mathbb{A}(\rho)} \neq \emptyset$, let $\langle \pi_0, \rho_0 \rangle$ be the least pair in $\mathfrak{M}^{\alpha}_{\mathbb{A}(\rho)}$. Then $(\alpha, \mathbb{A}(\rho), \mathsf{f}^{\alpha,\mathbb{A}(\rho)}_{\pi_0,\rho_0}) \in \mathfrak{F}^{\alpha}$. If $\langle \pi_0, \rho_0, \vec{\mathbf{D}}_0 \rangle \in X^{\alpha}_{\mathbb{A}(\rho)}$, then $(\alpha, \mathbb{A}(\rho), \mathfrak{D}) \in \mathfrak{S}^{\alpha}$, where $\mathfrak{D} := \langle V_{\pi_0,\rho_0}; \in; \vec{\mathbf{D}}_0 \rangle$.

3. Let $\mathbb{A}$ be a reflection pattern and $\rho \in \mathbf{dom}(\mathbb{A})$. Suppose further that $\mathbb{A}(\rho)$ is of the form $\mathbb{A}(\rho) = (\pi; \rho\text{-}\mathsf{P}_0; \lambda\text{-}\mathsf{P}_0; \mathbb{X}; \beta)$ or $\mathbb{A}(\rho) = (\pi; \rho\text{-}\mathsf{P}_0; \lambda\text{-}\mathsf{P}_0; \emptyset; \beta)$, where λ is a limit and $\beta \leq \gamma \leq \alpha$. Also assume $\mathfrak{M}^{\gamma}_{\mathbb{A}(\rho)} \neq \emptyset$. Let $\bar{\kappa} := \Psi^{\gamma}_{\mathbb{A}(\rho)}(\pi)$ and $\bar{\rho} := \Psi^{\gamma}_{\mathbb{A}(\rho)}(\rho)$.

3.1. Suppose that $\bar{\rho}$ is a limit $> \bar{\kappa}$. Let $\delta \in [\bar{\kappa}, \bar{\rho})$ and assume that α is the least ordinal $\xi' \geq \gamma$ such that $\delta \in C(\xi', \bar{\kappa})$.

If $\gamma > \beta$, then

$$\mathbb{C}(\delta) := (\bar{\kappa}; \mathsf{M}^{<\gamma}_{\mathbb{A}(\rho)}\text{-}\delta\text{-}\mathsf{P}_0; \mathsf{M}^{\gamma}_{\mathbb{A}(\rho)}; \mathbb{A}(\rho); \alpha + 1) \qquad \to 4.a$$

is a projection instance in $\mathfrak{R}^{\alpha+1}$ with interval $[\bar{\kappa}, \delta]$.

The function $\delta' \mapsto \mathbb{C}(\delta')$ with domain

$$\{\delta' \in [\bar{\kappa}, \bar{\rho}) :\ \exists \xi' \ [\delta' \in C(\xi', \bar{\kappa})]\}$$

is a reflection pattern.

For every $\xi \in C(\xi, \bar{\kappa}) \cap [\beta, \gamma)$ and every $\eta \in C(\xi, \bar{\kappa}) \cap (\rho, \lambda)$,

$$(\mathbb{C}(\delta), \xi, \eta)$$

is a reflection instance in $\mathfrak{R}^{\alpha+1}$ with interval $[\bar{\kappa}, \delta]$.

If $\gamma = \beta$, then

$$\mathbb{D}(\delta) := (\bar{\kappa}; \delta\text{-}\mathsf{P}_0; \bar{\rho}\text{-}\mathsf{P}_0; \mathbb{A}(\rho); \alpha + 1) \qquad \to 2.$$

is a projection instance and reflection instance in $\mathfrak{R}^{\alpha+1}$ with interval $[\bar{\kappa}, \delta]$.

The function $\delta' \mapsto \mathbb{D}(\delta')$ with domain $\{\delta' \in [\bar{\kappa}, \bar{\rho}] :\ \exists \xi' \ [\delta' \in C(\xi', \bar{\kappa})]\}$ is a reflection pattern.

3.2. Now assume that $\alpha = \gamma$ and $\bar{\rho}$ is a successor $\rho_0 + 1$.

If $\gamma > \beta$, then for every $0 < m < \omega$,

$$\mathbb{C}(m) := (\bar{\kappa}; \mathsf{M}^{<\gamma}_{\mathbb{A}(\rho)}\text{-}\rho_0\text{-}\mathsf{P}_m; \mathsf{M}^{\alpha}_{\mathbb{A}(\rho)}; \mathbb{A}(\rho); \alpha + 1) \qquad \to 12.a$$

is a projection instance in $\mathfrak{R}^{\alpha+1}$ with interval $[\bar{\kappa}, \rho_0]$. The function $n \mapsto \mathbb{C}(n)$ with domain $(0, \omega)$ is a reflection pattern.

For every $\xi \in C(\xi, \bar{\kappa}) \cap [\beta, \gamma)$ and every $\eta \in C(\xi, \bar{\kappa}) \cap (\rho, \lambda)$,

$$(\mathbb{C}(m), \xi, \eta)$$

is a reflection instance in $\mathfrak{R}^{\alpha+1}$ with interval $[\bar{\kappa}, \rho_0]$.

If $\gamma = \beta$, then for every $0 < m < \omega$,

$$\mathbb{D}(m) := (\bar{\kappa}; \rho_0\text{-}\mathsf{P}_m; \rho_0\text{-}\mathsf{P}_\infty; \mathbb{A}(\rho); \alpha + 1) \qquad \to 8.$$

is a reflection pattern and reflection instance in $\mathfrak{R}^{\alpha+1}$ with interval $[\bar{\kappa}, \rho_0]$.

The function $n \mapsto \mathbb{D}(n)$ with domain $(0, \omega)$ is a reflection pattern.

3.3. Now assume that $\alpha = \gamma$ and $\bar{\rho} = \bar{\kappa}$.

If $\gamma > \beta$, then for every $\xi \in C(\xi, \bar{\kappa}) \cap [\beta, \gamma)$ and $\eta \in C(\xi, \bar{\kappa}) \cap (\rho, \lambda)$,

$$\mathbb{I} \;:=\; (\bar{\kappa}; \mathsf{M}^{\xi}_{\mathbb{A}\,(\eta)}\text{-}\bar{\kappa}\text{-}\mathsf{P}_0; \mathsf{M}^{\alpha}_{\mathbb{A}\,(\rho)}; \mathbb{A}\,(\rho); \alpha + 1)$$

is a reflection instance with level $\alpha + 1$ and interval $[\bar{\kappa}, \bar{\kappa}]$, and

$$\mathbb{E} \;:=\; (\pi; \mathsf{M}^{\xi}_{\mathbb{A}\,(\eta)}; \bar{\kappa}; \mathbb{I}; \alpha + 1) \qquad \to 14.$$

is a projection instance in $\mathfrak{R}^{\alpha+1}$ with interval $[\pi, \eta]$. $\mathbb{I}$ is also a reflection pattern.

If $\gamma = \beta$, then

$$\mathbb{F} \;:=\; (\bar{\kappa}; \mathsf{RLC}; \mathbb{A}\,(\rho); \alpha + 1) \qquad \to 1.6$$

is a projection instance and reflection instance in $\mathfrak{R}^{\alpha+1}$ with interval $[\bar{\kappa}, \bar{\kappa}]$. $\mathbb{F}$ is also a reflection pattern.

4.a. Let $\mathbb{C}\,(\delta)$ be a projection instance in $\mathfrak{R}^{\alpha}$ of the form

$$\mathbb{C}\,(\delta) \;=\; (\pi; \mathsf{M}^{<\gamma}_{\mathbb{A}\,(\rho)}\text{-}\delta\text{-}\mathsf{P}_0; \mathsf{M}^{\zeta}_{\mathbb{Y}}; \mathbb{X}; \alpha_0),$$

where $\mathbf{dom}(\mathbb{A}\,) \neq (0, \omega)$. Let $\beta := o(\mathbb{A}\,(\rho))$. Then $X^{\alpha}_{\mathbb{C}\,(\delta)}$ consist of all triples $\langle \kappa, \sigma, \vec{\mathbf{D}} \rangle$ such that $\kappa \leq \sigma < \pi$, κ is inaccessible, and the following are satisfied:

1. $C(\alpha, \kappa) \cap \pi \;=\; \kappa$ and $\mathbb{C}\,(\delta), \alpha \in C(\alpha, \kappa)$.
2. $\langle V_{\kappa,\sigma}; \in; \vec{\mathbf{D}} \rangle \xrightarrow[\equiv]{\Pi^1_0} \langle V_{\pi,\delta}; \in; \mathfrak{C}(\alpha, \delta) \rangle$, where $\vec{\mathbf{D}} \subseteq V_\sigma$ is the $<_{\mathbf{L}}$-least such set.
3. If $\xi \in C(\xi, \kappa) \cap [\beta, \gamma)$ and $\vartheta \in C(\xi, \kappa) \cap \mathbf{dom}(\mathbb{A}\,)$ and $\vartheta > \rho$, then κ is σ-Π^1_0-reducible in
$$^{o}\mathfrak{M}^{\xi}_{\mathbb{A}\,(\vartheta)} \quad := \quad \{ \langle \kappa^*, \sigma^* \rangle :\; \exists \vartheta^* \left(\langle \kappa^*, \vartheta^* \rangle \in \mathfrak{M}^{\xi}_{\mathbb{A}\,(\vartheta)} \,\wedge\, \kappa^* \leq \sigma^* < \vartheta^* \right) \}.$$
4. If $\tau \in C(\tau, \kappa) \cap [\alpha_0, \alpha)$ and $\nu \in C(\tau, \kappa) \cap \mathbf{dom}(\mathbb{C}\,)$ and $\nu > \delta$, then κ is σ-Π^1_0-reducible in
$$^{o}\mathfrak{M}^{\tau}_{\mathbb{C}\,(\nu)} \quad := \quad \{ \langle \kappa^*, \sigma^* \rangle :\; \exists \nu^* \left(\langle \kappa^*, \nu^* \rangle \in \mathfrak{M}^{\tau}_{\mathbb{C}\,(\nu)} \,\wedge\, \kappa^* \leq \sigma^* < \nu^* \right) \}.$$

Set

$$\mathfrak{M}^{\alpha}_{\mathbb{C}\,(\delta)} := \{ \langle \kappa, \sigma \rangle :\; \exists \vec{\mathbf{D}}\, [\langle \kappa, \sigma, \vec{\mathbf{D}} \rangle \in X^{\alpha}_{\mathbb{C}\,(\delta)}] \}.$$

Suppose $\langle \kappa_0, \sigma_0 \rangle \in \mathfrak{M}^{\alpha}_{\mathbb{C}\,(\delta)}$. Then $\langle \kappa_0, \sigma_0, \vec{\mathbf{D}}_0 \rangle \in X^{\alpha}_{\mathbb{C}\,(\delta)}$ for some $\vec{\mathbf{D}}_0$ which is uniquely determined by $\langle \kappa_0, \sigma_0 \rangle$. Let j_0 be determined by

$$j_0 : \langle V_{\kappa_0,\sigma_0}; \in; \vec{\mathbf{D}}_0 \rangle \xrightarrow[\equiv]{\Pi^1_0} \langle V_{\pi,\sigma}; \in; \mathfrak{C}(\alpha, \sigma) \rangle.$$

Define

$$\bar{j}_0 \;:=\; j_0 \restriction [\kappa_0, \sigma_0) \;\cup\; \{ \langle \sigma_0, \sigma \rangle \}$$

and set

$$\mathsf{f}^{\alpha,\mathbb{C}\,(\delta)}_{\kappa_0,\sigma_0} \;:=\; \bar{j}_0^{\;-1} \restriction C(\alpha, \kappa_0).$$

$[\kappa_0, \sigma_0]$ is the interval assigned to $\mathsf{f}^{\alpha,\mathbb{C}\,(\delta)}_{\kappa_0,\sigma_0}$.

If $\mathfrak{M}^{\alpha}_{\mathbb{C}\,(\delta)} \neq \emptyset$, let $\langle \pi_0, \delta_0 \rangle$ be the least pair in $\mathfrak{M}^{\alpha}_{\mathbb{C}\,(\delta)}$. Let $\vec{\mathbf{D}}_0$ be such that $\langle \pi_0, \delta_0, \vec{\mathbf{D}}_0 \rangle \in X^{\alpha}_{\mathbb{C}\,(\delta)}$. Then $(\alpha, \mathbb{C}\,(\delta), \mathsf{f}^{\alpha,\mathbb{C}\,(\delta)}_{\pi_0,\delta_0}) \in \mathfrak{F}^{\alpha}$ and $(\alpha, \mathbb{C}\,(\delta), \mathfrak{D}) \in \mathfrak{G}^{\alpha}$, where $\mathfrak{D} := \langle V_{\pi_0,\delta_0}; \in; \vec{\mathbf{D}}_0 \rangle$.

4.b. Let $\mathbb{C}(\delta)$ be a projection instance in $\mathfrak{R}^\alpha$ of the form

$$\mathbb{C}(\delta) \;=\; (\pi; \mathsf{M}^{<\gamma}_{\mathbb{A}(m)}\text{-}\delta\text{-}\mathsf{P}_0; \mathsf{M}^{\zeta}_{\mathbb{Y}}; \mathbb{X}; \alpha_0),$$

where $\mathbf{dom}(\mathbb{A}) = (0, \omega)$. Let $\beta := o(\mathbb{A}(m))$. Then $X^\alpha_{\mathbb{C}(\delta)}$ consist of all triples $\langle \kappa, \sigma, \vec{\mathbf{D}}\rangle$ such that $\kappa \le \sigma < \pi$, κ is inaccessible, and the following are satisfied:

1. $C(\alpha, \kappa) \cap \pi \;=\; \kappa$ and $\mathbb{C}(\delta), \alpha \in C(\alpha, \kappa)$.
2. $\langle V_{\kappa,\sigma}; \in; \vec{\mathbf{D}}\rangle \;\xrightarrow[\equiv]{\Pi^1_0}\; \langle V_{\pi,\delta}; \in; \mathfrak{C}(\alpha, \delta)\rangle$, where $\vec{\mathbf{D}} \subseteq V_\sigma$ is the $<_{\mathbf{L}}$-least such set.
3. If $\xi \in C(\xi, \kappa) \cap [\beta, \gamma)$ and $m < k < \omega$, then κ is σ-Π^1_0-reducible in ${}^o\mathfrak{M}^{\xi}_{\mathbb{A}(k)}$
4. If $\tau \in C(\tau, \kappa) \cap [\alpha_0, \alpha)$ and $\nu \in C(\tau, \kappa) \cap \mathbf{dom}(\mathbb{C})$ and $\nu > \delta$, then κ is σ-Π^1_0-reducible in

$${}^o\mathfrak{M}^{\tau}_{\mathbb{C}(\nu)} \;:=\; \{\langle \kappa^*, \sigma^*\rangle :\; \exists \nu^* \big(\langle \kappa^*, \nu^*\rangle \in \mathfrak{M}^{\tau}_{\mathbb{C}(\nu)} \,\wedge\, \kappa^* \le \sigma^* < \nu^*\big)\}.$$

Set

$$\mathfrak{M}^{\alpha}_{\mathbb{C}(\delta)} := \{\langle \kappa, \sigma\rangle :\; \exists \vec{\mathbf{D}}\, [\langle \kappa, \sigma, \vec{\mathbf{D}}\rangle \in X^\alpha_{\mathbb{C}(\delta)}]\}.$$

Suppose $\langle \kappa_0, \sigma_0\rangle \in \mathfrak{M}^{\alpha}_{\mathbb{C}(\delta)}$. Then $\langle \kappa_0, \sigma_0, \vec{\mathbf{D}}_0\rangle \in X^\alpha_{\mathbb{C}(\delta)}$ for some $\vec{\mathbf{D}}_0$ which is uniquely determined by $\langle \kappa_0, \sigma_0\rangle$. Let j_0 be determined by

$$j_0 : \langle V_{\kappa_0,\sigma_0}; \in; \vec{\mathbf{D}}_0\rangle \;\xrightarrow[\equiv]{\Pi^1_0}\; \langle V_{\pi,\sigma}; \in; \mathfrak{C}(\alpha, \sigma)\rangle.$$

Define

$$\bar{j}_0 \;:=\; j_0 \restriction [\kappa_0, \sigma_0) \;\cup\; \{\langle \sigma_0, \sigma\rangle\}$$

and set

$$\mathsf{f}^{\alpha,\mathbb{C}(\delta)}_{\kappa_0,\sigma_0} \;:=\; \bar{j}_0^{\;-1} \restriction C(\alpha, \kappa_0).$$

$[\kappa_0, \sigma_0]$ is the interval assigned to $\mathsf{f}^{\alpha,\mathbb{C}(\delta)}_{\kappa_0,\sigma_0}$.

If $\mathfrak{M}^{\alpha}_{\mathbb{C}(\delta)} \neq \emptyset$, let $\langle \pi_0, \delta_0\rangle$ be the least pair in $\mathfrak{M}^{\alpha}_{\mathbb{C}(\delta)}$. Let $\vec{\mathbf{D}}_0$ be such that $\langle \pi_0, \delta_0, \vec{\mathbf{D}}_0\rangle \in X^\alpha_{\mathbb{C}(\delta)}$. Then $(\alpha, \mathbb{C}(\delta), \mathsf{f}^{\alpha,\mathbb{C}(\delta)}_{\pi_0,\delta_0}) \in \mathfrak{F}^\alpha$ and $(\alpha, \mathbb{C}(\delta), \mathfrak{D}) \in \mathfrak{S}^\alpha$, where $\mathfrak{D} := \langle V_{\pi_0,\delta_0}; \in; \vec{\mathbf{D}}_0\rangle$.

4.c. Let $\mathbb{C}(\delta)$ be a projection instance in $\mathfrak{R}^\alpha$ of the form

$$\mathbb{C}(\delta) \;=\; (\pi; \mathsf{M}^{<\gamma}_{\mathbb{A}}\text{-}\delta\text{-}\mathsf{P}_0; \mathsf{M}^{\zeta}_{\mathbb{Y}}; \mathbb{X}; \alpha_0),$$

where $\mathbb{A}$ is a constant reflection pattern. Let $\beta := o(\mathbb{A})$. Then $X^\alpha_{\mathbb{C}(\delta)}$ consist of all triples $\langle \kappa, \sigma, \vec{\mathbf{D}}\rangle$ such that $\kappa \le \sigma < \pi$, κ is inaccessible, and the following are satisfied:

1. $C(\alpha, \kappa) \cap \pi \;=\; \kappa$ and $\mathbb{C}(\delta), \alpha \in C(\alpha, \kappa)$.
2. $\langle V_{\kappa,\sigma}; \in; \vec{\mathbf{D}}\rangle \;\xrightarrow[\equiv]{\Pi^1_0}\; \langle V_{\pi,\delta}; \in; \mathfrak{C}(\alpha, \delta)\rangle$, where $\vec{\mathbf{D}} \subseteq V_\sigma$ is the $<_{\mathbf{L}}$-least such set.
3. If $\xi \in C(\xi, \kappa) \cap [\beta, \gamma)$, then κ is σ-Π^1_0-reducible in

$${}^o\mathfrak{M}^{\xi}_{\mathbb{A}} \;:=\; \{\langle \kappa^*, \sigma^*\rangle :\; \exists \rho^* \big(\langle \kappa^*, \rho^*\rangle \in \mathfrak{M}^{\xi}_{\mathbb{A}} \,\wedge\, \kappa^* \le \sigma^* < \rho^*\big)\}.$$

4. If $\tau \in C(\tau,\kappa) \cap [\alpha_0,\alpha)$ and $\nu \in C(\tau,\kappa) \cap \mathbf{dom}(\mathbb{C}\,)$ and $\nu > \delta$, then κ is σ-Π^1_0-reducible in

$$^{o}\mathfrak{M}^{\tau}_{\mathbb{C}\,(\nu)} \;:=\; \{\langle\kappa^*,\sigma^*\rangle :\ \exists\nu^*\,\big(\langle\kappa^*,\nu^*\rangle \in \mathfrak{M}^{\tau}_{\mathbb{C}\,(\nu)} \wedge \kappa^* \le \sigma^* < \nu^*\big)\}.$$

Set

$$\mathfrak{M}^{\alpha}_{\mathbb{C}\,(\delta)} := \{\langle\kappa,\sigma\rangle :\ \exists\vec{\mathbf{D}}\,[\langle\kappa,\sigma,\vec{\mathbf{D}}\rangle \in X^{\alpha}_{\mathbb{C}\,(\delta)}]\}.$$

Suppose $\langle\kappa_0,\sigma_0\rangle \in \mathfrak{M}^{\alpha}_{\mathbb{C}\,(\delta)}$. Then $\langle\kappa_0,\sigma_0,\vec{\mathbf{D}}_0\rangle \in X^{\alpha}_{\mathbb{C}\,(\delta)}$ for some $\vec{\mathbf{D}}_0$ which is uniquely determined by $\langle\kappa_0,\sigma_0\rangle$. Let j_0 be determined by

$$j_0 : \langle V_{\kappa_0,\sigma_0};\in;\vec{\mathbf{D}}_0\rangle \xrightarrow[\equiv]{\Pi^1_0} \langle V_{\pi,\sigma};\in;\mathfrak{C}(\alpha,\sigma)\rangle.$$

Define

$$\bar{j_0} \;:=\; j_0 \restriction [\kappa_0,\sigma_0) \,\cup\, \{\langle\sigma_0,\sigma\rangle\}$$

and set

$$\mathsf{f}^{\alpha,\mathbb{C}\,(\delta)}_{\kappa_0,\sigma_0} \;:=\; \bar{j_0}^{\,-1} \restriction C(\alpha,\kappa_0).$$

$[\kappa_0,\sigma_0]$ is the interval assigned to $\mathsf{f}^{\alpha,\mathbb{C}\,(\delta)}_{\kappa_0,\sigma_0}$.

If $\mathfrak{M}^{\alpha}_{\mathbb{C}\,(\delta)} \ne \emptyset$, let $\langle\pi_0,\delta_0\rangle$ be the least pair in $\mathfrak{M}^{\alpha}_{\mathbb{C}\,(\delta)}$. Let $\vec{\mathbf{D}}_0$ be such that $\langle\pi_0,\delta_0,\vec{\mathbf{D}}_0\rangle \in X^{\alpha}_{\mathbb{C}\,(\delta)}$. Then $(\alpha,\mathbb{C}\,(\delta),\mathsf{f}^{\alpha,\mathbb{C}\,(\delta)}_{\pi_0,\delta_0}) \in \mathfrak{F}^{\alpha}$ and $(\alpha,\mathbb{C}\,(\delta),\mathfrak{D}) \in \mathfrak{G}^{\alpha}$, where $\mathfrak{D} := \langle V_{\pi_0,\delta_0};\in;\vec{\mathbf{D}}_0\rangle$.

5.a. Suppose $\mathbb{C}\,(\delta)$ is a projection instance of the form $\mathbb{C}\,(\delta) = (\pi;\mathsf{M}^{<\gamma}_{\mathbb{A}\,(\rho)}\text{-}\delta\text{-}\mathsf{P}_0;\mathsf{M}^{\zeta}_{\mathbb{Y}};\mathbb{X};\alpha_0)$, where $\mathbf{dom}(\mathbb{A}\,) \ne (0,\omega)$, $\mathfrak{M}^{\alpha_1}_{\mathbb{C}(\delta)} \ne \emptyset$, $\bar\kappa = \Psi^{\alpha_1}_{\mathbb{C}(\delta)}(\pi)$, $\bar\delta = \Psi^{\alpha_1}_{\mathbb{C}(\delta)}(\delta)$, and $\alpha_0 \le \alpha_1 \le \alpha$. Let $[\mu,\rho]$ be the interval of $\mathbb{A}\,(\rho)$.

5.a.1. First suppose that $\bar\delta$ is a limit $> \bar\kappa$. Let $\delta' \in [\bar\kappa,\bar\delta)$ and assume that α is the least ordinal $\xi' \ge \alpha_1$ such that $\delta' \in C(\xi',\bar\kappa)$.

If $\alpha_1 > \alpha_0$, then

$$\mathbb{D}\,(\delta') \;:=\; (\bar\kappa;\mathsf{M}^{<\alpha_1}_{\mathbb{C}\,(\delta)}\text{-}\delta'\text{-}\mathsf{P}_0;\mathsf{M}^{\alpha_1}_{\mathbb{C}\,(\delta)};\mathbb{C}\,(\delta);\alpha+1) \qquad \to 4.a$$

is a projection instance in $\mathfrak{R}^{\alpha+1}$ with interval $[\bar\kappa,\delta']$.

The function $\sigma \mapsto \mathbb{D}\,(\sigma)$ with domain $\{\sigma \in [\bar\kappa,\bar\delta) :\ \exists\xi'\ [\sigma \in C(\xi',\bar\kappa)]\}$ is a reflection pattern.

For every $\xi \in C(\xi,\bar\kappa) \cap [\alpha_0,\alpha_1)$ and $\eta \in C(\xi,\bar\kappa) \cap \mathbf{dom}(\mathbb{C}\,)$ with $\eta > \delta$,

$$(\mathbb{D}\,(\delta'),\xi,\eta)$$

is a reflection instance in $\mathfrak{R}^{\alpha+1}$with interval $[\bar\kappa,\delta']$.

If $\alpha_1 = \alpha_0$, then

$$\mathbb{E}\,(\delta') \;:=\; (\bar\kappa;\mathsf{M}^{<\gamma}_{\mathbb{A}\,(\rho)}\text{-}\delta'\text{-}\mathsf{P}_0;\mathsf{M}^{\alpha}_{\mathbb{C}\,(\delta)};\mathbb{C}\,(\delta);\alpha+1) \qquad \to 4.a$$

is a projection instance in $\mathfrak{R}^{\alpha+1}$ with interval $[\bar\kappa,\delta']$.

The function $\sigma \mapsto \mathbb{E}\,(\sigma)$ with domain $\{\sigma \in [\bar\kappa,\bar\delta) :\ \exists\xi'\ [\sigma \in C(\xi',\bar\kappa)]\}$ is a reflection pattern.

For every $\xi \in C(\xi,\bar\kappa) \cap [o(\mathbb{A}\,(\rho)),\gamma)$ and $\eta \in C(\xi,\bar\kappa) \cap \mathbf{dom}(\mathbb{A}\,)$ with $\eta > \rho$,

$$(\mathbb{E}\,(\delta'),\xi,\eta)$$

is a reflection instance in $\mathfrak{R}^{\alpha+1}$ with interval $[\bar\kappa,\delta']$.

5.a.2. Now assume that $\bar{\delta}$ is a successor $\delta_0 + 1$ and $\alpha = \alpha_1$.

If $\alpha_1 > \alpha_0$, then for every $0 < m < \omega$,

$$\mathbb{D}\,(m) \;:=\; (\bar{\kappa}; \mathsf{M}^{<\alpha_1}_{\mathbb{C}\,(\delta)}\text{-}\delta_0\text{-}\mathsf{P}_m; \mathsf{M}^{\alpha_1}_{\mathbb{C}\,(\delta)}; \mathbb{C}\,(\delta); \alpha+1) \qquad \to 12.a$$

is a projection instance in $\mathfrak{R}^{\alpha+1}$ with interval $[\bar{\kappa}, \delta_0]$.

The function $n \mapsto \mathbb{D}\,(n)$ with domain $(0, \omega)$ is a reflection pattern.

For every $m \in (0, \omega)$, $\xi \in C(\xi, \bar{\kappa}) \cap [\alpha_0, \alpha_1)$, and $\eta \in C(\xi, \bar{\kappa}) \cap \mathbf{dom}(\mathbb{C}\,)$ with $\eta > \delta$,

$$(\mathbb{D}\,(m), \xi, \eta)$$

is a reflection instance in $\mathfrak{R}^{\alpha+1}$ with interval $[\bar{\kappa}, \delta_0]$.

If $\alpha_1 = \alpha_0$, then for every $0 < m < \omega$,

$$\mathbb{E}\,(m) \;:=\; (\bar{\kappa}; \mathsf{M}^{<\gamma}_{\mathbb{A}\,(\rho)}\text{-}\delta_0\text{-}\mathsf{P}_m; \mathsf{M}^{\alpha_1}_{\mathbb{C}\,(\delta)}; \mathbb{C}\,(\delta); \alpha+1) \qquad \to 12.a$$

is a projection instance in $\mathfrak{R}^{\alpha+1}$ with interval $[\bar{\kappa}, \delta_0]$.

The function $n \mapsto \mathbb{E}\,(n)$ with domain $(0, \omega)$ is a reflection pattern.

For every $m \in (0, \omega)$, $\xi \in C(\xi, \bar{\kappa}) \cap [o(\mathbb{A}\,(\rho)), \gamma)$, and $\eta \in C(\xi, \bar{\kappa}) \cap \mathbf{dom}(\mathbb{A}\,)$ with $\eta > \delta$,

$$(\mathbb{E}\,(m), \xi, \eta)$$

is a reflection instance in $\mathfrak{R}^{\alpha+1}$ with interval $[\bar{\kappa}, \delta_0]$.

5.a.3. Finally suppose $\bar{\delta} = \bar{\kappa}$ and $\alpha = \alpha_1$.

If $\alpha_1 > \alpha_0$, then for every $\xi \in C(\xi, \bar{\kappa}) \cap [\alpha_0, \alpha)$ and $\eta \in C(\xi, \bar{\kappa}) \cap \mathbf{dom}(\mathbb{C}\,)$ with $\eta > \delta$,

$$\mathbb{I} \;:=\; (\bar{\kappa}; \mathsf{M}^{\xi}_{\mathbb{C}\,(\eta)}\text{-}\bar{\kappa}\text{-}\mathsf{P}_0; \mathsf{M}^{\alpha_1}_{\mathbb{C}\,(\delta)}; \mathbb{C}\,(\delta); \alpha+1)$$

is a reflection instance in $\mathfrak{R}^{\alpha+1}$ with interval $[\bar{\kappa}, \bar{\kappa}]$. $\mathbb{I}$ is also a reflection pattern, and

$$\mathbb{F} := (\pi; \mathsf{M}^{\xi}_{\mathbb{C}\,(\eta)}; \mathbb{I}; \emptyset; \alpha+1) \qquad \to 14.$$

is a projection instance in $\mathfrak{R}^{\alpha+1}$ with interval $[\pi, \eta]$.

If $\alpha_1 = \alpha_0$, then for every $\tau \in C(\tau, \bar{\kappa}) \cap [o(\mathbb{A}\,(\rho)), \gamma)$ and $\eta \in C(\tau, \bar{\kappa}) \cap \mathbf{dom}(\mathbb{A}\,)$ with $\eta > \delta$,

$$\mathbb{J} \;:=\; (\bar{\kappa}; \mathsf{M}^{\tau}_{\mathbb{A}\,(\eta)}\text{-}\bar{\kappa}\text{-}\mathsf{P}_0; \mathsf{M}^{\alpha_1}_{\mathbb{C}\,(\delta)}; \mathbb{C}\,(\delta); \alpha+1) \qquad \to 14.$$

is a reflection instance in $\mathfrak{R}^{\alpha+1}$ with interval $[\bar{\kappa}, \bar{\kappa}]$. $\mathbb{J}$ is also a reflection pattern, and

$$\mathbb{G} \;:=\; (\mu; \mathsf{M}^{\tau}_{\mathbb{A}\,(\eta)}; \bar{\kappa}; \mathbb{J}; \alpha+1)$$

is a projection instance in $\mathfrak{R}^{\alpha+1}$ with interval $[\mu, \eta]$.

5.b. Suppose $\mathbb{C}\,(\delta)$ is a projection instance of the form $\mathbb{C}\,(\delta) \;=\; (\pi; \mathsf{M}^{<\gamma}_{\mathbb{A}\,(m)}\text{-}\delta\text{-}\mathsf{P}_0; \mathsf{M}^{\zeta}_{\mathbb{Y}}; \mathbb{X}; \alpha_0)$, where $\mathbf{dom}(\mathbb{A}\,) = (0, \omega)$, $\mathfrak{M}^{\alpha_1}_{\mathbb{C}\,(\delta)} \neq \emptyset$, $\bar{\kappa} = \Psi^{\alpha_1}_{\mathbb{C}\,(\delta)}(\pi)$, $\bar{\delta} = \Psi^{\alpha_1}_{\mathbb{C}\,(\delta)}(\delta)$, and $\alpha_0 \leq \alpha_1 \leq \alpha$.

5.b.1. First suppose that $\bar{\delta}$ is a limit $> \bar{\kappa}$. Let $\delta' \in [\bar{\kappa}, \bar{\delta})$ and assume that α is the least ordinal $\xi' \geq \alpha_1$ such that $\delta' \in C(\xi', \bar{\kappa})$.

If $\alpha_1 > \alpha_0$, then

$$\mathbb{D}\,(\delta') \;:=\; (\bar{\kappa}; \mathsf{M}^{<\alpha_1}_{\mathbb{C}\,(\delta)}\text{-}\delta'\text{-}\mathsf{P}_0; \mathsf{M}^{\alpha_1}_{\mathbb{C}\,(\delta)}; \mathbb{C}\,(\delta); \alpha+1) \qquad \to 4.a$$

is a projection instance in $\Re^{\alpha+1}$ with interval $[\bar{\kappa}, \delta']$.

The function $\sigma \mapsto \mathbb{D}\,(\sigma)$ with domain $\{\sigma \in [\bar{\kappa}, \bar{\delta}) : \; \exists \xi' \; [\sigma \in C(\xi', \bar{\kappa})]\}$ is a reflection pattern.

For every $\xi \in C(\xi, \bar{\kappa}) \cap [\alpha_0, \alpha_1)$ and $\eta \in C(\xi, \bar{\kappa}) \cap \mathbf{dom}(\mathbb{C}\,)$ with $\eta > \delta$,

$$(\mathbb{D}\,(\delta'), \xi, \eta)$$

is a reflection instance in $\Re^{\alpha+1}$ with interval $[\bar{\kappa}, \delta']$.

If $\alpha_1 = \alpha_0$, then

$$\mathbb{E}\,(\delta') \;:=\; (\bar{\kappa}; \mathsf{M}^{<\gamma}_{\mathbb{A}\,(m)}\text{-}\delta'\text{-}\mathsf{P}_0; \mathsf{M}^{\alpha}_{\mathbb{C}\,(\delta)}; \mathbb{C}\,(\delta); \alpha+1) \qquad \to 4.b$$

is a projection instance in $\Re^{\alpha+1}$ with interval $[\bar{\kappa}, \delta']$.

The function $\sigma \mapsto \mathbb{E}\,(\sigma)$ with domain $\{\sigma \in [\bar{\kappa}, \bar{\delta}) : \; \exists \xi' \; [\sigma \in C(\xi', \bar{\kappa})]\}$ is a reflection pattern.

For every $\xi \in C(\xi, \bar{\kappa}) \cap [o(\mathbb{A}\,(\rho)), \gamma)$ and $m < k < \omega$,

$$(\mathbb{E}\,(\delta'), \xi, k)$$

is a reflection instance in $\Re^{\alpha+1}$ with interval $[\bar{\kappa}, \delta']$.

5.b.2. Now assume that $\bar{\delta}$ is a successor $\delta_0 + 1$ and $\alpha = \alpha_1$.

If $\alpha_1 > \alpha_0$, then for every $0 < l < \omega$,

$$\mathbb{D}\,(l) \;:=\; (\bar{\kappa}; \mathsf{M}^{<\alpha_1}_{\mathbb{C}\,(\delta)}\text{-}\delta_0\text{-}\mathsf{P}_l; \mathsf{M}^{\alpha_1}_{\mathbb{C}\,(\delta)}; \mathbb{C}\,(\delta); \alpha+1) \qquad \to 12.a$$

is a projection instance in $\Re^{\alpha+1}$ with interval $[\bar{\kappa}, \delta_0]$.

The function $n \mapsto \mathbb{D}\,(n)$ with domain $(0, \omega)$ is a reflection pattern.

For every $l \in (0, \omega)$, $\xi \in C(\xi, \bar{\kappa}) \cap [\alpha_0, \alpha_1)$, and $\eta \in C(\xi, \bar{\kappa}) \cap \mathbf{dom}(\mathbb{C}\,)$ with $\eta > \delta$,

$$(\mathbb{D}\,(l), \xi, \eta)$$

is a reflection instance in $\Re^{\alpha+1}$ with interval $[\bar{\kappa}, \delta_0]$.

If $\alpha_1 = \alpha_0$, then for every $0 < l < \omega$,

$$\mathbb{E}\,(l) \;:=\; (\bar{\kappa}; \mathsf{M}^{<\gamma}_{\mathbb{A}\,(m)}\text{-}\delta_0\text{-}\mathsf{P}_l; \mathsf{M}^{\alpha_1}_{\mathbb{C}\,(\delta)}; \mathbb{C}\,(\delta); \alpha+1) \qquad \to 12.b$$

is a projection instance in $\Re^{\alpha+1}$ with interval $[\bar{\kappa}, \delta_0]$.

The function $n \mapsto \mathbb{E}\,(n)$ with domain $(0, \omega)$ is a reflection pattern.

For every $l \in (0, \omega)$, $\xi \in C(\xi, \bar{\kappa}) \cap [o(\mathbb{A}\,(m)), \gamma)$, and $m < k < \omega$,

$$(\mathbb{E}\,(l), \xi, k)$$

is a reflection instance in $\Re^{\alpha+1}$ with interval $[\bar{\kappa}, \delta_0]$.

5.b.3. Finally suppose $\bar{\delta} = \bar{\kappa}$ and $\alpha = \alpha_1$.

If $\alpha_1 > \alpha_0$, then for every $\xi \in C(\xi, \bar{\kappa}) \cap [\alpha_0, \alpha)$ and $\eta \in C(\xi, \bar{\kappa}) \cap \mathbf{dom}(\mathbb{C})$ with $\eta > \delta$,

$$\mathbb{I} := (\bar{\kappa}; \mathsf{M}^{\xi}_{\mathbb{C}(\eta)}\text{-}\bar{\kappa}\text{-}\mathsf{P}_0; \mathsf{M}^{\alpha_1}_{\mathbb{C}(\delta)}; \mathbb{C}(\delta); \alpha + 1)$$

is a reflection instance in $\mathfrak{R}^{\alpha+1}$ with interval $[\bar{\kappa}, \bar{\kappa}]$. $\mathbb{I}$ is also a reflection pattern, and the projection instance associated with $\mathbb{I}$ is

$$\mathbb{F} := (\pi; \mathsf{M}^{\xi}_{\mathbb{C}(\eta)}; \mathbb{I}; \emptyset; \alpha + 1) \qquad \to 14.$$

$\mathbb{F}$ is a projection instance in $\mathfrak{R}^{\alpha+1}$ with interval $[\pi, \eta]$.

If $\alpha_1 = \alpha_0$, then for every $\tau \in C(\tau, \bar{\kappa}) \cap [o(\mathbb{A}(\rho)), \gamma)$ and $m < k < \omega$,

$$\mathbb{J} := (\bar{\kappa}; \mathsf{M}^{\tau}_{\mathbb{A}(k)}\text{-}\bar{\kappa}\text{-}\mathsf{P}_0; \mathsf{M}^{\alpha_1}_{\mathbb{C}(\delta)}; \mathbb{C}(\delta); \alpha + 1)$$

is a reflection instance in $\mathfrak{R}^{\alpha+1}$ with interval $[\bar{\kappa}, \bar{\kappa}]$. $\mathbb{J}$ is also a reflection pattern, and

$$\mathbb{G} := (\mu; \mathsf{M}^{\tau}_{\mathbb{A}(k)}; \bar{\kappa}; \mathbb{J}; \alpha + 1) \qquad \to 14.$$

is a projection instance in $\mathfrak{R}^{\alpha+1}$ with the same interval as $\mathbb{A}(k)$.

5.c. Suppose $\mathbb{C}(\delta)$ is a projection instance of the form $\mathbb{C}(\delta) = (\pi; \mathsf{M}^{<\gamma}_{\mathbb{A}}\text{-}\delta\text{-}\mathsf{P}_0; \mathsf{M}^{\zeta}_{\mathbb{Y}}; \mathbb{X}; \alpha_0)$, where $\mathbb{A}$ is a constant reflection pattern. Further assume that $\mathfrak{M}^{\alpha_1}_{\mathbb{C}(\delta)} \neq \emptyset$, $\bar{\kappa} = \Psi^{\alpha_1}_{\mathbb{C}(\delta)}(\pi)$, $\bar{\delta} = \Psi^{\alpha_1}_{\mathbb{C}(\delta)}(\delta)$, and $\alpha_0 \leq \alpha_1 \leq \alpha$. Let $[\mu, \rho]$ be the interval of $\mathbb{A}$.

5.c.1. First suppose that $\bar{\delta}$ is a limit $> \bar{\kappa}$. Let $\delta' \in [\bar{\kappa}, \bar{\delta})$ and assume that α is the least ordinal $\xi' \geq \alpha_1$ such that $\delta' \in C(\xi', \bar{\kappa})$.

If $\alpha_1 > \alpha_0$, then

$$\mathbb{D}(\delta') := (\bar{\kappa}; \mathsf{M}^{<\alpha_1}_{\mathbb{C}(\delta)}\text{-}\delta'\text{-}\mathsf{P}_0; \mathsf{M}^{\alpha_1}_{\mathbb{C}(\delta)}; \mathbb{C}(\delta); \alpha + 1) \qquad \to 4.a$$

is a projection instance in $\mathfrak{R}^{\alpha+1}$ with interval $[\bar{\kappa}, \delta']$.

The function $\sigma \mapsto \mathbb{D}(\sigma)$ with domain $\{\sigma \in [\bar{\kappa}, \bar{\delta}) : \exists \xi' \; [\sigma \in C(\xi', \bar{\kappa})]\}$ is a reflection pattern.

For every $\xi \in C(\xi, \bar{\kappa}) \cap [\alpha_0, \alpha_1)$ and $\eta \in C(\xi, \bar{\kappa}) \cap \mathbf{dom}(\mathbb{C})$ with $\eta > \delta$,

$$(\mathbb{D}(\delta'), \xi, \eta)$$

is a reflection instance in $\mathfrak{R}^{\alpha+1}$ with interval $[\bar{\kappa}, \delta']$.

If $\alpha_1 = \alpha_0$, then

$$\mathbb{E}(\delta') := (\bar{\kappa}; \mathsf{M}^{<\gamma}_{\mathbb{A}}\text{-}\delta'\text{-}\mathsf{P}_0; \mathsf{M}^{\alpha}_{\mathbb{C}(\delta)}; \mathbb{C}(\delta); \alpha + 1) \qquad \to 4.c$$

is a projection instance in $\mathfrak{R}^{\alpha+1}$ with interval $[\bar{\kappa}, \delta']$.

The function $\sigma \mapsto \mathbb{E}(\sigma)$ with domain $\{\sigma \in [\bar{\kappa}, \bar{\delta}) : \exists \xi' \; [\sigma \in C(\xi', \bar{\kappa})]\}$ is a reflection pattern.

For every $\xi \in C(\xi, \bar{\kappa}) \cap [o(\mathbb{A}), \gamma)$,

$$(\mathbb{E}(\delta'), \xi)$$

is a reflection instance in $\mathfrak{R}^{\alpha+1}$ with interval $[\bar{\kappa}, \delta']$.

5.c.2. Now assume that $\bar{\delta}$ is a successor $\delta_0 + 1$ and $\alpha = \alpha_1$.

If $\alpha_1 > \alpha_0$, then for every $0 < m < \omega$,

$$\mathbb{D}\,(m) \;:=\; (\bar{\kappa}; \mathsf{M}^{<\alpha_1}_{\mathbb{C}\,(\delta)}\text{-}\delta_0\text{-}\mathsf{P}_m; \mathsf{M}^{\alpha_1}_{\mathbb{C}\,(\delta)}; \mathbb{C}\,(\delta); \alpha + 1) \qquad \to 12.a$$

is a projection instance in $\mathfrak{R}^{\alpha+1}$ with interval $[\bar{\kappa}, \delta_0]$.

The function $n \mapsto \mathbb{D}\,(n)$ with domain $(0, \omega)$ is a reflection pattern.

For every $m \in (0, \omega)$, $\xi \in C(\xi, \bar{\kappa}) \,\cap\, [\alpha_0, \alpha_1)$, and $\eta \in C(\xi, \bar{\kappa}) \,\cap\, \mathbf{dom}(\mathbb{C}\,)$ with $\eta > \delta$,

$$(\mathbb{D}\,(m), \xi, \eta)$$

is a reflection instance in $\mathfrak{R}^{\alpha+1}$ with interval $[\bar{\kappa}, \delta_0]$.

If $\alpha_1 = \alpha_0$, then for every $0 < m < \omega$,

$$\mathbb{E}\,(m) \;:=\; (\bar{\kappa}; \mathsf{M}^{<\gamma}_{\mathbb{A}}\text{-}\delta_0\text{-}\mathsf{P}_m; \mathsf{M}^{\alpha_1}_{\mathbb{C}\,(\delta)}; \mathbb{C}\,(\delta); \alpha + 1) \qquad \to 12.c$$

is a projection instance in $\mathfrak{R}^{\alpha+1}$ with interval $[\bar{\kappa}, \delta_0]$.

The function $n \mapsto \mathbb{E}\,(n)$ with domain $(0, \omega)$ is a reflection pattern.

For every $m \in (0, \omega)$, $\xi \in C(\xi, \bar{\kappa}) \,\cap\, [o(\mathbb{A}\,), \gamma)$,

$$(\mathbb{E}\,(m), \xi)$$

is a reflection instance in $\mathfrak{R}^{\alpha+1}$ with interval $[\bar{\kappa}, \delta_0]$.

5.c.3. Finally suppose $\bar{\delta} = \bar{\kappa}$ and $\alpha = \alpha_1$.

If $\alpha_1 > \alpha_0$, then for every $\xi \in C(\xi, \bar{\kappa}) \cap [\alpha_0, \alpha)$ and $\eta \in C(\xi, \bar{\kappa}) \cap \mathbf{dom}(\mathbb{C}\,)$ with $\eta > \delta$,

$$\mathbb{I} \;:=\; (\bar{\kappa}; \mathsf{M}^{\xi}_{\mathbb{C}\,(\eta)}\text{-}\bar{\kappa}\text{-}\mathsf{P}_0; \mathsf{M}^{\alpha_1}_{\mathbb{C}\,(\delta)}; \mathbb{C}\,(\delta); \alpha + 1)$$

is a reflection instance in $\mathfrak{R}^{\alpha+1}$ with interval $[\bar{\kappa}, \bar{\kappa}]$. $\mathbb{I}$ is also a reflection pattern, and

$$\mathbb{F} := (\pi; \mathsf{M}^{\xi}_{\mathbb{C}\,(\eta)}; \mathbb{I}; \emptyset; \alpha + 1) \qquad \to 14.$$

is a projection instance in $\mathfrak{R}^{\alpha+1}$ with interval $[\pi, \eta]$.

If $\alpha_1 = \alpha_0$, then for every $\tau \in C(\tau, \bar{\kappa}) \,\cap\, [o(\mathbb{A}\,(\rho)), \gamma)$

$$\mathbb{J} \;:=\; (\bar{\kappa}; \mathsf{M}^{\tau}_{\mathbb{A}}\text{-}\bar{\kappa}\text{-}\mathsf{P}_0; \mathsf{M}^{\alpha_1}_{\mathbb{C}\,(\delta)}; \mathbb{C}\,(\delta); \alpha + 1)$$

is a reflection instance in $\mathfrak{R}^{\alpha+1}$ with interval $[\bar{\kappa}, \bar{\kappa}]$. $\mathbb{J}$ is also a reflection pattern, and

$$\mathbb{G} \;:=\; (\mu; \mathsf{M}^{\tau}_{\mathbb{A}}; \bar{\kappa}; \mathbb{J}; \alpha + 1) \qquad \to 14.$$

is a projection instance in $\mathfrak{R}^{\alpha+1}$ with interval $[\mu, \rho]$.

6. Suppose $\mathbb{C}$ is a projection instance in of the form

$$\mathbb{C} \;:=\; (\pi; \rho\text{-}\mathsf{P}_m; \mathbb{X}; \beta),$$

where $m > 0$ and $\beta \leq \alpha$. Then $X^{\alpha}_{\mathbb{C}}$ consist of all triples $\langle \kappa, \sigma, \vec{\mathbf{D}} \rangle$ such that $\kappa \leq \sigma < \pi$, κ is inaccessible, and the following are satisfied:

1. $C(\alpha,\kappa)\cap\pi = \kappa$ and $\mathbb{C}, \alpha\in C(\alpha,\kappa)$.

2. $\langle V_{\kappa,\sigma};\in;\vec{\mathbf{D}}\rangle \overset{\Pi^1_m}{\underset{\equiv}{\longrightarrow}} \langle V_{\pi,\rho};\in;\mathfrak{C}(\alpha,\rho)\rangle$, where $\vec{\mathbf{D}}\subseteq V_\sigma$ is the $<_{\mathbf{L}}$-least such set.

3. If $\xi\in C(\xi,\kappa)\cap[\beta,\alpha)$, then κ is σ-Π^1_{m-1}-reducible in $\mathfrak{M}^{\xi}_{\mathbb{C}}$.

Set
$$\mathfrak{M}^{\alpha}_{\mathbb{C}} := \{\langle\kappa,\sigma\rangle:\ \exists\vec{\mathbf{D}}\,[\langle\kappa,\sigma,\vec{\mathbf{D}}\rangle\in X^{\alpha}_{\mathbb{C}}]\}.$$
Suppose $\langle\kappa_0,\sigma_0\rangle\in\mathfrak{M}^{\alpha}_{\mathbb{C}}$. Then $\langle\kappa_0,\sigma_0,\vec{\mathbf{D}}_0\rangle\in X^{\alpha}_{\mathbb{C}}$ for some $\vec{\mathbf{D}}_0$ which is uniquely determined by $\langle\kappa_0,\sigma_0\rangle$. Let j_0 be determined by
$$j_0:\langle V_{\kappa_0,\sigma_0};\in;\vec{\mathbf{D}}_0\rangle \overset{\Pi^1_m}{\underset{\equiv}{\longrightarrow}} \langle V_{\pi,\rho};\in;\mathfrak{C}(\alpha,\rho)\rangle.$$
Define
$$\bar{j_0} := j_0\restriction[\kappa_0,\sigma_0)\ \cup\ \{\langle\sigma_0,\rho\rangle\}$$
and set
$$\mathsf{f}^{\alpha,\mathbb{C}}_{\kappa_0,\sigma_0} := \bar{j_0}^{-1}\restriction C(\alpha,\kappa_0).$$
$[\pi,\rho]$ is the interval assigned to $\mathsf{f}^{\alpha,\mathbb{C}}_{\kappa_0,\sigma_0}$.

If $\mathfrak{M}^{\alpha}_{\mathbb{C}}\neq\emptyset$, let $\langle\pi_0,\delta_0\rangle$ be the least pair in $\mathfrak{M}^{\alpha}_{\mathbb{C}}$. Let $\vec{\mathbf{D}}_0$ be such that $\langle\pi_0,\delta_0,\vec{\mathbf{D}}_0\rangle\in X^{\alpha}_{\mathbb{C}}$. Then $(\alpha,\mathbb{C},\mathsf{f}^{\alpha,\mathbb{C}}_{\pi_0,\delta_0})\in\mathfrak{F}^{\alpha}$ and $(\alpha,\mathbb{C},\mathfrak{D})\in\mathfrak{S}^{\alpha}$, where $\mathfrak{D}:=\langle V_{\pi_0,\delta_0};\in;\vec{\mathbf{D}}_0\rangle$.

7. Suppose $\mathbb{C}$ is a projection instance of the form $\mathbb{C} = (\pi;\rho\text{-}\mathsf{P}_m;\mathbb{X};\beta)$ where $m>0$. Also assume $\mathfrak{M}^{\alpha_0}_{\mathbb{C}}\neq\emptyset$, $\bar\kappa=\Psi^{\alpha_0}_{\mathbb{C}}(\pi)$, $\bar\rho=\Psi^{\alpha_0}_{\mathbb{C}}(\rho)$, and $\beta\le\alpha_0\le\alpha$.

7.1. Suppose $\alpha_0>\beta$, $m-1>0$, and $\alpha=\alpha_0$. Then
$$\mathbb{D}_0 := (\bar\kappa;\mathsf{M}^{<\alpha_0}_{\mathbb{C}}\text{-}\bar\rho\text{-}\mathsf{P}_{m-1};\mathsf{M}^{\alpha_0}_{\mathbb{C}};\mathbb{C};\alpha+1) \qquad \to 10.c$$
is a projection instance in $\mathfrak{R}^{\alpha+1}$ with interval $[\bar\kappa,\bar\rho]$. $\mathbb{D}_0$ is also a reflection pattern.

For all $\tau\in C(\tau,\bar\kappa)\cap[\beta,\alpha_0)$,
$$(\mathbb{D}_0,\tau)$$
is a reflection instance in $\mathfrak{R}^{\alpha+1}$ with interval $[\bar\kappa,\bar\rho]$.

7.2 Suppose $\alpha_0=\beta$, $m-1>0$, and $\alpha=\alpha_0$. Then
$$\mathbb{E}_0 := (\bar\kappa;\bar\rho\text{-}\mathsf{P}_{m-1};\mathbb{C};\alpha+1) \qquad \to 6.$$
is a projection instance and reflection instance in $\mathfrak{R}^{\alpha+1}$ with interval $[\bar\kappa,\bar\rho]$. $\mathbb{E}_0$ is also a reflection pattern.

7.3. Suppose $\alpha_0>\beta$, $m-1=0$ and $\bar\rho$ is a limit $>\bar\kappa$. Assume further that $\delta\in[\bar\kappa,\bar\rho)$ and α is the least ordinal $\gamma\ge\alpha_0$ such that $\delta\in C(\bar\kappa,\gamma)$. Then
$$\mathbb{D}_1(\delta) := (\bar\kappa;\mathsf{M}^{<\alpha_0}_{\mathbb{C}}\text{-}\delta\text{-}\mathsf{P}_0;\mathsf{M}^{\alpha_0}_{\mathbb{C}};\mathbb{C};\alpha+1) \qquad \to 4.c$$
is a projection instance in $\mathfrak{R}^{\alpha+1}$ with interval $[\bar\kappa,\delta]$.

For every $\tau\in C(\tau,\bar\kappa)\cap[\beta,\alpha_0)$,
$$(\mathbb{D}_1(\delta),\tau)$$
is a reflection instance in $\mathfrak{R}^{\alpha+1}$ with interval $[\bar\kappa,\delta]$.

The function $\delta'\mapsto\mathbb{D}_1(\delta')$ with domain $\{\delta'\in[\bar\kappa,\bar\rho):\ \exists\xi\ [\delta'\in C(\xi,\bar\kappa)]\}$ is a reflection pattern.

7.4. Suppose $\alpha_0 = \beta$, $m-1=0$ and $\bar\rho$ is a limit $> \bar\kappa$. Assume further that $\delta \in [\bar\kappa, \bar\rho)$ and α is the least ordinal $\gamma \geq \alpha_0$ such that $\delta \in C(\bar\kappa, \gamma)$. Then

$$\mathbb{E}_1(\delta) \ := \ (\bar\kappa; \delta\text{-}\mathsf{P}_0; \bar\rho\text{-}\mathsf{P}_0; \mathbb{C}\,; \alpha+1) \qquad \to 2.$$

is a projection instance and reflection instance in $\Re^{\alpha+1}$ with interval $[\bar\kappa, \delta]$.

The function $\delta' \mapsto \mathbb{E}_1(\delta')$ with domain $\{\delta' \in [\bar\kappa, \bar\rho) : \ \exists \xi\ [\delta' \in C(\xi, \bar\kappa)]\}$ is a reflection pattern.

7.5. Suppose $\alpha = \alpha_0$, $\alpha_0 > \beta$, $m-1=0$ and $\bar\rho$ is a successor ρ_0+1. Then for every $0 < n < \omega$,

$$\mathbb{D}_2(n) \ := \ (\bar\kappa; \mathsf{M}_{\mathbb{C}}^{<\alpha_0}\text{-}\rho_0\text{-}\mathsf{P}_n; \mathsf{M}_{\mathbb{C}}^{\alpha_0}; \mathbb{C}\,; \alpha+1) \qquad \to 12.c$$

is a projection instance in $\Re^{\alpha+1}$ with interval $[\bar\kappa, \rho_0]$.

For every $\tau \in C(\tau, \bar\kappa) \cap [\beta, \alpha_0)$,

$$(\mathbb{D}_2(n), \tau)$$

is a reflection instance in $\Re^{\alpha+1}$ with interval $[\bar\kappa, \rho_0]$.

The function $n' \mapsto \mathbb{D}_2(n')$ with domain $(0, \omega)$ is a reflection pattern.

7.6. Suppose $\alpha = \alpha_0 = \beta$, $m-1=0$ and $\bar\rho$ is a successor ρ_0+1. Then for every $0 < n < \omega$,

$$\mathbb{E}_2(n) \ := \ (\bar\kappa; \rho_0\text{-}\mathsf{P}_n; \rho_0\text{-}\mathsf{P}_\infty; \mathbb{C}\,; \alpha+1) \qquad \to 8.$$

is a projection instance and reflection instance in $\Re^{\alpha+1}$ with interval $[\bar\kappa, \rho_0]$.

The function $n' \mapsto \mathbb{E}_2(n')$ with domain $(0, \omega)$ is a reflection pattern.

7.7. Suppose $\alpha_0 > \beta$, $\alpha = \alpha_0$, $m-1=0$, and $\bar\rho = \bar\kappa$. Then for every $\xi \in C(\xi, \bar\kappa) \cap [\beta, \alpha_0)$,

$$\mathbb{I} \ := \ (\bar\kappa; \mathsf{M}_{\mathbb{C}}^{\xi}\text{-}\bar\kappa\text{-}\mathsf{P}_0; \mathsf{M}_{\mathbb{C}}^{\alpha_0}; \mathbb{C}\,; \alpha+1)$$

is a reflection instance in $\Re^{\alpha+1}$ with interval $[\bar\kappa, \bar\kappa]$ and

$$\mathbb{F} \ := \ (\pi; \mathsf{M}_{\mathbb{C}}^{\xi}; \bar\kappa; \mathbb{I}; \alpha+1) \qquad \to 14.$$

is projection instance in $\Re^{\alpha+1}$ with interval $[\pi, \rho]$ (in fact, $\pi = \rho$).

$\mathbb{I}$ is also a reflection pattern.

7.8. Suppose $\alpha_0 = \alpha = \beta$, $m-1=0$, and $\bar\rho = \bar\kappa$. Then

$$\mathbb{G} \ := \ (\bar\kappa; \mathsf{RLC}; \mathbb{C}\,; \alpha+1) \qquad \to 1.6.$$

is a projection instance and reflection instance in $\Re^{\alpha+1}$ with interval $[\bar\kappa, \bar\kappa]$.

$\mathbb{G}$ is also a reflection pattern.

8. Suppose $\mathbb{C}$ is a reflection pattern with domain $(0, \omega)$ of the form

$$\mathbb{C}\,(m) = (\pi; \rho\text{-}\mathsf{P}_m; \rho\text{-}\mathsf{P}_\infty; \mathbb{X}; \beta),$$

where $\beta \leq \alpha$. Then $X^{\alpha}_{\mathbb{C}\,(m)}$ consists of all triples $\langle \kappa, \sigma, \vec{\mathbf{D}} \rangle$ such that $\kappa \leq \sigma < \pi$ and the following are satisfied:

1. $C(\alpha,\kappa)\cap\pi = \kappa$ and $\mathbb{C}(m),\alpha\in C(\alpha,\kappa)$.
2. $\langle V_{\kappa,\sigma};\in;\vec{\mathbf{D}}\rangle \xrightarrow[\equiv]{\Pi^1_m} \langle V_{\pi,\rho};\in;\mathfrak{C}(\alpha,\rho)\rangle$, where $\vec{\mathbf{D}}\subseteq V_\sigma$ is the $<_{\mathbf{L}}$-least such set.
3. If $\xi\in C(\xi,\kappa)\cap[\beta,\alpha)$ and $0<n<\omega$, then κ is σ-Π^1_m-reducible in $\mathfrak{M}^{\xi}_{\mathbb{C}(n)}$.

Set

$$\mathfrak{M}^{\alpha}_{\mathbb{C}(m)} := \{\langle\kappa,\sigma\rangle : \exists\vec{\mathbf{D}}\,[\langle\kappa,\sigma,\vec{\mathbf{D}}\rangle\in X^{\alpha}_{\mathbb{C}(m)}]\}.$$

Suppose $\langle\kappa_0,\sigma_0\rangle\in\mathfrak{M}^{\alpha}_{\mathbb{C}(m)}$. Then $\langle\kappa_0,\sigma_0,\vec{\mathbf{D}}_0\rangle\in X^{\alpha}_{\mathbb{C}(m)}$ for some $\vec{\mathbf{D}}_0$ which is uniquely determined by $\langle\kappa_0,\sigma_0\rangle$. Let j_0 be determined by

$$j_0 : \langle V_{\kappa_0,\sigma_0};\in;\vec{\mathbf{D}}_0\rangle \xrightarrow[\equiv]{\Pi^1_m} \langle V_{\pi,\rho};\in;\mathfrak{C}(\alpha,\rho)\rangle.$$

Define

$$\bar{j_0} := j_0\restriction[\kappa_0,\sigma_0)\ \cup\ \{\langle\sigma_0,\rho\rangle\}$$

and set

$$\mathsf{f}^{\alpha,\mathbb{C}(m)}_{\kappa_0,\sigma_0} := \bar{j_0}^{-1}\restriction C(\alpha,\kappa_0).$$

$[\pi,\rho]$ is the interval assigned to $\mathsf{f}^{\alpha,\mathbb{C}(m)}_{\kappa_0,\sigma_0}$.

If $\mathfrak{M}^{\alpha}_{\mathbb{C}(m)}\neq\emptyset$, let $\langle\pi_0,\delta_0\rangle$ be the least pair in $\mathfrak{M}^{\alpha}_{\mathbb{C}(m)}$. Let $\vec{\mathbf{D}}_0$ be such that $\langle\pi_0,\delta_0,\vec{\mathbf{D}}_0\rangle\in X^{\alpha}_{\mathbb{C}(m)}$. Then $(\alpha,\mathbb{C}(m),\mathsf{f}^{\alpha,\mathbb{C}(m)}_{\pi_0,\delta_0})\in\mathfrak{F}^{\alpha}$ and $(\alpha,\mathbb{C}(m),\mathfrak{D})\in\mathfrak{G}^{\alpha}$, where $\mathfrak{D}:=\langle V_{\pi_0,\delta_0};\in;\vec{\mathbf{D}}_0\rangle$.

9. Suppose $\mathbb{C}$ is a reflection pattern with domain $(0,\omega)$ of the form $\mathbb{C}(m)=(\pi;\rho\text{-}\mathsf{P}_m;\rho\text{-}\mathsf{P}_\infty;\mathbb{X};\beta)$. Also assume $\mathfrak{M}^{\alpha_0}_{\mathbb{C}(m)}\neq\emptyset$, $\bar{\kappa}=\Psi^{\alpha_0}_{\mathbb{C}(m)}(\pi)$, $\bar{\rho}=\Psi^{\alpha_0}_{\mathbb{C}(m)}(\rho)$, and $\beta\le\alpha_0=\alpha$.

9.1. Suppose $\alpha_0>\beta$. Then

$$\mathbb{D} := (\bar{\kappa};\mathsf{M}^{<\alpha_0}_{\mathbb{C}(m)}\text{-}\rho\text{-}\mathsf{P}_m;\mathsf{M}^{\alpha_0}_{\mathbb{C}(m)};\mathbb{C}(m);\alpha+1) \qquad \to 10.b$$

is a projection instance in $\mathfrak{R}^{\alpha+1}$ with interval $[\bar{\kappa},\bar{\rho}]$. $\mathbb{D}$ is also a reflection pattern.

For every $\tau\in C(\tau,\bar{\kappa})\cap[\beta,\alpha_0)$ and $m<n<\omega$,

$$(\mathbb{D},\tau,n)$$

is a reflection instance in $\mathfrak{R}^{\alpha+1}$ with interval $[\bar{\kappa},\bar{\rho}]$.

9.2 If $\alpha_0=\beta$, then

$$\mathbb{E} := (\bar{\kappa};\bar{\rho}\text{-}\mathsf{P}_m;\mathbb{C}(m);\alpha+1) \qquad \to 6.$$

is a projection instance and reflection instance in $\mathfrak{R}^{\alpha+1}$ with interval $[\bar{\kappa},\bar{\rho}]$. $\mathbb{E}$ is also a reflection pattern.

10.a. Suppose $\mathbb{C}$ is a reflection pattern and a projection instance in $\mathfrak{R}^{\alpha}$ of the form

$$\mathbb{C} = (\pi;\mathsf{M}^{<\gamma}_{\mathbb{B}(\vartheta)}\text{-}\rho\text{-}\mathsf{P}_m;\mathsf{M}^{\nu}_{\mathbb{Y}};\mathbb{Z};\beta)$$

where $m>0$ and $\mathbf{dom}(\mathbb{B})\neq(0,\omega)$. Let $\delta:=o(\mathbb{B}(\vartheta))$. Note that $\delta<\gamma$.

Then $X^{\alpha}_{\mathbb{C}}$ consists of all triples $\langle\kappa,\sigma,\vec{\mathbf{D}}\rangle$ such that $\kappa\le\sigma<\pi$ and the following are satisfied:

1. $C(\alpha,\kappa)\cap\pi \;=\; \kappa$ and $\mathbb{C}\,,\alpha\in C(\alpha,\kappa)$.
2. $\langle V_{\kappa,\sigma};\in;\vec{\mathbf{D}}\rangle \xrightarrow[\equiv]{\Pi^1_m} \langle V_{\pi,\rho};\in;\mathfrak{C}(\alpha,\rho)\rangle$, where $\vec{\mathbf{D}}\subseteq V_\sigma$ is the $<_{\mathbf{L}}$-least such set.
3. If $\xi\in C(\xi,\kappa)\,\cap\,[\delta,\gamma)$, $\vartheta'\in\mathbf{dom}(\mathbb{B}\,)$, and $\vartheta'>\vartheta$, then κ is σ-Π^1_{m-1}-reducible in

$$^{o}\mathfrak{M}^{\xi}_{\mathbb{B}\,(\vartheta')} \;\;:=\;\; \{\langle\kappa^*,\sigma^*\rangle :\; \exists\nu^*\,\big(\langle\kappa^*,\nu^*\rangle\in\mathfrak{M}^{\xi}_{\mathbb{B}\,(\vartheta')}\,\wedge\,\kappa^*\le\sigma^*<\nu^*\,\big)\}.$$

4. If $\tau\in C(\tau,\kappa)\,\cap\,[\beta,\alpha)$, then κ is σ-Π^1_{m-1}-reducible in $\mathfrak{M}^{\tau}_{\mathbb{C}}$.

Set

$$\mathfrak{M}^{\alpha}_{\mathbb{C}} := \{\langle\kappa,\sigma\rangle :\; \exists\vec{\mathbf{D}}\,[\langle\kappa,\sigma,\vec{\mathbf{D}}\rangle\in X^{\alpha}_{\mathbb{C}}]\}.$$

Suppose $\langle\kappa_0,\sigma_0\rangle\in\mathfrak{M}^{\alpha}_{\mathbb{C}}$. Then $\langle\kappa_0,\sigma_0,\vec{\mathbf{D}}_0\rangle\in X^{\alpha}_{\mathbb{C}}$ for some $\vec{\mathbf{D}}_0$ which is uniquely determined by $\langle\kappa_0,\sigma_0\rangle$. Let j_0 be determined by

$$j_0 : \langle V_{\kappa_0,\sigma_0};\in;\vec{\mathbf{D}}_0\rangle \xrightarrow[\equiv]{\Pi^1_m} \langle V_{\pi,\rho};\in;\mathfrak{C}(\alpha,\rho)\rangle.$$

Define

$$\bar{j_0} \;:=\; j_0\restriction[\kappa_0,\sigma_0)\;\cup\;\{\langle\sigma_0,\rho\rangle\}$$

and set

$$\mathsf{f}^{\alpha,\mathbb{C}}_{\kappa_0,\sigma_0} \;:=\; \bar{j_0}^{\,-1}\restriction C(\alpha,\kappa_0).$$

$[\pi,\rho]$ is the interval assigned to $\mathsf{f}^{\alpha,\mathbb{C}}_{\kappa_0,\sigma_0}$.

If $\mathfrak{M}^{\alpha}_{\mathbb{C}}\neq\emptyset$, let $\langle\pi_0,\delta_0\rangle$ be the least pair in $\mathfrak{M}^{\alpha}_{\mathbb{C}}$. Let $\vec{\mathbf{D}}_0$ be such that $\langle\pi_0,\delta_0,\vec{\mathbf{D}}_0\rangle\in X^{\alpha}_{\mathbb{C}}$. Then $(\alpha,\mathbb{C}\,,\mathsf{f}^{\alpha,\mathbb{C}}_{\pi_0,\delta_0})\in\mathfrak{F}^{\alpha}$ and $(\alpha,\mathbb{C}\,,\mathfrak{D})\in\mathfrak{S}^{\alpha}$, where $\mathfrak{D}:=\langle V_{\pi_0,\delta_0};\in;\vec{\mathbf{D}}_0\rangle$.

10.b. Suppose $\mathbb{C}$ is a reflection pattern and a projection instance in $\mathfrak{R}^{\alpha}$ of the form

$$\mathbb{C} \;=\; (\pi;\mathsf{M}^{<\gamma}_{\mathbb{B}\,(k)}\text{-}\rho\text{-}\mathsf{P}_m;\mathsf{M}^{\nu}_{\mathbb{Y}};\mathbb{Z};\beta)$$

where $m>0$ and $\mathbf{dom}(\mathbb{B}\,)=(0,\omega)$. Let $\delta:=o(\mathbb{B}\,(k))$. Note that $\delta<\gamma$.

Then $X^{\alpha}_{\mathbb{C}}$ consists of all triples $\langle\kappa,\sigma,\vec{\mathbf{D}}\rangle$ such that $\kappa\le\sigma<\pi$ and the following are satisfied:

1. $C(\alpha,\kappa)\cap\pi \;=\; \kappa$ and $\mathbb{C}\,,\alpha\in C(\alpha,\kappa)$.
2. $\langle V_{\kappa,\sigma};\in;\vec{\mathbf{D}}\rangle \xrightarrow[\equiv]{\Pi^1_m} \langle V_{\pi,\rho};\in;\mathfrak{C}(\alpha,\rho)\rangle$, where $\vec{\mathbf{D}}\subseteq V_\sigma$ is the $<_{\mathbf{L}}$-least such set.
3. If $\xi\in C(\xi,\kappa)\,\cap\,[\delta,\gamma)$, $n\in\mathbf{dom}(\mathbb{B}\,)$, and $n>k$, then κ is σ-Π^1_{m-1}-reducible in $\mathfrak{M}^{\xi}_{\mathbb{B}\,(n)}$.
4. If $\tau\in C(\tau,\kappa)\,\cap\,[\beta,\alpha)$, then κ is σ-Π^1_{m-1}-reducible in $\mathfrak{M}^{\tau}_{\mathbb{C}}$.

Set

$$\mathfrak{M}^{\alpha}_{\mathbb{C}} := \{\langle\kappa,\sigma\rangle :\; \exists\vec{\mathbf{D}}\,[\langle\kappa,\sigma,\vec{\mathbf{D}}\rangle\in X^{\alpha}_{\mathbb{C}}]\}.$$

Suppose $\langle\kappa_0,\sigma_0\rangle\in\mathfrak{M}^{\alpha}_{\mathbb{C}}$. Then $\langle\kappa_0,\sigma_0,\vec{\mathbf{D}}_0\rangle\in X^{\alpha}_{\mathbb{C}}$ for some $\vec{\mathbf{D}}_0$ which is uniquely determined by $\langle\kappa_0,\sigma_0\rangle$. Let j_0 be determined by

$$j_0 : \langle V_{\kappa_0,\sigma_0};\in;\vec{\mathbf{D}}_0\rangle \xrightarrow[\equiv]{\Pi^1_m} \langle V_{\pi,\rho};\in;\mathfrak{C}(\alpha,\rho)\rangle.$$

Define

$$\bar{j_0} \;:=\; j_0 \restriction [\kappa_0, \sigma_0) \;\cup\; \{\langle \sigma_0, \rho\rangle\}$$

and set

$$\mathsf{f}^{\alpha,\mathbb{C}}_{\kappa_0,\sigma_0} \;:=\; \bar{j_0}^{\,-1} \restriction C(\alpha,\kappa_0).$$

$[\pi,\rho]$ is the interval assigned to $\mathsf{f}^{\alpha,\mathbb{C}}_{\kappa_0,\sigma_0}$.

If $\mathfrak{M}^{\alpha}_{\mathbb{C}} \neq \emptyset$, let $\langle \pi_0, \delta_0\rangle$ be the least pair in $\mathfrak{M}^{\alpha}_{\mathbb{C}}$. Let $\vec{\mathbf{D}}_0$ be such that $\langle \pi_0, \delta_0, \vec{\mathbf{D}}_0\rangle \in X^{\alpha}_{\mathbb{C}}$. Then $(\alpha, \mathbb{C}, \mathsf{f}^{\alpha,\mathbb{C}}_{\pi_0,\delta_0}) \in \mathfrak{F}^{\alpha}$ and $(\alpha, \mathbb{C}, \mathfrak{D}) \in \mathfrak{S}^{\alpha}$, where $\mathfrak{D} := \langle V_{\pi_0,\delta_0}; \in; \vec{\mathbf{D}}_0\rangle$.

10.c. Suppose $\mathbb{C}$ is a reflection pattern and a projection instance in $\mathfrak{R}^{\alpha}$ of the form

$$\mathbb{C} \;=\; (\pi; \mathsf{M}^{<\gamma}_{\mathbb{B}}\text{-}\rho\text{-}\mathsf{P}_m; \mathsf{M}^{\nu}_{\mathbb{Y}}; \mathbb{Z}; \beta)$$

where $\mathbb{B}$ is a constant reflection pattern and $m > 0$ Let $\delta := o(\mathbb{B})$. Note that $\delta < \gamma$.

Then $X^{\alpha}_{\mathbb{C}}$ consists of all triples $\langle \kappa, \sigma, \vec{\mathbf{D}}\rangle$ such that $\kappa \le \sigma < \pi$ and the following are satisfied:

1. $C(\alpha,\kappa) \cap \pi \;=\; \kappa$ and $\mathbb{C}, \alpha \in C(\alpha,\kappa)$.
2. $\langle V_{\kappa,\sigma}; \in; \vec{\mathbf{D}}\rangle \xrightarrow[\equiv]{\Pi^1_m} \langle V_{\pi,\rho}; \in; \mathfrak{C}(\alpha,\rho)\rangle$, where $\vec{\mathbf{D}} \subseteq V_{\sigma}$ is the $<_{\mathbf{L}}$-least such set.
3. If $\xi \in C(\xi,\kappa) \,\cap\, [\delta,\gamma)$, then κ is σ-Π^1_{m-1}-reducible in $\mathfrak{M}^{\xi}_{\mathbb{B}}$.
4. If $\tau \in C(\tau,\kappa) \,\cap\, [\beta,\alpha)$, then κ is σ-Π^1_{m-1}-reducible in $\mathfrak{M}^{\tau}_{\mathbb{C}}$.

Set

$$\mathfrak{M}^{\alpha}_{\mathbb{C}} := \{\langle \kappa,\sigma\rangle :\; \exists \vec{\mathbf{D}}\,[\langle \kappa,\sigma,\vec{\mathbf{D}}\rangle \in X^{\alpha}_{\mathbb{C}}]\}.$$

Suppose $\langle \kappa_0, \sigma_0\rangle \in \mathfrak{M}^{\alpha}_{\mathbb{C}}$. Then $\langle \kappa_0, \sigma_0, \vec{\mathbf{D}}_0\rangle \in X^{\alpha}_{\mathbb{C}}$ for some $\vec{\mathbf{D}}_0$ which is uniquely determined by $\langle \kappa_0, \sigma_0\rangle$. Let j_0 be determined by

$$j_0 : \langle V_{\kappa_0,\sigma_0}; \in; \vec{\mathbf{D}}_0\rangle \xrightarrow[\equiv]{\Pi^1_m} \langle V_{\pi,\rho}; \in; \mathfrak{C}(\alpha,\rho)\rangle.$$

Define

$$\bar{j_0} \;:=\; j_0 \restriction [\kappa_0, \sigma_0) \;\cup\; \{\langle \sigma_0, \rho\rangle\}$$

and set

$$\mathsf{f}^{\alpha,\mathbb{C}}_{\kappa_0,\sigma_0} \;:=\; \bar{j_0}^{\,-1} \restriction C(\alpha,\kappa_0).$$

$[\pi,\rho]$ is the interval assigned to $\mathsf{f}^{\alpha,\mathbb{C}}_{\kappa_0,\sigma_0}$.

If $\mathfrak{M}^{\alpha}_{\mathbb{C}} \neq \emptyset$, let $\langle \pi_0, \delta_0\rangle$ be the least pair in $\mathfrak{M}^{\alpha}_{\mathbb{C}}$. Let $\vec{\mathbf{D}}_0$ be such that $\langle \pi_0, \delta_0, \vec{\mathbf{D}}_0\rangle \in X^{\alpha}_{\mathbb{C}}$. Then $(\alpha, \mathbb{C}, \mathsf{f}^{\alpha,\mathbb{C}}_{\pi_0,\delta_0}) \in \mathfrak{F}^{\alpha}$ and $(\alpha, \mathbb{C}, \mathfrak{D}) \in \mathfrak{S}^{\alpha}$, where $\mathfrak{D} := \langle V_{\pi_0,\delta_0}; \in; \vec{\mathbf{D}}_0\rangle$.

11.a. Suppose $\mathbb{C}$ is a reflection pattern and a projection instance in $\mathfrak{R}^{\alpha}$ of the form

$$\mathbb{C} \;=\; (\pi; \mathsf{M}^{<\gamma}_{\mathbb{B}(\vartheta)}\text{-}\rho\text{-}\mathsf{P}_m; \mathsf{M}^{\nu}_{\mathbb{Y}}; \mathbb{Z}; \beta)$$

where $m > 0$ and $\mathbf{dom}(\mathbb{B}) \neq (0,\omega)$. Also assume $\mathfrak{M}^{\alpha_0}_{\mathbb{C}} \neq \emptyset$, $\bar{\kappa} = \Psi^{\alpha_0}_{\mathbb{C}}(\pi)$, $\bar{\rho} = \Psi^{\alpha_0}_{\mathbb{C}}(\rho)$, and $\beta \le \alpha_0 \le \alpha$.

Let $\delta := o(\mathbb{B}(\vartheta))$.

11.a.1 If $\alpha_0 > \beta$, $\alpha = \alpha_0$, and $m-1>0$, then

$$\mathbb{D}_1 := (\bar\kappa; \mathsf{M}^{<\alpha_0}_{\mathbb{C}}\text{-}\bar\rho\text{-}\mathsf{P}_{m-1}; \mathsf{M}^{\alpha_0}_{\mathbb{C}}; \mathbb{C}\,; \alpha+1) \qquad \to 10.c$$

is a projection instance in $\mathfrak{R}^{\alpha+1}$ with interval $[\bar\kappa, \bar\rho]$. $\mathbb{D}_1$ is also a reflection pattern.

For every $\tau \in C(\tau, \bar\kappa) \cap [\beta, \alpha_0)$,

$$(\mathbb{D}_1, \tau)$$

is a reflection instance in $\mathfrak{R}^{\alpha+1}$ with interval $[\bar\kappa, \bar\rho]$.

11.a.2. If $\alpha_0 = \alpha = \beta$ and $m-1>0$, then

$$\mathbb{D}_2 := (\bar\kappa; \mathsf{M}^{<\gamma}_{\mathbb{B}(\vartheta)}\text{-}\bar\rho\text{-}\mathsf{P}_{m-1}; \mathsf{M}^{\alpha_0}_{\mathbb{C}}; \mathbb{C}\,; \alpha+1) \qquad \to 10.a$$

is a projection instance in $\mathfrak{R}^{\alpha+1}$ with interval $[\bar\kappa, \bar\rho]$. $\mathbb{D}_2$ is also a reflection pattern.

For every $\zeta \in C(\zeta, \bar\kappa) \cap [\delta, \gamma)$ and $\vartheta' \in C(\zeta, \bar\kappa) \cap \mathbf{dom}(\mathbb{B})$ with $\vartheta' > \vartheta$,

$$(\mathbb{D}_2, \zeta, \vartheta')$$

is a reflection instance in $\mathfrak{R}^{\alpha+1}$ with interval $[\bar\kappa, \bar\rho]$.

From now on we assume $m-1=0$.

11.a.3. Suppose $\alpha_0 > \beta$ and $\bar\rho$ is a limit $> \bar\kappa$. Assume further that $\eta \in [\bar\kappa, \bar\rho]$ and that α is the least ordinal $\alpha' \geq \alpha_0$ such that $\eta \in C(\alpha', \bar\kappa)$. Then

$$\mathbb{D}(\eta) \;=\; (\bar\kappa; \mathsf{M}^{<\alpha_0}_{\mathbb{C}}\text{-}\eta\text{-}\mathsf{P}_0; \mathsf{M}^{\alpha_0}_{\mathbb{C}}; \mathbb{C}\,; \alpha+1) \qquad \to 4.c$$

is a projection instance in $\mathfrak{R}^{\alpha+1}$ with interval $[\bar\kappa, \eta]$.

For every $\xi \in C(\xi, \bar\kappa) \cap [\beta, \alpha_0)$,

$$(\mathbb{D}(\eta), \xi)$$

is a reflection instance in $\mathfrak{R}^{\alpha+1}$ with interval $[\bar\kappa, \eta]$.

The function $\eta' \mapsto \mathbb{D}(\eta')$ with domain $\{\eta' \in [\bar\kappa, \bar\rho) : \exists \alpha' \big(\eta' \in C(\alpha', \bar\kappa)\big)\}$ is a reflection pattern.

11.a.4. Suppose $\alpha_0 > \beta$, $\alpha = \alpha_0$, and $\bar\rho$ is a successor $\rho_1 + 1$. Then for every $0 < n < \omega$,

$$\mathbb{E}(n) \;=\; (\bar\kappa; \mathsf{M}^{<\alpha_0}_{\mathbb{C}}\text{-}\rho_1\text{-}\mathsf{P}_n; \mathsf{M}^{\alpha_0}_{\mathbb{C}}; \mathbb{C}\,; \alpha+1) \qquad \to 12.c$$

is a projection instance in $\mathfrak{R}^{\alpha+1}$ with interval $[\bar\kappa, \rho_1]$.

For every $\xi \in C(\xi, \bar\kappa) \cap [\beta, \alpha_0)$,

$$(\mathbb{E}(n), \xi)$$

is a reflection instance in $\mathfrak{R}^{\alpha+1}$ with interval $[\bar\kappa, \rho_1]$.

The function $n' \mapsto \mathbb{E}(n')$ with domain $(0, \omega)$ is a reflection pattern.

11.a.5. Suppose $\alpha_0 > \beta$, $\alpha = \alpha_0$, and $\bar\rho = \bar\kappa$. Then for every $\xi \in C(\xi, \bar\kappa) \cap [\beta, \alpha_0)$,

$$\mathbb{I} \;=\; (\bar\kappa; \mathsf{M}^{\xi}_{\mathbb{C}}\text{-}\bar\kappa\text{-}\mathsf{P}_0; \mathsf{M}^{\alpha_0}_{\mathbb{C}}; \mathbb{C}\,; \alpha+1)$$

is a reflection instance in $\mathfrak{R}^{\alpha+1}$ with interval $[\bar\kappa, \bar\kappa]$ and

$$\mathbb{F} \;=\; (\pi; \mathsf{M}^{\xi}_{\mathbb{C}}\,; \bar\kappa; \mathbb{I}; \alpha+1) \qquad \to 14.$$

is a projection instance in $\mathfrak{R}^{\alpha+1}$ with interval $[\pi, \rho]$.

$\mathbb{I}$ is also a reflection pattern.

11.a.6. Suppose $\alpha_0 = \beta$ and $\bar\rho$ is a limit $> \bar\kappa$. Assume further that $\eta \in [\bar\kappa, \bar\rho]$ and that α is the least ordinal $\alpha' \geq \alpha_0$ such that $\eta \in C(\alpha', \bar\kappa)$. Then

$$\mathbb{G}(\eta) \;=\; (\bar\kappa; \mathsf{M}^{<\alpha_0}_{\mathbb{B}\,(\vartheta)}\text{-}\eta\text{-}\mathsf{P}_0; \mathsf{M}^{\alpha_0}_{\mathbb{C}}; \mathbb{C}\,; \alpha+1) \qquad \to 4.a$$

is a projection instance in $\mathfrak{R}^{\alpha+1}$ with interval $[\bar\kappa, \eta]$.

For every $\zeta \in C(\zeta, \bar\kappa) \cap [\delta, \gamma)$ and $\vartheta' \in C(\zeta, \bar\kappa) \cap \mathbf{dom}(\mathbb{B}\,)$ with $\vartheta' > \vartheta$,

$$(\mathbb{G}(\eta), \zeta, \vartheta')$$

is a reflection instance in $\mathfrak{R}^{\alpha+1}$ with interval $[\bar\kappa, \eta]$.

The function $\eta' \mapsto \mathbb{G}(\eta')$ with domain $\{\eta' \in [\bar\kappa, \bar\rho) : \; \exists\alpha' \; [\eta' \in C(\alpha', \bar\kappa)]\}$ is a reflection pattern.

11.a.7. Suppose $\alpha_0 = \alpha = \beta$ and $\bar\rho$ is a successor $\rho_1 + 1$. Then for every $0 < n < \omega$,

$$\mathbb{H}(n) \;=\; (\bar\kappa; \mathsf{M}^{<\gamma}_{\mathbb{B}\,(\vartheta)}\text{-}\rho_1\text{-}\mathsf{P}_n; \mathsf{M}^{\alpha_0}_{\mathbb{C}}; \mathbb{C}\,; \alpha+1) \qquad \to 12.$$

is a projection instance in $\mathfrak{R}^{\alpha+1}$ with interval $[\bar\kappa, \rho_1]$.

For every $\zeta \in C(\zeta, \bar\kappa) \cap [\delta, \gamma)$ and $\vartheta' \in C(\zeta, \bar\kappa) \cap \mathbf{dom}(\mathbb{B}\,)$ with $\vartheta' > \vartheta$,

$$(\mathbb{H}(n), \zeta, \vartheta')$$

is a reflection instance in $\mathfrak{R}^{\alpha+1}$ with interval $[\bar\kappa, \rho_1]$.

The function $n' \mapsto \mathbb{H}(n')$ with domain $(0, \omega)$ is a reflection pattern.

11.a.8. Suppose $\alpha_0 = \alpha = \beta$ and $\bar\rho = \bar\kappa$. Then for every $\zeta \in C(\zeta, \bar\kappa) \cap [\delta, \gamma)$ and $\vartheta' \in C(\zeta, \bar\kappa) \cap \mathbf{dom}(\mathbb{B}\,)$ with $\vartheta' > \vartheta$,

$$\mathbb{I}^* \;=\; (\bar\kappa; \mathsf{M}^{\zeta}_{\mathbb{B}\,(\vartheta')}\text{-}\bar\kappa\text{-}\mathsf{P}_0; \mathsf{M}^{\alpha_0}_{\mathbb{C}}; \mathbb{C}\,; \alpha+1)$$

is a reflection instance in $\mathfrak{R}^{\alpha+1}$ with interval $[\bar\kappa, \bar\kappa]$ and

$$\mathbb{F}^* \;=\; (\mu; \mathsf{M}^{\zeta}_{\mathbb{B}\,(\vartheta')}; \bar\kappa; \mathbb{I}^*; \alpha+1) \qquad \to 14.$$

is a projection instance in $\mathfrak{R}^{\alpha+1}$ with interval $[\mu, \nu]$, where $[\mu, \nu]$ is the interval of $\mathbb{B}\,(\vartheta')$.

$\mathbb{I}^*$ is also a reflection pattern.

11.b. Suppose $\mathbb{C}$ is a reflection pattern and a projection instance in $\mathfrak{R}^{\alpha}$ of the form

$$\mathbb{C} \;=\; (\pi; \mathsf{M}^{<\gamma}_{\mathbb{B}\,(k)}\text{-}\rho\text{-}\mathsf{P}_m; \mathsf{M}^{\nu}_{\mathbb{Y}}; \mathbb{Z}; \beta)$$

where $m > 0$ and $\mathbf{dom}(\mathbb{B}\,) = (0, \omega)$. Also assume $\mathfrak{M}^{\alpha_0}_{\mathbb{C}} \neq \emptyset$, $\bar\kappa = \Psi^{\alpha_0}_{\mathbb{C}}(\pi)$, $\bar\rho = \Psi^{\alpha_0}_{\mathbb{C}}(\rho)$, and $\beta \leq \alpha_0 \leq \alpha$.

Let $\delta := o(\mathbb{B}\,(k))$.

11.b.1 If $\alpha_0 > \beta$, $\alpha = \alpha_0$, and $m - 1 > 0$, then

$$\mathbb{D}_1 := (\bar\kappa; \mathsf{M}^{<\alpha_0}_{\mathbb{C}}\text{-}\bar\rho\text{-}\mathsf{P}_{m-1}; \mathsf{M}^{\alpha_0}_{\mathbb{C}}; \mathbb{C}\,; \alpha+1) \qquad \to 10.c$$

is a projection instance in $\mathfrak{R}^{\alpha+1}$ with interval $[\bar\kappa, \bar\rho]$. $\mathbb{D}_1$ is also a reflection pattern.

For every $\tau \in C(\tau, \bar\kappa) \cap [\beta, \alpha_0)$,

$$(\mathbb{D}_1, \tau)$$

is a reflection instance in $\mathfrak{R}^{\alpha+1}$ with interval $[\bar\kappa, \bar\rho]$.

11.b.2. If $\alpha_0 = \alpha = \beta$ and $m - 1 > 0$, then

$$\mathbb{D}_2 := (\bar{\kappa}; \mathsf{M}^{<\gamma}_{\mathbb{B}(k)}\text{-}\bar{\rho}\text{-}\mathsf{P}_{m-1}; \mathsf{M}^{\alpha_0}_{\mathbb{C}}; \mathbb{C}; \alpha + 1) \qquad \rightarrow 10.b$$

is a projection instance in $\Re^{\alpha+1}$ with interval $[\bar{\kappa}, \bar{\rho}]$. $\mathbb{D}_2$ is also a reflection pattern.

For every $\zeta \in C(\zeta, \bar{\kappa}) \cap [\delta, \gamma)$ and $l \in \mathbf{dom}(\mathbb{B})$ with $l > k$,

$$(\mathbb{D}_2, \zeta, l)$$

is a reflection instance in $\Re^{\alpha+1}$ with interval $[\bar{\kappa}, \bar{\rho}]$.

From now on we assume $m - 1 = 0$**.**

11.b.3. Suppose $\alpha_0 > \beta$ and $\bar{\rho}$ is a limit $> \bar{\kappa}$. Assume further that $\eta \in [\bar{\kappa}, \bar{\rho}]$ and that α is the least ordinal $\alpha' \geq \alpha_0$ such that $\eta \in C(\alpha', \bar{\kappa})$. Then

$$\mathbb{D}(\eta) = (\bar{\kappa}; \mathsf{M}^{<\alpha_0}_{\mathbb{C}}\text{-}\eta\text{-}\mathsf{P}_0; \mathsf{M}^{\alpha_0}_{\mathbb{C}}; \mathbb{C}; \alpha + 1) \qquad \rightarrow 4.c$$

is a projection instance in $\Re^{\alpha+1}$ with interval $[\bar{\kappa}, \eta]$.

For every $\xi \in C(\xi, \bar{\kappa}) \cap [\beta, \alpha_0)$,

$$(\mathbb{D}(\eta), \xi)$$

is a reflection instance in $\Re^{\alpha+1}$ with interval $[\bar{\kappa}, \eta]$.

The function $\eta' \mapsto \mathbb{D}(\eta')$ with domain $\{\eta' \in [\bar{\kappa}, \bar{\rho}) : \exists \alpha' \, [\eta' \in C(\alpha', \bar{\kappa})]\}$ is a reflection pattern.

11.b.4. Suppose $\alpha_0 > \beta$ and $\bar{\rho}$ is a successor $\rho_1 + 1$. Then for every $0 < n < \omega$,

$$\mathbb{E}(n) = (\bar{\kappa}; \mathsf{M}^{<\alpha_0}_{\mathbb{C}}\text{-}\rho_1\text{-}\mathsf{P}_n; \mathsf{M}^{\alpha_0}_{\mathbb{C}}; \mathbb{C}; \alpha + 1) \qquad \rightarrow 12.c$$

is a projection instance in $\Re^{\alpha+1}$ with interval $[\bar{\kappa}, \rho_1]$.

For every $\xi \in C(\xi, \bar{\kappa}) \cap [\beta, \alpha_0)$,

$$(\mathbb{E}(n), \xi)$$

is a reflection instance in $\Re^{\alpha+1}$ with interval $[\bar{\kappa}, \rho_1]$.

The function $n' \mapsto \mathbb{E}(n')$ with domain $(0, \omega)$ is a reflection pattern.

11.b.5. Suppose $\alpha_0 > \beta$, $\alpha = \alpha_0$, and $\bar{\rho} = \bar{\kappa}$. Then for every $\xi \in C(\xi, \bar{\kappa}) \cap [\beta, \alpha_0)$,

$$\mathbb{I} = (\bar{\kappa}; \mathsf{M}^{\xi}_{\mathbb{C}}\text{-}\bar{\kappa}\text{-}\mathsf{P}_0; \mathsf{M}^{\alpha_0}_{\mathbb{C}}; \mathbb{C}; \alpha + 1)$$

is a reflection instance in $\Re^{\alpha+1}$ with interval $[\bar{\kappa}, \bar{\kappa}]$ and

$$\mathbb{F} = (\pi; \mathsf{M}^{\xi}_{\mathbb{C}}; \bar{\kappa}; \mathbb{I}; \alpha + 1) \qquad \rightarrow 14.$$

is a projection instance in $\Re^{\alpha+1}$ with interval $[\pi, \rho]$.

$\mathbb{I}$ is also a reflection pattern.

11.b.6. Suppose $\alpha_0 = \beta$ and $\bar{\rho}$ is a limit $> \bar{\kappa}$. Assume further that $\eta \in [\bar{\kappa}, \bar{\rho}]$ and that α is the least ordinal $\alpha' \geq \alpha_0$ such that $\eta \in C(\alpha', \bar{\kappa})$. Then

$$\mathbb{G}(\eta) \;=\; (\bar{\kappa}; \mathsf{M}^{<\alpha_0}_{\mathbb{B}\,(k)}\text{-}\eta\text{-}\mathsf{P}_0; \mathsf{M}^{\alpha_0}_{\mathbb{C}}; \mathbb{C}\,; \alpha+1) \qquad \rightarrow 4.b$$

is a projection instance in $\mathfrak{R}^{\alpha+1}$ with interval $[\bar{\kappa}, \eta]$.

For every $\zeta \in C(\zeta, \bar{\kappa}) \cap [\delta, \gamma)$ and $l \in \mathbf{dom}(\mathbb{B}\,)$ with $l > k$,

$$(\mathbb{G}(\eta), \zeta, l)$$

is a reflection instance in $\mathfrak{R}^{\alpha+1}$ with interval $[\bar{\kappa}, \eta]$.

The function $\eta' \mapsto \mathbb{G}(\eta')$ with domain $\{\eta' \in [\bar{\kappa}, \bar{\rho}) : \exists \alpha' \, [\eta' \in C(\alpha', \bar{\kappa})]\}$ is a reflection pattern.

11.b.7. Suppose $\alpha_0 = \alpha = \beta$ and $\bar{\rho}$ is a successor $\rho_1 + 1$. Then for every $0 < n < \omega$,

$$\mathbb{H}(n) \;=\; (\bar{\kappa}; \mathsf{M}^{<\gamma}_{\mathbb{B}\,(k)}\text{-}\rho_1\text{-}\mathsf{P}_n; \mathsf{M}^{\alpha_0}_{\mathbb{C}}; \mathbb{C}\,; \alpha+1) \qquad \rightarrow 12.b$$

is a projection instance in $\mathfrak{R}^{\alpha+1}$ with interval $[\bar{\kappa}, \rho_1]$.

For every $\zeta \in C(\zeta, \bar{\kappa}) \cap [\delta, \gamma)$ and $l \in \mathbf{dom}(\mathbb{B}\,)$ with $l > k$,

$$(\mathbb{H}(n), \zeta, l)$$

is a reflection instance in $\mathfrak{R}^{\alpha+1}$ with interval $[\bar{\kappa}, \rho_1]$.

The function $n' \mapsto \mathbb{H}(n')$ with domain $(0, \omega)$ is a reflection pattern.

11.b.8. Suppose $\alpha_0 = \alpha = \beta$ and $\bar{\rho} = \bar{\kappa}$. Then for every $\xi \in C(\xi, \bar{\kappa}) \cap [\delta, \gamma)$ and $l \in \mathbf{dom}(\mathbb{B}\,)$ with $l > n$,

$$\mathbb{I}^* \;=\; (\bar{\kappa}; \mathsf{M}^{\xi}_{\mathbb{B}\,(l)}\text{-}\bar{\kappa}\text{-}\mathsf{P}_0; \mathsf{M}^{\alpha_0}_{\mathbb{C}}; \mathbb{C}\,; \alpha+1)$$

is a reflection instance in $\mathfrak{R}^{\alpha+1}$ with interval $[\bar{\kappa}, \bar{\kappa}]$ and

$$\mathbb{F}^* \;=\; (\mu; \mathsf{M}^{\xi}_{\mathbb{B}\,(l)}; \bar{\kappa}; \mathbb{I}^*; \alpha+1) \qquad \rightarrow 14.$$

is a projection instance in $\mathfrak{R}^{\alpha+1}$ with interval $[\mu, \vartheta]$, where $[\mu, \vartheta]$ is the interval of $\mathbb{B}\,(l)$.

$\mathbb{I}^*$ is also a reflection pattern.

11.c. Suppose $\mathbb{C}$ is a reflection pattern and a projection instance in $\mathfrak{R}^{\alpha}$ of the form

$$\mathbb{C} \;=\; (\pi; \mathsf{M}^{<\gamma}_{\mathbb{B}}\text{-}\rho\text{-}\mathsf{P}_m; \mathsf{M}^{\nu}_{\mathbb{Y}}; \mathbb{Z}; \beta)$$

where $m > 0$ and $\mathbb{B}$ is a constant reflection pattern. Also assume $\mathfrak{M}^{\alpha_0}_{\mathbb{C}} \neq \emptyset$, $\bar{\kappa} = \Psi^{\alpha_0}_{\mathbb{C}}(\pi)$, $\bar{\rho} = \Psi^{\alpha_0}_{\mathbb{C}}(\rho)$, and $\beta \leq \alpha_0 \leq \alpha$.

Let $\delta := o(\mathbb{B}\,)$.

11.c.1 If $\alpha_0 > \beta$, $\alpha = \alpha_0$, and $m - 1 > 0$, then

$$\mathbb{D}_1 := (\bar{\kappa}; \mathsf{M}^{<\alpha_0}_{\mathbb{C}}\text{-}\bar{\rho}\text{-}\mathsf{P}_{m-1}; \mathsf{M}^{\alpha_0}_{\mathbb{C}};\,; \mathbb{C}\,; \alpha+1) \qquad \rightarrow 10.c$$

is a projection instance in $\mathfrak{R}^{\alpha+1}$ with interval $[\bar{\kappa}, \bar{\rho}]$. $\mathbb{D}_1$ is also a reflection pattern.

For every $\tau \in C(\tau, \bar{\kappa}) \cap [\beta, \alpha_0)$,

$$(\mathbb{D}_1, \tau)$$

is a reflection instance in $\mathfrak{R}^{\alpha+1}$ with interval $[\bar{\kappa}, \bar{\rho}]$.

11.c.2. If $\alpha_0 = \alpha = \beta$ and $m-1>0$, then

$$\mathbb{D}_2 := (\bar{\kappa}; \mathsf{M}_{\mathbb{B}}^{<\gamma}\text{-}\bar{\rho}\text{-}\mathsf{P}_{m-1}; \mathsf{M}_{\mathbb{C}}^{\alpha_0}; \mathbb{C}\,; \alpha+1) \qquad \to 10.c$$

is a projection instance in $\Re^{\alpha+1}$ with interval $[\bar{\kappa}, \bar{\rho}]$. $\mathbb{D}_2$ is also a reflection pattern.

For every $\zeta \in C(\zeta, \bar{\kappa}) \cap [\delta, \gamma)$,

$$(\mathbb{D}_2, \zeta)$$

is a reflection instance in $\Re^{\alpha+1}$ with interval $[\bar{\kappa}, \bar{\rho}]$.

From now on we assume $m-1=0$.

11.c.3. Suppose $\alpha_0 > \beta$ and $\bar{\rho}$ is a limit $> \bar{\kappa}$. Assume further that $\eta \in [\bar{\kappa}, \bar{\rho}]$ and that α is the least ordinal $\alpha' \geq \alpha_0$ such that $\eta \in C(\alpha', \bar{\kappa})$. Then

$$\mathbb{D}(\eta) \;=\; (\bar{\kappa}; \mathsf{M}_{\mathbb{C}}^{<\alpha_0}\text{-}\eta\text{-}\mathsf{P}_0; \mathsf{M}_{\mathbb{C}}^{\alpha_0}; \mathbb{C}\,; \alpha+1) \qquad \to 4.c$$

is a projection instance in $\Re^{\alpha+1}$ with interval $[\bar{\kappa}, \eta]$.

For every $\xi \in C(\xi, \bar{\kappa}) \cap [\beta, \alpha_0)$,

$$(\mathbb{D}(\eta), \xi)$$

is a reflection instance in $\Re^{\alpha+1}$ with interval $[\bar{\kappa}, \eta]$.

The function $\eta' \mapsto \mathbb{D}(\eta')$ with domain $\{\eta' \in [\bar{\kappa}, \bar{\rho}) :\; \exists \alpha' \; [\eta' \in C(\alpha', \bar{\kappa})]\}$ is a reflection pattern.

11.c.4. Suppose $\alpha_0 > \beta$, $\alpha = \alpha_0$, and $\bar{\rho}$ is a successor $\rho_1 + 1$. Then for every $0 < n < \omega$,

$$\mathbb{E}(n) \;=\; (\bar{\kappa}; \mathsf{M}_{\mathbb{C}}^{<\alpha_0}\text{-}\rho_1\text{-}\mathsf{P}_n; \mathsf{M}_{\mathbb{C}}^{\alpha_0}; \mathbb{C}\,; \alpha+1) \qquad \to 12.c$$

is a projection instance in $\Re^{\alpha+1}$ with interval $[\bar{\kappa}, \rho_1]$.

For every $\xi \in C(\xi, \bar{\kappa}) \cap [\beta, \alpha_0)$,

$$(\mathbb{E}(n), \xi)$$

is a reflection instance in $\Re^{\alpha+1}$ with interval $[\bar{\kappa}, \rho_1]$.

The function $n' \mapsto \mathbb{E}(n')$ with domain $(0, \omega)$ is a reflection pattern.

11.c.5. Suppose $\alpha_0 > \beta$, $\alpha = \alpha_0$, and $\bar{\rho} = \bar{\kappa}$. Then for every $\xi \in C(\xi, \bar{\kappa}) \cap [\beta, \alpha_0)$,

$$\mathbb{I} \;=\; (\bar{\kappa}; \mathsf{M}_{\mathbb{C}}^{\xi}\text{-}\bar{\kappa}\text{-}\mathsf{P}_0; \mathsf{M}_{\mathbb{C}}^{\alpha_0}; \mathbb{C}\,; \alpha+1)$$

is a reflection instance in $\Re^{\alpha+1}$ with interval $[\bar{\kappa}, \bar{\kappa}]$ and

$$\mathbb{F} \;=\; (\pi; \mathsf{M}_{\mathbb{C}}^{\xi}; \bar{\kappa}; \mathbb{I}; \alpha+1) \qquad \to 14.$$

is a projection instance in $\Re^{\alpha+1}$ with interval $[\pi, \rho]$.

$\mathbb{I}$ is also a reflection pattern.

11.c.6. Suppose $\alpha_0 = \beta$ and $\bar{\rho}$ is a limit $> \bar{\kappa}$. Assume further that $\eta \in [\bar{\kappa}, \bar{\rho}]$ and that α is the least ordinal $\alpha' \geq \alpha_0$ such that $\eta \in C(\alpha', \bar{\kappa})$. Then

$$\mathbb{G}(\eta) \;=\; (\bar{\kappa}; \mathsf{M}_{\mathbb{B}}^{<\alpha_0}\text{-}\eta\text{-}\mathsf{P}_0; \mathsf{M}_{\mathbb{C}}^{\alpha_0}; \mathbb{C}\,; \alpha+1) \qquad \to 4.c$$

is a projection instance in $\Re^{\alpha+1}$ with interval $[\bar\kappa, \eta]$.

For every $\zeta \in C(\zeta, \bar\kappa) \cap [\delta, \gamma)$,

$$(\mathbb{G}(\eta), \zeta)$$

is a reflection instance in $\Re^{\alpha+1}$ with interval $[\bar\kappa, \eta]$.

The function $\eta' \mapsto \mathbb{G}(\eta')$ with domain $\{\eta' \in [\bar\kappa, \bar\rho) : \exists \alpha' \ [\eta' \in C(\alpha', \bar\kappa)]\}$ is a reflection pattern.

11.c.7. Suppose $\alpha_0 = \alpha = \beta$ and $\bar\rho$ is a successor $\rho_1 + 1$. Then for every $0 < n < \omega$,

$$\mathbb{H}(n) \ = \ (\bar\kappa; \mathsf{M}^{<\gamma}_{\mathbb{B}}\text{-}\rho_1\text{-}\mathsf{P}_n; \mathsf{M}^{\alpha_0}_{\mathbb{C}}; \mathbb{C}\,; \alpha+1) \qquad \to 12.c$$

is a projection instance in $\Re^{\alpha+1}$ with interval $[\bar\kappa, \rho_1]$.

For every $\zeta \in C(\zeta, \bar\kappa) \cap [\delta, \gamma)$,

$$(\mathbb{H}(n), \zeta)$$

is a reflection instance in $\Re^{\alpha+1}$ with interval $[\bar\kappa, \rho_1]$.

The function $n' \mapsto \mathbb{H}(n')$ with domain $(0, \omega)$ is a reflection pattern.

11.c.8. Suppose $\alpha_0 = \alpha = \beta$ and $\bar\rho = \bar\kappa$. Then for every $\xi \in C(\xi, \bar\kappa) \cap [\delta, \gamma)$,

$$\mathbb{I}^* \ = \ (\bar\kappa; \mathsf{M}^{\xi}_{\mathbb{B}}\text{-}\bar\kappa\text{-}\mathsf{P}_0; \mathsf{M}^{\alpha_0}_{\mathbb{C}}; \mathbb{C}\,; \alpha+1) \qquad \to 14.$$

is a reflection instance in $\Re^{\alpha+1}$ with interval $[\bar\kappa, \bar\kappa]$ and

$$\mathbb{F}^* \ = \ (\mu; \mathsf{M}^{\xi}_{\mathbb{B}}\,; \bar\kappa; \mathbb{I}^*; \alpha+1) \qquad \to 14.$$

is a projection instance in $\Re^{\alpha+1}$ with interval $[\mu, \vartheta]$, where $[\mu, \vartheta]$ is the interval of $\mathbb{B}$.

$\mathbb{I}^*$ is also a reflection pattern.

12.a. Suppose $\mathbb{C}$ is a reflection pattern with domain $(0, \omega)$ of the form

$$\mathbb{C}\,(m) \ = \ (\pi; \mathsf{M}^{<\gamma}_{\mathbb{B}\,(\rho)}\text{-}\delta_0\text{-}\mathsf{P}_m; \mathsf{M}^{\upsilon}_{\mathbb{Y}}; \mathbb{Z}; \beta)$$

where $\mathbf{dom}(\mathbb{B}\,) \neq (0, \omega)$ and $\beta \leq \alpha$. Let $\vartheta := o(\mathbb{B}\,(\rho))$. Note that $\vartheta < \gamma$.

Then $X^{\alpha}_{\mathbb{C}\,(m)}$ consists of all triples $\langle \kappa, \sigma, \vec{\mathbf{D}} \rangle$ such that $\kappa \leq \sigma < \pi$ and the following are satisfied:

1. $C(\alpha, \kappa) \cap \pi \ = \ \kappa$ and $\mathbb{C}\,(m), \alpha \in C(\alpha, \kappa)$.
2. $\langle V_{\kappa,\sigma}; \in; \vec{\mathbf{D}} \rangle \ \overset{\Pi^1_m}{\underset{\equiv}{\longrightarrow}} \ \langle V_{\pi,\delta_0}; \in; \mathfrak{C}(\alpha, \delta_0) \rangle$, where $\vec{\mathbf{D}} \subseteq V_\sigma$ is the $<_{\mathbf{L}}$-least such set.
3. If $\zeta \in C(\zeta, \kappa) \cap [\vartheta, \gamma)$, $\rho' \in C(\zeta, \kappa) \cap \mathbf{dom}(\mathbb{B}\,)$, and $\rho' > \rho$, then κ is σ-Π^1_{m-1}-reducible in ${}^{o}\mathfrak{M}^{\zeta}_{\mathbb{B}\,(\rho')}$.
4. If $\tau \in C(\tau, \kappa) \cap [\beta, \alpha)$ and $0 < n < \omega$, then κ is σ-Π^1_{m-1}-reducible in $\mathfrak{M}^{\tau}_{\mathbb{C}\,(n)}$.

Set

$$\mathfrak{M}^{\alpha}_{\mathbb{C}\,(m)} := \{\langle\kappa,\sigma\rangle :\ \exists\vec{\mathbf{D}}\,[\langle\kappa,\sigma,\vec{\mathbf{D}}\rangle \in X^{\alpha}_{\mathbb{C}\,(m)}]\}.$$

Suppose $\langle\kappa_0,\sigma_0\rangle \in \mathfrak{M}^{\alpha}_{\mathbb{C}\,(m)}$. Then $\langle\kappa_0,\sigma_0,\vec{\mathbf{D}}_0\rangle \in X^{\alpha}_{\mathbb{C}\,(m)}$ for some $\vec{\mathbf{D}}_0$ which is uniquely determined by $\langle\kappa_0,\sigma_0\rangle$. Let j_0 be determined by

$$j_0 : \langle V_{\kappa_0,\sigma_0};\in;\vec{\mathbf{D}}_0\rangle \xrightarrow[\equiv]{\Pi^1_m} \langle V_{\pi,\delta_0};\in;\mathfrak{C}(\alpha,\delta_0)\rangle.$$

Define

$$\bar{j_0} \ :=\ j_0 \restriction [\kappa_0,\sigma_0) \ \cup\ \{\langle\sigma_0,\delta_0\rangle\}$$

and set

$$\mathsf{f}^{\alpha,\mathbb{C}}_{\kappa_0,\sigma_0} \ :=\ \bar{j_0}^{\,-1} \restriction C(\alpha,\kappa_0).$$

$[\pi,\delta_0]$ is the interval assigned to $\mathsf{f}^{\alpha,\mathbb{C}\,(m)}_{\kappa_0,\sigma_0}$.

If $\mathfrak{M}^{\alpha}_{\mathbb{C}\,(m)} \neq \emptyset$, let $\langle\pi^*,\sigma^*\rangle$ be the least pair in $\mathfrak{M}^{\alpha}_{\mathbb{C}\,(m)}$. Let $\vec{\mathbf{D}}^*$ be such that $\langle\pi^*,\sigma^*,\vec{\mathbf{D}}^*\rangle \in X^{\alpha}_{\mathbb{C}\,(m)}$. Then $(\alpha,\mathbb{C}\,(m),\mathsf{f}^{\alpha,\mathbb{C}\,(m)}_{\pi^*,\sigma^*}) \in \mathfrak{F}^{\alpha}$ and $(\alpha,\mathbb{C}\,(m),\mathfrak{D}) \in \mathfrak{S}^{\alpha}$, where $\mathfrak{D} := \langle V_{\pi^*,\sigma^*};\in;\vec{\mathbf{D}}^*\rangle$.

12.b. Suppose $\mathbb{C}$ is a reflection pattern with domain $(0,\omega)$ of the form

$$\mathbb{C}\,(m) \ =\ (\pi;\mathsf{M}^{<\gamma}_{\mathbb{B}\,(n)}\text{-}\delta_0\text{-}\mathsf{P}_m;\mathsf{M}^{\upsilon}_{\mathbb{Y}};\mathbb{Z};\beta)$$

where $\mathbf{dom}(\mathbb{B}\,) = (0,\omega)$ and $\beta \le \alpha$. Let $\vartheta := o(\mathbb{B}\,(n))$. Note that $\vartheta < \gamma$.

Then $X^{\alpha}_{\mathbb{C}\,(m)}$ consists of all triples $\langle\kappa,\sigma,\vec{\mathbf{D}}\rangle$ such that $\kappa \le \sigma < \pi$ and the following are satisfied:

1. $C(\alpha,\kappa) \cap \pi \ =\ \kappa$ and $\mathbb{C}\,(m),\alpha \in C(\alpha,\kappa)$.
2. $\langle V_{\kappa,\sigma};\in;\vec{\mathbf{D}}\rangle \xrightarrow[\equiv]{\Pi^1_m} \langle V_{\pi,\delta_0};\in;\mathfrak{C}(\alpha,\delta_0)\rangle$, where $\vec{\mathbf{D}} \subseteq V_\sigma$ is the $<_{\mathbf{L}}$-least such set.
3. If $\zeta \in C(\zeta,\kappa)\ \cap\ [\vartheta,\gamma)$ and $n < n' < \omega$, then κ is σ-Π^1_{m-1}-reducible in ${}^{o}\mathfrak{M}^{\zeta}_{\mathbb{B}\,(n')}$.
4. If $\tau \in C(\tau,\kappa)\ \cap\ [\beta,\alpha)$ and $0 < n < \omega$, then κ is σ-Π^1_{m-1}-reducible in $\mathfrak{M}^{\tau}_{\mathbb{C}\,(n)}$.

Set

$$\mathfrak{M}^{\alpha}_{\mathbb{C}\,(m)} := \{\langle\kappa,\sigma\rangle :\ \exists\vec{\mathbf{D}}\,\langle\kappa,\sigma,\vec{\mathbf{D}}\rangle \in X^{\alpha}_{\mathbb{C}\,(m)}\}.$$

Suppose $\langle\kappa_0,\sigma_0\rangle \in \mathfrak{M}^{\alpha}_{\mathbb{C}\,(m)}$. Then $\langle\kappa_0,\sigma_0,\vec{\mathbf{D}}_0\rangle \in X^{\alpha}_{\mathbb{C}\,(m)}$ for some $\vec{\mathbf{D}}_0$ which is uniquely determined by $\langle\kappa_0,\sigma_0\rangle$. Let j_0 be determined by

$$j_0 : \langle V_{\kappa_0,\sigma_0};\in;\vec{\mathbf{D}}_0\rangle \xrightarrow[\equiv]{\Pi^1_m} \langle V_{\pi,\delta_0};\in;\mathfrak{C}(\alpha,\delta_0)\rangle.$$

Define

$$\bar{j_0} \ :=\ j_0 \restriction [\kappa_0,\sigma_0) \ \cup\ \{\langle\sigma_0,\delta_0\rangle\}$$

and set

$$\mathsf{f}^{\alpha,\mathbb{C}}_{\kappa_0,\sigma_0} \ :=\ \bar{j_0}^{\,-1} \restriction C(\alpha,\kappa_0).$$

$[\pi,\delta_0]$ is the interval assigned to $\mathsf{f}^{\alpha,\mathbb{C}\,(m)}_{\kappa_0,\sigma_0}$.

If $\mathfrak{M}^{\alpha}_{\mathbb{C}\,(m)} \neq \emptyset$, let $\langle\pi^*,\sigma^*\rangle$ be the least pair in $\mathfrak{M}^{\alpha}_{\mathbb{C}\,(m)}$. Let $\vec{\mathbf{D}}^*$ be such that $\langle\pi^*,\sigma^*,\vec{\mathbf{D}}^*\rangle \in X^{\alpha}_{\mathbb{C}\,(m)}$. Then $(\alpha,\mathbb{C}\,(m),\mathsf{f}^{\alpha,\mathbb{C}\,(m)}_{\pi^*,\sigma^*}) \in \mathfrak{F}^{\alpha}$ and $(\alpha,\mathbb{C}\,(m),\mathfrak{D}) \in \mathfrak{S}^{\alpha}$, where $\mathfrak{D} := \langle V_{\pi^*,\sigma^*};\in;\vec{\mathbf{D}}^*\rangle$.

12.c Suppose $\mathbb{C}$ is a reflection pattern with domain $(0,\omega)$ of the form

$$\mathbb{C}(m) = (\pi; \mathsf{M}_{\mathbb{B}}^{<\gamma}\text{-}\delta_0\text{-}\mathsf{P}_m; \mathsf{M}_{\mathbb{Y}}^{\rho}; \mathbb{Z}; \beta)$$

where $\beta \leq \alpha$ and $\mathbb{B}$ is a constant reflection pattern. Let $\vartheta := o(\mathbb{B})$. Note that $\vartheta < \gamma$. Then $X_{\mathbb{C}(m)}^{\alpha}$ consists of all triples $\langle \kappa, \sigma, \vec{\mathbf{D}} \rangle$ such that $\kappa \leq \sigma < \pi$ and the following are satisfied:

1. $C(\alpha,\kappa) \cap \pi = \kappa$ and $\mathbb{C}(m), \alpha \in C(\alpha,\kappa)$.
2. $\langle V_{\kappa,\sigma}; \in; \vec{\mathbf{D}} \rangle \overset{\Pi^1_m}{\underset{\equiv}{\longrightarrow}} \langle V_{\pi,\delta_0}; \in; \mathfrak{C}(\alpha,\delta_0) \rangle$, where $\vec{\mathbf{D}} \subseteq V_\sigma$ is the $<_{\mathbf{L}}$-least such set.
3. If $\zeta \in C(\zeta,\kappa) \cap [\vartheta,\gamma)$, then κ is σ-Π^1_{m-1}-reducible in ${}^{o}\mathfrak{M}_{\mathbb{B}}^{\zeta}$.
4. If $\tau \in C(\tau,\kappa) \cap [\beta,\alpha)$ and $0 < n < \omega$, then κ is σ-Π^1_{m-1}-reducible in $\mathfrak{M}_{\mathbb{C}(n)}^{\tau}$.

Set

$$\mathfrak{M}_{\mathbb{C}(m)}^{\alpha} := \{ \langle \kappa, \sigma \rangle : \exists \vec{\mathbf{D}} \, [\langle \kappa, \sigma, \vec{\mathbf{D}} \rangle \in X_{\mathbb{C}(m)}^{\alpha}] \}.$$

Suppose $\langle \kappa_0, \sigma_0 \rangle \in \mathfrak{M}_{\mathbb{C}(m)}^{\alpha}$. Then $\langle \kappa_0, \sigma_0, \vec{\mathbf{D}}_0 \rangle \in X_{\mathbb{C}(m)}^{\alpha}$ for some $\vec{\mathbf{D}}_0$ which is uniquely determined by $\langle \kappa_0, \sigma_0 \rangle$. Let j_0 be determined by

$$j_0 : \langle V_{\kappa_0,\sigma_0}; \in; \vec{\mathbf{D}}_0 \rangle \overset{\Pi^1_m}{\underset{\equiv}{\longrightarrow}} \langle V_{\pi,\delta_0}; \in; \mathfrak{C}(\alpha,\delta_0) \rangle.$$

Define

$$\bar{j_0} := j_0 \restriction [\kappa_0, \sigma_0) \cup \{ \langle \sigma_0, \delta_0 \rangle \}$$

and set

$$\mathsf{f}_{\kappa_0,\sigma_0}^{\alpha,\mathbb{C}} := \bar{j_0}^{-1} \restriction C(\alpha,\kappa_0).$$

$[\pi, \delta_0]$ is the interval assigned to $\mathsf{f}_{\kappa_0,\sigma_0}^{\alpha,\mathbb{C}(m)}$.

If $\mathfrak{M}_{\mathbb{C}(m)}^{\alpha} \neq \emptyset$, let $\langle \pi^*, \sigma^* \rangle$ be the least pair in $\mathfrak{M}_{\mathbb{C}(m)}^{\alpha}$. Let $\vec{\mathbf{D}}^*$ be such that $\langle \pi^*, \sigma^*, \vec{\mathbf{D}}^* \rangle \in X_{\mathbb{C}(m)}^{\alpha}$. Then $(\alpha, \mathbb{C}(m), \mathsf{f}_{\pi^*,\sigma^*}^{\alpha,\mathbb{C}(m)}) \in \mathfrak{F}^{\alpha}$ and $(\alpha, \mathbb{C}(m), \mathfrak{D}) \in \mathfrak{G}^{\alpha}$, where $\mathfrak{D} := \langle V_{\pi^*,\sigma^*}; \in; \vec{\mathbf{D}}^* \rangle$.

13.a. Suppose $\mathbb{C}$ is a reflection pattern with domain $(0,\omega)$ of the form

$$\mathbb{C}(m) = (\pi; \mathsf{M}_{\mathbb{B}(\rho)}^{<\gamma}\text{-}\delta_0\text{-}\mathsf{P}_m; \mathsf{M}_{\mathbb{Y}}^{\upsilon}; \mathbb{Z}; \beta)$$

where $\beta \leq \alpha$ and $\mathbf{dom}(\mathbb{B}) \neq (0,\omega)$.

Also assume $\mathfrak{M}_{\mathbb{C}(m)}^{\alpha_0} \neq \emptyset$, $\bar{\kappa} = \Psi_{\mathbb{C}(m)}^{\alpha_0}(\pi)$, $\bar{\delta} = \Psi_{\mathbb{C}(m)}^{\alpha_0}(\delta_0)$, and $\beta \leq \alpha_0 \leq \alpha$.

Let $\vartheta := o(\mathbb{B}(\rho))$. Note that $\vartheta < \gamma$.

13.a.1 If $\alpha_0 > \beta$, $\alpha = \alpha_0$, and $m - 1 > 0$, then

$$\mathbb{D}_1 := (\bar{\kappa}; \mathsf{M}_{\mathbb{C}(m)}^{<\alpha_0}\text{-}\bar{\delta}\text{-}\mathsf{P}_{m-1}; \mathsf{M}_{\mathbb{C}(m)}^{\alpha_0}; \mathbb{C}(m); \alpha+1) \qquad \to 10.b$$

is a projection instance in $\mathfrak{R}^{\alpha+1}$ with interval $[\bar{\kappa}, \bar{\delta}]$.

$\mathbb{D}_1$ is also a reflection pattern with interval $[\bar{\kappa}, \bar{\kappa} + \delta_0]$. For every $\tau \in C(\tau,\bar{\kappa}) \cap [\beta, \alpha_0)$ and $n \in \mathbf{dom}(\mathbb{C})$ with $n > m$,

$$(\mathbb{D}_1, \tau, n)$$

is a reflection instance in $\mathfrak{R}^{\alpha+1}$ with interval $[\bar{\kappa}, \bar{\delta}]$.

13.a.2. If $\alpha_0 = \alpha = \beta$ and $m - 1 > 0$, then

$$\mathbb{D}_2 := (\bar{\kappa}; \mathsf{M}^{<\gamma}_{\mathbb{B}(\rho)}\text{-}\bar{\delta}\text{-}\mathsf{P}_{m-1}; \mathsf{M}^{\alpha_0}_{\mathbb{C}(m)}; \mathbb{C}(m); \alpha + 1) \qquad \rightarrow 10.a$$

is a projection instance in $\Re^{\alpha+1}$ with interval $[\bar{\kappa}, \bar{\delta}]$.

$\mathbb{D}_2$ is also a reflection pattern.

For every $\zeta \in C(\zeta, \bar{\kappa}) \cap [\vartheta, \gamma)$ and $\rho' \in \mathbf{dom}(\mathbb{B}) \cap C(\zeta, \bar{\kappa})$ with $\rho' > \rho$,

$$(\mathbb{D}_2, \zeta, \rho')$$

is a reflection instance in $\Re^{\alpha+1}$ with interval $[\bar{\kappa}, \bar{\delta}]$.

In the following cases of item 13.a we assume $m - 1 = 0$**.**

13.a.3. Suppose $\alpha_0 > \beta$ and $\bar{\delta}$ is a limit $> \bar{\kappa}$. Assume further that $\eta \in [\bar{\kappa}, \bar{\delta}]$ and that α is the least ordinal $\alpha' \geq \alpha_0$ such that $\eta \in C(\alpha', \bar{\kappa})$. Then

$$\mathbb{D}(\eta) \;=\; (\bar{\kappa}; \mathsf{M}^{<\alpha_0}_{\mathbb{C}(m)}\text{-}\eta\text{-}\mathsf{P}_0; \mathsf{M}^{\alpha_0}_{\mathbb{C}(m)}; \mathbb{C}(m); \alpha + 1) \qquad \rightarrow 4.b$$

is a projection instance in $\Re^{\alpha+1}$ with interval $[\bar{\kappa}, \eta]$.

For every $\tau \in C(\tau, \bar{\kappa}) \cap [\beta, \alpha_0)$ and every $m < n < \omega$,

$$(\mathbb{D}(\eta), \tau, n)$$

is a reflection instance in $\Re^{\alpha+1}$ with interval $[\bar{\kappa}, \eta]$.

The function $\eta' \mapsto \mathbb{D}(\eta')$ with domain $\{\eta' \in [\bar{\kappa}, \bar{\delta}) : \exists \alpha' \, [\eta' \in C(\alpha', \bar{\kappa})]\}$ is a reflection pattern.

13.a.4. Suppose $\alpha_0 > \beta$, $\alpha = \alpha_0$, and $\bar{\delta}$ is a successor $\delta_1 + 1$. Then for every $0 < n < \omega$,

$$\mathbb{E}(n) \;=\; (\bar{\kappa}; \mathsf{M}^{<\alpha_0}_{\mathbb{C}(m)}\text{-}\delta_1\text{-}\mathsf{P}_n; \mathsf{M}^{\alpha_0}_{\mathbb{C}(m)}; \mathbb{C}(m); \alpha + 1) \qquad \rightarrow 12.b$$

is a projection instance in $\Re^{\alpha+1}$ with interval $[\bar{\kappa}, \delta_1]$.

For every $\tau \in C(\tau, \bar{\kappa}) \cap [\beta, \alpha_0)$ and $m < l < \omega$,

$$(\mathbb{E}(n), \tau, l)$$

is a reflection instance in $\Re^{\alpha+1}$ with interval $[\bar{\kappa}, \delta_1]$.

The function $n' \mapsto \mathbb{E}(n')$ with domain $(0, \omega)$ is a reflection pattern.

13.a.5. Suppose $\alpha_0 > \beta$, $\alpha = \alpha_0$, and $\bar{\delta} = \bar{\kappa}$. Then for every $\tau \in C(\tau, \bar{\kappa}) \cap [\beta, \alpha_0)$ and $m < n < \omega$,

$$\mathbb{I} \;=\; (\bar{\kappa}; \mathsf{M}^{\tau}_{\mathbb{C}(n)}\text{-}\bar{\kappa}\text{-}\mathsf{P}_0; \mathsf{M}^{\alpha_0}_{\mathbb{C}(m)}; \mathbb{C}(m); \alpha + 1)$$

is a reflection instance in $\Re^{\alpha+1}$ with interval $[\bar{\kappa}, \bar{\kappa}]$ and

$$\mathbb{F} \;=\; (\pi; \mathsf{M}^{\tau}_{\mathbb{C}(n)}; \bar{\kappa}; \mathbb{I}; \alpha + 1) \qquad \rightarrow 14.$$

is a projection instance in $\Re^{\alpha+1}$ with interval $[\pi, \delta_0]$.

$\mathbb{I}$ is also a reflection pattern.

13.a.6. Suppose $\alpha_0 = \beta$ and $\bar{\delta}$ is a limit $> \bar{\kappa}$. Assume further that $\eta \in [\bar{\kappa}, \bar{\delta}]$ and that α is the least ordinal $\alpha' \geq \alpha_0$ such that $\eta \in C(\alpha', \bar{\kappa})$. Then

$$\mathbb{G}(\eta) \;=\; (\bar{\kappa}; \mathsf{M}^{<\alpha_0}_{\mathbb{B}\,(\rho)}\text{-}\eta\text{-}\mathsf{P}_0; \mathsf{M}^{\alpha_0}_{\mathbb{C}}; \mathbb{C}\,; \alpha+1) \qquad \to 4.a$$

is a projection instance in $\Re^{\alpha+1}$ with interval $[\bar{\kappa}, \eta]$.

For every $\zeta \in C(\zeta, \bar{\kappa}) \cap [\vartheta, \gamma)$ and $\rho' \in \mathbf{dom}(\mathbb{B}\,) \cap C(\zeta, \bar{\kappa})$ with $\rho' > \rho$,

$$(\mathbb{G}(\eta), \zeta, \rho')$$

is a reflection instance in $\Re^{\alpha+1}$ with interval $[\bar{\kappa}, \eta]$.

The function $\eta' \mapsto \mathbb{G}(\eta')$ with domain $\{\eta' \in [\bar{\kappa}, \bar{\rho}) :\; \exists\alpha'\; [\eta' \in C(\alpha', \bar{\kappa})]\}$ is a reflection pattern.

13.a.7. Suppose $\alpha_0 = \alpha = \beta$ and $\bar{\delta}$ is a successor $\delta_1 + 1$. Then for every $0 < n < \omega$,

$$\mathbb{H}(n) \;=\; (\bar{\kappa}; \mathsf{M}^{<\gamma}_{\mathbb{B}\,(\rho)}\text{-}\delta_1\text{-}\mathsf{P}_n; \mathsf{M}^{\alpha_0}_{\mathbb{C}\,(m)}; \mathbb{C}\,(m); \alpha+1) \qquad \to 12.a$$

is a projection instance in $\Re^{\alpha+1}$ with interval $[\bar{\kappa}, \delta_1]$.

For every $\zeta \in C(\zeta, \bar{\kappa}) \cap [\vartheta, \gamma)$ and $\rho' \in \mathbf{dom}(\mathbb{B}\,) \cap C(\zeta, \bar{\kappa})$ with $\rho' > \rho$,

$$(\mathbb{H}(n), \zeta, \rho')$$

is a reflection instance in $\Re^{\alpha+1}$ with interval $[\bar{\kappa}, \delta_1]$.

The function $n' \mapsto \mathbb{H}(n')$ with domain $(0, \omega)$ is a reflection pattern.

13.a.8. Suppose $\alpha_0 = \alpha = \beta$ and $\bar{\delta} = \bar{\kappa}$. Then for every $\xi \in C(\xi, \bar{\kappa}) \cap [\vartheta, \gamma)$ and $\rho' \in \mathbf{dom}(\mathbb{B}\,) \cap C(\xi, \bar{\kappa})$ with $\rho' > \rho$,

$$\mathbb{I}^* \;=\; (\bar{\kappa}; \mathsf{M}^{\xi}_{\mathbb{B}\,(\rho')}\text{-}\bar{\kappa}\text{-}\mathsf{P}_0; \mathsf{M}^{\alpha_0}_{\mathbb{C}\,(m)}; \mathbb{C}\,(m); \alpha+1)$$

is a reflection instance in $\Re^{\alpha+1}$ with interval $[\bar{\kappa}, \bar{\kappa}]$ and

$$\mathbb{F}^* \;=\; (\mu; \mathsf{M}^{\xi}_{\mathbb{B}\,(\rho')}; \bar{\kappa}; \mathbb{I}^*; \alpha+1) \qquad \to 14.$$

is a projection instance in $\Re^{\alpha+1}$ with interval $[\mu, \nu]$, where $[\mu, \nu]$ is the interval of $\mathbb{B}\,(\rho')$.

$\mathbb{I}^*$ is also a reflection pattern.

13.b. Suppose $\mathbb{C}$ is a reflection pattern with domain $(0, \omega)$ of the form

$$\mathbb{C}\,(m) \;=\; (\pi; \mathsf{M}^{<\gamma}_{\mathbb{B}\,(k)}\text{-}\delta_0\text{-}\mathsf{P}_m; \mathsf{M}^{\upsilon}_{\mathbb{Y}}; \mathbb{Z}; \beta)$$

where $\beta \leq \alpha$ and $\mathbf{dom}(\mathbb{B}\,) = (0, \omega)$.

Also assume $\mathfrak{M}^{\alpha_0}_{\mathbb{C}\,(m)} \neq \emptyset$, $\bar{\kappa} = \Psi^{\alpha_0}_{\mathbb{C}\,(m)}(\pi)$, $\bar{\delta} = \Psi^{\alpha_0}_{\mathbb{C}\,(m)}(\delta_0)$, and $\beta \leq \alpha_0 \leq \alpha$.

Let $\vartheta := o(\mathbb{B}\,(k))$. Note that $\vartheta < \gamma$.

13.b.1 If $\alpha_0 > \beta$, $\alpha = \alpha_0$, and $m - 1 > 0$, then

$$\mathbb{D}_1 := (\bar{\kappa}; \mathsf{M}^{<\alpha_0}_{\mathbb{C}(m)}\text{-}\bar{\delta}\text{-}\mathsf{P}_{m-1}; \mathsf{M}^{\alpha_0}_{\mathbb{C}(m)}; \mathbb{C}(m); \alpha+1) \qquad \to 10.b$$

is a projection instance in $\mathfrak{R}^{\alpha+1}$ with interval $[\bar{\kappa}, \bar{\delta}]$.

$\mathbb{D}_1$ is also a reflection pattern. For every $\tau \in C(\tau, \bar{\kappa}) \cap [\beta, \alpha_0)$ and $n \in \mathbf{dom}(\mathbb{C})$ with $n > m$,

$$(\mathbb{D}_1, \tau, n)$$

is a reflection instance in $\mathfrak{R}^{\alpha+1}$ with interval $[\bar{\kappa}, \bar{\delta}]$.

13.b.2. If $\alpha_0 = \alpha = \beta$ and $m - 1 > 0$, then

$$\mathbb{D}_2 := (\bar{\kappa}; \mathsf{M}^{<\gamma}_{\mathbb{B}(k)}\text{-}\bar{\delta}\text{-}\mathsf{P}_{m-1}; \mathsf{M}^{\alpha_0}_{\mathbb{C}(m)}; \mathbb{C}(m); \alpha+1) \qquad \to 10.b$$

is a projection instance in $\mathfrak{R}^{\alpha+1}$ with interval $[\bar{\kappa}, \bar{\delta}]$.

$\mathbb{D}_2$ is also a reflection pattern.

For every $\zeta \in C(\zeta, \bar{\kappa}) \cap [\vartheta, \gamma)$ and $k < k' < \omega$,

$$(\mathbb{D}_2, \zeta, k')$$

is a reflection instance in $\mathfrak{R}^{\alpha+1}$ with interval $[\bar{\kappa}, \bar{\delta}]$.

In the following cases of item 13.b we assume $m - 1 = 0$.

13.b.3. Suppose $\alpha_0 > \beta$ and $\bar{\delta}$ is a limit $> \bar{\kappa}$. Assume further that $\eta \in [\bar{\kappa}, \bar{\delta}]$ and that α is the least ordinal $\alpha' \geq \alpha_0$ such that $\eta \in C(\alpha', \bar{\kappa})$. Then

$$\mathbb{D}(\eta) \; = \; (\bar{\kappa}; \mathsf{M}^{<\alpha_0}_{\mathbb{C}(m)}\text{-}\eta\text{-}\mathsf{P}_0; \mathsf{M}^{\alpha_0}_{\mathbb{C}(m)}; \mathbb{C}(m); \alpha+1) \qquad \to 4.b$$

is a projection instance in $\mathfrak{R}^{\alpha+1}$ with interval $[\bar{\kappa}, \eta]$.

For every $\tau \in C(\tau, \bar{\kappa}) \cap [\beta, \alpha_0)$ and every $m < n < \omega$,

$$(\mathbb{D}(\eta), \tau, n)$$

is a reflection instance in $\mathfrak{R}^{\alpha+1}$ with interval $[\bar{\kappa}, \eta]$.

The function $\eta' \mapsto \mathbb{D}(\eta')$ with domain $\{\eta' \in [\bar{\kappa}, \bar{\delta}) : \; \exists \alpha' \; [\eta' \in C(\alpha', \bar{\kappa})]\}$ is a reflection pattern.

13.b.4. Suppose $\alpha_0 > \beta$, $\alpha = \alpha_0$, and $\bar{\delta}$ is a successor $\delta_1 + 1$. Then for every $0 < n < \omega$,

$$\mathbb{E}(n) \; = \; (\bar{\kappa}; \mathsf{M}^{<\alpha_0}_{\mathbb{C}(m)}\text{-}\delta_1\text{-}\mathsf{P}_n; \mathsf{M}^{\alpha_0}_{\mathbb{C}(m)}; \mathbb{C}(m); \alpha+1) \qquad \to 12.b$$

is a projection instance in $\mathfrak{R}^{\alpha+1}$ with interval $[\bar{\kappa}, \delta_1]$.

For every $\tau \in C(\tau, \bar{\kappa}) \cap [\beta, \alpha_0)$ and $m < l < \omega$,

$$(\mathbb{E}(n), \tau, l)$$

is a reflection instance in $\mathfrak{R}^{\alpha+1}$ with interval $[\bar{\kappa}, \delta_1]$.

The function $n' \mapsto \mathbb{E}(n')$ with domain $(0, \omega)$ is a reflection pattern.

13.b.5. Suppose $\alpha_0 > \beta$, $\alpha = \alpha_0$, and $\bar{\delta} = \bar{\kappa}$. Then for every $\tau \in C(\tau, \bar{\kappa}) \cap [\beta, \alpha_0)$ and $m < n < \omega$,

$$\mathbb{I} \;=\; (\bar{\kappa}; \mathsf{M}^{\tau}_{\mathbb{C}\,(n)}\text{-}\bar{\kappa}\text{-}\mathsf{P}_0; \mathsf{M}^{\alpha_0}_{\mathbb{C}\,(m)}; \mathbb{C}\,(m); \alpha+1)$$

is a reflection instance in $\Re^{\alpha+1}$ with interval $[\bar{\kappa}, \bar{\kappa}]$ and

$$\mathbb{F} \;=\; (\pi; \mathsf{M}^{\tau}_{\mathbb{C}\,(n)}; \bar{\kappa}; \mathbb{I}; \alpha+1) \qquad \rightarrow 14.$$

is a projection instance in $\Re^{\alpha+1}$ with interval $[\pi, \delta_0]$.

$\mathbb{I}$ is also a reflection pattern.

13.b.6. Suppose $\alpha_0 = \beta$ and $\bar{\delta}$ is a limit $> \bar{\kappa}$. Assume further that $\eta \in [\bar{\kappa}, \bar{\delta}]$ and that α is the least ordinal $\alpha' \geq \alpha_0$ such that $\eta \in C(\alpha', \bar{\kappa})$. Then

$$\mathbb{G}(\eta) \;=\; (\bar{\kappa}; \mathsf{M}^{<\alpha_0}_{\mathbb{B}\,(k)}\text{-}\eta\text{-}\mathsf{P}_0; \mathsf{M}^{\alpha_0}_{\mathbb{C}}; \mathbb{C}\,; \alpha+1) \qquad \rightarrow 4.b$$

is a projection instance in $\Re^{\alpha+1}$ with interval $[\bar{\kappa}, \eta]$.

For every $\zeta \in C(\zeta, \bar{\kappa}) \cap [\vartheta, \gamma)$ and $k < k' < \omega$,

$$(\mathbb{G}(\eta), \zeta, k')$$

is a reflection instance in $\Re^{\alpha+1}$ with interval $[\bar{\kappa}, \eta]$.

The function $\eta' \mapsto \mathbb{G}(\eta')$ with domain $\{\eta' \in [\bar{\kappa}, \bar{\rho}) : \exists \alpha' \, [\eta' \in C(\alpha', \bar{\kappa})]\}$ is a reflection pattern.

13.b.7. Suppose $\alpha_0 = \beta$ and $\bar{\delta}$ is a successor $\delta_1 + 1$. Then for every $0 < n < \omega$,

$$\mathbb{H}(n) \;=\; (\bar{\kappa}; \mathsf{M}^{<\gamma}_{\mathbb{B}\,(k)}\text{-}\delta_1\text{-}\mathsf{P}_n; \mathsf{M}^{\alpha_0}_{\mathbb{C}\,(m)}; \mathbb{C}\,(m); \alpha+1) \qquad \rightarrow 12.a$$

is a projection instance in $\Re^{\alpha+1}$ with interval $[\bar{\kappa}, \delta_1]$.

For every $\zeta \in C(\zeta, \bar{\kappa}) \cap [\vartheta, \gamma)$ and $k < k' < \omega$,

$$(\mathbb{H}(n), \zeta, k')$$

is a reflection instance in $\Re^{\alpha+1}$ with interval $[\bar{\kappa}, \delta_1]$.

The function $n' \mapsto \mathbb{H}(n')$ with domain $(0, \omega)$ is a reflection pattern.

13.b.8. Suppose $\alpha_0 = \alpha = \beta$ and $\bar{\delta} = \bar{\kappa}$. Then for every $\xi \in C(\xi, \bar{\kappa}) \cap [\vartheta, \gamma)$ and $k < k' < \omega$,

$$\mathbb{I}^* \;=\; (\bar{\kappa}; \mathsf{M}^{\xi}_{\mathbb{B}\,(k')}\text{-}\bar{\kappa}\text{-}\mathsf{P}_0; \mathsf{M}^{\alpha_0}_{\mathbb{C}\,(m)}; \mathbb{C}\,(m); \alpha+1)$$

is a reflection instance in $\Re^{\alpha+1}$ with interval $[\bar{\kappa}, \bar{\kappa}]$ and

$$\mathbb{F}^* \;=\; (\mu; \mathsf{M}^{\xi}_{\mathbb{B}\,(k')}; \bar{\kappa}; \mathbb{I}^*; \alpha+1) \qquad \rightarrow 14.$$

is a projection instance in $\Re^{\alpha+1}$ with interval $[\mu, \nu]$, where $[\mu, \nu]$ is the interval of $\mathbb{B}\,(k')$.

$\mathbb{I}^*$ is also a reflection pattern.

13.c. Suppose $\mathbb{C}$ is a reflection pattern with domain $(0,\omega)$ of the form

$$\mathbb{C}\,(m) \;=\; (\pi; \mathsf{M}_{\mathbb{B}}^{<\gamma}\text{-}\delta_0\text{-}\mathsf{P}_m; \mathsf{M}_{\mathbb{Y}}^{\rho}; \mathbb{Z}; \beta)$$

where $\beta < \alpha$ and $\mathbb{B}$ is a constant reflection pattern.

Also assume $\mathfrak{M}_{\mathbb{C}\,(m)}^{\alpha_0} \neq \emptyset$, $\bar\kappa = \Psi_{\mathbb{C}\,(m)}^{\alpha_0}(\pi)$, $\bar\delta = \Psi_{\mathbb{C}\,(m)}^{\alpha_0}(\delta_0)$, and $\beta \le \alpha_0 \le \alpha$.

Let $\vartheta := o(\mathbb{B}\,)$. Note that $\vartheta < \gamma$.

13.c.1 If $\alpha_0 > \beta$, $\alpha = \alpha_0$, and $m-1 > 0$, then

$$\mathbb{D}_1 := (\bar\kappa; \mathsf{M}_{\mathbb{C}\,(m)}^{<\alpha_0}\text{-}\bar\delta\text{-}\mathsf{P}_{m-1}; \mathsf{M}_{\mathbb{C}\,(m)}^{\alpha_0}; \mathbb{C}\,(m); \alpha+1) \qquad \to 10.b$$

is a projection instance in $\mathfrak{R}^{\alpha+1}$ with interval $[\bar\kappa, \bar\delta]$. $\mathbb{D}_1$ is also a reflection pattern.

For every $\tau \in C(\tau,\bar\kappa) \cap [\beta, \alpha_0)$ and $n \in \mathbf{dom}(\mathbb{C}\,)$ with $n > m$,

$$(\mathbb{D}_1, \tau, n)$$

is a reflection instance in $\mathfrak{R}^{\alpha+1}$ with interval $[\bar\kappa, \bar\delta]$.

13.c.2. If $\alpha_0 = \alpha = \beta$ and $m-1 > 0$, then

$$\mathbb{D}_2 := (\bar\kappa; \mathsf{M}_{\mathbb{B}}^{<\gamma}\text{-}\bar\delta\text{-}\mathsf{P}_{m-1}; \mathsf{M}_{\mathbb{C}\,(m)}^{\alpha_0}; \mathbb{C}\,(m); \alpha+1) \qquad \to 10.c$$

is a projection instance in $\mathfrak{R}^{\alpha+1}$ with interval $[\bar\kappa, \bar\delta]$.

$\mathbb{D}_2$ is also a reflection pattern.

For every $\zeta \in C(\zeta, \bar\kappa) \,\cap\, [\vartheta, \gamma)$,

$$(\mathbb{D}_2, \zeta)$$

is a reflection instance in $\mathfrak{R}^{\alpha+1}$ with interval $[\bar\kappa, \bar\delta]$.

In the following cases of item 13.c we assume $m-1=0$.

13.c.3. Suppose $\alpha_0 > \beta$ and $\bar\delta$ is a limit $> \bar\kappa$. Assume further that $\eta \in [\bar\kappa, \bar\delta]$ and that α is the least ordinal $\alpha' \ge \alpha_0$ such that $\eta \in C(\alpha', \bar\kappa)$. Then

$$\mathbb{D}\,(\eta) \;=\; (\bar\kappa; \mathsf{M}_{\mathbb{C}\,(m)}^{<\alpha_0}\text{-}\eta\text{-}\mathsf{P}_0; \mathsf{M}_{\mathbb{C}\,(m)}^{\alpha_0}; \mathbb{C}\,(m); \alpha+1) \qquad \to 4.b$$

is a projection instance in $\mathfrak{R}^{\alpha+1}$ with interval $[\bar\kappa, \eta]$.

For every $\tau \in C(\tau, \bar\kappa) \,\cap\, [\beta, \alpha_0)$ and every $m < n < \omega$,

$$(\mathbb{D}\,(\eta), \tau, n)$$

is a reflection instance in $\mathfrak{R}^{\alpha+1}$ with interval $[\bar\kappa, \eta]$.

The function $\eta' \mapsto \mathbb{D}\,(\eta')$ with domain $\{\eta' \in [\bar\kappa, \bar\delta) :\ \exists\alpha'\ [\eta' \in C(\alpha', \bar\kappa)]\}$ is a reflection pattern.

13.c.4. Suppose $\alpha_0 > \beta$, $\alpha = \alpha_0$, and $\bar{\delta}$ is a successor $\delta_1 + 1$. Then for every $0 < n < \omega$,

$$\mathbb{E}\,(n) \;=\; (\bar{\kappa}; \mathsf{M}^{<\alpha_0}_{\mathbb{C}\,(m)}\text{-}\delta_1\text{-}\mathsf{P}_n; \mathsf{M}^{\alpha_0}_{\mathbb{C}\,(m)}; \mathbb{C}\,(m); \alpha+1) \qquad \to 12.b$$

is a projection instance in $\mathfrak{R}^{\alpha+1}$ with interval $[\bar{\kappa}, \delta_1]$.

For every $\tau \in C(\tau, \bar{\kappa}) \cap [\beta, \alpha_0)$ and $n < l < \omega$,

$$(\mathbb{E}\,(n), \tau, l)$$

is a reflection instance in $\mathfrak{R}^{\alpha+1}$ with interval $[\bar{\kappa}, \delta_1]$.

The function $n' \mapsto \mathbb{E}\,(n')$ with domain $(0, \omega)$ is a reflection pattern.

13.c.5. Suppose $\alpha_0 > \beta$, $\alpha = \alpha_0$, and $\bar{\delta} = \bar{\kappa}$. Then for every $\tau \in C(\tau, \bar{\kappa}) \cap [\beta, \alpha_0)$ and $0 < n < \omega$,

$$\mathbb{I} \;=\; (\bar{\kappa}; \mathsf{M}^{\tau}_{\mathbb{C}\,(n)}\text{-}\bar{\kappa}\text{-}\mathsf{P}_0; \mathsf{M}^{\alpha_0}_{\mathbb{C}\,(m)}; \mathbb{C}\,(m); \alpha+1)$$

is a reflection instance in $\mathfrak{R}^{\alpha+1}$ with interval $[\bar{\kappa}, \bar{\kappa}]$ and

$$\mathbb{F} \;=\; (\pi; \mathsf{M}^{\tau}_{\mathbb{C}\,(n)}; \bar{\kappa}; \mathbb{I}; \alpha+1) \qquad \to 14.$$

is a projection instance in $\mathfrak{R}^{\alpha+1}$ with interval $[\pi, \delta_0]$.

$\mathbb{I}$ is also a reflection pattern.

13.c.6. Suppose $\alpha_0 = \beta$ and $\bar{\delta}$ is a limit $> \bar{\kappa}$. Assume further that $\eta \in [\bar{\kappa}, \bar{\delta}]$ and that α is the least ordinal $\alpha' \geq \alpha_0$ such that $\eta \in C(\alpha', \bar{\kappa})$. Then

$$\mathbb{G}(\eta) \;=\; (\bar{\kappa}; \mathsf{M}^{<\alpha_0}_{\mathbb{B}}\text{-}\eta\text{-}\mathsf{P}_0; \mathsf{M}^{\alpha_0}_{\mathbb{C}}; \mathbb{C}\,; \alpha+1) \qquad \to 4.c$$

is a projection instance in $\mathfrak{R}^{\alpha+1}$ with interval $[\bar{\kappa}, \eta]$.

For every $\zeta \in C(\zeta, \bar{\kappa}) \cap [\vartheta, \gamma)$,

$$(\mathbb{G}(\eta), \zeta)$$

is a reflection instance in $\mathfrak{R}^{\alpha+1}$ with interval $[\bar{\kappa}, \eta]$.

The function $\eta' \mapsto \mathbb{G}(\eta')$ with domain $\{\eta' \in [\bar{\kappa}, \bar{\rho}) : \; \exists \alpha' \; [\eta' \in C(\alpha', \bar{\kappa})]\}$ is a reflection pattern.

13.c.7. Suppose $\alpha_0 = \alpha = \beta$ and $\bar{\delta}$ is a successor $\delta_1 + 1$. Then for every $0 < n < \omega$,

$$\mathbb{H}(n) \;=\; (\bar{\kappa}; \mathsf{M}^{<\gamma}_{\mathbb{B}}\text{-}\delta_1\text{-}\mathsf{P}_n; \mathsf{M}^{\alpha_0}_{\mathbb{C}\,(m)}; \mathbb{C}\,(m); \alpha+1) \qquad \to 12.c$$

is a projection instance in $\mathfrak{R}^{\alpha+1}$ with interval $[\bar{\kappa}, \delta_1]$.

For every $\zeta \in C(\zeta, \bar{\kappa}) \cap [\vartheta, \gamma)$,

$$(\mathbb{H}(n), \zeta)$$

is a reflection instance in $\mathfrak{R}^{\alpha+1}$ with interval $[\bar{\kappa}, \delta_1]$.

The function $n' \mapsto \mathbb{H}(n')$ with domain $(0, \omega)$ is a reflection pattern.

13.c.8. Suppose $\alpha_0 = \alpha = \beta$ and $\bar{\delta} = \bar{\kappa}$. Then for every $\xi \in C(\xi, \bar{\kappa}) \cap [\vartheta, \gamma)$,

$$\mathbb{I}^* \;=\; (\bar{\kappa}; \mathsf{M}^{\xi}_{\mathbb{B}}\text{-}\bar{\kappa}\text{-}\mathsf{P}_0; \mathsf{M}^{\alpha_0}_{\mathbb{C}\,(m)}; \mathbb{C}\,(m); \alpha+1)$$

is a reflection instance in $\mathfrak{R}^{\alpha+1}$ with interval $[\bar{\kappa}, \bar{\kappa}]$ and

$$\mathbb{F}^* \;=\; (\mu; \mathsf{M}^{\xi}_{\mathbb{B}}; \bar{\kappa}; \mathbb{I}^*; \alpha+1) \qquad \to 14.$$

is a projection instance in $\mathfrak{R}^{\alpha+1}$ with interval $[\mu, \upsilon]$, where $[\mu, \upsilon]$ is the interval of $\mathbb{B}$.

$\mathbb{I}^*$ is also a reflection pattern.

14. Suppose $\mathbb{C}$ is a projection instance of the form

$$\mathbb{C} \;=\; (\pi; \mathsf{M}^{\xi}_{\mathbb{X}}; \bar{\kappa}; \mathbb{I}; \beta)$$

with $\beta \le \alpha$, where $\mathbb{I}$ is a reflection instance of the form

$$\mathbb{I} \;=\; (\bar{\kappa}; \mathsf{M}^{\xi}_{\mathbb{X}}\text{-}\bar{\kappa}\text{-}\mathsf{P}_0; \mathsf{M}^{\zeta}_{\mathbb{Y}}; \mathbb{Z}; \beta).$$

Let $[\pi, \rho]$ be the interval of $\mathbb{X}$. Then $\mathfrak{M}^{\alpha}_{\mathbb{C}}$ consists of all pairs $\langle \pi_0, \rho_0 \rangle$ such that $\pi_0 \le \rho_0 < \bar{\kappa}$ satisfying

1. $C(\alpha, \pi_0) \cap \bar{\kappa} = \pi_0$.
2. $\alpha, \mathbb{C} \in C(\alpha, \kappa)$.
3. $\langle \pi_0, \rho_0 \rangle \in \mathfrak{M}^{\xi}_{\mathbb{X}}$.

Set $X^{\alpha}_{\mathbb{C}} \;:=\; \{\langle \pi_0, \rho_0, \emptyset \rangle :\; \langle \pi_0, \rho_0 \rangle \in \mathfrak{M}^{\alpha}_{\mathbb{C}}\}$.

If $\langle \pi', \rho' \rangle \in \mathfrak{M}^{\alpha}_{\mathbb{C}}$ then $\langle \pi', \rho' \rangle \in \mathfrak{M}^{\xi}_{\mathbb{X}}$ and we define

$$\mathsf{f}^{\alpha,\mathbb{C}}_{\pi',\rho'} \;:=\; \mathsf{f}^{\xi,\mathbb{X}}_{\pi',\rho'}.$$

$\mathsf{f}^{\alpha,\mathbb{C}}_{\pi',\rho'}$ has the same interval as $\mathsf{f}^{\xi,\mathbb{X}}_{\pi',\rho'}$.

If $\mathfrak{M}^{\alpha}_{\mathbb{C}} \neq \emptyset$, let $\langle \pi_0, \rho_0 \rangle$ be the least pair in $\mathfrak{M}^{\alpha}_{\mathbb{C}}$. Let $\vec{\mathbf{D}}_0$ be such that $\langle \pi_0, \rho_0, \vec{\mathbf{D}}_0 \rangle \in X^{\xi}_{\mathbb{X}}$. Then $(\alpha, \mathbb{C}, \mathsf{f}^{\alpha,\mathbb{C}}_{\pi_0,\rho_0}) \in \mathfrak{F}^{\alpha}$ and $(\alpha, \mathbb{C}, \mathfrak{D}) \in \mathfrak{S}^{\alpha}$, where $\mathfrak{D} := \langle V_{\pi_0,\rho_0}; \in; \vec{\mathbf{D}}_0 \rangle$.

15. Suppose $\mathbb{C}$ is a projection instance in $\mathfrak{R}^{\alpha}$ of the form

$$\mathbb{C} \;=\; (\pi; \mathsf{M}^{\xi}_{\mathbb{X}}; \bar{\kappa}; \mathbb{I}; \beta),$$

where $\mathbb{I}$ is a reflection instance of the form

$$\mathbb{I} \;=\; (\bar{\kappa}; \mathsf{M}^{\xi}_{\mathbb{X}}\text{-}\bar{\kappa}\text{-}\mathsf{P}_0; \mathsf{M}^{\zeta}_{\mathbb{Y}}; \mathbb{Z}; \beta).$$

Let $[\pi, \rho]$ be the interval of $\mathbb{X}$.

Also assume $\beta \le \alpha_0 \le \alpha$, $\mathfrak{M}^{\alpha_0}_{\mathbb{C}} \neq \emptyset$, $\bar{\pi} = \Psi^{\alpha_0}_{\mathbb{C}}(\pi)$, and $\bar{\rho} = \Psi^{\alpha_0}_{\mathbb{C}}(\rho)$.

Note that $\langle \bar{\pi}, \bar{\rho} \rangle \in \mathfrak{M}^{\xi}_{\mathbb{X}}$. The new projection instances, reflection instances, and reflection patterns pertaining to $\langle \bar{\pi}, \bar{\rho} \rangle, \mathbb{C}$ are defined according to the shape of $\mathbb{X}$ analogously to the cases 1.-14. In order to clarify this definition we shall look at one particular example. Suppose that $\mathbb{X}$ is of the form

$$\mathbb{X} = \mathbb{A}\,(\rho) = (\pi; \rho\text{-}\mathsf{P}_0; \lambda\text{-}\mathsf{P}_0; \mathbb{U}; \gamma),$$

where λ is a limit.

Analogously to item 3, we then define the following projection instances and reflection instances.

15.1. First suppose that $\bar\rho$ is a limit $> \bar\pi$. Let $\delta \in [\bar\pi, \bar\rho)$ and assume that α is the least ordinal $\alpha' \geq \alpha_0$ such that $\delta \in C(\alpha', \bar\pi)$.

If $\xi > \gamma$, then

$$\mathbb{D}\,(\delta) \;:=\; (\bar\pi; \mathsf{M}^{<\xi}_{\mathbb{A}\,(\rho)}\text{-}\delta\text{-}\mathsf{P}_0; \mathsf{M}^{\alpha_0}_{\mathbb{C}}; \mathbb{C}\,; \alpha+1) \qquad \to 4.a$$

is a projection instance in $\Re^{\alpha+1}$ with interval $[\bar\pi, \delta]$.

The function $\delta' \mapsto \mathbb{D}\,(\delta')$ with domain

$$\{\delta' \in [\bar\pi, \bar\rho) :\; \exists \xi' \; [\delta' \in C(\xi', \bar\pi)]\}$$

is a reflection pattern.

For every $\zeta \in C(\zeta, \bar\pi) \,\cap\, [\gamma, \xi)$ and every $\eta \in C(\zeta, \bar\pi) \,\cap\, (\rho, \lambda)$,

$$(\mathbb{D}\,(\delta), \zeta, \eta)$$

is a reflection instance in $\Re^{\alpha+1}$ with interval $[\bar\pi, \delta]$.

If $\xi = \gamma$, then

$$\widetilde{\mathbb{D}}\,(\delta) \;:=\; (\bar\pi; \delta\text{-}\mathsf{P}_0; \bar\rho\text{-}\mathsf{P}_0; \mathbb{C}\,; \alpha+1) \qquad \to 2.$$

is a projection instance and reflection instance in $\Re^{\alpha+1}$ with interval $[\bar\pi, \delta]$.

The function $\delta' \mapsto \widetilde{\mathbb{D}}\,(\delta')$ with domain $\{\delta' \in [\bar\pi, \bar\rho] :\; \exists \xi' \; [\delta' \in C(\xi', \bar\pi)]\}$ is a reflection pattern.

15.2. Now assume that $\bar\rho$ is a successor $\rho_0 + 1$ and $\alpha = \alpha_0$.

If $\xi > \gamma$, then for every $0 < m < \omega$,

$$\widehat{\mathbb{D}}\,(m) \;:=\; (\bar\pi; \mathsf{M}^{<\xi}_{\mathbb{A}\,(\rho)}\text{-}\rho_0\text{-}\mathsf{P}_m; \mathsf{M}^{\alpha_0}_{\mathbb{C}}; \mathbb{C}\,; \alpha+1) \qquad \to 12.a$$

is a projection instance in $\Re^{\alpha+1}$ with interval $[\bar\pi, \rho_0]$. The function $n \mapsto \widehat{\mathbb{D}}\,(n)$ with domain $(0, \omega)$ is a reflection pattern.

For every $\zeta \in C(\zeta, \bar\pi) \,\cap\, [\gamma, \xi)$ and every $\eta \in C(\zeta, \bar\pi) \,\cap\, (\rho, \lambda)$,

$$(\widehat{\mathbb{D}}\,(m), \zeta, \eta)$$

is a reflection instance in $\Re^{\alpha+1}$ with interval $[\bar\pi, \rho_0]$.

If $\xi = \gamma$, then for every $0 < m < \omega$,

$$\mathbb{D}^*(m) \;:=\; (\bar\pi; \rho_0\text{-}\mathsf{P}_m; \rho_0\text{-}\mathsf{P}_\infty; \mathbb{C}\,; \alpha+1) \qquad \to 8.$$

is a reflection pattern and reflection instance in $\Re^{\alpha+1}$ with interval $[\bar\pi, \rho_0]$.

The function $n \mapsto \mathbb{D}^*(n)$ with domain $(0, \omega)$ is a reflection pattern.

15.3. Finally assume $\bar\rho = \bar\pi$ and $\alpha = \alpha_0$.

If $\xi > \gamma$, then for every $\zeta \in C(\zeta, \bar\pi) \,\cap\, [\gamma, \xi)$ and $\eta \in C(\zeta, \bar\pi) \,\cap\, (\rho, \lambda)$,

$$\widehat{\mathbb{I}} \;:=\; (\bar\pi; \mathsf{M}^{\zeta}_{\mathbb{A}\,(\eta)}\text{-}\bar\pi\text{-}\mathsf{P}_0; \mathsf{M}^{\alpha_0}_{\mathbb{C}}; \mathbb{C}\,; \alpha+1)$$

is a reflection instance in $\Re^{\alpha+1}$ with interval $[\bar{\pi}, \bar{\pi}]$, and

$$\widehat{\mathbb{F}} := (\pi; \mathsf{M}^{\zeta}_{\mathbb{A}(\eta)}; \bar{\pi}; \widehat{\mathbb{I}}; \alpha) \qquad \to 14.$$

is a projection instance in $\Re^{\alpha+1}$ with interval $[\pi, \eta]$. $\widehat{\mathbb{I}}$ is also a reflection pattern.

If $\xi = \gamma$, then

$$\mathbb{G} := (\bar{\pi}; \mathsf{RLC}; \mathbb{C}; \alpha+1) \qquad \to 1.6$$

is a projection instance and reflection instance in $\Re^{\alpha+1}$ with interval $[\bar{\pi}, \bar{\pi}]$. $\mathbb{G}$ is also a reflection pattern.

Definition 3.3 Here we define the notion of a reflection instance being *associated with* or *pertaining to* a projection instance. Let $\mathbb{K}$ be a reflection instance. Then a projection instance $\mathbb{X}$ is is associated with $\mathbb{K}$ or the projection instance pertaining to $\mathbb{K}$ iff one of the following holds:

1. $\mathbb{K}$ is also a projection instance and $\mathbb{K} = \mathbb{X}$.
2. $\mathbb{K}$ has one of the forms $(\mathbb{X}, \xi)$ or $(\mathbb{X}, \xi, \eta)$.
3. $\mathbb{K}$ is of the form $(\bar{\kappa}; \mathsf{M}^{\xi}_{\mathbb{Y}}\text{-}\bar{\kappa}\text{-}\mathsf{P}_0; \dots)$ and $\mathbb{X}$ is of the form $(\pi; \mathsf{M}^{\xi}_{\mathbb{X}}; \bar{\kappa}; \mathbb{K}; \dots)$.

Note that for every reflection instance there is exactly one projection instance associated with it.

4 Structure Theory

Lemma 4.1 *(i)* $\alpha \le \alpha' \wedge \eta \le \eta' \implies C(\alpha, \eta) \subseteq C(\alpha', \eta')$.

(ii) If π is a regular cardinal $> \omega$ and $\beta < \pi$, then $|C(\alpha, \beta)| < \pi$, where $|C(\alpha, \beta)|$ denotes the cardinality of $C(\alpha, \beta)$.

(iii) $\lambda \in Lim \implies C(\alpha, \lambda) = \bigcup_{\eta<\lambda} C(\alpha, \eta) \wedge C(\lambda, \alpha) = \bigcup_{\eta<\lambda} C(\eta, \alpha)$.

Lemma 4.2 *Let $\pi > \omega$ be a regular cardinal. Set*

$$\mathcal{C}^{\alpha}_{\pi} := \{\upsilon < \pi : C(\alpha, \upsilon) \cap \pi = \upsilon\}.$$

Then $\mathcal{C}^{\alpha}_{\pi}$ is closed and unbounded in π. Moreover, if $\alpha \in C(\alpha, \pi)$, then

$$\tilde{\mathcal{C}}^{\alpha}_{\pi} := \{\upsilon < \pi : C(\alpha, \upsilon) \cap \pi = \upsilon \wedge \alpha \in C(\alpha, \upsilon)\}$$

is closed and unbounded in π.

Proof: Closedness follows from Lemma 4.1,(iii). For unboundedness, let $\beta < \pi$. Define $\beta_0 := \sup(C(\alpha, \beta+1)) \cap \pi$, $\beta_{n+1} := \sup(C(\alpha, \beta_n)) \cap \pi$ and $\beta_\infty := \sup_n \beta_n$. Due to $\pi > \omega$ and regularity of π, we get $|C(\alpha, \beta)| < \pi$. Thus $\beta_0 < \pi$, again by the regularity of π. Inductively one verifies $\beta_n < \pi$ for all $n < \omega$, and hence $\beta_\infty < \pi$. From Lemma 4.1,(iii) it follows $C(\alpha, \beta_\infty) = \bigcup_n C(\alpha, \beta_n)$. Hence we get

$$C(\alpha, \beta_\infty) \cap \pi = \bigcup_n (C(\alpha, \beta_n) \cap \pi) = \sup_n \beta_{n+1} = \beta_\infty.$$

If $\alpha \in C(\alpha, \pi)$, then $\alpha \in C(\alpha, \rho)$ for some $\rho < \pi$ by Lemma 4.1,(iii). Therefore $\tilde{\mathcal{C}}^{\alpha}_{\pi}$ is closed and unbounded in π as well. □

Corollary 4.3 *If $\mathbb{X} \in \Re^\alpha$ is of either form $(\pi; \mathsf{RSC}; \dots)$ or $(\pi; \mathsf{RLC}; \dots)$, and $\alpha \in C(\alpha, \pi)$, then $\pi \in \mathbf{dom}(\Psi^\alpha_\mathbb{X})$ and $\Psi^\alpha_\mathbb{X}(\pi) < \pi$.*

Proof: This follows from Lemma 4.2. □

Lemma 4.4 *Let $\mathbb{X} \in \Re^\alpha$ with interval $[\pi, \delta]$ and $\Psi^\alpha_\mathbb{X}(\pi) < \pi$. Define $f : \mathbf{dom}(\Psi^\alpha_\mathbb{X}) \cup \pi \longrightarrow \pi$ by $f(\nu) = \nu$ if $\nu < \pi$ and $f(\nu) = \Psi^\alpha_\mathbb{X}(\nu)$ if $\nu \in \mathbf{dom}(\Psi^\alpha_\mathbb{X})$.*

1. *If $\nu \in \mathbf{dom}(f)$ and $\nu = \eta + \gamma$, then $f(\nu) = f(\eta) + f(\gamma)$.*
2. *If $\nu \in \mathbf{dom}(f)$ and $\nu = \varphi\eta\gamma$, then $f(\nu) = \varphi f(\eta) f(\gamma)$.*

Proof: The function $\Psi^\alpha_\mathbb{X}$ is the inverse of a partial elementary isomorphism and therefore commutes with $+$ and φ. □

Lemma 4.5 *Let $\mathbb{X} \in \Re^\alpha$ with interval $[\pi, \delta]$ and $\Psi^\alpha_\mathbb{X}(\pi) < \pi$.*

(i) If $\mathbb{X}$ is not of the form $(\pi; \mathsf{RSC}, \dots)$, then $\Psi^\alpha_\mathbb{X}(\pi)$ is a limit cardinal.

(ii) Let $\mathbb{X} = (\pi; \mathsf{RSC}; \dots)$ with $\pi = \mu^+$, where μ is a cardinal. Then $\mu < \Psi^\alpha_\mathbb{X}(\pi) < \pi$, and for all $\xi, \zeta < \Psi^\alpha_\mathbb{X}$ it holds $\varphi\xi\zeta < \Psi^\alpha_\mathbb{X}(\pi)$.

Proof: (i): In this case π is limit cardinal. Hence $\mu < \Psi^\alpha_\mathbb{X}(\pi)$ implies $\mu^+ \in C(\alpha, \Psi^\alpha_\mathbb{X}(\pi)) \cap \pi$, yielding $\mu^+ < \Psi^\alpha_\mathbb{X}(\pi)$.

(ii): We proceed by induction on α. Note that $\pi \in C(\alpha, \Psi^\alpha_\mathbb{X}(\pi))$. By (i) and by induction hypothesis, π is not of the form $\Psi^\beta_\mathbb{B}(\rho)$ with $\beta, \rho, \mathbb{B} \in C(\alpha, \Psi^\alpha_\mathbb{X}(\pi))$ and $\beta < \alpha$. Therefore the only possibility for π to enter the set $C(\alpha, \Psi^\alpha_\mathbb{X}(\pi))$ is by way of $\mu \in C(\alpha, \Psi^\alpha_\mathbb{X})$. Hence $\mu < \Psi^\alpha_\mathbb{X}(\pi)$.

If $\xi, \zeta < \Psi^\alpha_\mathbb{X}(\pi)$ then $\varphi\xi\zeta \in C(\alpha, \Psi^\alpha_\mathbb{X}(\pi)) \cap \pi$, thus $\varphi\xi\zeta < \Psi^\alpha_\mathbb{X}(\pi)$. □

Lemma 4.6 *Let $\mu < \mathbf{I}$ be a cardinal and $\mathbb{X} \in \Re^\alpha$ with interval $[\pi, \delta]$ and $\Psi^\alpha_\mathbb{X}(\pi) < \pi$. Also assume that $\mathbb{X}$ is not of the form $(\pi; \mathsf{RSC}, \dots)$.*

(i) $\mu^+ < \Psi^\alpha_\mathbb{X}(\pi)$ iff $\mu < \Psi^\alpha_\mathbb{X}(\pi)$.

(ii) $\Psi^\alpha_\mathbb{X}(\pi) < \mu^+$ iff $\Psi^\alpha_\mathbb{X}(\pi) \leq \mu$.

(iii) $\Psi^\alpha_\mathbb{X}(\pi) \neq \mu^+$.

Proof: This follows from Lemma 4.5,(i). □

Lemma 4.7 *Let $\mu < \mathbf{I}$ be a cardinal and $\mathbb{X} = (\kappa; \mathsf{RSC}; \dots) \in \Re^\alpha$. Also assume $\alpha \in C(\alpha, \kappa)$.*

(i) $\mu^+ < \Psi^\alpha_\mathbb{X}(\kappa)$ iff $\mu^+ < \kappa$.

(ii) $\Psi^\alpha_\mathbb{X}(\kappa) < \mu^+$ iff $\kappa \leq \mu^+$.

(iii) $\Psi^\alpha_\mathbb{X}(\kappa) \neq \mu^+$.

Proof: This follows from Lemma 4.5,(ii). □

Lemma 4.8 *If $\mathbb{X}$ is a projection instance in $\mathfrak{R}^\alpha$ with interval $[\pi,\delta]$, then $\mathbb{X} \in C(\alpha,\pi)$.*

Proof: Use induction on α. □

Lemma 4.9 *If $\eta \in C(\alpha,\rho) \cap \mathbf{I}$, then $\eta^+ \in C(\alpha,\rho)$.*

Proof: Let $\eta \in C_n(\alpha,\rho) \cap \mathbf{I}$. The assertion is proved by induction on n. Only the case when η is not a limit cardinal needs consideration. If $\eta =_{NF} \omega^\xi + \delta$ or $\eta =_{NF} \varphi\xi\delta$ with $\xi, \delta \in C(\alpha,\rho)$ then $\eta^+ = \max(\xi^+, \delta^+)$, and therefore the assertion follows from the inductive assumption. If $\eta = \Psi^\beta_{\mathbb{X}}(\pi)$ with $\beta, \mathbb{X}, \pi \in C_n(\alpha,\rho)$ and $\beta < \alpha$, then $\eta^+ = \pi$ according to Lemma 4.5, and hence $\eta^+ \in C(\alpha,\rho)$. The remaining cases are trivial. □

Theorem 4.10 (Existence) *Let $\mathbb{X} \in \mathfrak{R}^\alpha$ be a projection instance with interval $[\pi,\rho]$. If $\alpha \in C(\alpha,\pi)$, then $\mathfrak{M}^\alpha_{\mathbb{X}} \neq \emptyset$.*

Proof: We distinguish cases according to the shape of $\mathbb{X}$.

Case 1: Let $\mathbb{X} = (\pi; \triangle; \ldots; \gamma)$, where $\triangle$ is RSC or RLC. Then π is a regular cardinal $> \omega$ as can be easily verified by induction on α. By Lemma 4.1,(iii), there exists $\delta < \pi$ such that $\alpha, \mathbb{X} \in C(\alpha,\delta)$. Thus, by Lemma 4.2, there exists $\upsilon < \pi$ such that $\alpha, \mathbb{X} \in C(\alpha,\upsilon)$ and $C(\alpha,\upsilon) \cap \pi = \upsilon$. In particular $\mathfrak{M}^\alpha_{\mathbb{X}} \neq \emptyset$.

Case 2: Let $\mathbb{X} = \mathbb{A}(\bar\rho) = (\pi; \bar\rho\text{-}\mathsf{P}_0; \lambda\text{-}\mathsf{P}_0; \mathbb{Y}; \gamma)$. Then λ is a limit, $\pi \le \bar\rho < \lambda$ and π is η-Π^1_0-reducible for all $\pi \le \eta < \lambda$.

Suppose $\mathbb{X} \in \mathfrak{R}^\alpha$ and $\alpha \in C(\alpha,\pi)$. Then $\alpha \ge \gamma$. By induction on α, we shall show that $F(\alpha)$ holds, where $F(\alpha)$ is the assertion

$$\forall \xi \in C(\xi,\pi) \cap [\gamma,\alpha)\, \forall \rho' \in C(\xi,\pi) \cap [\pi,\lambda)\, \forall \vartheta' \in C(\xi,\pi) \cap (\rho',\lambda) \quad \pi \text{ is } \rho'\text{-}\Pi^1_0\text{-reducible in } {}^o\mathfrak{M}^\xi_{\mathbb{A}(\vartheta')}. \tag{21}$$

(21) is trivial when $\alpha = \gamma$. If α is a limit $> \gamma$ and $\xi \in C(\xi,\pi) \cap [\gamma,\alpha)$, then $\xi+1 \in C(\xi+1,\pi) \cap [\gamma,\alpha)$, and thus the assertion is immediate by applying the induction hypothesis to $\xi+1$.

Now assume $\alpha = \alpha_0 + 1 > \gamma$. Then $\alpha_0 \in C(\alpha_0,\pi)$ and by induction hypothesis we have $F(\alpha_0)$. Thus in order to show $F(\alpha)$ it suffices to verify that for every $\rho \in C(\alpha_0,\pi) \cap [\pi,\lambda)$ and $\vartheta \in C(\alpha_0,\pi) \cap (\rho,\lambda)$,

$$\pi \text{ is } \rho\text{-}\Pi^1_0\text{-reducible in } {}^o\mathfrak{M}^{\alpha_0}_{\mathbb{A}(\vartheta)}. \tag{22}$$

Now let $\rho \in C(\xi,\pi) \cap [\pi,\lambda)$ and $\vartheta \in C(\alpha_0,\pi) \cap (\rho,\lambda)$. From $F(\alpha_0)$ we get

$$\forall \xi \in C(\xi,\pi) \cap [\gamma,\alpha_0)\, \forall \vartheta' \in C(\xi,\pi) \cap (\vartheta,\lambda) \quad \big(\vartheta \in C(\xi,\pi) \;\Rightarrow\; \pi \text{ is } \vartheta\text{-}\Pi^1_0\text{-reducible in } {}^o\mathfrak{M}^\xi_{\mathbb{A}(\vartheta')}\big). \tag{23}$$

Let $G(\alpha_0,\pi,\vartheta)$ be the statement of (23). We would like to express $G(\alpha_0,\pi,\vartheta)$ via a Π^1_0 statement within a structure of the form $\langle V_{\pi,\vartheta+1}; \in, \mathfrak{V}; \vec{Y}\rangle$, where $\vec{Y}$ is a sequence of predicates $\vec{Y} \subseteq V_\vartheta$. Note that all the ordinals involved in the statement $G(\alpha_0,\pi,\vartheta)$ belong to $C(\alpha_0,\pi)$ and that the sets $\mathfrak{M}^\xi_{\mathbb{A}(\vartheta')}$ addressed in $G(\alpha_0,\pi,\vartheta)$ are subsets of V_π. Let $\ulcorner\alpha_0\urcorner, \ulcorner\pi\urcorner, \ulcorner\vartheta\urcorner$ be codes in V_π for α_0, π, ϑ, respectively, as members of $C(\alpha_0, \pi)$. We can then express $G(\alpha_0, \pi, \vartheta)$ via a Π^1_0 formula in the structure $\langle V_{\vartheta+1}; \in; \mathfrak{V}; \ulcorner\alpha_0\urcorner; \ulcorner\pi\urcorner; \ulcorner\vartheta\urcorner; \mathfrak{C}(\alpha_0, \vartheta)\rangle$, say by

$$\langle V_{\pi,\vartheta+1}; \in; \mathfrak{V}; \ulcorner\alpha_0\urcorner; \ulcorner\pi\urcorner; \ulcorner\vartheta\urcorner; \mathfrak{C}(\alpha_0, \vartheta)\rangle \models \hat{G}(\ulcorner\alpha_0\urcorner, \ulcorner\pi\urcorner, \ulcorner\vartheta\urcorner). \tag{24}$$

Assume further that

$$V_{\pi,\rho} \models \psi[P] \tag{25}$$

for some $P \subseteq V_\pi$ and Π^1_0 formula ψ.

Let $\hat{\mathcal{C}}^{\alpha_0}_{\pi,\mathbb{X}} = \{\sigma < \pi : \ C(\alpha_0, \sigma) \cap \pi = \sigma \ \wedge \ \alpha_0, \mathbb{X} \in C(\alpha_0, \sigma)\}$. From (25), (24), Lemma 4.2 and the fact that $\mathbb{X} \in C(\alpha_0, \pi)$ (using Lemma 4.8), we obtain

$$\begin{array}{rcl}\langle V_{\pi,\vartheta+1}; \in; \mathfrak{V}; \ulcorner\alpha_0\urcorner; \ulcorner\pi\urcorner; \ulcorner\vartheta\urcorner; \ulcorner\rho\urcorner; \mathfrak{C}(\alpha_0, \vartheta); P; \hat{\mathcal{C}}^{\alpha_0}_{\pi,\mathbb{X}}\rangle & \models & \hat{G}(\ulcorner\alpha_0\urcorner, \ulcorner\pi\urcorner, \ulcorner\vartheta\urcorner) \ \wedge \\ & & \text{“}V_{\pi,\rho} \models \psi[P]\text{”} \ \wedge \\ & & \text{“}\hat{\mathcal{C}}^{\alpha_0}_{\pi,\mathbb{X}} \text{ is unbounded in } \pi\text{”}.\end{array} \tag{26}$$

Employing the $\vartheta + 1$-Π^1_0-reducibility of π, there exist $0 < \pi_0 \leq \vartheta_0 < \pi$ and $D, Q, Z \subseteq V_{\vartheta_0}$ such that

$$\langle V_{\pi_0,\vartheta_0+1}; \in; \vec{a}; D, Q, Z\rangle \xrightarrow[\equiv]{\Pi^1_0} \langle V_{\pi,\vartheta+1}; \in; \vec{a}; \mathfrak{C}(\alpha_0, \vartheta), P, \hat{\mathcal{C}}^{\alpha_0}_{\pi,\mathbb{X}}\rangle, \tag{27}$$

where $\vec{a} := \ulcorner\alpha_0\urcorner; \ulcorner\pi\urcorner; \ulcorner\vartheta\urcorner; \ulcorner\rho\urcorner$. The latter yields

$$\begin{array}{rcl}\langle V_{\pi_0,\vartheta_0+1}; \in; \mathfrak{V}; \ulcorner\alpha_0\urcorner; \ulcorner\pi\urcorner; \ulcorner\vartheta\urcorner; \ulcorner\rho\urcorner; D; Q; Z\rangle & \models & \hat{G}(\ulcorner\alpha_0\urcorner, \ulcorner\pi\urcorner, \ulcorner\vartheta\urcorner) \ \wedge \\ & & \text{“}V_{\pi_0,\rho_0} \models \psi[Q]\text{”} \ \wedge \\ & & \text{“}Z \text{ is unbounded in } \pi_0\text{”},\end{array} \tag{28}$$

where ρ_0 is the decoding of $\ulcorner\rho\urcorner$ in the latter structure. Since $Z = V_{\pi_0} \cap \hat{\mathcal{C}}^{\alpha_0}_{\pi,\mathbb{X}}$ and Z is unbounded in π_0 we get $C(\alpha_0, \pi_0) \cap \pi = \pi_0$ and $\alpha_0, \mathbb{X} \in C(\alpha_0, \pi_0)$. (27) also yields $D \cap V_{\pi_0} = \mathfrak{C}(\alpha_0, \vartheta) \cap V_{\pi_0}$ and thus $\rho, \vartheta \in C(\alpha_0, \pi_0)$. Moreover, (27) and (28) yield that $\langle \pi_0, \vartheta_0\rangle \in \mathfrak{M}^{\alpha_0}_{\mathbb{A}(\vartheta)}$. Furthermore, it holds $\pi_0 \leq \rho_0 < \vartheta_0$ and ρ is definable in $\langle V_{\vartheta+1}; \in; \mathfrak{V}; \ulcorner\alpha_0\urcorner; \ulcorner\pi\urcorner; \ulcorner\vartheta\urcorner; \mathfrak{C}(\alpha_0, \vartheta)\rangle$, and hence from (27) it follows

$$\langle V_{\pi_0,\rho_0}; \in; Q, \rangle \xrightarrow[\equiv]{\Pi^1_0} \langle V_{\pi,\rho}; \in; P\rangle. \tag{29}$$

The upshot is that π is ρ-Π^1_0-reducible in ${}^{o}\mathfrak{M}^{\alpha_0}_{\mathbb{A}(\vartheta)}$. This completes the proof of (21).

As $\bar{\rho} \in C(\gamma, \pi)$ and $\mathbb{X} = \mathbb{A}(\bar{\rho})$, one readily sees that $F(\alpha + 1)$ implies $\mathfrak{M}^{\alpha}_{\mathbb{X}} \neq \emptyset$.

Case 3: These are the remaining cases. They are dealt with by similar considerations as in Case 2. □

Corollary 4.11 (Existence) *Let $\mathbb{X} \in \Re^\alpha$ be a projection instance with interval $[\pi, \delta]$. If $\alpha \in C(\alpha, \pi)$, then $\Psi^\alpha_{\mathbb{X}}(\delta) < \pi$*

Proof: This follows from Theorem 4.10. □

Definition 4.12 Let $\mathbb{U}$ and $\mathbb{W}$ be projection instances. Then let

$$\mathbb{U} \lhd \mathbb{W}$$

if one of the following holds:

- There exists a reflection pattern $\mathbb{A}$ and $\delta, \eta \in \mathbf{dom}(\mathbb{A})$ such that $\mathbb{U} = \mathbb{A}(\delta)$ and $\mathbb{W} = \mathbb{A}(\eta)$ and $\delta < \eta$.
- $\mathbb{U} = (\pi; \mathsf{M}^{\xi}_{\mathbb{X}}; \dots)$ and $\mathbb{W} = (\pi; \mathsf{M}^{\zeta}_{\mathbb{Y}}; \dots)$ and one of the following is satisfied:
    1. $\xi < \zeta$
    2. $\xi = \zeta$ and there exists a reflection pattern $\mathbb{A}$ and $\delta, \eta \in \mathbf{dom}(\mathbb{A})$ such that $\mathbb{X} = \mathbb{A}(\delta)$ and $\mathbb{Y} = \mathbb{A}(\eta)$ and $\delta < \eta$.

We write $\mathbb{U} \unlhd \mathbb{W}$ if $\mathbb{U} \lhd \mathbb{W}$ or $\mathbb{U} = \mathbb{W}$. $\mathbb{U}$ and $\mathbb{W}$ are said to be *relatives*, or $\mathbb{U} \sim \mathbb{W}$, if $\mathbb{U} \unlhd \mathbb{W} \ \vee \ \mathbb{W} \unlhd \mathbb{U}$. We also extend the relation $\unlhd$ to reflection instances. For reflection instances $\mathbb{H}$ and $\mathbb{K}$, $\mathbb{H} \unlhd \mathbb{K}$ holds if one of the following is satisfied:

1. $\mathbb{H}$ and $\mathbb{K}$ are both projection instances and $\mathbb{H} \unlhd \mathbb{K}$ holds.
2. $\mathbb{H}$ and $\mathbb{K}$ are of the form $(\mathbb{X}, \xi)$ and $(\mathbb{Y}, \zeta)$, respectively, and $\mathbb{X} \unlhd \mathbb{Y} \ \wedge \ \xi \leq \zeta$.
3. $\mathbb{H}$ and $\mathbb{K}$ are of the form $(\mathbb{X}, \xi, \eta)$ and $(\mathbb{Y}, \zeta, \rho)$, respectively, and $\mathbb{X} \unlhd \mathbb{Y} \wedge \xi \leq \zeta \wedge \eta \leq \rho$.

$\mathbb{H}$ and $\mathbb{K}$ are said to be *relatives* if they are both projection instances and are relatives as projection instances, or they are of the form $(\mathbb{X}, \xi)$ and $(\mathbb{Y}, \zeta)$, respectively, and $\mathbb{X} \sim \mathbb{Y}$, or they are of the form $(\mathbb{X}, \xi, \eta)$ and $(\mathbb{Y}, \zeta, \rho)$, respectively, and $\mathbb{X} \sim \mathbb{Y}$, or they are of the form $(\pi; \mathsf{M}^{\xi}_{\mathbb{X}}; \dots)$ and $(\pi; \mathsf{M}^{\zeta}_{\mathbb{Y}}; \dots)$, respectively, and $\mathbb{X} \sim \mathbb{Y}$.

Lemma 4.13 *Let $\mathbb{C}$ be a reflection pattern in $\mathfrak{R}^{\alpha}$ and $\eta', \eta'' \in \mathbf{dom}(\mathbb{C})$. Let $\mathbb{U} = \mathbb{C}(\eta')$ and $\mathbb{W} = \mathbb{C}(\eta'')$. Let $[\bar{\kappa}, \bar{\rho}]$ be the interval of $\mathbb{U}$. Suppose $\Psi^{\alpha}_{\mathbb{U}}(\bar{\kappa}), \Psi^{\alpha}_{\mathbb{W}}(\bar{\kappa}) < \bar{\kappa}$. Then*

$$\Psi^{\alpha}_{\mathbb{U}}(\bar{\kappa}) < \Psi^{\alpha}_{\mathbb{W}}(\bar{\kappa}) \quad \textit{iff} \quad \mathbb{U} \lhd \mathbb{W} \ \wedge \ \eta' \in C(\alpha, \Psi^{\alpha}_{\mathbb{W}}(\bar{\kappa})).$$

Proof: Assume $\eta' < \eta''$ and $\eta' \in C(\alpha, \Psi^{\alpha}_{\mathbb{W}}(\bar{\kappa}))$. Then we have $\mathbb{U} \lhd \mathbb{W}$. We need to show $\Psi^{\alpha}_{\mathbb{U}}(\bar{\kappa}) < \Psi^{\alpha}_{\mathbb{W}}(\bar{\kappa})$. One has to distinguish several cases arising from the shape of $\mathbb{C}$.

Let $\mathbb{C}(\delta)$ be a projection instance in $\mathfrak{R}^{\alpha}$ of the form

$$\mathbb{C}(\delta) \ = \ (\pi; \mathsf{M}^{<\gamma}_{\mathbb{A}(\rho)}\text{-}\delta\text{-}\mathsf{P}_0; \mathsf{M}^{\zeta}_{\mathbb{Y}}; \mathbb{X}; \alpha_0),$$

where $\mathbf{dom}(\mathbb{A}) \neq (0, \omega)$. Let $\beta := o(\mathbb{A}(\rho))$. Then $X^{\alpha}_{\mathbb{C}(\eta'')}$ consist of all triples $\langle \kappa, \sigma, \vec{\mathbf{D}} \rangle$ such that $\kappa \leq \sigma < \bar{\kappa}$, κ is inaccessible, and the following are satisfied:

1. $C(\alpha, \kappa) \cap \bar{\kappa} \ = \ \kappa$ and $\mathbb{C}(\eta''), \alpha \in C(\alpha, \kappa)$.
2. $\langle V_{\kappa,\sigma}; \in; \vec{\mathbf{D}} \rangle \ \xrightarrow[\equiv]{\Pi^1_0} \ \langle V_{\bar{\kappa},\eta''}; \in; \mathfrak{C}(\alpha, \eta'') \rangle$, where $\vec{\mathbf{D}} \subseteq V_{\sigma}$ is the $<_{\mathbf{L}}$-least such set.
3. If $\xi \in C(\xi, \kappa) \cap [\beta, \gamma)$ and $\vartheta \in C(\xi, \kappa) \cap \mathbf{dom}(\mathbb{A})$ and $\vartheta > \rho$, then κ is σ-Π^1_0-reducible in

$$^{o}\mathfrak{M}^{\xi}_{\mathbb{A}(\vartheta)} \ := \ \{ \langle \kappa^*, \sigma^* \rangle : \ \exists \vartheta^* \left(\langle \kappa^*, \vartheta^* \rangle \in \mathfrak{M}^{\xi}_{\mathbb{A}(\vartheta)} \ \wedge \ \kappa^* \leq \sigma^* < \vartheta^* \right) \}.$$

4. If $\tau \in C(\tau, \kappa) \cap [\alpha_0, \alpha)$ and $\nu \in C(\tau, \kappa) \cap \mathbf{dom}(\mathbb{C})$ and $\nu > \eta''$, then κ is σ-Π^1_0-reducible in

$$^o\mathfrak{M}^{\tau}_{\mathbb{C}(\nu)} \;\; := \;\; \{\langle \kappa^*, \sigma^* \rangle : \; \exists \nu^* \left(\langle \kappa^*, \nu^* \rangle \in \mathfrak{M}^{\tau}_{\mathbb{C}(\nu)} \; \wedge \; \kappa^* \leq \sigma^* < \nu^* \right\}.$$

We have $\langle \Psi^{\alpha}_{\mathbb{W}}(\bar{\kappa}), \Psi^{\alpha}_{\mathbb{W}}(\eta'') \rangle \in \mathfrak{M}^{\alpha}_{\mathbb{C}(\eta'')}$. Since $\eta' \in C(\alpha, \Psi^{\alpha}_{\mathbb{W}}(\bar{\kappa}))$, $\eta' < \eta''$ and

$$\langle V_{\Psi^{\alpha}_{\mathbb{W}}(\bar{\kappa}), \Psi^{\alpha}_{\mathbb{W}}(\eta'')}; \in; \vec{\mathbf{D}} \rangle \xrightarrow[\equiv]{\Pi^1_0} \langle V_{\bar{\kappa}, \eta''}; \in; \mathfrak{C}(\alpha, \eta'') \rangle$$

(where $\vec{\mathbf{D}} \subseteq V_{\Psi^{\alpha}_{\mathbb{W}}(\eta'')}$ is the $<_{\mathbf{L}}$-least such set) it follows that there exist $\kappa \leq \sigma < \Psi^{\alpha}_{\mathbb{W}}(\bar{\kappa})$ such that κ is inaccessible, and the following are satisfied:

1. $C(\alpha, \kappa) \cap \bar{\kappa} = \kappa$ and $\mathbb{C}(\eta'), \alpha \in C(\alpha, \kappa)$.

2. $\langle V_{\kappa, \sigma}; \in; \vec{\mathbf{D}}_1 \rangle \xrightarrow[\equiv]{\Pi^1_0} \langle V_{\bar{\kappa}, \eta'}; \in; \mathfrak{C}(\alpha, \eta') \rangle$, where $\vec{\mathbf{D}}_1 \subseteq V_\sigma$ is the $<_{\mathbf{L}}$-least such set.

3. If $\xi \in C(\xi, \kappa) \cap [\beta, \gamma)$ and $\vartheta \in C(\xi, \kappa) \cap \mathbf{dom}(\mathbb{A})$ and $\vartheta > \rho$, then κ is σ-Π^1_0-reducible in

$$^o\mathfrak{M}^{\xi}_{\mathbb{A}(\vartheta)} \;\; := \;\; \{\langle \kappa^*, \sigma^* \rangle : \; \exists \vartheta^* \left(\langle \kappa^*, \vartheta^* \rangle \in \mathfrak{M}^{\xi}_{\mathbb{A}(\vartheta)} \; \wedge \; \kappa^* \leq \sigma^* < \vartheta^* \right\}.$$

4. If $\tau \in C(\tau, \kappa) \cap [\alpha_0, \alpha)$ and $\nu \in C(\tau, \kappa) \cap \mathbf{dom}(\mathbb{C})$ and $\nu > \eta'$, then κ is σ-Π^1_0-reducible in

$$^o\mathfrak{M}^{\tau}_{\mathbb{C}(\nu)} \;\; := \;\; \{\langle \kappa^*, \sigma^* \rangle : \; \exists \nu^* \left(\langle \kappa^*, \nu^* \rangle \in \mathfrak{M}^{\tau}_{\mathbb{C}(\nu)} \; \wedge \; \kappa^* \leq \sigma^* < \nu^* \right\}.$$

As a result, we get $\Psi^{\alpha}_{\mathbb{U}}(\bar{\kappa}) < \Psi^{\alpha}_{\mathbb{W}}(\bar{\kappa})$.

The remaining cases follow by similar arguments. □

Lemma 4.14 *Let $\mathbb{U} \in \mathfrak{R}^{\alpha}$ and $\mathbb{W} \in \mathfrak{R}^{\beta}$ be projection instances with intervals $[\pi, \delta]$ and $[\kappa, \eta]$, respectively. Suppose $\Psi^{\alpha}_{\mathbb{U}}(\pi) < \pi$ and $\Psi^{\beta}_{\mathbb{W}}(\kappa) < \kappa$. Let $\eta' \in \mathbf{dom}(\Psi^{\beta}_{\mathbb{W}})$. If $\Psi^{\beta}_{\mathbb{W}}(\eta') \in [\Psi^{\alpha}_{\mathbb{U}}(\pi), \Psi^{\alpha}_{\mathbb{U}}(\delta)]$, then $\beta = \alpha$ and $\mathbb{W} = \mathbb{U}$, or $\kappa = \Psi^{\alpha}_{\mathbb{U}}(\rho)$, $\mathbb{W} = (\Psi^{\alpha}_{\mathbb{U}}(\rho); \mathsf{RSC}; \mathbb{U}; \alpha + 1)$, and $\beta > \alpha$ for some $\rho \in \mathbf{dom}(\Psi^{\alpha}_{\mathbb{U}})$.*

Proof: By induction on α. □

Theorem 4.15 (<-Comparison) *Let $\mathbb{U} \in \mathfrak{R}^{\alpha}$ and $\mathbb{W} \in \mathfrak{R}^{\beta}$ be projection instances with intervals $[\pi, \delta]$ and $[\kappa, \eta]$, respectively. Suppose $\Psi^{\alpha}_{\mathbb{U}}(\pi) < \pi$ and $\Psi^{\beta}_{\mathbb{W}}(\kappa) < \kappa$. Let $\delta' \in \mathbf{dom}(\Psi^{\alpha}_{\mathbb{U}})$ and $\eta' \in \mathbf{dom}(\Psi^{\beta}_{\mathbb{W}})$.*

Then

(A) $\Psi^{\alpha}_{\mathbb{U}}(\delta') < \Psi^{\beta}_{\mathbb{W}}(\eta')$

holds iff one of the following obtains:

(a) $\pi \leq \Psi^{\beta}_{\mathbb{W}}(\kappa)$.

(b) $\alpha < \beta \; \wedge \; \Psi^{\alpha}_{\mathbb{U}}(\delta') < \kappa \leq \Psi^{\alpha}_{\mathbb{U}}(\delta)$.

(c) $\alpha < \beta \ \wedge \ \mathbb{U}, \alpha \in C(\beta, \Psi^{\beta}_{\mathbb{W}}(\kappa)) \ \wedge \ \Psi^{\alpha}_{\mathbb{U}}(\delta) < \kappa$.

(d) $\beta \le \alpha \ \wedge \ \big[\mathbb{W} \notin C(\alpha, \Psi^{\alpha}_{\mathbb{U}}(\pi)) \vee \beta \notin C(\alpha, \Psi^{\alpha}_{\mathbb{U}}(\pi)) \vee \pi \le \Psi^{\beta}_{\mathbb{W}}(\eta')\big]$.

(e) $\alpha = \beta \ \wedge \ \mathbb{U} \in C(\beta, \Psi^{\beta}_{\mathbb{W}}(\kappa)) \ \wedge \ \mathbb{U} \lhd \mathbb{W}$.

(f) $\alpha = \beta \ \wedge \ \mathbb{U} = \mathbb{W} \ \wedge \ \delta' < \eta'$.

Moreover, if $\Psi^{\alpha}_{\mathbb{U}}(\delta') = \Psi^{\beta}_{\mathbb{W}}(\eta')$, *then* $\alpha = \beta$ *and* $\mathbb{U} = \mathbb{W}$ *and* $\delta' = \eta'$.

Proof: We proceed by induction on $\alpha \# \beta$. We first address the equivalence

$$(A) \Leftrightarrow (a) \vee (b) \vee (c) \vee (d) \vee (e) \vee (f).$$

Obviously the equivalence holds when $\pi \le \Psi^{\beta}_{\mathbb{W}}(\kappa)$. Thus we may assume

$$\Psi^{\beta}_{\mathbb{W}}(\kappa) < \pi \tag{30}$$

from now on, that is to say that (a) is false. We shall distinguish several cases.

Case 1. $\alpha < \beta$.

Case 1.1. $\Psi^{\alpha}_{\mathbb{U}}(\pi) \le \Psi^{\beta}_{\mathbb{W}}(\kappa)$.

As $\mathbb{U}, \alpha \in C(\alpha, \Psi^{\alpha}_{\mathbb{U}}(\pi))$ we get $\mathbb{U}, \pi, \delta, \alpha \in C(\beta, \Psi^{\beta}_{\mathbb{W}}(\kappa))$, and hence $\Psi^{\alpha}_{\mathbb{U}}(\pi), \Psi^{\alpha}_{\mathbb{U}}(\delta) \in C(\beta, \Psi^{\beta}_{\mathbb{W}}(\kappa))$.

If $\Psi^{\alpha}_{\mathbb{U}}(\delta) < \kappa$ then from the above we get $\Psi^{\alpha}_{\mathbb{U}}(\delta) < \Psi^{\beta}_{\mathbb{W}}(\kappa)$, and hence (A) and (c) are true.

If $\kappa \le \Psi^{\alpha}_{\mathbb{U}}(\delta)$ then $\Psi^{\alpha}_{\mathbb{U}}(\pi) < \kappa \le \Psi^{\alpha}_{\mathbb{U}}(\delta)$. Lemma 4.14 then yields $\kappa = \Psi^{\alpha}_{\mathbb{U}}(\rho)$ for some $\rho \in \mathbf{dom}(\Psi^{\alpha}_{\mathbb{U}})$ and $\mathbb{W} = (\kappa; \mathsf{RSC}; \mathbb{U}; \alpha + 1)$. Then (A) implies $\kappa > \Psi^{\alpha}_{\mathbb{U}}(\delta')$ and thus (b) holds. On the other hand, if (b) holds one easily verifies (A).

In any case equivalence obtains.

Case 1.2. $\Psi^{\beta}_{\mathbb{W}}(\kappa) < \Psi^{\alpha}_{\mathbb{U}}(\pi)$.

$\Psi^{\alpha}_{\mathbb{U}}(\pi) \le \Psi^{\beta}_{\mathbb{W}}(\eta)$ would yield $\alpha > \beta$ by Lemma 4.14 and is therefore ruled out. Hence we must have $\Psi^{\beta}_{\mathbb{W}}(\eta) < \Psi^{\alpha}_{\mathbb{U}}(\pi)$ as well. Thus (A) is false. Also (a),(d),(e),(f) are automatically false. Thus to show equivalence, we have to refute (b) and (c). (b) would yield $\Psi^{\alpha}_{\mathbb{U}}(\pi) < \Psi^{\beta}_{\mathbb{W}}(\kappa)$ by Lemma 4.14 which collides with $\Psi^{\beta}_{\mathbb{W}}(\eta) < \Psi^{\alpha}_{\mathbb{U}}(\pi)$. (c) would imply $\Psi^{\alpha}_{\mathbb{U}}(\pi) \in C(\beta, \Psi^{\beta}_{\mathbb{W}}(\kappa)) \cap \kappa$, yielding the contradiction $\Psi^{\alpha}_{\mathbb{U}}(\pi) < \Psi^{\beta}_{\mathbb{W}}(\kappa)$.

Case 2. $\beta < \alpha$.

Case 2.1. $\Psi^{\alpha}_{\mathbb{U}}(\pi) < \Psi^{\beta}_{\mathbb{W}}(\kappa)$.

First we show (d). As $\beta, \mathbb{W} \in C(\alpha, \Psi^{\alpha}_{\mathbb{U}}(\pi))$ would imply $\Psi^{\beta}_{\mathbb{W}}(\kappa) \in C(\alpha, \Psi^{\alpha}_{\mathbb{U}}(\pi)) \cap \pi$, and thus $\Psi^{\beta}_{\mathbb{W}}(\kappa) < \Psi^{\alpha}_{\mathbb{U}}(\pi)$, we must have (d).

Next we show that (A) holds. From $\Psi^{\beta}_{\mathbb{W}}(\kappa) \le \Psi^{\alpha}_{\mathbb{U}}(\delta')$ we would get $\Psi^{\alpha}_{\mathbb{U}}(\pi) < \Psi^{\beta}_{\mathbb{W}}(\kappa) \le \Psi^{\alpha}_{\mathbb{U}}(\delta')$, and hence the contradiction $\beta > \alpha$ by Lemma 4.14. As a result, $\Psi^{\alpha}_{\mathbb{U}}(\delta') < \Psi^{\beta}_{\mathbb{W}}(\kappa) \le \Psi^{\beta}_{\mathbb{W}}(\eta')$, and therefore (A) holds.

Case 2.2. $\Psi^{\beta}_{\mathbb{W}}(\kappa) \le \Psi^{\alpha}_{\mathbb{U}}(\pi)$.

Note this implies $\beta, \mathbb{W} \in C(\alpha, \Psi^{\alpha}_{\mathbb{U}}(\pi))$ and thus $\Psi^{\beta}_{\mathbb{W}}(\kappa) \in C(\alpha, \Psi^{\alpha}_{\mathbb{U}}(\pi))$ so that we actually have $\Psi^{\beta}_{\mathbb{W}}(\kappa) < \Psi^{\alpha}_{\mathbb{U}}(\pi)$ in this case.

First assume that (A) holds. Then $\Psi^{\beta}_{\mathbb{W}}(\kappa) < \Psi^{\alpha}_{\mathbb{U}}(\pi) < \Psi^{\beta}_{\mathbb{W}}(\eta)$. By Lemma 4.14 the latter yields $\pi = \Psi^{\beta}_{\mathbb{W}}(\rho)$ for some $\rho \in \mathbf{dom}(\Psi^{\beta}_{\mathbb{W}})$. One verifies then that $\pi \le \Psi^{\beta}_{\mathbb{W}}(\eta)$ must hold as well. Hence we have proved (d).

Now assume (d). Since $\Psi^{\beta}_{\mathbb{W}}(\kappa) < \Psi^{\alpha}_{\mathbb{U}}(\pi)$ and $\beta < \alpha$ we get $\eta, \mathbb{W} \in C(\alpha, \Psi^{\alpha}_{\mathbb{U}}(\pi))$. As a result $\pi \leq \Psi^{\beta}_{\mathbb{W}}(\eta')$, yielding (A). Since (a),(b),(c),(e),(f) are ruled out, we have shown the desired equivalence.

Case 3. $\alpha = \beta$.

Case 3.1. $\pi < \kappa$.

Then $\Psi^{\alpha}_{\mathbb{U}}(\pi) \leq \Psi^{\beta}_{\mathbb{W}}(\kappa)$ would imply $\pi < \Psi^{\beta}_{\mathbb{W}}(\kappa)$, yielding (a) contrary to (30). Therefore $\Psi^{\beta}_{\mathbb{W}}(\kappa) < \Psi^{\alpha}_{\mathbb{U}}(\pi)$ must hold. From Lemma 4.14 we can further conclude that $\Psi^{\beta}_{\mathbb{W}}(\eta') < \Psi^{\alpha}_{\mathbb{U}}(\pi)$. The latter also implies $\beta, \mathbb{W} \in C(\alpha, \Psi^{\alpha}_{\mathbb{U}}(\pi))$. As a result, (A) as well as (a)-(f) are all false, which shows equivalence in this case.

Case 3.2. $\kappa < \pi$.

$\Psi^{\beta}_{\mathbb{W}}(\kappa) \leq \Psi^{\alpha}_{\mathbb{U}}(\pi)$ then implies $\kappa < \Psi^{\alpha}_{\mathbb{U}}(\pi)$, and thus (A) as well as (a)-(f) would be false.

Suppose $\Psi^{\alpha}_{\mathbb{U}}(\pi) < \Psi^{\beta}_{\mathbb{W}}(\kappa)$. Owing to Lemma 4.14 $\Psi^{\alpha}_{\mathbb{U}}(\pi) < \Psi^{\beta}_{\mathbb{W}}(\kappa) \leq \Psi^{\alpha}_{\mathbb{U}}(\delta)$ is impossible, whence $\Psi^{\alpha}_{\mathbb{U}}(\delta) < \Psi^{\beta}_{\mathbb{W}}(\kappa)$ must hold as well. The latter implies (A). As $\beta, \mathbb{W} \in C(\alpha, \Psi^{\alpha}_{\mathbb{U}}(\pi))$ would yield $\kappa \in C(\alpha, \Psi^{\alpha}_{\mathbb{U}}(\pi)) \cap \pi$ and hence the contradiction $\kappa < \Psi^{\alpha}_{\mathbb{U}}(\pi)$, we also get (d). As a result, equivalence obtains.

Case 3.3. $\kappa = \pi$.

Case 3.3.1. One easily sees that if $\mathbb{U} = (\mathbb{I}; \mathsf{RLC}; \emptyset; 0)$ or $\mathbb{U} = (\omega^{+(n)}; \mathsf{RSC}; \emptyset; 0)$, then $\mathbb{U} = \mathbb{W}$ must hold and (A) as well as (a)-(f) are false.

Case 3.3.2. Let $\mathbb{U}$ be of the form $(\Psi^{\xi}_{\mathbb{X}}(\rho)^{+(n)}; \mathsf{RSC}; \mathbb{X}; \xi + 1)$ for some $n > 0$. We want to show $\mathbb{U} = \mathbb{W}$. Let $[\tau, \rho]$ be the interval of $\mathbb{X}$.

First suppose that $\mathbb{W}$ is of the form $(\Psi^{\xi'}_{\mathbb{X}'}(\rho')^{+(m)}; \mathsf{RSC}; \mathbb{X}'; \xi'+1)$. Let $[\tau', \rho']$ be the interval of $\mathbb{X}'$. If $\Psi^{\xi}_{\mathbb{X}}(\tau) = \Psi^{\xi'}_{\mathbb{X}'}(\tau')$, then the induction hypothesis would yield $\mathbb{X} = \mathbb{X}'$ and consequently $\mathbb{U} = \mathbb{W}$. If $\Psi^{\xi}_{\mathbb{X}}(\tau) < \Psi^{\xi'}_{\mathbb{X}'}(\tau') \leq \Psi^{\xi}_{\mathbb{X}}(\rho)$ then $\mathbb{X} = \mathbb{X}'$ or $\mathbb{X}'$ is of the form $(\Psi^{\xi}_{\mathbb{X}}; \mathsf{RSC}; \ldots; \xi + 1)$ by Lemma 4.14. However, the latter is incompatible with $(\Psi^{\xi}_{\mathbb{X}}(\rho)^{+(n)}; \mathsf{RSC}; \mathbb{X}; \xi + 1)$ being a projection instance. Therefore $\mathbb{X} = \mathbb{X}'$ must obtain, yielding $\mathbb{U} = \mathbb{W}$.

Next assume that $\mathbb{W}$ is of the form $(\Psi^{\xi'}_{\mathbb{X}'}(\mu'); \mathsf{RSC}; \mathbb{X}'; \xi' + 1)$, where μ' is a regular cardinal. Let $[\tau', \rho']$ be the interval of $\mathbb{X}'$. The constellations $\Psi^{\xi'}_{\mathbb{X}'}(\tau') \leq \Psi^{\xi}_{\mathbb{X}}(\rho) \leq \Psi^{\xi'}_{\mathbb{X}'}(\rho')$ or $\Psi^{\xi}_{\mathbb{X}}(\tau) \leq \Psi^{\xi'}_{\mathbb{X}'}(\rho') \leq \Psi^{\xi}_{\mathbb{X}}(\rho)$ would yield $\mathbb{X} = \mathbb{X}'$ owing to Lemma 4.14 and the induction hypothesis. From $\Psi^{\xi}_{\mathbb{X}}(\rho) < \Psi^{\xi'}_{\mathbb{X}'}(\tau')$ we would get the contradiction $\Psi^{\xi}_{\mathbb{X}}(\rho)^{+(n)} < \Psi^{\xi'}_{\mathbb{X}'}(\tau') \leq \Psi^{\xi'}_{\mathbb{X}'}(\mu')$ since $\Psi^{\xi'}_{\mathbb{X}'}(\tau')$ is an inaccessible cardinal. From $\Psi^{\xi'}_{\mathbb{X}'}(\rho') < \Psi^{\xi}_{\mathbb{X}}(\tau)$ we would get a similar contradiction. As a result, $\mathbb{X} = \mathbb{X}'$ which entails $\mathbb{U} = \mathbb{W}$.

Finally assume $\mathbb{W}$ is not of the form $(\kappa; \mathsf{RSC}; \ldots)$. However, then κ would be a limit cardinal, contradicting $\kappa = \pi = \Psi^{\xi}_{\mathbb{X}}(\rho)^{+(n)}$. We have now covered all possible cases and thus $\mathbb{U} = \mathbb{W}$. From $\mathbb{U} = \mathbb{W}$ and $\alpha = \beta$ we get that (A) and (a)-(b) are false.

Case 3.3.3. Let $\mathbb{U}$ be of the form $(\Psi^{\xi}_{\mathbb{X}}(\rho); \ldots)$. By similar arguments as in the previous case one can show that $\mathbb{W}$ has to be of the form $(\Psi^{\xi'}_{\mathbb{X}'}(\rho'); \ldots)$ as well. Since $\Psi^{\xi}_{\mathbb{X}}(\rho) = \Psi^{\xi'}_{\mathbb{X}'}(\rho')$ the induction hypothesis yields $\xi = \xi'$, $\mathbb{X} = \mathbb{X}'$, and $\rho = \rho'$. As a consequence, $\mathbb{U}$ and $\mathbb{W}$ are related. If $\mathbb{U} = \mathbb{W}$, then (A) holds if and only if (f) holds. As (a)-(e) are ruled out, this shows equivalence in the case $\mathbb{U} = \mathbb{W}$.

Now assume $\mathbb{U} \neq \mathbb{W}$. Let $[\pi, \rho]$ and $[\pi, \rho']$ be the intervals of $\mathbb{U}$ and $\mathbb{W}$, respectively. Firstly, suppose $\Psi^{\alpha}_{\mathbb{U}}(\pi) < \Psi^{\alpha}_{\mathbb{W}}(\pi)$. Note that $\Psi^{\alpha}_{\mathbb{W}}(\pi) \leq \Psi^{\alpha}_{\mathbb{U}}(\rho)$ is impossible by Lemma 4.14. Thus $\Psi^{\alpha}_{\mathbb{U}}(\rho) < \Psi^{\alpha}_{\mathbb{W}}(\pi)$. Therefore (A) holds and (e) holds as well by Lemma 4.13, establishing equivalence.

By the same arguments $\Psi^{\alpha}_{\mathbb{U}}(\pi) > \Psi^{\alpha}_{\mathbb{W}}(\pi)$ would imply that (A) and (a)-(d) are false.

If $\Psi^\alpha_{\mathbb{U}}(\pi) = \Psi^\alpha_{\mathbb{W}}(\pi)$ then $\mathbb{U} = \mathbb{W}$ by Lemma 4.13, contradicting our assumption.

Case 3.3.4. Let $\mathbb{U}$ be of the form $(\Xi; \rho\text{-}\mathsf{P}_0; \mathbf{I}\text{-}\mathsf{P}_0; \emptyset; \alpha)$. Then $\pi = \kappa = \Xi$. One easily checks that $\mathbb{W}$ also has to be of the form $(\Xi; \rho'\text{-}\mathsf{P}_0; \mathbf{I}\text{-}\mathsf{P}_0; \emptyset; \alpha)$. If $\rho < \rho'$ one gets $\mathbb{U} \lhd \mathbb{W}$ and (A) and (e) obtain, yielding the equivalence of (A) and (a)$\vee \ldots \vee$(f). If $\rho > \rho'$ one gets that (A) as well as (a)-(f) are false. If $\rho = \rho'$ then $\mathbb{W} = \mathbb{U}$ and (A) is equivalent to (f), yielding the equivalence of (A) and (a)$\vee \ldots \vee$(f). □

Corollary 4.16 *Let $\mathbb{X}$ be a projection instance with $\mathbb{X} \in \Re^\alpha$, $\delta \in \mathbf{Veb}_{Cl}$ and $\delta \in \mathbf{dom}(\Psi^\alpha_{\mathbb{X}})$. Suppose $\rho \leq \Psi^\alpha_{\mathbb{X}}(\delta)$. Then we have*

$$\Psi^\alpha_{\mathbb{X}}(\delta) \in C(\beta, \rho) \quad \Longleftrightarrow \quad \alpha < \beta \ \wedge \ \alpha, \delta, \mathbb{X} \in C(\beta, \rho).$$

Proof: "$\Leftarrow$" is obvious. To show "$\Rightarrow$", we prove

$$\Psi^\alpha_{\mathbb{X}}(\delta) \in C_n(\beta, \rho) \ \Rightarrow \ \alpha < \beta \ \wedge \ \alpha, \delta, \mathbb{X} \in C_n(\beta, \rho)$$

by induction on n. For $n = 0$ this follows from $\Psi^\alpha_{\mathbb{X}}(\delta) \notin C_0(\beta, \rho)$. Now suppose $\Psi^\alpha_{\mathbb{X}}(\delta) \in C_n(\beta, \rho)$ and $n = m + 1$. Then, due to Lemma 4.5 and Lemma 4.6 there exist $\gamma, \delta', \mathbb{Y} \in C_m(\beta, \rho)$ with $\gamma < \beta$, $\delta' \in \mathbf{dom}(\Psi^\gamma_{\mathbb{Y}})$ and $\Psi^\gamma_{\mathbb{Y}}(\delta') = \Psi^\alpha_{\mathbb{X}}(\delta)$. By Theorem 4.15 we thus obtain $\gamma = \alpha$, $\delta = \delta'$ and $\mathbb{Y} = \mathbb{X}$. Therefore we get $\alpha < \beta$ and $\alpha, \delta, \mathbb{X} \in C_n(\beta, \rho)$. □

Definition 4.17 $\alpha =_{NF} \omega^{\alpha_1} + \cdots + \omega^{\alpha_n}$ means that the right hand side exhibits the Cantor normal form of α.

$\alpha =_{NF} \varphi\beta\gamma$ stands for $\alpha = \varphi\beta\gamma$ and $\beta, \gamma < \alpha$.

Lemma 4.18

(i) $\alpha =_{NF} \omega^{\alpha_1} + \cdots + \omega^{\alpha_n} \implies [\alpha \in C(\zeta, \rho) \iff \alpha_1, \ldots, \alpha_n \in C(\zeta, \rho)]$.

(ii) $\alpha =_{NF} \varphi\alpha_1\alpha_2 \implies [\alpha \in C(\zeta, \rho) \iff \alpha_1, \alpha_2 \in C(\zeta, \rho)]$.

(iii) Let $\mu < \Xi$ be a cardinal. Then we have $\mu \in C(\zeta, \rho) \iff \mu^+ \in C(\zeta, \rho)]$.

Proof: (i) Using induction on n, one shows that $\alpha \in C_n(\zeta, \rho)$ implies $\alpha_1, \ldots, \alpha_n \in C_n(\zeta, \rho)$. Similarly one proves (ii) and (iii). □

5 The ordinal representation system

In the previous section we showed the uniqueness of representations of the form $\Psi^\alpha_{\mathbb{X}}(\delta)$, providing that $\alpha \in C(\alpha, \pi)$ and $\delta \in \mathbf{dom}(\Psi^\alpha_{\mathbb{X}}) \cap \mathbf{Veb}_{Cl}$. By viewing $\Psi^\alpha_{\mathbb{X}}(\delta)$ as an expression to denote an ordinal, we will arrive at an ordinal representation system.

Definition 5.1 $\eta =_{NF} \omega^\alpha + \beta$ stands for $\eta = \omega^\alpha + \beta \wedge \alpha < \eta \wedge \beta < \omega^{\alpha+1}$. $\nu =_{NF} \rho^+$ stands for $\nu = \rho^+$ and ρ is a cardinal. $\rho =_{NF} \Psi^\alpha_{\mathbb{X}}(\delta)$ stands for $\delta \in \mathbf{dom}(\Psi^\alpha_{\mathbb{X}})$ and $\rho = \Psi^\alpha_{\mathbb{X}}(\delta)$ and $\Psi^\alpha_{\mathbb{X}}(\delta) < \pi$, where $\mathbb{X}$ has an interval of the form $[\pi, \nu]$. $\eta =_{NF} \varphi\alpha\beta$ stands for $\eta = \varphi\alpha\beta \wedge \alpha, \beta < \varphi\alpha\beta$.

In what follows, $\mathbb{X} \in \mathcal{OT}$ is meant to convey that every ordinal occurring in the projection instance $\mathbb{X}$ is an element of $\mathcal{OT}$.

Definition 5.2 The set of ordinal representations, $\mathcal{OT}$, is inductively defined as follows:

($\mathcal{OT}0$) $0, \Xi, \mathbf{I} \in \mathcal{OT}$.

For every $0 < n < \omega$, $\omega^{+(n)}, \Xi^{+(n)} \in \mathcal{OT}$.

($\mathcal{OT}1$) If $\eta =_{NF} \omega^\alpha + \beta$ and $\alpha, \beta \in \mathcal{OT}$, then $\eta \in \mathcal{OT}$.

($\mathcal{OT}2$) If $\eta =_{NF} \varphi\alpha\beta$, $\alpha > 0$, $\eta < \mathbf{I}$ and $\alpha, \beta \in \mathcal{OT}$, then $\eta \in \mathcal{OT}$.

($\mathcal{OT}3$) If $\mathbb{X} \in \Re^\alpha$ is a projection instance with interval $[\pi, \rho]$, $\rho \in \mathbf{dom}(\Psi^\alpha_\mathbb{X})$, $\alpha \in C(\alpha, \pi)$, $\mathbb{X}, \alpha, \rho \in \mathcal{OT}$, and $\mathbb{X}$ is not of the form $(\pi, \mathsf{RSC}; \dots)$, then for every $0 < n < \omega$, $\Psi^\alpha_\mathbb{X}(\rho)^{+(n)} \in \mathcal{OT}$.

($\mathcal{OT}4$) If $\mathbb{X} \in \Re^\alpha$, $\alpha, \rho, \mathbb{X} \in \mathcal{OT}$, $\rho \in \mathbf{dom}(\Psi^\alpha_\mathbb{X})$, $\alpha \in C(\alpha, \pi)$, and $\rho \in \mathbf{Veb}_{Cl}$, then $\Psi^\alpha_\mathbb{X}(\rho) \in \mathcal{OT}$.

Theorem 5.3 *Different ordinal representations denote different ordinals.*

Proof: This follows from Theorem 4.15. □

Convention. For the remainder of this paper, all ordinals are assumed to be in $\mathcal{OT}$.

The $<$-comparison theorem 4.15 provides an inductive way of comparing ordinals in $\mathcal{OT}$ with regard to the relation $<$. However, an obstacle to arriving at a recursive ordinal representation system is raised by deciding whether $\alpha \in C(\alpha, \pi)$.

5.1 A procedure for deciding $\alpha \in C(\gamma, \delta)$

Definition 5.4 By induction on the definition of $\alpha \in \mathcal{OT}$, $K_\delta(\alpha)$ is defined as follows.

(K1) $K_\delta(0) = K_\delta(\Xi) = K_\delta(\mathbf{I}) = K_\delta(\omega^{+(n)}) == K_\delta(\Xi^{+(n)}) = \emptyset$.

(K2) If $\alpha =_{NF} \omega^{\alpha_1} + \alpha_2$ or $\alpha =_{NF} \varphi\alpha_1\alpha_2$, then $K_\delta(\alpha) = \bigcup_{1 \le i \le 2} K_\delta(\alpha_i)$.

(K3) If $\alpha = \Psi^\beta_\mathbb{X}(\rho)^{+(n)}$, where $\mathbb{X} \in \Re^\beta$ is a projection instance with interval $[\pi, \rho]$, $\rho \in \mathbf{dom}(\Psi^\alpha_\mathbb{X})$, $\beta \in C(\beta, \pi)$, $\mathbb{X}, \beta, \rho \in \mathcal{OT}$, and $\mathbb{X}$ is not of the form $(\pi, \mathsf{RSC}; \dots)$, then

$$K_\delta(\alpha) = \begin{cases} \emptyset & \textit{if } \alpha < \delta \\ K_\delta(\beta) \cup K_\delta(\mathbb{X}) \cup K_\delta(\rho) \cup \{\beta\} & \textit{else.} \end{cases}$$

(K4) If $\alpha = \Psi^\beta_\mathbb{X}(\rho)$, where $\mathbb{X} \in \Re^\beta$ is a projection instance with $\rho \in \mathbf{dom}(\Psi^\alpha_\mathbb{X})$, $\beta \in C(\beta, \pi)$, $\mathbb{X}, \beta, \rho \in \mathcal{OT}$, then

$$K_\delta(\alpha) = \begin{cases} \emptyset & \textit{if } \alpha < \delta \\ K_\delta(\beta) \cup K_\delta(\mathbb{X}) \cup K_\delta(\rho) \cup \{\beta\} & \textit{else.} \end{cases}$$

Lemma 5.5 *If $\alpha \in \mathcal{OT}$ and δ, γ are arbitrary ordinals, then*

$$\alpha \in C(\gamma, \delta) \quad \textit{iff} \quad K_\delta(\alpha) < \gamma.$$

Proof: This is straightforwardly verified by induction on $\alpha \in \mathcal{OT}$. □

Theorem 5.6 *Each ordinal of $\mathcal{OT}$ can be identified with the unique term denoting it, and thus $\mathcal{OT}$ can be coded as a set of natural numbers. Under this coding, $\mathcal{OT}$ is a primitive recursive set and, moreover, the ordering relation inherited from the ordinals is also primitive recursive.*

Proof: This follows from Corollary 4.11, Theorem 4.15 and Lemma 5.5. □

6 The lifting of a projection function

A projection function $\Psi^{\alpha}_{\mathbb{X}}$ with interval $[\pi, \delta]$ is the inverse p^{-1} of a function

$$p : \langle V_{\kappa,\sigma}; \in; \vec{\mathbf{D}}\rangle \xrightarrow[\equiv]{\Pi^1_n} \langle V_{\pi,\delta}; \in; \mathfrak{C}(\alpha, \delta)\rangle$$

where $\kappa = \Psi^{\alpha}_{\mathbb{X}}(\pi)$ and $\sigma = \Psi^{\alpha}_{\mathbb{X}}(\delta)$ The domain of $\Psi^{\alpha}_{\mathbb{X}}$ is then the set $C(\alpha, \kappa) \cap [\pi, \delta]$. Thus, in general, $\Psi^{\alpha}_{\mathbb{X}}$ is only defined on a proper subset of $[\pi, \delta]$. Since p^{-1} is the inverse of a partial isomorphism we could actually extend the function $\Psi^{\alpha}_{\mathbb{X}}$ so that it becomes defined on more points of the interval $[\pi, \delta]$. The latter is not only a possibility but also a necessity if we are to use the projection function in the cut elimination procedure for the infinitary proof system. On the other hand it is, in general, not possible to extend $\Psi^{\alpha}_{\mathbb{X}}$ to the whole interval $[\pi, \delta]$ in a sensible way. We also want the extension to be definable in a primitive recursive way on ordinals from $\mathcal{OT}$. It turns out that the crucial property that we require of an extension $\hat{\Psi}^{\alpha}_{\mathbb{X}}$ of $\Psi^{\alpha}_{\mathbb{X}}$ is that every ordinal of the collapsed interval which belongs to $\mathcal{OT}$ (i.e. $\mathcal{OT} \cap [\Psi^{\alpha}_{\mathbb{X}}(\pi), \Psi^{\alpha}_{\mathbb{X}}(\delta)]$) is the image of an ordinal in $\mathcal{OT} \cap [\pi, \delta]$ under $\hat{\Psi}^{\alpha}_{\mathbb{X}}$.

Convention. From now on we shall only deal with ordinals and projection instances belonging to $\mathcal{OT}$. Functions $\Psi^{\alpha}_{\mathbb{X}}$ will always be assumed to satisfy $\Psi^{\alpha}_{\mathbb{X}}(\pi) < \pi$, where $\mathbb{X}$ has an interval of the form $[\pi, \delta]$.

Definition 6.1 Let $\mathbb{X}$ be a projection instance with interval $[\pi, \delta]$. Let $\kappa = \Psi^{\alpha}_{\mathbb{X}}(\pi)$ and $\eta = \Psi^{\alpha}_{\mathbb{X}}(\delta)$. The function

$$\ell^{\alpha}_{\mathbb{X}} : \mathcal{OT} \cap [0, \eta] \longrightarrow \mathcal{OT} \cap [0, \delta]$$

is defined as follows:

1. $\ell^{\alpha}_{\mathbb{X}}(\sigma) = \sigma$ for every $\sigma < \kappa$.

2. If $\sigma = \Psi^{\alpha}_{\mathbb{X}}(\rho)$, where $\rho \in \mathbf{dom}(\Psi^{\alpha}_{\mathbb{X}})$ then $\ell^{\alpha}_{\mathbb{X}}(\sigma) = \rho$.

3. $\ell^{\alpha}_{\mathbb{X}}(\sigma) = \omega^{\ell^{\alpha}_{\mathbb{X}}(\sigma_1)} + \ell^{\alpha}_{\mathbb{X}}(\sigma_2)$ if $\sigma =_{NF} \omega^{\sigma_1} + \sigma_2$ and $\sigma \geq \kappa$.

4. $\ell^{\alpha}_{\mathbb{X}}(\sigma) = \varphi\ell^{\alpha}_{\mathbb{X}}(\sigma_1)\ell^{\alpha}_{\mathbb{X}}(\sigma_2)$ if $\sigma =_{NF} \varphi\sigma_1\sigma_2$ and $\sigma \geq \kappa$.

5. Suppose $\sigma = \Psi^{\beta}_{\mathbb{Y}}(\nu) \geq \kappa$, where $\beta > \alpha$. Then $\nu \in \mathbf{Veb}_{Cl}$ and hence $\nu = \Psi^{\alpha}_{\mathbb{X}}(\tau)$ for some τ. But then there exists a projection instance $\mathbb{Z} \in \Re^{<\alpha}$ of the form $(\tau; \mathsf{RSC}; \dots)$. We then define
$$\ell^{\alpha}_{\mathbb{X}}(\sigma) = \Psi^{\beta}_{\mathbb{Z}}(\tau).$$

Finally, we define the function $\hat{\Psi}^{\alpha}_{\mathbb{X}}$ with $\mathbf{dom}(\hat{\Psi}^{\alpha}_{\mathbb{X}}) = \mathbf{ran}(\ell^{\alpha}_{\mathbb{X}})$ by letting

$$\hat{\Psi}^{\alpha}_{\mathbb{X}}(\rho) = (\ell^{\alpha}_{\mathbb{X}})^{-1}(\rho).$$

Owing to its definition, $\hat{\Psi}^{\alpha}_{\mathbb{X}}$ is an extension of $\Psi^{\alpha}_{\mathbb{X}}$ and $\left(\hat{\Psi}^{\alpha}_{\mathbb{X}}\right)^{-1} = \ell^{\alpha}_{\mathbb{X}}$.

Lemma 6.2 *Under the assumptions of Definition 6.1 the mapping $\sigma \mapsto \ell^\alpha_{\mathbb{X}}(\sigma)$ is defined for all $\sigma \in \mathcal{OT} \cap [0,\delta]$. Moreover, $\ell^\alpha_{\mathbb{X}}$ is order-preserving and thus $\hat{\Psi}^\alpha_{\mathbb{X}}$ is order-preserving as well.*

Proof: This ensues from the way $\ell^\alpha_{\mathbb{X}}$ has been defined. A formal proof proceeds by induction on the generation of σ. □

Definition 6.3 Suppose $\mathbb{U}$ is a projection instance of the form $(\bar\kappa; \mathsf{M}^{<\gamma}_{\mathbb{V}}\text{-}\delta\text{-}\mathsf{P}_n; \dots)$ with interval $[\bar\kappa, \delta]$, where $\mathbb{V}$ has an interval $[\mu, \rho]$. With the help of the lifting functions of Definition 6.1 we would like to define a lifting function

$$\ell_{\mathbb{U},\mathbb{V}} : \mathcal{OT} \cap [0,\delta] \longrightarrow \mathcal{OT} \cap [0,\rho].$$

The definition of $\ell_{\mathbb{U},\mathbb{V}}$ proceeds by induction on the complexity of $\mathbb{U}$ as follows:

1. Suppose
$$\mathbb{U} = (\bar\kappa; \mathsf{M}^{<\alpha}_{\mathbb{C}(\delta)}\text{-}\delta'\text{-}\mathsf{P}_0; \mathsf{M}^{\alpha}_{\mathbb{C}(\delta)}; \dots),$$
where $\mathbb{V} = \mathbb{C}(\delta)$ is a projection instance of the form $\mathbb{C}(\delta) = (\pi; \mathsf{M}^{<\gamma}_{\mathbb{A}(\rho)}\text{-}\delta\text{-}\mathsf{P}_0; \dots)$ and $\bar\kappa = \Psi^\alpha_{\mathbb{C}(\delta)}(\pi)$. Then we define
$$\ell_{\mathbb{U},\mathbb{V}} := \ell^\alpha_{\mathbb{C}(\delta)}.$$

2. Suppose
$$\mathbb{U} = (\bar\kappa; \mathsf{M}^{<\gamma}_{\mathbb{A}(\rho)}\text{-}\delta'\text{-}\mathsf{P}_0; \mathsf{M}^{\alpha}_{\mathbb{C}(\delta)}; \dots)$$
and $\mathbb{V} = \mathbb{A}(\rho)$, where $\mathbb{C}(\delta)$ is a projection instance of the form $\mathbb{C}(\delta) = (\pi; \mathsf{M}^{<\gamma}_{\mathbb{A}(\rho)}\text{-}\delta\text{-}\mathsf{P}_0; \dots)$ and $\bar\kappa = \Psi^\alpha_{\mathbb{C}(\delta)}(\pi)$. Then $\ell_{\mathbb{C}(\delta),\mathbb{V}}$ is already defined and we let
$$\ell_{\mathbb{U},\mathbb{V}} := \ell_{\mathbb{C}(\delta),\mathbb{V}} \circ \ell^\alpha_{\mathbb{C}(\delta)}.$$

3. The general form of the first case is that
$$\mathbb{U} = (\bar\kappa; \mathsf{M}^{<\alpha}_{\mathbb{V}}\text{-}\delta'\text{-}\mathsf{P}_n; \mathsf{M}^{\alpha}_{\mathbb{V}}; \dots),$$
where $\mathbb{V}$ is a projection instance of the form $(\pi; \mathsf{M}^{<\gamma}_{\mathbb{Y}}\text{-}\delta\text{-}\mathsf{P}_m; \dots)$ and $\bar\kappa = \Psi^\alpha_{\mathbb{V}}(\pi)$. Then we define
$$\ell_{\mathbb{U},\mathbb{V}} := \ell^\alpha_{\mathbb{V}}.$$

4. The general form of the second case is that
$$\mathbb{U} = (\bar\kappa; \mathsf{M}^{<\gamma}_{\mathbb{V}}\text{-}\delta'\text{-}\mathsf{P}_n; \mathsf{M}^{\alpha}_{\mathbb{Z}}; \dots)$$
and $\mathbb{Z} \neq \mathbb{V}$, where $\mathbb{Z}$ is a projection instance of the form $(\pi; \mathsf{M}^{<\gamma}_{\mathbb{V}}\text{-}\delta\text{-}\mathsf{P}_m; \dots)$ and $\bar\kappa = \Psi^\alpha_{\mathbb{Z}}(\pi)$. Then $\ell_{\mathbb{Z},\mathbb{V}}$ is already defined and we let
$$\ell_{\mathbb{U},\mathbb{V}} := \ell_{\mathbb{Z},\mathbb{V}} \circ \ell^\alpha_{\mathbb{Z}}.$$

7 The Calculus $RS(\mathcal{OT})$

It is well known that the axioms of Peano Arithmetic, **PA**, can be derived in a sequent calculus, $\mathbf{PA}_\omega$, augmented by an infinitary rule, the so–called *ω–rule*[2]

$$\frac{\Gamma, A(\bar{n}) \text{ for all } n}{\Gamma, \forall x A(x)}.$$

An ordinal analysis for **PA** is then attained as follows:

- Each **PA**–proof can be "unfolded" into a $\mathbf{PA}_\omega$–proof of the same sequent.
- Each such $\mathbf{PA}_\omega$–proof can be transformed into a cut–free $\mathbf{PA}_\omega$–proof of the same sequent of length $< \varepsilon_0$.

In order to obtain a similar result for set theories like **KP**, we have to work a bit harder. Guided by the ordinal analysis of **PA**, we would like to invent an infinitary rule which, when added to **KP**, enables us to eliminate cuts. As opposed to the natural numbers, it is not clear how to associate a canonical name with each element of the set–theoretic universe. However, within the confines of the constructible universe, which is made from the ordinals, it is pretty obvious how to "name" sets once we have names for ordinals at our disposal. Recall that $\mathbf{L}_\alpha$, the α^{th} level of Gödel's constructible hierarchy $\mathbf{L}$, is defined by $\mathbf{L}_0 = \emptyset$, $\mathbf{L}_\lambda = \bigcup\{\mathbf{L}_\beta : \beta < \lambda\}$ for limits λ, and $\mathbf{L}_{\beta+1} = \big\{X : X \subseteq \mathbf{L}_\beta;\ X \text{ definable over } \langle \mathbf{L}_\beta, \in \rangle\big\}$. So any element of $\mathbf{L}$ of level α is definable withn $\mathbf{L}_\alpha$ from elements of $\mathbf{L}$ with levels $< \alpha$.

7.1 The Language of $RS(\mathcal{OT})$

Henceforth, we shall restrict ourselves to ordinals from $\mathcal{OT}$.

Recall that variables $\mathbb{I}, \mathbb{J}, \mathbb{K}, \mathbb{I}^*, \mathbb{K}^*$ (also with indices) are assumed to range over reflection instances.

Definition 7.1 The *$\mathcal{L}_M$-formulae* and the *$RS(\mathcal{OT})$–terms* are generated simultaneously. Each term has a *level* which is an ordinal of $\mathcal{OT}$. We shall denote the level of an $RS(\mathcal{OT})$–term t by $|t|$; $t \in \mathcal{T}(\alpha)$ stands for $|t| < \alpha$ and $t \in \mathcal{T}$ for $t \in \mathcal{T}(\mathbf{I})$.

Whenever $\mathbb{K}$ is a reflection instance in $\mathcal{OT}$ with interval $[\pi, \delta]$ and $t_1, \ldots, t_r$ $(r \geq 1)$ are distinct terms such that $\pi \leq |t_i| < \delta$ or t_i is $\mathbb{L}_\delta$, where $1 \leq i \leq r$, then we expand the language of set theory, $\mathcal{L}$, by a new relation symbol $M_{\mathbb{K}}^{t_1,\ldots,t_r}$. The augmented language will be denoted by $\mathcal{L}_M$.

The *atomic formulae* of $\mathcal{L}_M$ are those of either form $(a \in b)$, $\neg(a \in b)$, $M_{\mathbb{K}}^{t_1,\ldots,t_r}(a_1, \ldots, a_r)$, or $\neg M_{\mathbb{K}}^{t_1,\ldots,t_r}(a_1, \ldots, a_r)$, where $a_1, \ldots, a_r$ are free variables. The *$\mathcal{L}_M$–formulae* are obtained from atomic ones by closing off under $\wedge, \vee, (\exists x \in a), (\forall x \in a), \exists x$, and $\forall x$.

The *$RS(\mathcal{OT})$–terms* and their *levels* are generated as follows.

1. For each $\alpha \leq \mathbf{I}$, $\mathbb{L}_\alpha$ is an $RS(\mathcal{OT})$–term of level α.
2. Let $\alpha < \mathbf{I}$. The formal expression $[x \in \mathbb{L}_\alpha : F(x, s_1, \cdots s_n)^{\mathbb{L}_\alpha}]$ is an $RS(\mathcal{OT})$–term of level α if $F(a, b_1, \cdots, b_n)$ is an $\mathcal{L}_M$–formula with all free variables exhibited and $s_1, \cdots, s_n$ are $RS(\mathcal{OT})$–terms with levels $< \alpha$.

[2] $\bar{n}$ stands for the n^{th} numeral

The $RS(\mathcal{OT})$*–formulae* are the expressions of the form $F(s_1, \dots, s_n)^{\mathbb{L}_{\mathbf{I}}}$, where $F(a_1, \dots, a_n)$ is an $\mathcal{L}_M$–formula, $s_1, \dots, s_n \in \mathcal{T}$, and $|s_1|, \dots, |s_n| < \mathbf{I}$.

For technical convenience, we let $\neg A$ be the formula in negation normal form which arises from A by (i) putting $\neg$ in front of each atomic formula, (ii) replacing $\wedge, \vee, (\forall x \in a), (\exists x \in a)$ by $\vee, \wedge, (\exists x \in a), (\forall x \in a)$, respectively, and (iii) dropping double negations.

Convention: In the sequel, $RS(\mathcal{OT})$–formulae will be referred to as formulae. The same usage applies to $RS(\mathcal{OT})$–terms.

Definition 7.2 For every ordinal α of the representation system we define a set of ordinals $k(\alpha)$ as follows:

(k1) $k(0) = k(\Xi) = k(\mathbf{I}) = k(\omega^{+(n)}) == k(\Xi^{+(n)}) = \emptyset$.

(k2) If $\alpha =_{NF} \omega^{\alpha_1} + \alpha_2$ or $\alpha =_{NF} \varphi\alpha_1\alpha_2$, then $k(\alpha) = \bigcup_{1 \leq i \leq 2} k(\alpha_i)$.

(k3) If $\alpha = \Psi^{\beta}_{\mathbb{X}}(\rho)^{+(n)}$, then $k(\alpha) = k(\Psi^{\beta}_{\mathbb{X}}(\rho))$.

(k4) If $\alpha = \Psi^{\beta}_{\mathbb{X}}(\rho)$, where $\mathbb{X} \in \Re^{\beta}$ is a projection instance with interval $[\pi, \delta]$ and $\rho \in \mathbf{dom}(\Psi^{\alpha}_{\mathbb{X}})$, then $k(\alpha) = \{\alpha, \Psi^{\beta}_{\mathbb{X}}(\pi), \Psi^{\beta}_{\mathbb{X}}(\delta)\}$.

Definition 7.3 If ℓ is a projection instance, a reflection instance, a term, or a formula we want to define the set $k(\ell)$ of relevant ordinals occurring in ℓ. This is done inductively.

1. If $\mathbb{X}$ is a projection instance not of the form $(\pi; \mathsf{M}^{\xi}_{\mathbb{Y}}; \dots)$, then $k(\mathbb{X}) = \{\pi, \delta\}$, where $\mathbb{X}$ has an interval of the form $[\pi, \delta]$.
2. If $\mathbb{X}$ is a projection instance of the form $(\pi; \mathsf{M}^{\xi}_{\mathbb{Y}}; \dots)$, then $k(\mathbb{X}) = \{\pi, \xi\}$, where $\mathbb{X}$ has an interval of the form $[\pi, \delta]$.
3. If $\mathbb{K}$ is a reflection instance of the form $(\mathbb{Y}, \xi)$, then $k(\mathbb{K}) = k(\mathbb{Y}) \cup \{\xi\}$.
4. If $\mathbb{K}$ is a reflection instance of the form $(\mathbb{Y}, \xi, \gamma)$ then $k(\mathbb{K}) = k(\mathbb{Y}) \cup \{\xi, \gamma\}$.
5. If t is a term $\mathbb{L}_\alpha$ then $k(t) = k(\alpha)$.
6. If t is of the form $[x \in \mathbb{L}_\alpha : F(x, s_1, \cdots s_n)^{\mathbb{L}_\alpha}]$, where $F(a, b_1, \dots, b_n)$ is an $\mathcal{L}_M$-formula with all free variables exhibited, then $k(t) = \{\alpha\} \cup k(s_1) \cup \dots \cup k(s_n)$.
7. $k(s \in t) = k(\neg(s \in t)) = k(s) \cup k(t)$.
8. $k(M^{t_1,\dots,t_r}_{\mathbb{K}}(s_1, \dots, s_r)) = k(\neg M^{t_1,\dots,t_r}_{\mathbb{K}}(s_1, \dots, s_r)) = k(t_1) \cup \dots \cup k(t_r) \cup k(s_1) \cup \dots \cup k(s_r) \cup k(\mathbb{K})$.
9. $k(A \wedge B) = k(A \vee B) = k(A) \cup k(B)$.
10. $k((\forall x \in s)F(s)) = k((\exists x \in s)F(s)) = k(F(\mathbb{L}_0)) \cup k(s)$.

For technical convenience, we put $k(0) := k(1) := \emptyset$.

We set $|\ell| := max(k(\ell) \cup \{0\})$ and $|0| := |1| := 0$.

If $\mathcal{A}$ is a finite set consisting of objects of the above kind, put

$$k(\mathcal{A}) := \bigcup\{k(\ell) : \ell \in \mathcal{A}\}$$

and

$$|k(\mathcal{A})| := sup\{|k(\ell)| : \ell \in \mathcal{A}\}.$$

If ℓ is a projection instance, a reflection instance, a term, or a formula, we set

$$c(\ell) := \{\alpha : \alpha \text{ occurs in } \ell\ \}.$$

Here any occurrence of α (also those inside of terms and reflection patterns) has to be considered.

Definition 7.4 We use the symbol $\equiv$ to convey syntactical identity. For terms s, t with $|s| < |t|$ we set

$$s \mathrel{\overset{\circ}{\in}} t \equiv \begin{cases} B(s) & \text{if} \quad t \equiv [x \in \mathbb{L}_\beta : B(x)] \\ s \notin \mathbb{L}_0 & \text{if} \quad t \equiv \mathbb{L}_\beta. \end{cases}$$

Observe that $s \in t$ and $s \mathrel{\overset{\circ}{\in}} t$ have the same truth value under the intended interpretation in the constructible hierarchy.

7.2 The meaning of the predicates $M_{\mathbb{K}}^{t_1,\cdots,t_r}$

To explain the meaning of the relation symbols $M_{\mathbb{K}}^{t_1,\cdots,t_r}$ we assign a set of r-tuples, $\hat{\mathcal{M}}_{\mathbb{K}}^{t_1,\cdots,t_r}$, to $M_{\mathbb{K}}^{t_1,\cdots,t_r}$. But beforehand we need to introduce a substitution operation on terms.

Definition 7.5 Let $\mathbb{X}$ be projection instance in $\mathfrak{R}^\alpha$ with interval $[\pi, \delta]$. Furthermore suppose $\Psi_{\mathbb{X}}^{\alpha} \downarrow$. Let t be an $RS(\mathcal{OT})$-term such that $|t| \leq \delta$. The partial mapping $t \mapsto t^{\mathbb{X},\alpha}$ is defined by induction on the generation of t as follows:

1. If $t \equiv \mathbb{L}_\eta$ with $\eta < \pi$ then $t^{\mathbb{X},\alpha} := t$.

2. If $t \equiv \mathbb{L}_\eta$ with $\eta \geq \pi$ and $\eta \in \mathbf{dom}(\hat{\Psi}_{\mathbb{X}}^{\alpha})$, then $t^{\mathbb{X},\alpha} := \mathbb{L}_{\hat{\Psi}_{\mathbb{X}}^{\alpha}(\eta)}$.

3. If t is of the form $[x \in \mathbb{L}_\eta : F(x, s_1, \cdots s_n)^{\mathbb{L}_\eta}]$ with $\eta < \pi$ then $t^{\mathbb{X},\alpha} := t$.

4. Let t be of the form $[x \in \mathbb{L}_\eta : F(x, s_1, \cdots s_n)^{\mathbb{L}_\eta}]$ with $\eta \geq \pi$ such that $\eta \in \mathbf{dom}(\hat{\Psi}_{\mathbb{X}}^{\alpha})$ and $s_i^{\mathbb{X},\alpha}$ is defined for $1 \leq i \leq r$, then

$$t^{\mathbb{X},\alpha} \ := \ [x \in \mathbb{L}_{\hat{\Psi}_{\mathbb{X}}^{\alpha}(\eta)} : F\big(x, s_1^{\mathbb{X},\alpha}, \cdots s_n^{\mathbb{X},\alpha}\big)^{\mathbb{L}_{\hat{\Psi}_{\mathbb{X}}^{\alpha}(\eta)}}]$$

We shall write $t \in \mathbf{dom}(\hat{\Psi}_{\mathbb{X}}^{\alpha})$ or $\hat{\Psi}_{\mathbb{X}}^{\alpha}(t) \downarrow$ if $t^{\mathbb{X},\alpha}$ is defined, and in the latter case we also denote $t^{\mathbb{X},\alpha}$ by $\hat{\Psi}_{\mathbb{X}}^{\alpha}(t)$.

Definition 7.6 Suppose $\mathbb{U}$ is a projection instance of the form $(\bar{\kappa}; \mathsf{M}_{\mathbb{V}}^{<\gamma}\text{-}\delta\text{-}\mathsf{P}_n; \dots)$ with interval $[\bar{\kappa}, \delta]$, where $\mathbb{V}$ has an interval $[\mu, \rho]$. Suppose t be an $RS(\mathcal{OT})$-term such that $|t| \leq \delta$. With the help of the lifting function $\ell_{\mathbb{U},\mathbb{V}}$ of Definition 6.3 we can then lift the term t to a term $\ell_{\mathbb{U},\mathbb{V}}(t)$ with $|\ell_{\mathbb{U},\mathbb{V}}(t)| \leq \rho$ in the obvious way. Formally, this is done by induction on the generation of t similar to Definition 7.5.

Definition 7.7 Let $\mathbb{K}$ be a reflection instance and $M_{\mathbb{K}}^{t_1,\dots,t_r}$ be a relation symbol.

1. If $\mathbb{K}$ is of either form $(\pi; \mathsf{RLC}; \dots)$, $(\pi; \mathsf{RSC}; \dots)$, or $(\pi; 0\text{-}\mathsf{P}_m; \dots)$ with interval $[\pi, \pi]$. Then $r = 1$ and $t_1 \equiv \mathbb{L}_\pi$. We then put

$$\hat{\mathcal{M}}_{\mathbb{K}}^{t_1} \;:=\; \{\mathbb{L}_\eta : \eta < \pi\}.$$

2. If $\mathbb{K}$ is of the form $(\pi; \delta\text{-}\mathsf{P}_m; \dots; \beta)$, where $\delta > 0$, then

$$\hat{\mathcal{M}}_{\mathbb{K}}^{t_1,\dots,t_r} \;:=\; \{(t_1^{\mathbb{X},\alpha}, \dots, t_r^{\mathbb{X},\alpha}) : \beta \leq \alpha;\ \Psi_{\mathbb{X}}^{\alpha}\downarrow;\ \mathbb{X} \text{ is projection instance with interval } [\pi, \rho], \text{ where } \rho \geq \delta \text{ and } \delta, t_1, \dots, t_r \in \mathbf{dom}(\hat{\Psi}_{\mathbb{X}}^{\alpha})\}.$$

3. Suppose $\mathbb{K} = (\mathbb{C}\,(\delta), \xi, \eta)$ and

$$\mathbb{C}\,(\delta) \;=\; (\bar{\kappa}; \mathsf{M}_{\mathbb{A}\,(\rho)}^{<\gamma}\text{-}\bar{\delta}\text{-}\mathsf{P}_0; \mathsf{M}_{\mathbb{Y}}^{\zeta}; \mathbb{Z}; \alpha_0),$$

   where $\bar{\kappa} = \Psi_{\mathbb{A}\,(\rho)}^{\gamma}(\pi)$, $[\pi, \rho]$ is the interval of $\mathbb{A}\,(\rho)$, and $\mathbf{dom}(\mathbb{A}\,) \neq (0, \omega)$.

   Put $s_1 := \ell_{\mathbb{C}\,(\delta),\mathbb{A}\,(\rho)}(t_1), \dots, s_r := \ell_{\mathbb{C}\,(\delta),\mathbb{A}\,(\rho)}(t_r)$. Let $\mathcal{F}_1$ be the set of all functions $\hat{\Psi}_{\mathbb{X}}^{\alpha}$, where $\mathbb{X}$ is of either form $\mathbb{A}\,(\rho')$ or $(\pi; \mathsf{M}_{\mathbb{A}\,(\rho')}^{\xi'}; \dots; \gamma')$ such that $\Psi_{\mathbb{X}}^{\alpha}\downarrow$, $\Psi_{\mathbb{X}}^{\alpha}(\pi) < \bar{\kappa}$, $\xi' \geq \xi$, $\rho' \geq \eta$, and $\xi, \rho \in C(\xi, \Psi_{\mathbb{X}}^{\alpha}(\pi))$ as well as $s_1, \dots, s_r \in \mathbf{dom}(\hat{\Psi}_{\mathbb{X}}^{\alpha})$. Then

$$\hat{\mathcal{M}}_{\mathbb{K}}^{t_1,\dots,t_r} \;:=\; \{(s_1^{\mathbb{X},\alpha}, \dots, s_r^{\mathbb{X},\alpha}) : \hat{\Psi}_{\mathbb{X}}^{\alpha} \in \mathcal{F}_1\}.$$

4. Suppose $\mathbb{K} = (\mathbb{C}\,(m), \xi, \eta)$, where

$$\mathbb{C}\,(m) \;=\; (\bar{\kappa}; \mathsf{M}_{\mathbb{A}\,(\rho)}^{<\gamma}\text{-}\delta\text{-}\mathsf{P}_m; \mathsf{M}_{\mathbb{Y}}^{\zeta}; \mathbb{Z}; \alpha_0),$$

   $[\pi, \rho]$ is the interval of $\mathbb{A}\,(\rho)$, and $\mathbf{dom}(\mathbb{A}\,) \neq (0, \omega)$.

   Put $s_1 := \ell_{\mathbb{C}\,(m),\mathbb{A}\,(\rho)}(t_1), \dots, s_r := \ell_{\mathbb{C}\,(m),\mathbb{A}\,(\rho)}(t_r)$. Let $\mathcal{F}_2$ be the set of all functions $\hat{\Psi}_{\mathbb{X}}^{\alpha}$, where $\mathbb{X}$ is of either form $\mathbb{A}\,(\rho')$ or $(\pi; \mathsf{M}_{\mathbb{A}\,(\rho')}^{\xi'}; \dots; \gamma')$ such that $\Psi_{\mathbb{X}}^{\alpha}\downarrow$, $\Psi_{\mathbb{X}}^{\alpha}(\pi) < \bar{\kappa}$, $\xi' \geq \xi$, $\rho' \geq \eta$, and $\xi, \rho \in C(\xi, \Psi_{\mathbb{X}}^{\alpha}(\pi))$ as well as $s_1, \dots, s_r \in \mathbf{dom}(\hat{\Psi}_{\mathbb{X}}^{\alpha})$. Then

$$\hat{\mathcal{M}}_{\mathbb{K}}^{t_1,\dots,t_r} \;:=\; \{(s_1^{\mathbb{X},\alpha}, \dots, s_r^{\mathbb{X},\alpha}) : \hat{\Psi}_{\mathbb{X}}^{\alpha} \in \mathcal{F}_2\}.$$

5. Suppose $\mathbb{K} = (\bar{\kappa}; \mathsf{M}_{\mathbb{A}\,(\eta)}^{\xi}\text{-}\bar{\kappa}\text{-}\mathsf{P}_0; \mathsf{M}_{\mathbb{Y}}^{\zeta}; \mathbb{Z}; \alpha_0)$, where $[\pi, \eta]$ is the interval of $\mathbb{A}\,(\eta)$ and $\mathbf{dom}(\mathbb{A}\,) \neq (0, \omega)$. Then $r = 1$ and $t_1 \equiv \mathbb{L}_{\bar{\kappa}}$.

   Let $\mathcal{F}_3$ be the set of all functions $\hat{\Psi}_{\mathbb{X}}^{\alpha}$, where $\mathbb{X}$ is of either form $\mathbb{A}\,(\rho')$ or $(\pi; \mathsf{M}_{\mathbb{A}\,(\rho')}^{\xi'}; \dots; \gamma')$ such that $\Psi_{\mathbb{X}}^{\alpha}\downarrow$, $\Psi_{\mathbb{X}}^{\alpha}(\pi) < \bar{\kappa}$, $\xi' \geq \xi$, $\rho' \geq \eta$, and $\xi, \eta \in C(\xi, \Psi_{\mathbb{X}}^{\alpha}(\pi))$. We then put

$$\hat{\mathcal{M}}_{\mathbb{K}}^{t_1} \;:=\; \{\mathbb{L}_{\Psi_{\mathbb{X}}^{\alpha}(\pi)} : \Psi_{\mathbb{X}}^{\alpha} \in \mathcal{F}_3\}.$$

6. Suppose $\mathbb{K} = (\mathbb{C}(\delta), \xi, k)$, where

$$\mathbb{C}(\delta) \;=\; (\bar{\kappa}; \mathsf{M}^{<\gamma}_{\mathbb{A}(m)}\text{-}\delta\text{-}\mathsf{P}_0; \mathsf{M}^{\zeta}_{\mathbb{Y}}; \mathbb{Z}; \alpha_0),$$

$[\pi, \rho]$ is the interval of $\mathbb{A}(m)$, and $\mathbf{dom}(\mathbb{A}) = (0, \omega)$.

Put $s_1 := \ell_{\mathbb{C}(\delta),\mathbb{A}(m)}(t_1), \ldots, s_r := \ell_{\mathbb{C}(\delta),\mathbb{A}(m)}(t_r)$. Let $\mathcal{F}_4$ be the set of all functions $\hat{\Psi}^{\alpha}_{\mathbb{X}}$, where $\mathbb{X}$ is of either form $\mathbb{A}(m')$ or $(\pi; \mathsf{M}^{\xi'}_{\mathbb{A}(m')}; \ldots; \gamma')$ such that $\Psi^{\alpha}_{\mathbb{X}}\downarrow$, $\Psi^{\alpha}_{\mathbb{X}}(\pi) < \bar{\kappa}$, $\xi' \geq \xi$, $m' \geq k$, and $\xi \in C(\xi, \Psi^{\alpha}_{\mathbb{X}}(\pi))$ as well as $s_1, \ldots, s_r \in \mathbf{dom}(\hat{\Psi}^{\alpha}_{\mathbb{X}})$. Then

$$\hat{\mathcal{M}}^{t_1,\ldots,t_r}_{\mathbb{K}} \;:=\; \{(s_1^{\mathbb{X},\alpha}, \ldots, s_r^{\mathbb{X},\alpha}) : \; \hat{\Psi}^{\alpha}_{\mathbb{X}} \in \mathcal{F}_4\}.$$

7. Suppose $\mathbb{K} = (\mathbb{C}(m), \xi, k)$, where

$$\mathbb{C}(m) \;=\; (\bar{\kappa}; \mathsf{M}^{<\gamma}_{\mathbb{A}(l)}\text{-}\delta\text{-}\mathsf{P}_m; \mathsf{M}^{\zeta}_{\mathbb{Y}}; \mathbb{Z}; \alpha_0),$$

$[\pi, \rho]$ is the interval of $\mathbb{A}(l)$, and $\mathbf{dom}(\mathbb{A}) = (0, \omega)$.

Put $s_1 := \ell_{\mathbb{C}(m),\mathbb{A}(l)}(t_1), \ldots, s_r := \ell_{\mathbb{C}(m),\mathbb{A}(l)}(t_r)$. Let $\mathcal{F}_5$ be the set of all functions $\hat{\Psi}^{\alpha}_{\mathbb{X}}$, where $\mathbb{X}$ is of either form $\mathbb{A}(k)$ or $(\pi; \mathsf{M}^{\xi'}_{\mathbb{A}(k)}; \ldots; \gamma')$ such that $\Psi^{\alpha}_{\mathbb{X}}\downarrow$, $\Psi^{\alpha}_{\mathbb{X}}(\pi) < \bar{\kappa}$, $\xi' \geq \xi$, $k \geq l$, and $\xi \in C(\xi, \Psi^{\alpha}_{\mathbb{X}}(\pi))$ as well as $s_1, \ldots, s_r \in \mathbf{dom}(\hat{\Psi}^{\alpha}_{\mathbb{X}})$. Then

$$\hat{\mathcal{M}}^{t_1,\ldots,t_r}_{\mathbb{K}} \;:=\; \{(s_1^{\mathbb{X},\alpha}, \ldots, s_r^{\mathbb{X},\alpha}) : \; \hat{\Psi}^{\alpha}_{\mathbb{X}} \in \mathcal{F}_5\}.$$

8. Suppose $\mathbb{K} = (\bar{\kappa}; \mathsf{M}^{\xi}_{\mathbb{A}(l)}\text{-}\bar{\kappa}\text{-}\mathsf{P}_0; \mathsf{M}^{\zeta}_{\mathbb{Y}}; \mathbb{Z}; \alpha_0)$, where $[\pi, \rho]$ is the interval of $\mathbb{A}(l)$ and $\mathbf{dom}(\mathbb{A}) = (0, \omega)$. Then $r = 1$ and $t_1 \equiv \mathbb{L}_{\bar{\kappa}}$.

Let $\mathcal{F}_6$ be the set of all functions $\hat{\Psi}^{\alpha}_{\mathbb{X}}$, where $\mathbb{X}$ is of either form $\mathbb{A}(k)$ or $(\pi; \mathsf{M}^{\xi'}_{\mathbb{A}(k)}; \ldots; \gamma')$ such that $\Psi^{\alpha}_{\mathbb{X}}\downarrow$, $\Psi^{\alpha}_{\mathbb{X}}(\pi) < \bar{\kappa}$, $\xi' \geq \xi$, $k \geq l$, and $\xi \in C(\xi, \Psi^{\alpha}_{\mathbb{X}}(\pi))$. We then put

$$\hat{\mathcal{M}}^{t_1}_{\mathbb{K}} \;:=\; \{\mathbb{L}_{\Psi^{\alpha}_{\mathbb{X}}(\pi)} : \; \Psi^{\alpha}_{\mathbb{X}} \in \mathcal{F}_6\}.$$

9. Suppose $\mathbb{K} = (\mathbb{C}(\delta), \xi)$, where

$$\mathbb{C}(\delta) \;=\; (\bar{\kappa}; \mathsf{M}^{<\gamma}_{\mathbb{A}}\text{-}\delta\text{-}\mathsf{P}_0; \mathsf{M}^{\zeta}_{\mathbb{Y}}; \mathbb{Z}; \alpha_0),$$

where $\mathbb{A}$ is a constant projection instance with interval $[\pi, \rho]$.

Put $s_1 := \ell_{\mathbb{C}(\delta),\mathbb{A}}(t_1), \ldots, s_r := \ell_{\mathbb{C}(\delta),\mathbb{A}}(t_r)$. Let $\mathcal{F}_7$ be the set of all functions $\hat{\Psi}^{\alpha}_{\mathbb{X}}$, where $\mathbb{X}$ is of either form $\mathbb{A}$ or $(\pi; \mathsf{M}^{\xi'}_{\mathbb{A}}; \ldots; \gamma')$ such that $\Psi^{\alpha}_{\mathbb{X}}\downarrow$, $\Psi^{\alpha}_{\mathbb{X}}(\pi) < \bar{\kappa}$, $\xi' \geq \xi$, and $\xi \in C(\xi, \Psi^{\alpha}_{\mathbb{X}}(\pi))$ as well as $s_1, \ldots, s_r \in \mathbf{dom}(\hat{\Psi}^{\alpha}_{\mathbb{X}})$. Then

$$\hat{\mathcal{M}}^{t_1,\ldots,t_r}_{\mathbb{K}} \;:=\; \{(s_1^{\mathbb{X},\alpha}, \ldots, s_r^{\mathbb{X},\alpha}) : \; \hat{\Psi}^{\alpha}_{\mathbb{X}} \in \mathcal{F}_7\}.$$

10. Suppose $\mathbb{K} = (\mathbb{C}(m), \xi)$, where

$$\mathbb{C}(m) \;=\; (\bar{\kappa}; \mathsf{M}^{<\gamma}_{\mathbb{A}}\text{-}\delta\text{-}\mathsf{P}_m; \mathsf{M}^{\zeta}_{\mathbb{Y}}; \mathbb{Z}; \alpha_0),$$

where $\mathbb{A}$ is a constant projection instance with interval $[\pi, \rho]$.

Put $s_1 := \ell_{\mathbb{C}(m),\mathbb{A}}(t_1), \ldots, s_r := \ell_{\mathbb{C}(m),\mathbb{A}}(t_r)$. Let $\mathcal{F}_8$ be the set of all functions $\hat{\Psi}^{\alpha}_{\mathbb{X}}$, where $\mathbb{X}$ is of either form $\mathbb{A}$ or $(\pi; \mathsf{M}^{\xi'}_{\mathbb{A}}; \ldots; \gamma')$ such that $\Psi^{\alpha}_{\mathbb{X}}\downarrow$, $\Psi^{\alpha}_{\mathbb{X}}(\pi) < \bar{\kappa}$, $\xi' \geq \xi$, and $\xi \in C(\xi, \Psi^{\alpha}_{\mathbb{X}}(\pi))$ as well as $s_1, \ldots, s_r \in \mathbf{dom}(\hat{\Psi}^{\alpha}_{\mathbb{X}})$. Then

$$\hat{\mathcal{M}}^{t_1,\ldots,t_r}_{\mathbb{K}} \;:=\; \{(s_1^{\mathbb{X},\alpha}, \ldots, s_r^{\mathbb{X},\alpha}) : \; \hat{\Psi}^{\alpha}_{\mathbb{X}} \in \mathcal{F}_8\}.$$

11. Suppose $\mathbb{K} = (\bar{\kappa}; \mathsf{M}^{\xi}_{\mathbb{A}}\text{-}\bar{\kappa}\text{-}\mathsf{P}_0; \mathsf{M}^{\zeta}_{\mathbb{Y}}; \mathbb{Z}; \alpha_0)$, where $\mathbb{A}$ is a constant projection instance with interval $[\pi, \rho]$

    Let $\mathcal{F}_9$ be the set of all functions $\hat{\Psi}^{\alpha}_{\mathbb{X}}$, where $\mathbb{X}$ is of either form $\mathbb{A}$ or $(\pi; \mathsf{M}^{\xi'}_{\mathbb{A}}; \dots; \gamma')$ such that $\Psi^{\alpha}_{\mathbb{X}} \downarrow$, $\Psi^{\alpha}_{\mathbb{X}}(\pi) < \bar{\kappa}$, $\xi' \geq \xi$, and $\xi \in C(\xi, \Psi^{\alpha}_{\mathbb{X}}(\pi))$. We then put

$$\hat{\mathcal{M}}^{t_1}_{\mathbb{K}} \ := \ \{\mathbb{L}_{\Psi^{\alpha}_{\mathbb{X}}(\pi)} : \ \Psi^{\alpha}_{\mathbb{X}} \in \mathcal{F}_9\}.$$

12. Suppose $\mathbb{K} = (\mathbb{C}, \xi, \eta)$, where

$$\mathbb{C} \ = \ (\bar{\kappa}; \mathsf{M}^{<\gamma}_{\mathbb{A}(\rho)}\text{-}\delta\text{-}\mathsf{P}_m; \mathsf{M}^{\zeta}_{\mathbb{Y}}; \mathbb{Z}; \alpha_0),$$

    where $\mathbb{C}$ is a constant projection instance, $[\pi, \rho]$ is the interval of $\mathbb{A}(\rho)$, $\mathbf{dom}(\mathbb{A}) \neq (0, \omega)$, and $m > 0$.

    Put $s_1 := \ell_{\mathbb{C},\mathbb{A}(\rho)}(t_1), \dots, s_r := \ell_{\mathbb{C},\mathbb{A}(\rho)}(t_r)$. Let $\mathcal{F}_{10}$ be the set of all functions $\hat{\Psi}^{\alpha}_{\mathbb{X}}$, where $\mathbb{X}$ is of either form $\mathbb{A}(\rho')$ or $(\pi; \mathsf{M}^{\xi'}_{\mathbb{A}(\rho')}; \dots; \gamma')$ such that $\Psi^{\alpha}_{\mathbb{X}} \downarrow$, $\Psi^{\alpha}_{\mathbb{X}}(\pi) < \bar{\kappa}$, $\xi' \geq \xi$, $\rho' \geq \eta$, and $\xi, \rho \in C(\xi, \Psi^{\alpha}_{\mathbb{X}}(\pi))$ as well as $s_1, \dots, s_r \in \mathbf{dom}(\hat{\Psi}^{\alpha}_{\mathbb{X}})$. Then

$$\hat{\mathcal{M}}^{t_1,\dots,t_r}_{\mathbb{K}} \ := \ \{(s^{\mathbb{X},\alpha}_1, \dots, s^{\mathbb{X},\alpha}_r) : \ \hat{\Psi}^{\alpha}_{\mathbb{X}} \in \mathcal{F}_{10}\}.$$

13. Suppose $\mathbb{K} = (\mathbb{C}, \xi, l)$, where

$$\mathbb{C} \ = \ (\bar{\kappa}; \mathsf{M}^{<\gamma}_{\mathbb{A}(k)}\text{-}\delta\text{-}\mathsf{P}_m; \mathsf{M}^{\zeta}_{\mathbb{Y}}; \mathbb{Z}; \alpha_0),$$

    where $\mathbb{C}$ is a constant projection instance, $[\pi, \rho]$ is the interval of $\mathbb{A}(k)$, $\mathbf{dom}(\mathbb{A}) = (0, \omega)$, and $m > 0$.

    Put $s_1 := \ell_{\mathbb{C},\mathbb{A}(k)}(t_1), \dots, s_r := \ell_{\mathbb{C},\mathbb{A}(k)}(t_r)$. Let $\mathcal{F}_{11}$ be the set of all functions $\hat{\Psi}^{\alpha}_{\mathbb{X}}$, where $\mathbb{X}$ is of either form $\mathbb{A}(k')$ or $(\pi; \mathsf{M}^{\xi'}_{\mathbb{A}(k')}; \dots; \gamma')$ such that $\Psi^{\alpha}_{\mathbb{X}} \downarrow$, $\Psi^{\alpha}_{\mathbb{X}}(\pi) < \bar{\kappa}$, $\xi' \geq \xi$, $k' \geq l$, and $\xi \in C(\xi, \Psi^{\alpha}_{\mathbb{X}}(\pi))$ as well as $s_1, \dots, s_r \in \mathbf{dom}(\hat{\Psi}^{\alpha}_{\mathbb{X}})$. Then

$$\hat{\mathcal{M}}^{t_1,\dots,t_r}_{\mathbb{K}} \ := \ \{(s^{\mathbb{X},\alpha}_1, \dots, s^{\mathbb{X},\alpha}_r) : \ \hat{\Psi}^{\alpha}_{\mathbb{X}} \in \mathcal{F}_{11}\}.$$

14. Suppose $\mathbb{K} = (\mathbb{C}, \xi)$, where

$$\mathbb{C} \ = \ (\bar{\kappa}; \mathsf{M}^{<\gamma}_{\mathbb{A}}\text{-}\delta\text{-}\mathsf{P}_m; \mathsf{M}^{\zeta}_{\mathbb{Y}}; \mathbb{Z}; \alpha_0),$$

    where $\mathbb{C}$ and $\mathbb{A}$ are constant projection instances, $[\pi, \rho]$ is the interval of $\mathbb{A}$, and $m > 0$.

    Put $s_1 := \ell_{\mathbb{C},\mathbb{A}}(t_1), \dots, s_r := \ell_{\mathbb{C},\mathbb{A}}(t_r)$. Let $\mathcal{F}_{12}$ be the set of all functions $\hat{\Psi}^{\alpha}_{\mathbb{X}}$, where $\mathbb{X}$ is of either form $\mathbb{A}$ or $(\pi; \mathsf{M}^{\xi'}_{\mathbb{A}}; \dots; \gamma')$ such that $\Psi^{\alpha}_{\mathbb{X}} \downarrow$, $\Psi^{\alpha}_{\mathbb{X}}(\pi) < \bar{\kappa}$, $\xi' \geq \xi$, and $\xi \in C(\xi, \Psi^{\alpha}_{\mathbb{X}}(\pi))$ as well as $s_1, \dots, s_r \in \mathbf{dom}(\hat{\Psi}^{\alpha}_{\mathbb{X}})$. Then

$$\hat{\mathcal{M}}^{t_1,\dots,t_r}_{\mathbb{K}} \ := \ \{(s^{\mathbb{X},\alpha}_1, \dots, s^{\mathbb{X},\alpha}_r) : \ \hat{\Psi}^{\alpha}_{\mathbb{X}} \in \mathcal{F}_{12}\}.$$

7.3 The Rules of $RS(\mathcal{OT})$

Next we introduce a calculus, $RS(\mathcal{OT})$, with infinitary rules. $A, B, C, \ldots, F(t), G(t), \ldots$ will range over $RS(\mathcal{OT})$–formulae. We denote by upper case Greek letters $\Gamma, \Delta, \Lambda, \ldots$ finite sets of $RS(\mathcal{OT})$–formulae. The intended meaning of $\Gamma = \{A_1, \cdots, A_n\}$ is the disjunction $A_1 \vee \cdots \vee A_n$. Γ, A stands for $\Gamma \cup \{A\}$ etc.. We also use the shorthands $r \neq s := \neg(r = s)$ and $r \notin t := \neg(r \in t)$.

An $\mathcal{L}_{RS}$–formula is said to be $\Delta_0(\alpha)$ or $\Pi_0(\alpha)$ or $\Sigma_0(\alpha)$ if it contains only terms with levels $< \alpha$. For $k > 0$, an $\mathcal{L}_{RS}$–formula A is $\Pi_k(\alpha)$ if it is $\Pi_{k-1}(\alpha)$ or of the form

$$(\forall x_1 \in \mathbb{L}_\alpha) \cdots (Q_k x_k \in \mathbb{L}_\alpha) F(x_1, \ldots, x_k),$$

where the k quantifiers in front are alternating and $F(\mathbb{L}_0, \ldots, \mathbb{L}_0)$ is $\Delta_0(\alpha)$. Analogously, one defines $\Sigma_k(\alpha)$–formulae.

An $\mathcal{L}_{RS}$–formula is said to be $\Sigma(\alpha)$ if it belongs to the smallest collection of formulae which is obtained from the $\Delta_0(\alpha)$-formulae by closing of under $\vee, \wedge$ and quantifiers of the form $(\exists x \in \mathbb{L}_\alpha)$.

Given an $\mathcal{L}_{RS}$–formulae A and terms s, t, we denote by $A^{(s,t)}$ the formula which arises from A by replacing all the quantifiers $(\exists x \in t)$ and $(\forall x \in t)$ by $(\exists x \in s)$ and $(\forall x \in s)$, respectively. To economize on subscripts, we also write $A^{(s,\alpha)}$ for $A^{(s,\mathbb{L}_\alpha)}$ and $A^{(\beta,\alpha)}$ instead of $A^{(\mathbb{L}_\beta,\mathbb{L}_\alpha)}$.

Definition 7.8 The *logical rules* of $RS(\mathcal{OT})$ are:

$(\wedge)$ $$\frac{\Gamma, A \quad \Gamma, A'}{\Gamma, A \wedge A'}$$

$(\vee)$ $$\frac{\Gamma, A_i}{\Gamma, A_0 \vee A_1} \quad \text{if } i = 0 \text{ or } i = 1$$

$(\forall)$ $$\frac{\Gamma, s \mathring{\in} t \rightarrow F(s) \text{ for all } s \in \mathcal{T}(|t|)}{\Gamma, (\forall x \in t) F(x)}$$

$(\exists)$ $$\frac{\Gamma, s \mathring{\in} t \wedge F(s)}{\Gamma, (\exists x \in t) F(x)} \quad \text{if } s \in \mathcal{T}(|t|)$$

$(\notin)$ $$\frac{\Gamma, s \mathring{\in} t \rightarrow r \neq s \text{ for all } s \in \mathcal{T}(|t|)}{\Gamma, r \notin t}$$

$(\in)$ $$\frac{\Gamma, s \mathring{\in} t \ \wedge \ r = s}{\Gamma, r \in t} \quad \text{if } s \in \mathcal{T}(|t|)$$

$(\neg M_{\mathbb{K}}^{t_1,\ldots,t_r})$ $$\frac{\Gamma, \bigvee_{1 \le i \le r} s_i \neq p_i \text{ for all } (s_1, \ldots, s_r) \in \hat{\mathcal{M}}_{\mathbb{K}}^{t_1,\ldots,t_r} \text{ with } |s_i| \le |p_i| \text{ where } 1 \le i \le r}{\Gamma, \neg M_{\mathbb{K}}^{t_1,\ldots,t_r}(p_1, \ldots, p_r)}$$

$(M_{\mathbb{K}}^{t_1,\ldots,t_r})$ $$\frac{\Gamma, \bigwedge_{1 \le i \le r} s_i = p_i}{\Gamma, M_{\mathbb{K}}^{t_1,\ldots,t_r}(p_1, \ldots, p_r)} \quad \text{if } (s_1, \ldots, s_r) \in \hat{\mathcal{M}}_{\mathbb{K}}^{t_1,\ldots,t_r} \text{ and } |s_i| \le |p_i| \text{ where } 1 \le i \le r$$

(Cut) $$\frac{\Gamma, A \qquad \Gamma, \neg A}{\Gamma}$$

Definition 7.9 The *reflection rules* of $RS(\mathcal{OT})$: To introduce the reflection rules of $RS(\mathcal{OT})$ we first associate a collection of formulae, $\mathcal{F}(\mathbb{K})$, with each reflection instance $\mathbb{K}$.

1. If $\mathbb{K}$ is of one of the forms $(\pi;\mathsf{RLC};\ldots)$, $(\pi;\mathsf{RSC};\ldots)$, $(\pi;\mathsf{M}^{\alpha}_{\mathbb{X}}\text{-}\pi\text{-}\mathsf{P}_0;\ldots)$, $(\pi;\pi\text{-}\mathsf{P}_0;\ldots)$, or $\big((\pi;\mathsf{M}^{<\alpha}_{\mathbb{X}}\text{-}\pi\text{-}\mathsf{P}_0;\ldots),\ldots\big)$, then $\mathcal{F}(\mathbb{K})$ consists of the $\Pi_2(\pi)$-formulae.
2. If $\mathbb{K}$ is of the form $(\pi;\pi\text{-}\mathsf{P}_m;\ldots)$, where $m>0$, then $\mathcal{F}(\mathbb{K})$ consists of the $\Pi_{m+2}(\pi)$-formulae.
3. If $\mathbb{K}$ is of the form $(\pi;\rho\text{-}\mathsf{P}_m;\ldots)$, where $\rho>\pi$, then $\mathcal{F}(\mathbb{K})$ consists of the $\Pi_m(\rho)$-formulae.
4. If $\mathbb{K}$ is of the form $\big((\pi;\mathsf{M}^{<\alpha}_{\mathbb{X}}\text{-}\pi\text{-}\mathsf{P}_m;\ldots),\ldots\big)$, then $\mathcal{F}(\mathbb{K})$ consists of the $\Pi_{m+2}(\pi)$-formulae.
5. If $\mathbb{K}$ is of the form $\big((\pi;\mathsf{M}^{<\alpha}_{\mathbb{X}}\text{-}\rho\text{-}\mathsf{P}_m;\ldots),\ldots\big)$, where $\rho>\pi$, then $\mathcal{F}(\mathbb{K})$ consists of the $\Pi_m(\rho)$-formulae.

Note that when $[\pi,\delta]$ is the interval of $\mathbb{K}$, then any formula A in $\mathcal{F}(\mathbb{K})$ is of the form $G(q_1,\ldots,q_n,t_1,\ldots,t_r)$, where $G(a_1,\ldots,a_n,b_1,\ldots,b_r)$ is a Δ_0 formula of $\mathcal{L}_M$ with all free variables among the ones shown such that $q_1,\ldots,q_n,t_1,\ldots,t_r$ are distinct terms, $|q_1|,\ldots,|q_n|<\pi$ and for all $1\le i\le r$, $\pi\le|t_i|<\delta$ or $t_i\equiv\mathbb{L}_\delta$.

In the following, when we refer to a formula A of $\mathcal{F}(\mathbb{K})$ we will exhibit it in the form $F(t_1,\ldots,t_r)$, where the list $t_1,\ldots,t_r$ includes all the distinct terms of A of levels $\ge\pi$.

If $\mathbb{K}$ is a reflection instance with interval $[\pi,\delta]$, then the reflection rules associated with $\mathbb{K}$ are of the form

$$(Ref_{\mathbb{K}})\quad \frac{\Gamma, F(t_1,\ldots,t_r)}{\Gamma,(\exists y_1{\in}\mathbb{L}_\pi)\ldots(\exists y_r{\in}\mathbb{L}_\pi)[M^{t_1,\ldots,t_r}_{\mathbb{K}}(y_1,\ldots,y_r)\wedge F(y_1,\ldots,y_r)]}$$

where $F(t_1,\ldots,t_r)\in\mathcal{F}(\mathbb{K})$.

7.4 $\mathcal{H}$–controlled derivations

If we dropped the reflection rules from $RS(\mathcal{OT})$, the remaining calculus would enjoy full cut elimination owing to the symmetries of the pairs of rules $\langle(\wedge),(\vee)\rangle$, $\langle(\forall),(\exists)\rangle$, $\langle(\notin),(\in)\rangle$, $\langle(M^{\vec{t}}_{\mathbb{K}}),(\neg M^{\vec{t}}_{\mathbb{K}})\rangle$. However, partial cut elimination for $RS(\mathcal{OT})$ can be attained by delimiting a collection of derivations of a very uniform kind.

To define uniform derivations, we shall find it useful to apply the notion of operator controlled derivations of [2].

Definition 7.10 Let $\mathcal{P}(On)=\{X : X \text{ is a set of ordinals}\}$.

A class function

$$\mathcal{H}:\mathcal{P}(On)\to\mathcal{P}(On)$$

will be called *operator* if the following conditions are met for all $X,X'\in\mathcal{P}(On)$:

(H0) $0\in\mathcal{H}(X)$.

(H1) For $\alpha=_{NF}\omega^{\alpha_1}+\cdots+\omega^{\alpha_n}$,

$$\alpha\in\mathcal{H}(X)\iff\alpha_1,...,\alpha_n\in\mathcal{H}(X).$$

(In particular, (H1) implies that $\mathcal{H}(X)$ will be closed under $+$ and $\sigma\mapsto\omega^\sigma$, i.e., if $\alpha,\beta\in\mathcal{H}(X)$, then $\alpha+\beta,\omega^\alpha\in\mathcal{H}(X)$.)

(H2) $X \subseteq \mathcal{H}(X)$

(H3) $X' \subseteq \mathcal{H}(X) \implies \mathcal{H}(X') \subseteq \mathcal{H}(X)$.

Definition 7.11 (i) When f is a mapping $f : On^k \longrightarrow On$, then $\mathcal{H}$ is said to be *closed* under f, if, for all $X \in P(On)$ and $\alpha_1, \dots, \alpha_k \in \mathcal{H}(X)$,

$$f(\alpha_1, \dots, \alpha_k) \in \mathcal{H}(X).$$

(ii) $\alpha \in \mathcal{H} := \alpha \in \mathcal{H}(\emptyset)$.

(iii) $X \subseteq \mathcal{H} := X \subseteq \mathcal{H}(\emptyset)$.

(iv) For $s \in \mathcal{T}$ let $\mathcal{H}[s]$ denote the operator

$$\big(X \mapsto \mathcal{H}(k(s) \cup X)\big)_{X \in P(On)}.$$

(v) If $\mathcal{A}$ is set consisting of terms, formulae, and possibly elements from $\{0, 1\}$, then

$$\mathcal{H}[\mathcal{A}](X) := \mathcal{H}(k(\mathcal{A}) \cup X).$$

We shall also write $\mathcal{H}[\mathcal{A}, s_1, \dots, s_n]$ for $\mathcal{H}[\mathcal{A} \cup \{s_1, \dots, s_n\}]$, and occasionally $\mathcal{H}[\mathcal{A}, \pi]$ instead of $\mathcal{H}[\mathcal{A}, \mathbb{L}_\pi]$.

(vi) If ℓ is a projection instance, a reflection instance, a term, or a formula, we set $\ell \in \mathcal{H} := c(\ell) \subseteq \mathcal{H}$.

The next Lemma garners some simple properties of operators.

Lemma 7.12 *If $\mathcal{H}$ is an operator, then:*

(i) $\mathcal{H}[\mathcal{A}]$ *is an operator.*

(ii) $k(\mathcal{A}) \subseteq \mathcal{H} \implies \mathcal{H}[\mathcal{A}] = \mathcal{H}$.

(iii) $\forall X, X' \in \mathcal{P}(On)[X' \subseteq X \implies \mathcal{H}(X') \subseteq \mathcal{H}(X)]$.

Definition 7.13 To each $RS(\mathcal{OT})$–formula A we assign either a (possibly infinite) disjunction $\bigvee(A_\iota)_{\iota \in J}$ or conjunction $\bigwedge(A_\iota)_{\iota \in J}$ of $RS(\mathcal{OT})$–formulae. This assignment will be indicated by $A \cong \bigvee(A_\iota)_{\iota \in J}$ and $A \cong \bigwedge(A_\iota)_{\iota \in J}$, respectively.

- $r \in t \cong \bigvee(s \mathring{\in} t \wedge r = s)_{s \in \mathcal{T}(|t|)}$
- $M_{\mathbb{K}}^{t_1, \dots, t_r}(p_1, \dots, p_r) \cong \bigvee(p_1 = s_1 \wedge \dots \wedge p_r = s_r)_{(s_1, \dots, s_r) \in I}$ where
  $I := \{(s_1, \dots, s_r) \in \hat{\mathcal{M}}_{\mathbb{K}}^{t_1, \dots, t_r} : |s_1| \leq |p_1|; \dots ; |s_r| \leq |p_r|\}$.
- $(\exists x \in t) F(x) \cong \bigvee(s \mathring{\in} t \wedge F(s))_{s \in \mathcal{T}(|t|)}$
- $A_0 \vee A_1 \cong \bigvee(A_\iota)_{\iota \in \{0,1\}}$
- $\neg A \cong \bigwedge(\neg A_\iota)_{\iota \in J}$, if $A \cong \bigvee(A_\iota)_{\iota \in J}$.

Using this representation of formulae, we can define the *subformulae* of a formula as follows.[3] When $A \cong \bigwedge(A_\iota)_{\iota\in J}$ or $A \cong \bigvee(A_\iota)_{\iota\in J}$, then B is a subformula of A if $B \equiv A$ or, for some $\iota \in J$, B is a subformula of A_ι.

Since we also want to keep track of the complexity of cuts appearing in derivations, we endow each formula with an ordinal rank.

Definition 7.14 The *rank* of formulae and terms is determined as follows.

1. $rk(\mathbb{L}_\alpha) := \omega \cdot \alpha$.
2. $rk([x \in \mathbb{L}_\alpha : F(x)]) := \max\{\omega \cdot \alpha + 1, rk(F(\mathbb{L}_0)) + 2\}$.
3. $rk(s \in t) := rk(s \notin t) := \max\{rk(s) + 6, rk(t) + 1\}$.
4. $rk(M_{\mathbb{K}}^{t_1,\dots,t_r}(p_1,\dots,p_r)) := rk(\neg M_{\mathbb{K}}^{t_1,\dots,t_r}(p_1,\dots,p_r)) :=$ $\max(rk(p_1) + 6, \dots, rk(p_r) + 6) + r$.
5. $rk((\exists x \in t)F(x)) := rk((\forall x \in t)F(x)) := \max\{rk(t), rk(F(\mathbb{L}_0)) + 2\}$.
6. $rk(A \wedge B) := rk(A \vee B) := \max\{rk(A), rk(B)\} + 1$.

There is plenty of leeway in designing the actual rank of a formula. However, it is crucial that it should satisfy the following property.

Lemma 7.15 *If $A \cong \bigvee(A_\iota)_{\iota\in J}$ or $A \cong \bigwedge(A_\iota)_{\iota\in J}$, then*

$$(\forall \iota \in J)\ [rk(A_\iota) < rk(A)].$$

A proof for Lemma 7.15 is given in [2], Lemma 1.9. □

Using the formula representation of Definition 7.13, notwithstanding the many rules of $RS(\mathcal{OT})$, the notion of $\mathcal{H}$–controlled derivability can be defined concisely. We shall use $J \restriction \alpha$ to denote the set $\{\iota \in J : |\iota| < \alpha\}$.

Definition 7.16 Let $\mathcal{H}$ be an operator and let Γ be a finite set of $RS(\mathcal{OT})$–formulae. $\mathcal{H} \overset{\alpha}{\underset{\rho}{\vdash}} \Gamma$ is defined by recursion on α via

$$\{\alpha\} \cup k(\Gamma) \subseteq \mathcal{H}$$

and the following inductive clauses:

$$(\bigvee) \qquad \frac{\mathcal{H} \overset{\alpha_0}{\underset{\rho}{\vdash}} \Lambda, A_{\iota_0}}{\mathcal{H} \overset{\alpha}{\underset{\rho}{\vdash}} \Lambda, \bigvee(A_\iota)_{\iota\in J}} \qquad \begin{array}{r}\alpha_0 < \alpha \\ \iota_0 \in J \restriction \alpha\end{array}$$

$$(\bigwedge) \qquad \frac{\mathcal{H}[\iota] \overset{\alpha_\iota}{\underset{\rho}{\vdash}} \Lambda, A_\iota \textit{ for all } \iota \in J}{\mathcal{H} \overset{\alpha}{\underset{\rho}{\vdash}} \Lambda, \bigwedge(A_\iota)_{\iota\in J}} \qquad |\iota| \le \alpha_\iota < \alpha$$

$$(Cut) \qquad \frac{\mathcal{H} \overset{\alpha_0}{\underset{\rho}{\vdash}} \Lambda, \mathcal{B} \qquad \mathcal{H} \overset{\alpha_0}{\underset{\rho}{\vdash}} \Lambda, \neg\mathcal{B}}{\mathcal{H} \overset{\alpha}{\underset{\rho}{\vdash}} \Lambda} \qquad \begin{array}{r}\alpha_0 < \alpha \\ rk(B) < \rho\end{array}$$

$$(Ref_{\mathbb{K}}) \qquad \frac{\mathcal{H} \overset{\alpha_0}{\underset{\rho}{\vdash}} \Lambda, F(\vec{t}\,)}{\mathcal{H} \overset{\alpha}{\underset{\rho}{\vdash}} \Lambda, (\exists \vec{y} \in \mathbb{L}_\pi)[M_{\mathbb{K}}^{\vec{t}}(\vec{y}) \wedge F(\vec{y}\,)]} \qquad \alpha_0 + 1, \pi < \alpha$$

[3] That this constitutes a legitimite inductive definition will follow from Lemma 7.15.

The following observations are easily established by induction on α.

Lemma 7.17 *(i)* $\mathcal{H} \overset{\alpha}{\underset{\rho}{\vdash}} \Gamma \ \wedge\ \alpha \le \alpha' \in \mathcal{H} \ \wedge\ \rho \le \rho' \ \wedge\ k(\Lambda) \subseteq \mathcal{H} \implies \mathcal{H} \overset{\alpha'}{\underset{\rho'}{\vdash}} \Gamma, \Lambda\,.$

(ii) $\mathcal{H} \overset{\alpha}{\underset{\rho}{\vdash}} \Gamma, A \vee B \implies \mathcal{H} \overset{\alpha}{\underset{\rho}{\vdash}} \Gamma, A, B\,.$

(iii) $\mathcal{H} \overset{\alpha}{\underset{\rho}{\vdash}} \Gamma, (\forall x \in \mathbb{L}_\beta)F(x) \ \wedge\ \gamma \in \mathcal{H} \ \wedge\ \gamma \le \beta \implies \mathcal{H} \overset{\alpha}{\underset{\rho}{\vdash}} \Gamma, (\forall x \in \mathbb{L}_\gamma)F(x)\,.$

(iv) $\mathcal{H} \overset{\alpha}{\underset{\rho}{\vdash}} \Gamma, (\exists x \in \mathbb{L}_\beta)F(x) \ \wedge\ \gamma \in \mathcal{H} \ \wedge\ \beta \le \gamma \implies \mathcal{H} \overset{\alpha}{\underset{\rho}{\vdash}} \Gamma, (\exists x \in \mathbb{L}_\gamma)F(x)\,.$

8 Predicative Cut Elimination and Bounding

Cuts in $RS(\mathcal{OT})$–derivations whose cut formulae have not been introduced previously by a reflection inference will be called *uncritical*. Applying the usual cut elimination procedure for infinitary logic, uncritical cuts can be replaced by cuts with lesser rank. In this section we will deal with elimination of uncritical cuts in $\mathcal{L}_{RS}$ and its quantitative aspects. Since these results have literally the same proofs as their counterparts in [2], we refrain from repeating them here.

Besides cut elimination results, we show that existential quantifiers in $\mathcal{L}_{RS}$–derivations can always be "bounded" by the length of the derivation.

Lemma 8.1 (Inversion)

$$\mathcal{H} \overset{\alpha}{\underset{\rho}{\vdash}} \Gamma, \bigwedge(A_\iota)_{\iota\in J} \implies (\forall \iota \in J)\mathcal{H}[\iota] \overset{\alpha}{\underset{\rho}{\vdash}} \Gamma, A_\iota$$

Proof: Use induction on α. □

The next Lemma relates the rank of a formula A, to its level, $|A|$ (see 7.3).

Lemma 8.2 *Let A, B be formulae and s, t be terms.*

(i) $rk(A) = \omega \cdot |A| + n$ *for some* $n < \omega$.

(ii) $rk(s) = \omega \cdot |s| + m$ *for some* $m < \omega$.

(iii) $|A| < |B| \implies rk(A) < rk(B)$.

(iv) $|s| < |t| \implies rk(s) < rk(t)$.

Proof: See [2] , Lemma 1.9. □

Definition 8.3 Let $Reg := \{\pi \in \mathcal{OT} :\ \pi > \omega;\ \pi$ is a regular ordinal$\}$.

Lemma 8.4 (Reduction Lemma) *Let* $A \cong \bigvee(A_\iota)_{\iota\in J}$. *Assume* $\rho \notin Reg$, *where* $\rho := rk(A)$. *Then:*

$$\mathcal{H} \overset{\alpha}{\underset{\rho}{\vdash}} \Lambda, \neg A \ \wedge\ \mathcal{H} \overset{\beta}{\underset{\rho}{\vdash}} \Gamma, A \implies \mathcal{H} \overset{\alpha+\beta}{\underset{\rho}{\vdash}} \Lambda, \Gamma$$

Proof: Use induction on β. For details see [2] , Lemma 3.14. □

Theorem 8.5 (Predicative cut elimination) *Let $\mathcal{H}$ be closed under φ. If $\mathcal{H} \overset{\beta}{\underset{\rho+\omega^\alpha}{\vdash}} \Gamma$ and $[\rho, \rho+\omega^\alpha[\cap Reg = \emptyset$ and $\alpha \in \mathcal{H}$, then*

$$\mathcal{H} \overset{\varphi\alpha\beta}{\underset{\rho}{\vdash}} \Gamma .$$

Proof: By main induction on α and subsidiary induction on β (cf. [2] , Theorem 3.16). $\square$

Corollary 8.6 $\quad \mathcal{H} \overset{\beta}{\underset{\rho+1}{\vdash}} \Gamma \ \wedge\ \rho \notin Reg \implies \mathcal{H} \overset{\omega^\beta}{\underset{\rho}{\vdash}} \Gamma .$

Lemma 8.7 (Bounding Lemma) *Let $\mu \in Reg$ and $\beta \in \mathcal{H}$. If $\alpha \leq \beta < \mu$ and $B \in \Sigma_1(\mu)$, then*

$$\mathcal{H} \overset{\alpha}{\underset{\rho}{\vdash}} \Gamma, B \implies \mathcal{H} \overset{\alpha}{\underset{\rho}{\vdash}} \Gamma, B^{(\beta,\mu)} .$$

Proof by induction on α. Since $\alpha < \mu$, such B cannot be the principal formula of a reflection inference $(Ref_{\mathbb{H}})$.

If B is not the principal formula of the last inference, the assertion follows by using the inductive assumption on its premisses and reapplying the same inference. Let B be the principal formula of the last inference, which then must be $(\exists)$. So B has the form $(\exists x \in \mathbb{L}_\mu)F(x)$ with $\Delta_0(\mu)$–formula $F(\mathbb{L}_0)$. Also,

$$\mathcal{H} \overset{\alpha_0}{\underset{\rho}{\vdash}} \Gamma, B, s \mathrel{\mathring{\in}} \mathbb{L}_\mu \wedge F(s)$$

for some $\alpha_0 < \alpha$ and $s \in \mathcal{T}(\mu)$ with $|s| < \alpha$. By the induction hypothesis,

$$\mathcal{H} \overset{\alpha_0}{\underset{\rho}{\vdash}} \Gamma, B^{(\beta,\mu)}, s \mathrel{\mathring{\in}} \mathbb{L}_\mu \wedge F(s) .$$

Since $|s| < \beta, \mu$, we have $s \mathrel{\mathring{\in}} \mathbb{L}_\beta \equiv s \mathrel{\mathring{\in}} \mathbb{L}_\mu$. Thus, applying $(\exists)$, the assertion follows. $\square$

9 Embeddings

The aim of this Section is to embed **KPi** $+ \exists M$ (M is transitive $\wedge\ M \prec_1 V$) into $RS(\mathcal{OT})$.

9.1 Basic embeddings

Regarding proofs, we will be drawing on [2] when the proof is almost literally the same.

Definition 9.1 For $\Gamma = \{A_1, \ldots , A_n\}$ let

$$no(\Gamma) := \omega^{rk(A_1)} \# \cdots \# \omega^{rk(A_n)}.$$

We define

$$\Vdash \Gamma \quad :\iff \quad \text{for all operators } \mathcal{H},\ \mathcal{H}[\Gamma] \overset{no(\Gamma)}{\underset{0}{\vdash}} \Gamma,$$

$$\Vdash^* \Gamma \quad :\iff \quad \text{for all operators } \mathcal{H},\ \mathcal{H}[\Gamma] \overset{no(\Gamma)}{\underset{no(\Gamma)}{\vdash}} \Gamma$$

and

$$\Vdash^\xi_\rho \Gamma \quad :\iff \quad \text{for all operators } \mathcal{H},\ \mathcal{H}[\Gamma] \overset{no(\Gamma)\#\xi}{\underset{\rho}{\vdash}} \Gamma .$$

Lemma 9.2 *Let $s \subseteq t$ stand for the formula $(\forall x \in s)(x \in t)$.*

(i) $\Vdash A, \neg A$.

(ii) $\Vdash s \notin s$.

(iii) $\Vdash s \subseteq s$.

(iv) $\Vdash s \mathrel{\mathring{\notin}} t, s \mathrel{\mathring{\in}} t$ *for* $s \in \mathcal{T}(|t|)$.

(v) $\Vdash s \neq t, t = s$.

Proof: [2], Lemma 2.4, Lemma 2.5. ☐

Lemma 9.3 *Let $[s \neq t]$ be $\neg(s \subseteq t), \neg(t \subseteq s)$.*

$$\Vdash [s_1 \neq t_1], \ldots, [s_n \neq t_n], \neg A(s_1, \ldots, s_n), A(t_1, \ldots, t_n).$$

Proof: [2], Lemma 2.7. ☐

Corollary 9.4 (Equality and Extensionality)

$$\Vdash s_1 \neq t_1, \ldots, s_n \neq t_n, \neg A(s_1, \ldots, s_n), A(t_1, \ldots, t_n).$$

Proof: [2], Theorem 2.9. ☐

Lemma 9.5 (Foundation)

$$\Vdash (\forall x \in \mathbb{L}_\alpha)[(\forall y \in x)F(y) \to F(x)] \;\longrightarrow\; (\forall x \in \mathbb{L}_\alpha)F(x).$$

Proof: Fix an operator $\mathcal{H}$. Let $A \equiv (\forall x \in \mathbb{L}_\alpha)[(\forall y \in x)F(y) \to F(x)]$. First, we show by induction on $|s|$, that if $s \in \mathcal{T}(\alpha)$ then

$$(+) \qquad \mathcal{H}[A, s] \,\Big|\!\frac{\omega^{rk(A)}\#\omega^{|s|+1}}{0}\, \neg A, F(s)\,.$$

So assume that

$$\mathcal{H}[A, t] \,\Big|\!\frac{\omega^{rk(A)}\#\omega^{|t|+1}}{0}\, \neg A, F(t)$$

for all $t \in \mathcal{T}(|s|)$. Using $(\vee)$, this yields

$$\mathcal{H}[A, s, t] \,\Big|\!\frac{\omega^{rk(A)}\#\omega^{|t|+1}+1}{0}\, \neg A, t \mathrel{\mathring{\in}} s \to F(t)$$

for all $t \in \mathcal{T}(|s|)$, and hence

$$(1) \qquad \mathcal{H}[A, s] \,\Big|\!\frac{\omega^{rk(A)}\#\omega^{|s|}+2}{0}\, \neg A, (\forall x \in s)F(s)$$

via $(\forall)$. Set $\eta := \omega^{rk(A)}\#\omega^{|s|} + 2$. By Lemma 9.2(i), $\mathcal{H}[A, s] \,\big|\!\frac{\eta}{0}\, \neg F(s), F(s)$; therefore, using (1) and $(\wedge)$,

$$\mathcal{H}[A, s] \,\Big|\!\frac{\eta+1}{0}\, \neg A, (\forall y \in s)F(y) \wedge \neg F(s), F(s)\,.$$

From the latter we obtain

$$\mathcal{H}[A, s] \,\Big|\!\frac{\eta+2}{0}\, \neg A, s \mathrel{\mathring{\in}} \mathbb{L}_\alpha \wedge [(\forall y \in s)F(y) \wedge \neg F(s)], F(s)\,,$$

and hence $\mathcal{H}[A, s] \,\big|\!\frac{\eta+3}{0}\, \neg A, (\exists x \mathrel{\mathring{\in}} \mathbb{L}_\alpha)[(\forall y \in x)F(y) \wedge \neg F(x)], F(s)$ via $(\exists)$. This shows (+).

Finally, (+) enables us to deduce $\mathcal{H}[A, s] \,\big|\!\frac{\omega^{rk(A)}\#\omega^{|s|+1}+1}{0}\, \neg A, s \mathrel{\mathring{\in}} \mathbb{L}_\alpha \to F(s)$ from which the assertion follows by applying $(\forall)$ and $(\vee)$. ☐

Lemma 9.6 (Infinity Axiom) *If λ be a limit ordinal $> \omega$, then*

$$\Vdash \textit{(Infinity Axiom)}^{\mathbb{L}_\lambda},$$

i.e.,

$$\Vdash (\exists x \in \mathbb{L}_\lambda)[z \neq \emptyset \wedge (\forall y \in x)(\exists z \in x)(y \in z)].$$

Proof: [2], Theorem 2.9. □

Lemma 9.7 (Δ_0–Separation) *Let $A[a, b_1, \dots, b_n]$ be a Δ_0–formula of $\mathcal{L}_M$. If $\lambda \in Lim$ and $s, t_1, \dots, t_n \in \mathcal{T}(\lambda)$, then*

$$\Vdash (\exists y \in \mathbb{L}_\lambda)[(\forall x \in y)(x \in s \wedge A[s, t_1, \dots, t_n]) \wedge (\forall x \in s)(A[x, t_1, \dots, t_n] \to x \in y)].$$

More concisely, we can express this by " $\Vdash (\Delta_0$*–separation*$)^{\mathbb{L}_\lambda}$ *".*

Proof: [2], Theorem 2.9. □

Lemma 9.8 (Pair and Union) *Assume $\lambda \in Lim$ and $s, t \in \mathcal{T}(\lambda)$.*

(i) $\Vdash (\exists z \in \mathbb{L}_\lambda)(s \in z \wedge t \in z)$.

(ii) $\Vdash (\exists z \in \mathbb{L}_\lambda)(\forall y \in s)(\forall x \in y)(x \in z)$.

Proof: [2], Theorem 2.9. □

Conventions. We shall write $\exists x^\zeta$ and $\forall x^\zeta$ instead of $(\exists x \in \mathbb{L}_\zeta)$ and $(\forall x \in \mathbb{L}_\zeta)$, respectively.

Definition 9.9 The sequent calculus GML^* ("GML" stands for *"Grundmengenlehre"*) is defined as follows. The language of GML^* is $\mathcal{L}_M$, that is to say, in addition to the symbol $\in$, it has the predicate symbols $M_{\mathbb{K}}^{t_1,\dots,t_r}$ of $\mathcal{L}_M$. With the exception of Δ_0–collection, GML^* has the same axiom schemes as **KP**. (However, it is understood that the axiom schemes are defined with regard to $\mathcal{L}_M$. To be precise, GML^* comprises the axiom scheme of $\Delta_0(\mathcal{L}_M)$–separation, whereas $\Delta_0(\mathcal{L}_M)$–collection is not an axiom scheme of GML^*.) In addition, GML^* has 9 axioms expressing closure of the set-theoretic universe under the rudimentary functions $F_0, \dots, F_8$ of [3], VI. Lemma 1.11. Moreover, GML^* has the following axioms pertaining to the predicates $M_{\mathbb{K}}^{t_1,\dots,t_r}$: If $\vec{s}$ is a subsequence of $\vec{t}$, and $\vec{a}$ is a sequence of variables of the same length as $\vec{t}$, and $\vec{b}$ is a subsequence of $\vec{a}$ which arises from $\vec{a}$ in the same way as $\vec{s}$ arises from $\vec{t}$, then

$$\Gamma, \neg M_{\mathbb{K}}^{\vec{t}}(\vec{a}), M_{\mathbb{K}}^{\vec{s}}(\vec{b}) \tag{31}$$

is an axiom of GLM^*.

The theory GML has the language $\mathcal{L}$ and the same axioms as GML^* but with respect to the restricted language.

The sequent calculus LK^* ("LK" stands for *"Logischer Kalkül"*) has the same language as GML^* and the axioms (31) but no set-theoretic axioms.

Lemma 9.10 *Assume $\omega < \rho \leq \mathbf{I}$ is a limit ordinal. Let $\Gamma[\vec{a}\,] = \{A_1[\vec{a}\,], \ldots, A_k[\vec{a}\,]\}$ be a set of $\mathcal{L}_M$–formulae, where $\vec{a} = a_1, \ldots, a_n$. If $GML^* \vdash \Gamma[\vec{a}\,]$, then there exists $m < \omega$ such that, for all $\vec{s} = s_1, \ldots, s_n \in \mathcal{T}(\rho)$,*

$$\mathcal{H}[\Gamma[\vec{s}\,]^{\mathbb{L}_\rho}, \rho] \,\Big|\frac{\omega^{\omega\cdot\rho+m}}{\omega\cdot(\rho+1)}\, \Gamma[\vec{s}\,]^{\mathbb{L}_\rho}\,.$$

Here $\Gamma[\vec{s}\,]^{\mathbb{L}_\rho}$ stands for $\{A_1[\vec{s}\,]^{\mathbb{L}_\rho}, \ldots, A_k[\vec{s}\,]^{\mathbb{L}_\rho}\}$.

Proof by induction on GML^* derivations. As to the axioms of GML^*, the claim follows easily from previous results of this Section. The inferences of GML^* are dealt with in the same manner as in [2], Theorem 3.12. □

Lemma 9.11 *Assume $\rho < \mathbf{I}$. Let $\Gamma[\vec{a}\,] = \{A_1[\vec{a}\,], \ldots, A_k[\vec{a}\,]\}$ be a set of $\mathcal{L}$–formulae, where $\vec{a} = a_1, \ldots, a_n$. If $\mathbf{KP} \vdash \Gamma[\vec{a}\,]$, then there exists $m < \omega$ such that, for all $\vec{s} = s_1, \ldots, s_n \in \mathcal{T}(\rho^+)$,*

$$\mathcal{H}[\Gamma[\vec{s}\,]^{\mathbb{L}_{\rho^+}}, \rho] \,\Big|\frac{\omega^{\omega\cdot\rho^++m}}{\omega\cdot(\rho^++1)}\, \Gamma[\vec{s}\,]^{\mathbb{L}_{\rho^+}}\,.$$

Here $\Gamma[\vec{s}\,]^{\mathbb{L}_{\rho^+}}$ stands for $\{A_1[\vec{s}\,]^{\mathbb{L}_{\rho^+}}, \ldots, A_k[\vec{s}\,]^{\mathbb{L}_{\rho^+}}\}$.

Proof by induction on **KP** derivations in the same manner as in [2], Theorem 3.12. □

Lemma 9.12 *Assume $0 < \rho \leq \mathbf{I}$. Let $\Gamma[\vec{a}\,] = \{A_1[\vec{a}\,], \ldots, A_k[\vec{a}\,]\}$ be a set of $\mathcal{L}_M$–formulae, where $\vec{a} = a_1, \ldots, a_n$. If $LK^* \vdash \Gamma[\vec{a}\,]$, then for all $\vec{s} = s_1, \ldots, s_n \in \mathcal{T}(\rho)$,*

$$\Vdash^* \Gamma[\vec{s}\,]^{\mathbb{L}_\rho}$$

where $\Gamma[\vec{s}\,]^{\mathbb{L}_\rho}$ stands for $\{A_1[\vec{s}\,]^{\mathbb{L}_\rho}, \ldots, A_k[\vec{s}\,]^{\mathbb{L}_\rho}\}$.

Proof by induction on LK derivations. □

9.2 A variant

Rather than the theory $\mathbf{KPi} + \exists M\,(M$ is transitive $\wedge\ M \prec_1 V)$ we are going to embed a variant of it into the infinitary system.

Lemma 9.13 *There exists a sentence Mod_{GML} such that GML proves that for every transitive set M with $M \neq \emptyset$, M is a model of GML iff $M \models Mod_{GML}$.*

Proof: This follows from [3], VI, Lemma 1.11 and Lemma 1.6 or [21], Theorem 2.4. □

Lemma 9.14 *There is a Σ_1-formula $Sat_1(a)$ such that for all transitive sets M which are models of GML the following holds:*

$$M \prec_1 V \quad \textit{iff} \quad \forall a \in M\,\big[\,Sat_1(a) \;\rightarrow\; Sat_1(a)^M\,\big].$$

Proof: By [3], VI, Lemma 1.15 or [21], Lemma 2.4. □

Corollary 9.15 *Let $\chi(M)$ be the conjunction of the following formulas*

1. M is transitive

2. $\exists u{\in}M\, u \in M$

3. $M \models Mod_{GML}$

4. $\forall a \in M\, \big[\, Sat_1(a) \;\rightarrow\; Sat_1(a)^M\, \big]$

The theories $\mathbf{KPi} + \exists M\,(M$ *is transitive* $\wedge\; M \prec_1 V)$ *and* $\mathbf{KPi} + \exists M\,\chi(M)$ *prove the same theorems.*

Proof: This follows from Lemma 9.13 and Lemma 9.14. □

Theorem 9.16 *Let* $\Gamma[\vec{a}\,] = \{A_1[\vec{a}\,], \ldots, A_k[\vec{a}\,]\}$ *be a set of* $\mathcal{L}$*–formulae with* $\vec{a} = a_1, \ldots, a_n$*. Whenever* $\mathbf{KPi} + \exists M\,(M$ *is transitive* $\wedge\; M \prec_1 V) \vdash \Gamma[\vec{a}\,]$*, then there exists* $m < \omega$ *such that, for all* $\vec{s} = s_1, \ldots, s_n \in \mathcal{T}$*,*

$$\mathcal{H}[\Gamma[\vec{s}\,]^{\mathbb{L}_\mathbf{I}}] \;\Big|\frac{\mathbf{I}\cdot\omega^m}{\mathbf{I}+m}\; \Gamma[\vec{s}\,]^{\mathbb{L}_\mathbf{I}}\,.$$

Proof: Compared to Lemma 9.10, there are three new types of axioms to take care of, namely Δ_0-collection, $\forall x \exists y\big[x{\in}y \wedge y \text{ is admissible}\big]$, and $\exists M\,(M \text{ is transitive } \wedge\; M \prec_1 V)$.

We shall address Δ_0-collection first. Let $F(b,\vec{a})$ be a Δ_0-formula with all free variables exhibited. By Lemma 9.2 we have

$$\Vdash \neg(\forall x{\in}p)(\exists y{\in}\mathbb{L}_\mathbf{I})F(y,\vec{s}),\; (\forall x{\in}p)(\exists y{\in}\mathbb{L}_\mathbf{I})F(y,\vec{s}).$$

Using an inference $(Ref_{\mathbb{J}})$ with $\mathbb{J} = (\mathbf{I}; \mathsf{RLC}; \emptyset; 0)$ followed by two inferences $(\vee)$ we arrive at

$$\mathcal{H}[A] \;\Big|\frac{\mathbf{I}\cdot\omega^m}{\mathbf{I}+m}\; (\forall x{\in}p)(\exists y{\in}\mathbb{L}_\mathbf{I})F(y,\vec{s}) \;\rightarrow\; (\exists z{\in}\mathbb{L}_\mathbf{I})(\forall x{\in}p)(\exists y{\in}z)F(y,\vec{s})$$

for some m, where $A \equiv (\forall x{\in}p)(\exists y{\in}\mathbb{L}_\mathbf{I})F(y,\vec{s})$.

Next we deal with $\forall x \exists y\big[x{\in}y \wedge y \text{ is admissible}\big]$. By [21], one can pick a Π_3-sentence D such that $\mathbf{KP} \vdash D$ and $GML \vdash \forall y\big[y \text{ is admissible} \leftrightarrow D^y\big]$. As a result, we may take D^z to express that z is admissible.

Lemma 9.11 yields that there exists $m < \omega$ such that, for all terms $|s| < \mathbf{I}$,

$$\mathcal{H}[D^s] \;\Big|\frac{\omega^{\omega\cdot|s|^+ + m}}{\omega\cdot(|s|^+ + 1)}\; s{\in}\mathbb{L}_{|s|^+} \;\wedge\; D^{\mathbb{L}_{|s|^+}}\,.$$

From the latter we get

$$\mathcal{H} \;\Big|\frac{\mathbf{I}\cdot\omega^n}{\mathbf{I}+n}\; (\forall x{\in}\mathbb{L}_\mathbf{I})(\exists y \in \mathbb{L}_\mathbf{I})\,\big[x{\in}y \;\wedge\; D^y\big]$$

for some $n < \omega$, using inferences $(\wedge), (\exists), (\vee), (\forall)$ in that order.

Finally we address the axiom $\exists M\,(M \text{ is transitive } \wedge\; M \prec_1 V)$. By Corollary 9.15 this axiom can be expressed by $\exists M\,\chi(M)$. Let $Sat_1(a) \equiv \exists z A(a,z)$, where $A(a,b)$ is a Δ_0-formula of $\mathcal{L}$. Furthermore, let $B(c) \equiv \big[(c \;\; \textit{is transitive}\;) \wedge (\exists u{\in}c)(u{\in}c) \wedge Mod_{GML}\big]$. By Lemma 9.11 there exists $m_0 < \omega$, such that

$$\mathcal{H} \;\Big|\frac{\omega^{\Xi+m_0}}{\Xi+\omega}\; B(\mathbb{L}_\Xi)\,. \tag{32}$$

Let s, p be $RS(\mathcal{OT})$-terms such that $|s| < \Xi$ and $|p| < \mathbf{I}$. Put $\nu_{s,p} := no(\{\neg A(s,p), A(s,p)\})$. Starting from

$$\Vdash \neg A(s,p), A(s,p),$$

one applies an inference $(Ref_{\mathbb{K}_p})$ with $\mathbb{K}_p := (\Xi; |p| + 1\text{-}\mathsf{P}_0; \mathbf{I}\text{-}\mathsf{P}_0; \emptyset; 0)$ to get

$$\mathcal{H}[s,p] \, \Big|\!\frac{\nu_{s,p}\#\Xi}{0} \, \neg A(s,p), \; (\exists z \in \mathbb{L}_\Xi)[M^p_{\mathbb{K}}(z) \, \wedge \, A(s,z)] \, .$$

Since Lemma 9.12 yields

$$\mathcal{H}[s,p] \, \Big|\!\frac{\rho_{s,p}}{\rho_{s,p}} \, \neg(\exists z \in \mathbb{L}_\Xi)[M^p_{\mathbb{K}}(z) \, \wedge \, A(s,z)], \; (\exists z \in \mathbb{L}_\Xi) A(s,z) \, ,$$

where $\rho_{s,p} := no(\{\neg(\exists z \in \mathbb{L}_\Xi)[M^p_{\mathbb{K}}(z) \, \wedge \, A(s,z)], (\exists z \in \mathbb{L}_\Xi)A(s,z)\})$, we may apply (Cut) to obtain

$$\mathcal{H}[s,p] \, \Big|\!\frac{\rho_{s,p}+1}{\rho_{s,p}+1} \, \neg A(s,p), \; (\exists z \in \mathbb{L}_\Xi) A(s,z) \, .$$

The latter yields $\mathcal{H}[s,p] \, \Big|\!\frac{\rho_{s,p}+2}{\rho_{s,p}+1} \, p \mathrel{\mathring{\in}} \mathbb{L}_{\mathbf{I}} \to \neg A(s,p), \; (\exists z \in \mathbb{L}_\Xi) A(s,z)$ via $(\vee)$. Using $(\forall)$ we get

$$\mathcal{H}[s] \, \Big|\!\frac{\mathbf{I}}{\mathbf{I}} \, (\forall z {\in} \mathbb{L}_{\mathbf{I}}) \neg A(s,z), \; (\exists z \in \mathbb{L}_\Xi) A(s,z) \, ,$$

and thus, via two inferences $(\vee)$,

$$\mathcal{H}[s] \, \Big|\!\frac{\mathbf{I}+2}{I} \, Sat_1(s)^{\mathbb{L}_{\mathbf{I}}} \to Sat_1(s)^{\mathbb{L}_\Xi} \, ;$$

and hence $\mathcal{H}[s] \, \Big|\!\frac{\mathbf{I}+3}{\mathbf{I}} \, s \mathrel{\mathring{\in}} \mathbb{L}_\Xi \to \big(Sat_1(s)^{\mathbb{L}_{\mathbf{I}}} \to Sat_1(s)^{\mathbb{L}_\Xi}\big)$. Applying $(\forall)$ we arrive at

$$\mathcal{H} \, \Big|\!\frac{\mathbf{I}+4}{\mathbf{I}} \, (\forall x {\in} \mathbb{L}_\Xi)\big(Sat_1(x)^{\mathbb{L}_{\mathbf{I}}} \to Sat_1(x)^{\mathbb{L}_\Xi}\big) \, . \tag{33}$$

From (32) and (33) we obtain

$$\mathcal{H} \, \Big|\!\frac{\mathbf{I}+5}{\mathbf{I}} \, \chi(\mathbb{L}_\Xi) \, , \tag{34}$$

and thus, via inferences $(\wedge)$ and $(\exists)$, the desired

$$\mathcal{H} \, \Big|\!\frac{\mathbf{I}+7}{\mathbf{I}} \, (\exists u {\in} \mathbb{L}_{\mathbf{I}}) \chi(u)$$

ensues □

10 Strengthening reflection

Definition 10.1 Let $\mathbb{K}$ be a reflection instance with interval $[\pi, \delta]$. A formula A is said to be in $\Sigma_1(\mathcal{F}(\mathbb{K}))$ if it belongs to the smallest collection of formulae which comprises $\mathcal{F}(\mathbb{K})$ and is closed under quantifiers of the form $(\exists x \in \mathbb{L}_\delta)$.

A formula A is said to be in $\Sigma_1^{\vee,\wedge}(\mathcal{F}(\mathbb{K}))$ if it belongs to the smallest collection of formulae which contains $\mathcal{F}(\mathbb{K})$ and is closed under quantifiers of the form $(\exists x \in \mathbb{L}_\delta)$ and the connectives $\vee$ and $\wedge$.

Lemma 10.2 *Let* $\mathbb{K}$ *be a reflection instance with interval* $[\pi, \delta]$ *and* $F_1(\vec{t}), \ldots, F_m(\vec{t})$ *be* $\Sigma_1(\mathcal{F}(\mathbb{K}))$*-formulae, where* $\vec{t}$ *consists of the terms* t_i *with* $\pi \leq |t_i| \leq \delta$*. Assume*

$$\mathcal{H} \left|\frac{\alpha}{\rho}\right. \Lambda, F_1(\vec{t}), \ldots, F_m(\vec{t})$$

and $k(\vec{t}), k(\mathbb{K}) \subseteq \mathcal{H}$*. Then*

$$\mathcal{H} \left|\frac{\alpha \# \pi \cdot 2}{\max(\rho,\pi)+1}\right. \Lambda, (\exists \vec{y} {\in} \mathbb{L}_\pi)[M^{\vec{t}}_{\mathbb{K}}(\vec{y}) \wedge F_1(\vec{y})], \ldots, (\exists \vec{y} {\in} \mathbb{L}_\pi)[M^{\vec{t}}_{\mathbb{K}}(\vec{y}) \wedge F_m(\vec{y})]\,.$$

Proof: We proceed by induction on α.

If none of the formulae $F_j(\vec{t})$ is the main formula of the last inference, then the assertion follows from the induction hypothesis by subsequently applying the same inference.

Now suppose that a formula $F_j(\vec{t})$ is the main formula of the last inference. If $F_j(\vec{t})$ is in $\mathcal{F}(\mathbb{K})$, then the assertion follows by applying the induction hypothesis and subsequently an inference $(Ref_{\mathbb{K}})$. If $F_j(\vec{t})$ is not in $\mathcal{F}(\mathbb{K})$, then it is of the form $(\exists x \in \mathbb{L}_\delta) G_j(\vec{t}, x)$ and the last inference was $(\exists)$. As a result, we have a scenario of the form

$$\mathcal{H} \left|\frac{\alpha_0}{\rho}\right. \Lambda', s \mathring{\in} \mathbb{L}_\delta \wedge G_j(\vec{t}, s)$$

where $|s| < \delta$, $\alpha_0 < \alpha$ and $\Lambda' \subseteq \Lambda, F_1(\vec{t}), \ldots, F_m(\vec{t})$. Note that $s \mathring{\in} \mathbb{L}_\delta$ is the same formula as $s {\notin} \mathbb{L}_0$. Also note that in the latter case $\mathbb{L}_\delta$ appears among the terms $\vec{t}$, say $\mathbb{L}_\delta \equiv t_j$. The induction hypothesis then yields

$$\mathcal{H} \left|\frac{\alpha_0 \# \pi \cdot 2}{\bar{\rho}}\right. \Lambda, \Theta, (\exists \vec{y} {\in} \mathbb{L}_\pi)(\exists x {\in} \mathbb{L}_\pi)[M^{\vec{t},s}_{\mathbb{K}}(\vec{y}, x) \wedge x {\notin} \mathbb{L}_0 \wedge G_j(\vec{y}, x)]\,, \tag{35}$$

where $\Theta = \{(\exists \vec{y} {\in} \mathbb{L}_\pi)[M^{\vec{t}}_{\mathbb{K}}(\vec{y}) \wedge F_1(\vec{y})], \ldots, (\exists \vec{y} {\in} \mathbb{L}_\pi)[M^{\vec{t}}_{\mathbb{K}}(\vec{y}) \wedge F_m(\vec{y})]\}$. From Lemma 9.12 we obtain

$$\begin{aligned} \Vdash^* \neg(\exists \vec{y} {\in} \mathbb{L}_\pi)(\exists x \in \mathbb{L}_\pi)[M^{\vec{t},s}_{\mathbb{K}}(\vec{y}, x) \wedge x {\notin} \mathbb{L}_0 \wedge G_j(\vec{y}, x)], \\ (\exists \vec{y} {\in} \mathbb{L}_\pi)[M^{\vec{t}}_{\mathbb{K}}(\vec{y}) \wedge (\exists x \in y_j) G_j(\vec{y}, x)]. \end{aligned} \tag{36}$$

Note that $(\exists \vec{y} {\in} \mathbb{L}_\pi)[M^{\vec{t}}_{\mathbb{K}}(\vec{y}) \wedge (\exists x \in y_j) G_j(\vec{y}, x)]$ is the same formula as $(\exists \vec{y} {\in} \mathbb{L}_\pi)[M^{\vec{t}}_{\mathbb{K}}(\vec{y}) \wedge F_j(\vec{y})]$. Thus (Cut) applied to (35) and (36) provides the desired result. □

Lemma 10.3 *Let* $\mathbb{K}$ *be a reflection instance with interval* $[\pi, \delta]$ *and* $G(\vec{t})$ *be a* $\Sigma_1^{\vee,\wedge}(\mathcal{F}(\mathbb{K}))$*-formula, where* $\vec{t}$ *consists of the terms* t_i *with* $\pi \leq |t_i| \leq \delta$ *occurring in this formula.*

Assume

$$\mathcal{H} \left|\frac{\alpha}{\rho}\right. \Lambda, G(\vec{t})\,,$$

where and $k(\vec{t}), k(\mathbb{K}) \subseteq \mathcal{H}$*. Then*

$$\mathcal{H} \left|\frac{\alpha \# \pi \cdot \omega^{\omega \cdot (\delta+1)}}{\max(\rho, \pi + \omega \cdot (\delta+1))}\right. \Lambda, (\exists \vec{y} {\in} \mathbb{L}_\pi)[M^{\vec{t}}_{\mathbb{K}}(\vec{y}) \wedge G(\vec{y})]\,.$$

Proof: By pulling quantifiers $(\exists x {\in} \mathbb{L}_\delta)$ in front of the formula $G(\vec{t})$ we can render it a $\Sigma_1(\mathcal{F}(\mathbb{K}))$-formula, say $F(\vec{t})$. By Lemma 9.12 we then get

$$\Vdash^* \neg G(\vec{t}), F(\vec{t}).$$

Hence from $\mathcal{H} \overset{\alpha}{\underset{\rho}{\vdash}} \Lambda, G(\vec{t})$ we obtain

$$\mathcal{H} \overset{\alpha'}{\underset{\tau}{\vdash}} \Lambda, F(\vec{t})$$

via (Cut), where $\alpha' := \max(\alpha, no(F(\vec{t}))\#no(G(\vec{t}))) + 1$ and $\tau := \max(\rho, \pi + \omega \cdot (\delta + 1))$. Using Lemma 10.2, we get

$$\mathcal{H} \overset{\alpha'\#\alpha'\#\alpha'}{\underset{\tau}{\vdash}} \Lambda, (\exists \vec{y} \in \mathbb{L}_\pi)[M_{\mathbb{K}}^{\vec{t}}(\vec{y}) \wedge F(\vec{y})]\,. \tag{37}$$

Again using Lemma 9.12, we have

$$\Vdash^* \neg(\exists \vec{y} \in \mathbb{L}_\pi)[M_{\mathbb{K}}^{\vec{t}}(\vec{y}) \wedge F(\vec{y})], (\exists \vec{y} \in \mathbb{L}_\pi)[M_{\mathbb{K}}^{\vec{t}}(\vec{y}) \wedge G(\vec{y})]. \tag{38}$$

Thus (Cut) applied to (37) and (38) yields

$$\mathcal{H} \overset{\alpha\#\pi\cdot\omega^{\omega\cdot(\delta+1)}}{\underset{\tau}{\vdash}} \Lambda, (\exists \vec{y} \in \mathbb{L}_\pi)[M_{\mathbb{K}}^{\vec{t}}(\vec{y}) \wedge G(\vec{y})]\,.$$

□

11 The Operators $\mathcal{H}_\gamma$

In order to be able to remove critical cuts, i.e. cuts which were introduced by reflection inferences, we have to forgo arbitrary operators. We shall need operators $\mathcal{H}$ such that an $\mathcal{H}$–controlled derivation that satisfies certain extra conditions can be "collapsed" into a derivation with much smaller ordinal labels.

Definition 11.1 The operator $\mathcal{H}_\delta$ is defined by

$$\mathcal{H}_\delta(X) = \bigcap\{C(\alpha,\beta) : \; X \subseteq C(\alpha,\beta) \wedge \delta < \alpha\}$$

Lemma 11.2 *Let $\mathbb{K}$ be a reflection instance with $\mathbb{X}$ being its associated projection instance. Let $i(\mathbb{K})$ be the first point of the interval of $\mathbb{K}$.*

(i) $\mathcal{H}_\delta$ is an operator.

(ii) $\delta < \delta' \implies \mathcal{H}_\delta(X) \subseteq \mathcal{H}_{\delta'}(X)$.

(iii) $\mathcal{H}_\delta$ is closed under $(\alpha, \beta \mapsto \alpha + \beta)$, $(\alpha \mapsto \omega^\alpha)$, $(\alpha, \beta \mapsto \varphi\alpha\beta)_{\alpha,\beta<\mathbf{I}}$, and $(\sigma \mapsto \sigma^+)_{\sigma<\mathbf{I}}$.

(iv) $\{\alpha, \eta\} \cup k(\mathbb{K}) \subseteq \mathcal{H}_\delta(X) \wedge \alpha \leq \delta \wedge \mathbb{K} \in \Re^\alpha \wedge \eta \in \mathbf{dom}(\hat{\Psi}_{\mathbb{X}}^\alpha) \implies \hat{\Psi}_{\mathbb{X}}^\alpha(\eta) \in \mathcal{H}_\delta(X)$.

(v) If μ is a cardinal and $\mu \leq \eta \leq \mu^+ < \mathbf{I} \wedge \eta \in \mathcal{H}_\delta(X) \implies \mu, \mu^+ \in \mathcal{H}_\delta(X)$.

(vi) $\{\alpha, \eta\} \cup k(\mathbb{K}) \subseteq C(\delta, \rho) \wedge \alpha < \delta \wedge \mathbb{K} \in \Re^\alpha \wedge \eta \in \mathbf{dom}(\hat{\Psi}_{\mathbb{X}}^\alpha) \implies \hat{\Psi}_{\mathbb{X}}^\alpha(\eta) \in C(\delta, \rho)$.

(vii) $\{\alpha\} \cup k(\mathbb{K}) \subseteq C(\delta, \rho) \wedge \alpha < \delta \wedge \mathbb{K} \in \Re^\alpha \wedge \rho \leq i(\mathbb{K}) \implies \mathbb{K} \in C(\delta, \rho)$.

Proof: (i) follows from Lemma 4.18. (ii) holds by Lemma 4.1(i). (iii) follows from closure of any $C(\alpha,\beta)$ under these functions.

(iv): From $\{\alpha\}\cup k(\mathbb{K}) \subseteq \mathcal{H}_\delta(X)$, $X \subseteq C(\alpha',\rho)$ and $\alpha \le \delta < \alpha'$, it follows $\alpha, \pi \in C(\alpha',\rho)$, where π is the first point in the interval of $\mathbb{K}$. If $\pi < \rho$, then $\hat{\Psi}^{\alpha}_{\mathbb{K}}(\eta) \in C(\alpha',\rho)$ follows from $\hat{\Psi}^{\alpha}_{\mathbb{K}}(\eta) < \rho$. Now suppose $\rho \le \pi$. If $\mathbb{K}$ is of the form $(\tau;\mathsf{RSC};\emptyset;0)$, $(\tau;\mathsf{RLC};\emptyset;0)$, or $(\Xi;\nu\text{-}\mathsf{P}_0;\mathbf{I}\text{-}\mathsf{P}_0;\emptyset;\sigma)$, then $\mathbb{K} \in C(\alpha',\rho)$ is immediate, and hence $\hat{\Psi}^{\alpha}_{\mathbb{K}}(\eta) \in C(\alpha',\rho)$. If $\mathbb{K}$ is not of one of these shapes, then $\pi = \Psi^{\beta}_{\mathbb{G}}$ for some β and $\mathbb{G}$ and by Corollary 4.16 it follows $\beta, \mathbb{G} \in C(\alpha',\rho)$. By ferreting out all the possibilities of Definition 3.2 and taking into account that $k(\mathbb{K}) \subseteq \mathcal{H}_\delta(X)$, one concludes that $\mathbb{K} \in C(\alpha',\rho)$. Therefore we get $\hat{\Psi}^{\alpha}_{\mathbb{K}}(\eta) \in C(\alpha',\rho)$. As a result, $\hat{\Psi}^{\alpha}_{\mathbb{K}}(\eta) \in \mathcal{H}_\delta(X)$.

(v): Suppose $X \subseteq C(\alpha,\beta)$ with $\delta < \alpha$. Then we have to show $\mu \in C(\alpha,\beta)$. Note that $\eta \in C(\alpha,\beta)$. By induction on n, one verifies

$$(*) \quad \mu \le \eta \le \mu^+ \wedge \eta \in C_n(\alpha,\beta) \implies \mu \in C(\alpha,\beta),$$

yielding $\mu, \mu^+ \in C(\alpha,\beta)$. Let $\eta = \Psi^{\xi}_{\mathbb{K}}(\sigma)$ with $\xi, \mathbb{K} \in C_{n-1}(\alpha,\beta)$, then it follows from Lemma 4.5 and Lemma 4.7 that $\mathbb{K}$ is of the form $(\mu^+;\mathsf{RSC};\dots)$. Hence $\mu^+ \in C(\alpha,\beta)$ which also yields $\mu \in C(\alpha,\beta)$ by Lemma 4.18. The other cases follow easily from the induction hypothesis.

(vi) and (vii) where proved in the course of proving (i). □

Roughly speaking, the process of collapsing or projecting down a proof tree, which we will be using in the next section, involves pruning, grafting, and relabelling the tree with smaller ordinals. The relabelling will be done by applying variants of the functions $\xi \mapsto \hat{\Psi}^{\xi}_{\mathbb{K}}(\pi)$ to the ordinal labels of the original tree. We are compelled to pass to variants of these functions because the latter may not preserve the order of the ordinals of the given tree, and further if $\mathbb{K}$ has interval $[\pi,\delta]$, $\Psi^{\xi}_{\mathbb{K}}(\pi) < \pi$ may fail to be the case for some ordinal ξ of the tree. But that the relabelling be done in an order preserving way is necessary if this procedure is to transform proof trees into proof trees.

To handle the aforementioned difficulties, we will be needing several technical results, the meaning of which will emerge only gradually in the proofs of Theorem 12.1 and Theorem 12.3. I have preferred to ban these "side calculations" from the proofs of the main theorems since the danger is to be feared that they may obscure the central ideas underlying the cut elimination and projection procedure.

Definition 11.3 (i) $NF(\alpha,\beta)$ means that $\alpha_n \ge \beta_1$ if $\alpha = \omega^{\alpha_1} + \cdots + \omega^{\alpha_n}$ and $\beta = \omega^{\beta_1} + \cdots + \omega^{\beta_m}$ are the respective Cantor normal forms.

(ii) If $\mathfrak{W}$ is a projection instance or a reflection instance with interval $[\pi,\delta]$ we put $i(\mathfrak{W}) := \pi$.

(iii) $\mathbb{X} \bowtie \mathbb{Y}$ stands for $i(\mathbb{X}) = i(\mathbb{Y})$. The same notation applies to reflection instances.

Definition 11.4 Let $\mathbb{X}$ be a projection instance in $\mathfrak{R}^{\alpha}$. We define a set of ordinals, $\widetilde{\mathfrak{M}}^{\alpha}_{\mathbb{X}}$, as follows.

1. If $\mathbb{X}$ is of either form $(\pi;\mathsf{RLC};\dots)$, $(\pi;\mathsf{RSC};\dots)$, or $(\pi;0\text{-}\mathsf{P}_m;\dots)$ with interval $[\pi,\pi]$, then $\widetilde{\mathfrak{M}}^{\alpha}_{\mathbb{X}}$ consists of all ordinals $\Psi^{\alpha'}_{\mathbb{X}}(\pi)$ such that $\Psi^{\alpha'}_{\mathbb{X}}(\pi)\downarrow$, $\alpha' \ge \alpha$, and $\alpha \in C(\alpha', \Psi^{\alpha'}_{\mathbb{X}}(\pi))$.
2. Suppose $\mathbb{X} = \mathbb{C}(\delta)$, where $[\pi,\delta]$ is the interval of $\mathbb{C}(\delta)$, and $\mathbf{dom}(\mathbb{C}) \ne (0,\omega)$. Then $\widetilde{\mathfrak{M}}^{\alpha}_{\mathbb{X}}$ consists of all ordinals $\Psi^{\alpha'}_{\mathbb{X}'}(\pi)$, where $\mathbb{X}'$ is of either form $\mathbb{C}(\rho')$ or $(\pi;\mathsf{M}^{\xi'}_{\mathbb{C}(\rho')};\dots)$ such that $\Psi^{\alpha'}_{\mathbb{X}'}\downarrow$, $\xi' \ge \alpha$, $\rho' \ge \rho$, and $\alpha, \rho \in C(\xi, \Psi^{\alpha'}_{\mathbb{X}'}(\pi))$.

3. Suppose $\mathbb{X} = \mathbb{C}\,(m)$, where $[\pi, \delta]$ is the interval of $\mathbb{C}\,(m)$, and $\mathbf{dom}(\mathbb{C}\,) = (0, \omega)$. Then $\widetilde{\mathfrak{M}}^{\alpha}_{\mathbb{X}}$ consists of all ordinals $\Psi^{\alpha'}_{\mathbb{X}'}(\pi)$, where $\mathbb{X}'$ is of either form $\mathbb{C}\,(m')$ or $(\pi; \mathsf{M}^{\xi'}_{\mathbb{C}\,(m')}; \dots)$ such that $\Psi^{\alpha'}_{\mathbb{X}'}\downarrow$, $\xi' \geq \alpha$, $m' \geq m$, and $\alpha \in C(\xi, \Psi^{\alpha'}_{\mathbb{X}'}(\pi))$.

4. Suppose $\mathbb{X}$ is a constant reflection pattern with the interval $[\pi, \delta]$. Then $\widetilde{\mathfrak{M}}^{\alpha}_{\mathbb{X}}$ consists of all ordinals $\Psi^{\alpha'}_{\mathbb{X}'}(\pi)$, where $\mathbb{X}'$ is of either form $\mathbb{X}$ or $(\pi; \mathsf{M}^{\xi'}_{\mathbb{X}}; \dots)$ such that $\Psi^{\alpha'}_{\mathbb{X}'}\downarrow$, $\xi' \geq \alpha$, and $\alpha \in C(\xi, \Psi^{\alpha'}_{\mathbb{X}'}(\pi))$.

Every $\kappa \in \widetilde{\mathfrak{M}}^{\alpha}_{\mathbb{X}}$ is of the form $\Psi^{\alpha'}_{\mathbb{X}'}(\pi)$, where $\pi = i(\mathbb{X})$ and $\alpha', \mathbb{X}'$ are uniquely determined by κ. We put $\mathbf{p}_\kappa := \hat{\Psi}^{\alpha'}_{\mathbb{X}'}$. For terms t such that $|t| \leq \rho'$, where $[\pi, \rho']$ is the interval of $\mathbb{X}'$ and $\hat{\Psi}^{\alpha'}_{\mathbb{X}'}(t)\downarrow$ (cf. Definition 7.5) we let

$$\mathbf{p}_\kappa(t) \;:=\; \hat{\Psi}^{\alpha'}_{\mathbb{X}'}(t) = t^{\alpha', \mathbb{X}'}.$$

Also, if t is a term of the form $\mathbb{L}_\delta$ we sometimes use $\mathbf{p}_\kappa(\delta)$ as short for $\mathbf{p}_\kappa(\mathbb{L}_\delta)$.

Convention. In the following we shall be using the shorthand $\Psi^{\alpha}_{\mathbb{X}}$ for $\Psi^{\alpha}_{\mathbb{X}}(\tau)$, where $i(\mathbb{X}) = \tau$.

Definition 11.5 Let $\mathbb{K}$ be a reflection instance with projection instance $\mathbb{X}$. Let

$$\begin{aligned}\mathfrak{B}(\mathcal{A}; \mathbb{K}\,; \gamma) \;\;:\Longleftrightarrow\;\; & k(\mathbb{K}\,) \cup \{\gamma\} \subseteq \mathcal{H}_\gamma[\mathcal{A}] \;\wedge\; \mathbb{K}\, \in \mathfrak{R}^\gamma \\ & \wedge\, k(\mathcal{A}) \subseteq \bigcap \{C(\gamma+1, \Psi^{\gamma+1}_{\mathbb{Y}}) :\; \mathbb{Y} \bowtie \mathbb{X}\} \\ & \wedge\, \mathbb{K}\, \in C(\gamma+1, i(\mathbb{K}\,)).\end{aligned}$$

Lemma 11.6 *Let $\mathbb{K}$ be a reflection instance with projection instance $\mathbb{X}$. Assume $\mathfrak{B}(\mathcal{A}; \mathbb{K}\,; \gamma)$ and let $[\bar{\kappa}, \bar{\kappa}+\rho]$ be the interval of $\mathbb{K}$. Suppose $\pi \in \widetilde{\mathfrak{M}}^{\hat{\alpha}}_{\mathbb{X}}$, $\alpha \in \mathcal{H}_\gamma[\mathcal{A}]$, and $NF(\gamma, \omega^{\bar{\kappa}\cdot\alpha})$, where $\hat{\alpha} := \gamma + \omega^{\bar{\kappa}\cdot\alpha}$. For arbitrary α_0, let $\hat{\alpha_0} := \gamma + \omega^{\bar{\kappa}\cdot\alpha_0}$.*

(i) $\mathcal{H}_\gamma[\mathcal{A}](\emptyset) \subseteq C(\gamma+1, \Psi^{\gamma+1}_{\mathbb{X}})$ *and* $\mathcal{H}_\gamma[\mathcal{A}](\emptyset) \cap \bar{\kappa} \subseteq \Psi^{\gamma+1}_{\mathbb{X}}$.

(ii) $\Psi^{\hat{\alpha}+\pi}_{\mathbb{X}} \in \mathcal{H}_{\hat{\alpha}+\pi}[\mathcal{A}, \pi]$.

(iii) $\{\alpha_0\} \cup k(\mathbb{Y}) \subseteq \mathcal{H}_\gamma[\mathcal{A}] \;\wedge\; \alpha_0 < \alpha \;\wedge\; \mathbb{Y} \bowtie \mathbb{X} \implies \Psi^{\hat{\alpha_0}+\pi}_{\mathbb{Y}} < \Psi^{\hat{\alpha}+\pi}_{\mathbb{X}}$.

(iv) Suppose $t \in \mathcal{T}$, $|t| \leq \alpha_t < \alpha$, and $\alpha_t \in \mathcal{H}_\gamma[\mathcal{A}, t]$. If $\gamma_t := \gamma + \omega^{\bar{\kappa}\cdot\alpha_t + |t|}$ and $\beta_t := \gamma_t + \omega^{\bar{\kappa}\cdot\alpha_t}$, then

$$\mathfrak{B}(\mathcal{A} \cup \{t\}; \mathbb{K}\,; \gamma_t) \quad \textit{and} \quad \beta_t \in \mathcal{H}_{\gamma_t}[\mathcal{A}, t].$$

If in addition $t \in \mathcal{T}(\pi)$, then also

$$\Psi^{\beta_t+\pi}_{\mathbb{X}} < \Psi^{\hat{\alpha}+\pi}_{\mathbb{X}} \quad \textit{and} \quad \pi \in \widetilde{\mathfrak{M}}^{\beta_t}_{\mathbb{X}}.$$

(v) If $\mathbb{K} \bowtie \mathbb{K}^*$, $\mathbb{K}^* \in \mathfrak{R}^\gamma$ and $k(\mathbb{K}^*) \subseteq \mathcal{H}_\gamma[\mathcal{A}]$ then $\mathfrak{B}(\mathcal{A}; \mathbb{K}^*; \gamma)$.

Proof: (i) follows from $k(\mathcal{A}) \subseteq C(\gamma+1, \Psi^{\gamma+1}_{\mathbb{X}})$ in view of the definition of $\mathcal{H}_\gamma[\mathcal{A}]$.

(ii): Since $\{\gamma, \alpha\} \cup k(\mathbb{K}\,) \subseteq \mathcal{H}_{\hat{\alpha}}[\mathcal{A}]$, it follows $\gamma, \alpha, \bar{\kappa} \in \mathcal{H}_{\hat{\alpha}+\pi}[\mathcal{A}, \pi]$. As $\pi \in \mathcal{H}_{\hat{\alpha}+\pi}[\mathcal{A}, \pi]$, we get $\hat{\alpha}+\pi \in \mathcal{H}_{\hat{\alpha}+\pi}[\mathcal{A}, \pi]$, and hence, using Lemma 11.2(iv), we obtain $\Psi^{\hat{\alpha}+\pi}_{\mathbb{X}} \in \mathcal{H}_{\hat{\alpha}+\pi}[\mathcal{A}, \pi]$.

(iii): $\hat{\alpha}+\pi \in C(\hat{\alpha}+\pi, \Psi^{\hat{\alpha}+\pi}_{\mathbb{X}})$ and $NF(\gamma, \omega^{\bar{\kappa}\cdot\alpha})$ imply $\gamma \in C(\hat{\alpha}+\pi, \Psi^{\hat{\alpha}+\pi}_{\mathbb{X}})$ by 4.18. Therefore,

$$\{\alpha_0\} \cup k(\mathbb{Y}) \subseteq \mathcal{H}_\gamma[\mathcal{A}] \subseteq C(\gamma+1, \Psi^{\gamma+1}_{\mathbb{X}}) \subseteq C(\hat{\alpha}+\pi, \Psi^{\hat{\alpha}+\pi}_{\mathbb{X}}).$$

Thence, $\hat{\alpha_0}+\pi \in C(\hat{\alpha}+\pi, \Psi_{\mathbb{X}}^{\hat{\alpha}+\pi}) \cap \hat{\alpha}+\pi$ and $k(\mathbb{Y}) \subseteq C(\hat{\alpha}+\pi, \Psi_{\mathbb{X}}^{\hat{\alpha}+\pi})$. Therefore $\Psi_{\mathbb{Y}}^{\hat{\alpha_0}+\pi} \in C(\hat{\alpha}+\pi, \Psi_{\mathbb{X}}^{\hat{\alpha}+\pi})$ by Lemma 11.2(vi). As $\mathbb{K} \in C(\gamma+1, \bar{\kappa})$, $k(\mathbb{Y}) \subseteq C(\gamma+1, \bar{\kappa})$, and $\hat{\alpha_0}+\pi \in C(\hat{\alpha}+\pi, \bar{\kappa})$, we get $\mathbb{Y}, \hat{\alpha_0}+\pi \in C(\hat{\alpha}+\pi, \bar{\kappa})$, and hence $\Psi_{\mathbb{Y}}^{\hat{\alpha_0}+\pi} < \bar{\kappa}$. Together with $\Psi_{\mathbb{Y}}^{\hat{\alpha_0}+\pi} \in C(\hat{\alpha}+\pi, \Psi_{\mathbb{X}}^{\hat{\alpha}+\pi})$ we get $\Psi_{\mathbb{Y}}^{\hat{\alpha_0}+\pi} < \Psi_{\mathbb{X}}^{\hat{\alpha}+\pi}$.

(iv): $\{\gamma\} \cup k(\mathbb{K}) \subseteq \mathcal{H}_\gamma[\mathcal{A}]$ ensures $\gamma_t, \beta_t \in \mathcal{H}_{\gamma_t}[\mathcal{A}; t]$.

Let $\mathbb{Y} \bowtie \mathbb{X}$. As $\gamma, \mathbb{Y}, \bar{\kappa}, \alpha_t, |t| \in C(\gamma_t, \bar{\kappa})$ it follows $\gamma_t, \mathbb{Y} \in C(\gamma_t, \Psi_{\mathbb{Y}}^{\gamma_t})$. Since $NF(\gamma, \omega^{\bar{\kappa}\cdot\alpha})$ and $\alpha_t < \alpha$ yield $NF(\gamma, \omega^{\bar{\kappa}\cdot\alpha_t+|t|})$ we can deduce $\gamma \in C(\gamma_t, \Psi_{\mathbb{Y}}^{\gamma_t})$, and hence $\Psi_{\mathbb{Y}}^{\gamma} \in C(\gamma_t, \Psi_{\mathbb{Y}}^{\gamma_t})$. As a result, $C(\gamma+1, \Psi_{\mathbb{X}}^{\gamma+1}) \subseteq C(\gamma_t, \Psi_{\mathbb{Y}}^{\gamma_t})$. This shows $\mathfrak{B}(\mathcal{A} \cup \{t\}; \mathbb{K}; \gamma_t)$.

Now suppose $t \in \mathcal{T}(\pi)$. From $NF(\gamma, \omega^{\bar{\kappa}\cdot\alpha})$ it follows $\gamma \in C(\hat{\alpha}, \Psi_{\mathbb{X}}^{\hat{\alpha}})$ and hence $k(\mathcal{A} \cup \{t\}) \subseteq C(\hat{\alpha}, \pi)$ as $\Psi_{\mathbb{X}}^{\hat{\alpha}} \le \pi$ holds because of $\pi \in \widetilde{\mathfrak{M}}_{\mathbb{X}}^{\hat{\alpha}}$. Whence, $\beta_t \in C(\hat{\alpha}, \pi) \cap \hat{\alpha}$. This implies

$$\beta_t + \pi \in C(\hat{\alpha}+\pi, \Psi_{\mathbb{X}}^{\hat{\alpha}+\pi}) \cap \hat{\alpha}+\pi;$$

thus

$$\Psi_{\mathbb{X}}^{\beta_t+\pi} < \Psi_{\mathbb{X}}^{\hat{\alpha}+\pi}.$$

Finally, from $\beta_t \in C(\hat{\alpha}, \pi) \cap \hat{\alpha}$ and $\pi \in \widetilde{\mathfrak{M}}_{\mathbb{X}}^{\hat{\alpha}}$ we obtain $\pi \in \widetilde{\mathfrak{M}}_{\mathbb{X}}^{\beta_t}$.

(v) is obvious. ☐

Definition 11.7 *(i) Let $Card$ denote the set of cardinals $\le \mathbf{I}$.*

(ii) Let $\mathfrak{A}(\mathcal{A}; \mathbb{K}; \gamma; \mu)$ stand for

$$\mathfrak{B}(\mathcal{A}; \mathbb{K}; \gamma) \ \wedge \ \mu \in \mathcal{H}_\gamma[\mathcal{A}] \ \wedge \ \mu \in Card \ \wedge \ \pi \le \mu$$
$$\wedge \ \pi \in \bigcap \big\{ C(\delta, \Psi_{\mathbb{Z}}^{\delta}) : \delta > \gamma;\ \mathbb{Z} \in \Re^{\delta};\ \pi < i(\mathbb{Z}) \big\}$$

where $\pi := i(\mathbb{K})$.

Lemma 11.8 *Assume $\mathfrak{A}(\mathcal{A}; \mathbb{K}; \gamma; \mu)$, $NF(\gamma, \omega^{\mu\cdot\alpha})$, and $\alpha \in \mathcal{H}_\gamma[\mathcal{A}]$. For arbitrary β, let $\hat{\beta} := \gamma + \omega^{\mu\cdot\beta}$. Let $\mathbb{X}$ be the projection instance pertaining to $\mathbb{K}$ and $\pi = i(\mathbb{K})$. Then the following properties hold:*

(i) $\Psi_{\mathbb{X}}^{\hat{\alpha}} \in \mathcal{H}_{\hat{\alpha}}[\mathcal{A}] \ \wedge \ \Psi_{\mathbb{X}}^{\hat{\alpha}} \in \widetilde{\mathfrak{M}}_{\mathbb{X}}^{\hat{\alpha_0}} \cap \pi$.

(ii) $\mathcal{H}_\gamma[\mathcal{A}](\emptyset) \subseteq C(\gamma+1, \Psi_{\mathbb{X}}^{\gamma+1})$ *and* $\Psi_{\mathbb{X}}^{\gamma+1} < \pi$.

(iii) $\alpha_0 \in \mathcal{H}_\gamma[\mathcal{A}] \wedge \alpha_0 < \alpha \implies \Psi_{\mathbb{X}}^{\hat{\alpha_0}} < \Psi_{\mathbb{X}}^{\hat{\alpha}}$.

(iv) Suppose $k(\mathbb{K}^*) \subseteq \mathcal{H}_\gamma[\mathcal{A}]$, $\mathbb{K}^* \in \Re^\gamma$, $\mathbb{K} \bowtie \mathbb{K}^*$, and $t \in \mathcal{T}(\pi)$. If $\gamma_t = \gamma + \omega^{\mu\cdot\alpha+|t|}$, then

$$\mathfrak{A}(\mathcal{A} \cup \{t\}; \mathbb{K}^*; \gamma_t; \mu).$$

(v) If $\alpha_0 < \alpha$, $\{\alpha_0\} \cup k(\mathbb{J}) \subseteq \mathcal{H}_\gamma[\mathcal{A}]$, $\mathbb{J} \in \Re^\gamma$, $\tau = i(\mathbb{J})$, and $\pi \le \tau \le \mu$, then

$$\mathfrak{A}(\mathcal{A}; \mathbb{J}; \gamma; \mu) \ \wedge \ \mathfrak{A}(\mathcal{A}; \mathbb{J}; \hat{\alpha_0}; \mu).$$

Proof: (i): $\hat{\alpha} \in \mathcal{H}_{\hat{\alpha}}[\mathcal{A}]$ is obvious. Therefore, $\Psi_{\mathbb{X}}^{\hat{\alpha}} \in \mathcal{H}_{\hat{\alpha}}[\mathcal{A}]$ by 11.2(iv). Since $\mathcal{H}_\gamma[\mathcal{A}](\emptyset) \subseteq C(\gamma+1, \Psi_{\mathbb{X}}^{\gamma+1}) \subseteq C(\hat{\alpha}, \pi)$, we get $\hat{\alpha} \in C(\hat{\alpha}, \pi)$. We also have $\mathbb{K} \in C(\hat{\alpha}, \pi)$. Thus $\Psi_{\mathbb{X}}^{\hat{\alpha}} \in \widetilde{\mathfrak{M}}_{\mathbb{X}}^{\hat{\alpha}} \cap \pi$.

(ii): The first part is immediate as $k(\mathcal{A}) \subseteq C(\gamma+1, \Psi_{\mathbb{X}}^{\gamma+1})$. As $\mathfrak{B}(\mathcal{A}; \mathbb{K}; \gamma)$, it follows $\mathbb{X} \in C(\gamma+1, \pi)$. Hence $\mathbb{X}, \gamma \in C(\gamma+1, \pi)$, and thus $\Psi_{\mathbb{X}}^{\gamma+1} < \pi$.

(iii): Since $\hat{\alpha} \in C(\hat{\alpha}, \pi)$ by (ii), and $\mathbb{K} \in C(\hat{\alpha}, \pi)$ due to $\mathfrak{B}(\mathcal{A}; \mathbb{K}; \gamma)$, it follows $\hat{\alpha}, \mathbb{K} \in C(\hat{\alpha}, \Psi^{\hat{\alpha}}_{\mathbb{X}})$. $NF(\gamma, \omega^{\mu \cdot \alpha})$ involves $\gamma \in C(\hat{\alpha}, \Psi^{\hat{\alpha}}_{\mathbb{X}})$, thus $\Psi^{\gamma+1}_{\mathbb{X}} \in C(\hat{\alpha}, \Psi^{\hat{\alpha}}_{\mathbb{X}})$. From (ii) we get $\Psi^{\gamma+1}_{\mathbb{X}} < \pi$. Therefore, $\Psi^{\gamma+1}_{\mathbb{X}} < \Psi^{\hat{\alpha}}_{\mathbb{X}}$. In view of (ii), this yields $\mathcal{H}_\gamma[\mathcal{A}](\emptyset) \subseteq C(\hat{\alpha}, \Psi^{\hat{\alpha}}_{\mathbb{X}})$ and hence $\hat{\alpha_0} \in C(\hat{\alpha}, \Psi^{\hat{\alpha}}_{\mathbb{X}})$. $\Psi^{\hat{\alpha_0}}_{\mathbb{X}} < \pi$ follows by replacing α with α_0 in the proof of (i). Consequently, in view of the above, $\Psi^{\hat{\alpha_0}}_{\mathbb{X}} < \Psi^{\hat{\alpha}}_{\mathbb{X}}$.

(iv): Let $\mathbb{Y}$ be the projection instance pertaining to $\mathbb{K}^*$.

$\alpha, \mu, \gamma \in \mathcal{H}_\gamma[\mathcal{A}]$ guarantees $\mu, \gamma, |t|, \alpha_t \in \mathcal{H}_{\gamma_t}[\mathcal{A}, t]$. Therefore,

$$\gamma_t \in \mathcal{H}_{\gamma_t}[\mathcal{A}, t].$$

We claim that

$$(*) \quad k(\mathcal{A} \cup \{t\}) \subseteq C(\gamma_t + 1, \Psi^{\gamma_t+1}_{\mathbb{Y}})$$

By (ii), $\alpha, \gamma \in C(\gamma_t+1, \pi)$ and hence $\gamma_t \in C(\gamma_t+1, \pi)$, which implies $\gamma_t \in C(\gamma_t+1, \Psi^{\gamma_t+1}_{\mathbb{Y}})$. As $NF(\gamma, \omega^{\mu \cdot \alpha})$, this shows $\gamma \in C(\gamma_t + 1, \Psi^{\gamma_t+1}_{\mathbb{Y}})$. Since $\mathbb{X} \in C(\gamma + 1, \pi)$, $\mathbb{X} \bowtie \mathbb{Y}$, and $k(\mathbb{Y}) \subseteq C(\gamma + 1, \pi)$ we get $\mathbb{Y} \in C(\gamma + 1, \pi)$, and hence $\gamma, \mathbb{Y} \in C(\gamma_t + 1, \Psi^{\gamma_t+1}_{\mathbb{Y}})$. The latter yields $\Psi^{\gamma+1}_{\mathbb{Y}} \in C(\gamma_t + 1, \Psi^{\gamma_t+1}_{\mathbb{Y}})$. As $\gamma + 1, \mathbb{Y} \in C(\gamma + 1, \pi)$, we also get $\Psi^{\gamma+1}_{\mathbb{Y}} < \pi$. As a result, $\Psi^{\gamma+1}_{\mathbb{Y}} < \Psi^{\gamma_t+1}_{\mathbb{Y}}$. So we obtain $k(\mathcal{A}) \subseteq C(\gamma_t, \Psi^{\gamma_t+1}_{\mathbb{Y}})$ and hence $(*)$.

As $k(\mathbb{K}^*) \subseteq C(\gamma + 1, \pi)$, $\mathbb{K} \bowtie \mathbb{K}^*$, $\mathbb{K} \in \mathfrak{R}^\gamma$, $\mathbb{K} \in C(\gamma + 1, \pi)$ it follows $\mathbb{K}^* \in C(\gamma + 1, \pi)$. Therefore from $(*)$ and $\mathcal{H}_\gamma[\mathcal{A}] \subseteq \mathcal{H}_{\gamma_t}[\mathcal{A}, t]$ and $\gamma_t \in \mathcal{H}_{\gamma_t}[\mathcal{A}, t]$, we get $\mathfrak{A}(\mathcal{A} \cup \{t\}; \mathbb{K}^*; \gamma_t; \mu)$.

(v): We first have to verify $\mathbb{J} \in C(\gamma + 1, \tau)$. Since $\{\tau\} \cup k(\mathbb{J}) \subseteq C(\gamma + 1, \pi)$, $\mathbb{J} \in \mathfrak{R}^\gamma$, and $\pi \leq \tau$, we get $\mathbb{J} \in C(\gamma + 1, \pi)$ by Lemma 11.2,(vii). A fortiori we have $\mathbb{J} \in C(\gamma + 1, \tau)$.

Let $\mathbb{U}$ be the projection instance pertaining to $\mathbb{J}$. To verify $\mathfrak{B}(\mathcal{A}; \mathbb{J}; \gamma)$, assume $\mathbb{W} \bowtie \mathbb{U}$. We have to show

$$k(\mathcal{A}) \subseteq C(\gamma + 1, \Psi^{\gamma+1}_{\mathbb{W}}). \tag{39}$$

If $i(\mathbb{W}) = \pi$ then $\mathbb{W} \bowtie \mathbb{X}$ and (39) follows from $\mathfrak{B}(\mathcal{A}; \mathbb{K}; \gamma)$. If $i(\mathbb{W}) > \pi$ then $\pi \in C(\gamma + 1, \Psi^{\gamma+1}_{\mathbb{W}})$ because of $\mathfrak{A}(\mathcal{A}; \mathbb{K}; \gamma; \mu)$. We then get $\pi < \Psi^{\gamma+1}_{\mathbb{W}}$. Thus as $k(\mathcal{A}) \subseteq C(\gamma + 1, \pi)$ we get $k(\mathcal{A}) \subseteq C(\gamma + 1, \Psi^{\gamma+1}_{\mathbb{W}})$, verifying (39).

Next we verify

$$\tau \in \bigcap \big\{ C(\delta, \Psi^{\delta}_{\mathbb{Z}}) : \delta > \gamma; \mathbb{Z} \in \mathfrak{R}^\delta; \tau < i(\mathbb{Z}) \big\}.$$

Above we have already proved $\mathbb{J} \in C(\gamma + 1, \pi)$. If now $\delta > \gamma$ and $\mathbb{Z} \in \mathfrak{R}^\delta$ and $\kappa > \tau$, where $\kappa = i(\mathbb{Z})$, then $\pi \in C(\delta, \Psi^{\delta}_{\mathbb{Z}})$; whence $\mathbb{J} \in C(\delta, \Psi^{\delta}_{\mathbb{Z}})$ and therefore $\tau \in C(\delta, \Psi^{\delta}_{\mathbb{Z}})$.

$\mathfrak{A}(\mathcal{A}; \mathbb{J}; \gamma; \mu)$ and $\mathfrak{A}(\mathcal{A}; \mathbb{J}; \hat{\alpha_0}; \mu)$ are now immediate by the above. □

12 Impredicative cut elimination and projection

In general, the usual cut elimination procedure does not apply when the cut formula has been introduced by a reflection inference. A rather innocent scenario arises when $\mathbb{K}$ is a reflection pattern with interval $[\pi, \pi]$ and

$$\mathcal{H} \,\big|\!\frac{\alpha}{\pi+1}\, \Gamma$$

results from

$$\frac{\mathcal{H} \,\big|\!\frac{\xi_0}{\pi}\, \Gamma, F(\mathbb{L}_\pi)}{\mathcal{H} \,\big|\!\frac{\xi}{\pi}\, \Gamma, \exists z^\pi [M_{\mathbb{K}}(z) \wedge F(z)]} \,(Ref_{\mathbb{K}})$$

and

$$\frac{\mathcal{H}[s] \mid\frac{\xi_s}{\pi} \Gamma, M_{\mathbb{K}}(s) \rightarrow \neg F(s) \text{ for all } |s|<\pi}{\mathcal{H} \mid\frac{\xi}{\pi} \Gamma, \forall z^{\pi}[M_{\mathbb{K}}(z) \rightarrow \neg F(z)]}(\forall)$$

using (Cut), where $F(\mathbb{L}_\pi)$ is a $\mathcal{F}(\mathbb{K})$-formula. In this situation, the usual procedure of replacing a cut with cuts of lesser rank does not work. In order to overcome this problem, the proof tree has to undergo more radical transformations.

Theorem 12.1 *Let $\mathbb{K}$ be a reflection instance with interval $[\pi, \delta]$. Let $\mathbb{X}$ be the projection instance pertaining to $\mathbb{K}$. Suppose $\mathfrak{B}(\mathcal{A}; \mathbb{K}; \gamma)$ and $NF(\gamma, \omega^{\pi\cdot\alpha})$. Suppose further that $\mathbb{K}$ is* **not** *of one of the forms $(\pi; \mathsf{RSC}; \dots)$ or $(\pi; \mathsf{RLC}; \dots)$ or $(\pi; \mathsf{M}^{\alpha}_{\mathbb{Z}}\text{-}\pi\text{-}\mathsf{P}_0; \dots)$ or $(\pi; \pi\text{-}\mathsf{P}_0; \lambda\text{-}\mathsf{P}_0; \dots)$. Let $\Gamma[\vec{t}]$ be a set of $\mathcal{F}(\mathbb{K})$-formulae, where $\vec{t} = t_1, \dots, t_j$. Assume*

$$\mathcal{H}_\gamma[\mathcal{A}] \mid\frac{\alpha}{\upsilon} \Gamma[\vec{t}],$$

where $\upsilon \geq \pi + 1$ and that the following conditions are satisfied:

1. *All reflection inferences $(Ref_{\mathbb{J}})$ occurring in the derivation satisfy $\mathbb{J} \in \Re^\gamma$.*
2. *For every reflection inference $(Ref_{\mathbb{J}'})$ occurring in the derivation whose interval is of the form $[\kappa', \eta']$ with $\kappa' > \pi$ there exists a reflection instance $\mathbb{K}' \in \Re^\gamma$ with interval $[\pi, \delta']$ such that $\delta' > \eta'$, or $\delta' = \eta'$ and $\Pi_3(\delta') \subseteq \mathcal{F}(\mathbb{K}')$.*
3. *For every instance of (Cut) occurring in the derivations with cut formulae A and $\neg A$ there exists a reflection instance $\mathbb{K}' \in \Re^\gamma$ with interval $[\pi, \delta']$ such that $A, \neg A \in \mathcal{F}(\mathbb{K}')$.*

Then, for all $\kappa \in \widetilde{\mathfrak{M}}^{\tilde{\alpha}}_{\mathbb{X}}$,

$$\mathcal{H}_{\tilde{\alpha}+\kappa}[\mathcal{A}, \kappa] \mid\frac{\Psi^{\tilde{\alpha}+\kappa}_{\mathbb{X}}}{\Psi^{\tilde{\alpha}+\kappa}_{\mathbb{X}}} \Gamma[\mathbf{p}_\kappa(\vec{t})],$$

where $\mathbf{p}_\kappa(\vec{t})$ stands for $\mathbf{p}_\kappa(t_1), \dots, \mathbf{p}_\kappa(t_j)$[4] *and $\tilde{\alpha} = \gamma + \omega^{\pi\cdot\alpha}$, and moreover, all reflection inferences $Ref_{\mathbb{F}}$ occurring in the latter derivation satisfy $\mathbb{F} \in \Re^{\max(\tilde{\alpha}, o(\kappa)+1)}$.*

Proof: We proceed by induction on α and distinguish cases according to the last inference of the derivation.

Note that because of the requirement that $\mathbb{K}$ is not of one of the forms $(\pi; \mathsf{RSC}; \dots)$, $(\pi; \mathsf{RLC}; \dots)$, $(\pi; \mathsf{M}^{\alpha}_{\mathbb{Z}}\text{-}\pi\text{-}\mathsf{P}_0; \dots)$, or $(\pi; \pi\text{-}\mathsf{P}_0; \lambda\text{-}\mathsf{P}_0; \dots)$, it follows that all reflection inferences $(Ref_{\mathbb{J}'})$ of the form $[\kappa', \eta']$ with $\kappa' < \pi$ satisfy $\eta' < \pi$.

Case 1: The last inference is $(\forall)$ with principal formula $(\forall x \in s)F(x, \vec{t}) \in \Gamma[\vec{t}]$, where $|s| < \pi$ or s is one of the terms $\vec{t}$. Then, for all $|r| < |s|$, there exists α_r satisfying $|r| \leq \alpha_r < \alpha$ and

$$\mathcal{H}_\gamma[\mathcal{A}, r] \mid\frac{\alpha_r}{\upsilon} \Gamma[\vec{t}], r \mathrel{\mathring{\in}} s \rightarrow F(x, \vec{t}). \tag{40}$$

Define $\gamma_r := \gamma + \omega^{\pi\cdot\alpha_r + |r|}$ and $\beta_r := \gamma_t + \omega^{\pi\cdot\alpha_r}$. Then $NF(\gamma_r, \omega^{\pi\cdot\alpha_r})$. Also $\mathfrak{B}(\mathcal{A} \cup \{r\}; \mathbb{K}; \gamma_r)$ by 11.6(iv). Therefore, using the induction hypothesis on (40) and putting $\vec{t_\kappa} := \mathbf{p}_\kappa(\vec{t})$, $s_\kappa := \mathbf{p}_\kappa(s)$, $r_\kappa := \mathbf{p}_\kappa(r)$, we obtain

$$\mathcal{H}_{\beta_r+\kappa}[\mathcal{A}, r, \kappa] \mid\frac{\Psi^{\beta_r+\kappa}_{\mathbb{X}}}{\Psi^{\beta_r+\kappa}_{\mathbb{X}}} \Gamma[\vec{t_\kappa}], r_\kappa \mathrel{\mathring{\in}} s_\kappa \rightarrow F(r_\kappa, \vec{t_\kappa}) \tag{41}$$

[4]Recall that the functions $\mathbf{p}_\kappa$ were defined in Definition 11.4.

holds for all $\kappa \in \widetilde{\mathfrak{M}}^{\beta_r}_{\mathbb{X}}$ and $|r| < |s|$ for which $\mathbf{p}_\kappa(r)$ is defined. If $\kappa \in \widetilde{\mathfrak{M}}^{\tilde{\alpha}}_{\mathbb{X}}$, $|r| < |s|$ and $\mathbf{p}_\kappa(r)$ is defined, then, by Lemma 11.6(iv), $\kappa \in \widetilde{\mathfrak{M}}^{\beta_r}$ and $\Psi^{\beta_r+\kappa}_{\mathbb{X}} < \Psi^{\tilde{\alpha}+\kappa}_{\mathbb{X}}$. Also note that every term r' such that $|r'| < |s_\kappa|$ is of the form r_κ for a uniquely determined term r with $|r| < |s|$. Therefore from (41) we can conclude

$$\mathcal{H}_{\tilde{\alpha}+\kappa}[\mathcal{A},\kappa] \,\Big|\frac{\Psi^{\tilde{\alpha}+\kappa}_{\mathbb{X}}}{\Psi^{\tilde{\alpha}+\kappa}_{\mathbb{X}}}\, \Gamma[\vec{t}_\kappa], (\forall x \in s_\kappa)F(x, \vec{t}_\kappa)$$

by means of $(\forall)$. Since $\Gamma[\vec{t}_\kappa], (\forall x \in s_\kappa)F(x, \vec{t}_\kappa) = \Gamma[\vec{t}_\kappa]$, this provides the desired result.

Case 2: The last inference is of the form $(\bigwedge)$ but does not fall under the previous case. The assertion then follows by similar considerations as in the previous case.

Case 3: The last inference is $(\exists)$ with principal formula $(\exists x \in s)F(x, \vec{t}) \in \Gamma[\vec{t}]$ where $|s| < \pi$ or s is one of the terms $\vec{t}$. Then, for some term $|r_0| < |s|$ and ordinal $\alpha_0 < \alpha$ with $|r_0| < \alpha_0$ it holds

$$\mathcal{H}_\gamma[\mathcal{A}] \,\Big|\frac{\alpha_0}{\upsilon}\, \Gamma[\vec{t}], r_0 \mathrel{\mathring{\in}} s \wedge F(r_0, \vec{t}).$$

Hence, by the induction hypothesis , for all $\kappa \in \widetilde{\mathfrak{M}}^{\tilde{\alpha_0}}_{\mathbb{X}}$,

$$\mathcal{H}_{\tilde{\alpha_0}+\kappa}[\mathcal{A},\kappa] \,\Big|\frac{\Psi^{\tilde{\alpha_0}+\kappa}_{\mathbb{X}}}{\Psi^{\tilde{\alpha_0}+\kappa}_{\mathbb{X}}}\, \Gamma[\vec{t}_\kappa], \mathbf{p}_\kappa(r_0) \mathrel{\mathring{\in}} \mathbf{p}_\kappa(s) \;\wedge\; F(\mathbf{p}_\kappa(r_0), \vec{t}_\kappa).$$

The conditions on r_0 ensure that $|\mathbf{p}_\kappa(r_0)| < \Psi^{\tilde{\alpha}+\kappa}_{\mathbb{X}}$. As $\widetilde{\mathfrak{M}}^{\tilde{\alpha}}_{\mathbb{X}} \subseteq \widetilde{\mathfrak{M}}^{\tilde{\alpha_0}}_{\mathbb{X}}$ and $\Psi^{\tilde{\alpha_0}+\pi}_{\mathbb{X}} < \Psi^{\tilde{\alpha}+\pi}_{\mathbb{X}}$ hold, applying $(\exists)$ yields

$$\mathcal{H}_{\tilde{\alpha}+\kappa}[\mathcal{A},\kappa] \,\Big|\frac{\Psi^{\tilde{\alpha}+\kappa}_{\mathbb{X}}}{\Psi^{\tilde{\alpha}+\kappa}_{\mathbb{X}}}\, \Gamma[\vec{t}_\kappa], (\exists x \in \mathbf{p}_\kappa(s))F(x, \vec{t}_\kappa)$$

which is the desired result.

Case 4: The last inference is $(\bigvee)$ but does not fall under the previous case. The assertion then follows by similar (or simpler in the case of $(\vee)$) considerations as in the previous case.

Case 5: The last inference is $(Ref_{\mathbb{K}'})$ with principal formula

$$(\exists \vec{y} \in \mathbb{L}_\pi)[M^{\vec{s}}_{\mathbb{K}'}(\vec{y}) \;\wedge\; F(\vec{y})] \in \Gamma[\vec{t}]$$

and $F(\vec{s}) \in \mathcal{F}(\mathbb{K}')$, where $\mathbb{K}'$'s interval is of the form $[\pi, \delta']$. Then $\mathbb{K}'$ is a relative of $\mathbb{K}$ and there exists $\alpha_0 < \alpha$ such that we have a scenario

$$\mathcal{H}_\gamma[\mathcal{A}] \,\Big|\frac{\alpha_0}{\upsilon}\, \Gamma[\vec{t}], F(\vec{s}).$$

From $\mathbb{K}$ and $\mathbb{K}'$ we can compose a common maximum relative $\mathbb{K}^*$ such that $\mathbb{K} \trianglelefteq \mathbb{K}^*$ and $\mathbb{K}' \trianglelefteq \mathbb{K}^*$ as well as $\mathbb{K}^* \in \Re^\gamma$ and $\mathbb{K}^* \in \mathcal{H}_\gamma[\mathcal{A}]$. Let $\mathbb{Y}$ be the projection instance pertaining to $\mathbb{K}^*$. From the induction hypothesis we then obtain, for all $\tau \in \widetilde{\mathfrak{M}}^{\tilde{\alpha_0}}_{\mathbb{Y}}$,

$$\mathcal{H}_{\tilde{\alpha_0}+\tau}[\mathcal{A},\tau] \,\Big|\frac{\Psi^{\tilde{\alpha_0}+\tau}_{\mathbb{Y}}}{\Psi^{\tilde{\alpha_0}+\tau}_{\mathbb{Y}}}\, \Gamma[\vec{t}_\tau], F(\vec{s}_\tau), \tag{42}$$

where $\vec{t}_\tau := \mathbf{p}_\tau(\vec{t})$ and $\vec{s}_\tau := \mathbf{p}_\tau(\vec{s})$. In the sequel, fix $\kappa \in \widetilde{\mathfrak{M}}^{\hat{\alpha}}_{\mathbb{X}}$. Notice that for $\tau \in \widetilde{\mathfrak{M}}^{\tilde{\alpha_0}}_{\mathbb{Y}} \cap \kappa$ we have $\mathbf{p}_\tau(\delta') < \kappa$, $|\vec{s}_\tau| < \mathbf{p}_\tau(\delta')$, and $\Vdash M^{\vec{s}}_{\mathbb{K}'}(\vec{s}_\tau)$. From (42) we thus get

$$\mathcal{H}_{\tilde{\alpha_0}+\tau}[\mathcal{A},\tau] \,\Big|\frac{\Psi^{\tilde{\alpha_0}+\tau}_{\mathbb{Y}}+\omega}{\Psi^{\tilde{\alpha_0}+\tau}_{\mathbb{Y}}}\, \bigvee\Gamma[\vec{t}_\tau], (\exists \vec{y} \in \in \mathbb{L}_\kappa)[M^{\vec{s}}_{\mathbb{K}'}(\vec{y}) \;\wedge\; F(\vec{y})] \tag{43}$$

for all $\tau \in \widetilde{\mathfrak{M}}_{\mathbb{Y}}^{\tilde{\alpha_0}} \cap \kappa$.

To proceed further we have to distinguish cases according to the shape of $\mathbb{X}$. Note that $o(\mathbb{X}) < \tilde{\alpha}$.

Subcase 5.1: $\mathbb{X}$ is of the form $\mathbb{A}\,(\delta) = (\pi; \delta\text{-}\mathsf{P}_0; \lambda\text{-}\mathsf{P}_0; \dots)$ and δ is a limit. Then there is a reflection pattern $\mathbb{C}$ of the form $\mathbb{C}\,(\sigma) := (\kappa; \mathsf{M}^{<\zeta}_{\mathbb{A}\,(\gamma)}\text{-}\sigma\text{-}\mathsf{P}_0; \dots)$ with $\gamma \geq \delta$, $\mathbf{p}_\kappa(\delta) \in \mathbf{dom}(\mathbb{C}\,)$ and $\tilde{\alpha} \leq \zeta$. Since δ is a limit we can pick $\mu < \delta$ such that the levels of all terms from $\vec{t}$ are $< \mu$. Further note that $\Gamma[\vec{t}]$ is a set of $\mathcal{F}(\mathbb{K}\,)$ formulae, i.e., a set of $\Pi_0(\delta)$-formulae. As a result, $\Gamma[\vec{t}]$ is a set of $\Pi_0(\mu)$-formulae. Put

$$\mathbb{J} := (\mathbb{C}\,(\mathbf{p}_\kappa(\mu)), \tilde{\alpha_0}, \delta_0)$$

where $\delta_0 := \max(\delta, \delta')$. Then $k(\mathbb{J}) \subseteq \mathcal{H}_{\tilde{\alpha_0}+\kappa}[\mathcal{A}, \kappa]$. Let $\vec{q}$ be terms of level $< \kappa$. In view of Corollary 9.4, we get

$$\Vdash \vec{t_\tau} \neq \vec{q}, \bigwedge \neg\Gamma[\vec{t_\tau}], \bigvee \Gamma[\vec{q}].$$

Put $B := (\exists \vec{y} \in \mathbb{L}_\kappa)[M^{\vec{s}}_{\mathbb{K}'}(\vec{y}) \,\wedge\, F(\vec{y}\,)]$. Using (43) and (Cut), we get with

$$f(\vec{q}, \tau) \;\; := \;\; no(\vec{t_\tau} \neq \vec{q}, \bigwedge \neg\Gamma[\vec{t_\tau}], \bigvee \Gamma[\vec{q}]),$$

$$\mathcal{H}_{\tilde{\alpha_0}+\tau}[\mathcal{A}, \tau, \vec{q}\,] \; \overset{\Psi_{\mathbb{Y}}^{\tilde{\alpha_0}+\tau}\#f(\vec{q},\tau)}{\underset{\Psi_{\mathbb{Y}}^{\tilde{\alpha_0}+\tau}\#f(\vec{q},\tau)}{\vdash}} \; \vec{t_\tau} \neq \vec{q}, \bigvee \Gamma[\vec{q}],\, B \tag{44}$$

for all $\tau \in \widetilde{\mathfrak{M}}_{\mathbb{Y}}^{\tilde{\alpha_0}} \cap \kappa$ and $|\vec{q}| < \kappa$. From (44), applying inferences $(\vee)$ and subsequently an inference $(\neg M^{\vec{t_\kappa}}_{\mathbb{J}})$, we get

$$\mathcal{H}_{\tilde{\alpha_0}+\kappa}[\mathcal{A}, \kappa, \vec{q}\,] \; \overset{\Psi_{\mathbb{Y}}^{\tilde{\alpha_0}+\kappa}}{\underset{\Psi_{\mathbb{Y}}^{\tilde{\alpha_0}+\kappa}}{\vdash}} \; \neg M^{\vec{t_\kappa}}_{\mathbb{J}}(\vec{q}), \bigvee \Gamma[\vec{q}],\, B \tag{45}$$

for all terms $\vec{q}$ with levels $< \kappa$. Putting to use $(\vee)$ and subsequently $(\forall)$, we arrive at

$$\mathcal{H}_{\tilde{\alpha_0}+\kappa}[\mathcal{A}, \kappa] \; \overset{\Psi_{\mathbb{Y}}^{\tilde{\alpha_0}+\kappa}+\omega}{\underset{\Psi_{\mathbb{Y}}^{\tilde{\alpha_0}+\kappa}}{\vdash}} \; (\forall \vec{v} \in \mathbb{L}_\kappa)\big(M^{\vec{t_\kappa}}_{\mathbb{J}}(\vec{v}\,) \to \bigvee \Gamma[\vec{v}]\big),\, B. \tag{46}$$

Furthermore,

$$\Vdash \Gamma[\vec{t_\kappa}\,], \bigwedge \neg\Gamma[\vec{t_\kappa}\,]$$

holds by Lemma 9.2(i). As $\bigwedge \neg\Gamma[\vec{t_\kappa}]$ is a $\Pi_0(\mathbf{p}_\kappa(\mu))$-formula as well, we can apply an inference $(Ref_{\mathbb{J}})$, yielding[5]

$$\mathcal{H}_{\tilde{\alpha_0}+\kappa}[\mathcal{A}, \kappa] \; \overset{\Psi_{\mathbb{X}}^{\tilde{\alpha_0}+\kappa}}{\underset{\Psi_{\mathbb{X}}^{\tilde{\alpha_0}+\kappa}}{\vdash}} \; \Gamma[\vec{t_\kappa}], (\exists \vec{v} \in \mathbb{L}_\kappa)\big(M^{\vec{t_\kappa}}_{\mathbb{J}}(\vec{v}) \,\wedge\, \bigwedge \neg\Gamma[\vec{v}\,]\big)\,. \tag{47}$$

Since $\Psi_{\mathbb{Y}}^{\tilde{\alpha_0}+\kappa} + \omega, \Psi_{\mathbb{X}}^{\tilde{\alpha_0}+\kappa} < \Psi_{\mathbb{X}}^{\tilde{\alpha}+\kappa}$, (Cut) can be applied to (46) and (47). Hence,

$$\mathcal{H}_{\tilde{\alpha}+\kappa}[\mathcal{A}, \kappa] \; \overset{\Psi_{\mathbb{X}}^{\tilde{\alpha}+\kappa}}{\underset{\Psi_{\mathbb{X}}^{\tilde{\alpha}+\kappa}}{\vdash}} \; \Gamma[\vec{t_\kappa}\,],\, B \;\; (= \Gamma[\vec{t_\kappa}\,]). \tag{48}$$

[5]This is exactly the place, where the removal of an instance of $(Ref_{\mathbb{K}'})$ forces us to introduce a new inference $(Ref_{\mathbb{J}})$. Note that $\mathbb{J} \in \Re^{o(\kappa)}$.

Subcase 5.2: $\mathbb{X}$ is of the form $\mathbb{A}(\delta) = (\pi; \delta\text{-}\mathsf{P}_0; \lambda\text{-}\mathsf{P}_0; \dots)$ and δ is a successor $\mu + 1$. Then there is a reflection pattern $\mathbb{C}$ with $\mathbf{dom}(\mathbb{C}) = (0, \omega)$ of the form $\mathbb{C}(m) := (\kappa; \mathsf{M}^{<\zeta}_{\mathbb{A}(\gamma)}\text{-}\mathbf{p}_\kappa(\mu)\text{-}\mathsf{P}_m; \dots)$, where $\tilde{\alpha} \le \zeta$ and $\gamma \ge \delta$. Note that $\Gamma[\vec{t}\,]$ is a set of $\mathcal{F}(\mathbb{K})$ formulae, i.e., a set of $\Pi_0(\delta)$-formulae. Therefore $\mathbb{L}_\delta$ does not occur in $\Gamma[\vec{t}\,]$. Put $\mu_\kappa := \mathbf{p}_\kappa(\mu)$. Let $D(s_1^*, \dots, s_l^*, s_1, \dots, s_k)$ be the formula $\bigvee \Gamma[\vec{t}_\kappa]$, where $s_1, \dots, s_k$ consists of the terms of levels $\ge \kappa$ and $< \mu_\kappa$ which occur in D, whereas $s_1^*, \dots, s_l^*$ are the terms of level μ_κ occurring in D. Let $\tilde{D}(p_1, \dots, p_j, s_1, \dots, s_k)$ be the formula (where all terms with levels $\ge \kappa$ are exhibited) arising from $D(s_1^*, \dots, s_l^*, s_1, \dots, s_k)$ by the following transformations:

- In the first step, replace every occurrence of the form $s_i^* \in a$ by $(\exists y \in a) s_i^* = y$ for all $1 \le i \le l$.
- After the first step, replace every occurrence of the form $a \in s_i^*$ by $F_i(a)$ if s_i^* is of the form $[x \in \mathbb{L}_{\mu_\kappa} : F_i(x)]$.
- Render the formula in prenex normal form.

Note that the foregoing transformations are valid steps on the basis of GML^*. Therefore we get

$$\mathcal{H}_{\tilde{\alpha_0}+\kappa}[\mathcal{A}, \kappa] \;\frac{\;\omega^{\omega\cdot\lambda'}\;}{\omega\cdot\lambda'}\; \Gamma[\vec{t}_\kappa],\ \neg\tilde{D}(p_1, \dots, p_j, s_1, \dots, s_k) \tag{49}$$

by Lemma 9.10, where $\lambda' := \delta_\kappa$.

Note that $\neg\tilde{D}(p_1, \dots, p_j, s_1, \dots, s_k)$ is a $\Pi_n(\mu_\kappa)$-formula for a sufficiently large n. Put

$$\mathbb{J} := (\mathbb{C}(n), \tilde{\alpha_0}, \delta_0)$$

where $\delta_0 := \max(\delta, \delta')$. Then $k(\mathbb{J}) \subseteq \mathcal{H}_{\tilde{\alpha_0}+\kappa}[\mathcal{A}, \kappa]$.

Recall that for $\tau \in \widetilde{\mathfrak{M}}^{\tilde{\alpha_0}}_{\mathbb{Y}}$, $\vec{t}_\tau = \mathbf{p}_\tau(\vec{t}\,)$ and $\vec{s}_\tau = \mathbf{p}_\tau(\vec{s}\,)$. Let $\vec{p}_\tau := \mathbf{p}_\tau(\vec{p}\,)$. Using Lemma 9.10, we get that for all $\tau \in \widetilde{\mathfrak{M}}^{\tilde{\alpha_0}}_{\mathbb{Y}} \cap \kappa$ and terms $\vec{q}, \vec{a}$ with levels $< \kappa$,

$$\mathcal{H}_{\tilde{\alpha_0}+\kappa}[\mathcal{A}, \vec{q}, \vec{a}, \tau] \;\frac{\;\omega^{\omega\cdot\rho}\;}{\omega^{\omega\cdot\rho}}\; \vec{p}_\tau \ne \vec{q},\ \vec{s}_\tau \ne \vec{a},\ \neg\bigwedge\Gamma[\vec{t}_\tau],\ \tilde{D}(\vec{q}, \vec{a}),$$

where $\rho := \omega^{\max(|\vec{q},\vec{a}|,\tau)+\omega}$. Put $B := (\exists \vec{y} \in \mathbb{L}_\kappa)[M^{\vec{s}}_{\mathbb{K}'}(\vec{y}) \wedge F(\vec{y})]$. Using (43) and (Cut), we then get

$$\mathcal{H}_{\tilde{\alpha_0}+\tau}[\mathcal{A}, \tau, \vec{q}, \vec{a}\,] \;\frac{\;\Psi^{\tilde{\alpha_0}+\tau}_{\mathbb{Y}}\#\rho\;}{\Psi^{\tilde{\alpha_0}+\tau}_{\mathbb{Y}}\#\rho}\; \vec{p}_\tau \ne \vec{q},\ \vec{s}_\tau \ne \vec{a},\ \tilde{D}(\vec{q}, \vec{a}),\ B \tag{50}$$

for all $\tau \in \widetilde{\mathfrak{M}}^{\tilde{\alpha_0}}_{\mathbb{Y}} \cap \kappa$ and $|\vec{q}, \vec{a}| < \kappa$. Applying inferences $(\vee)$ and subsequently an inference $(\neg M^{\vec{p},\vec{s}}_{\mathbb{J}})$ to (50) we get

$$\mathcal{H}_{\tilde{\alpha_0}+\kappa}[\mathcal{A}, \kappa, \vec{q}, \vec{a}\,] \;\frac{\;\Psi^{\tilde{\alpha_0}+\kappa}_{\mathbb{Y}}\;}{\Psi^{\tilde{\alpha_0}+\kappa}_{\mathbb{Y}}}\; \neg M^{\vec{p},\vec{s}}_{\mathbb{J}}(\vec{q}, \vec{a}), \tilde{D}(\vec{q}, \vec{a}),\ B \tag{51}$$

for all terms $\vec{q}, \vec{a}$ with levels $< \kappa$. Putting to use $(\vee)$ and subsequently $(\forall)$, we arrive at

$$\mathcal{H}_{\tilde{\alpha_0}+\kappa}[\mathcal{A}, \kappa] \;\frac{\;\Psi^{\tilde{\alpha_0}+\kappa}_{\mathbb{Y}}+\omega\;}{\Psi^{\tilde{\alpha_0}+\kappa}_{\mathbb{Y}}}\; (\forall \vec{v} \in \mathbb{L}_\kappa)\,(\forall \vec{w} \in \mathbb{L}_\kappa)\,\big(M^{\vec{p},\vec{s}}_{\mathbb{J}}(\vec{v}, \vec{w}) \to \tilde{D}(\vec{v}, \vec{w}\,)\big),\ B. \tag{52}$$

If we apply an inference $(Ref_{\mathbb{J}})$ to (49) we obtain

$$\mathcal{H}_{\tilde{\alpha_0}+\kappa}[\mathcal{A},\kappa] \,\Big|^{\Psi_{\mathbb{X}}^{\tilde{\alpha_0}+\kappa}}_{\Psi_{\mathbb{X}}^{\tilde{\alpha_0}+\kappa}}\, \Gamma[\vec{t}_\kappa], (\exists \vec{v} \in \mathbb{L}_\kappa)\, (\exists \vec{w} \in \mathbb{L}_\kappa)\, \big(M_{\mathbb{J}}^{\vec{p},\vec{s}}(\vec{v},\vec{w}) \;\wedge\; \neg \tilde{D}(\vec{v},\vec{w})\big). \tag{53}$$

Since $\Psi_{\mathbb{Y}}^{\tilde{\alpha_0}+\kappa} + \omega, \Psi_{\mathbb{X}}^{\tilde{\alpha_0}+\kappa} < \Psi_{\mathbb{X}}^{\tilde{\alpha}+\kappa}$, (Cut) can be applied to (52) and (53). Hence,

$$\mathcal{H}_{\tilde{\alpha}+\kappa}[\mathcal{A},\kappa] \,\Big|^{\Psi_{\mathbb{X}}^{\tilde{\alpha}+\kappa}}_{\Psi_{\mathbb{X}}^{\tilde{\alpha}+\kappa}}\, \Gamma[\vec{t}_\kappa],\, B \;\; (= \Gamma[\vec{t}_\kappa]). \tag{54}$$

Subcase 5.3: $\mathbb{X}$ is of the form $\mathbb{A}\,(m) = (\pi; \delta\text{-}\mathsf{P}_m; \delta\text{-}\mathsf{P}_\infty; \dots)$ with $\mathbf{dom}(\mathbb{A}\,) = (0,\omega)$. Then there is a reflection pattern $\mathbb{C}$ of the form $\mathbb{C} \; := \; (\kappa; \mathsf{M}^{<\zeta}_{\mathbb{A}\,(m')}\text{-}\mathbf{p}_\kappa(\delta)\text{-}\mathsf{P}_{m'}; \dots)$ with $m' \geq m$ and $\tilde{\alpha} \leq \zeta$. Further note that $\Gamma[\vec{t}\,]$ is a set of $\mathcal{F}(\mathbb{K}\,)$ formulae, i.e., a set of $\Pi_m(\delta)$-formulae if $\delta > 0$ or a set of $\Pi_{m+2}(\delta)$-formulae if $\delta = 0$.

$\mathbb{Y}$ is of the form $(\pi; \delta\text{-}\mathsf{P}_n; \delta\text{-}\mathsf{P}_\infty; \dots)$. Put

$$\mathbb{J} := (\mathbb{C}\,, \tilde{\alpha_0}, n).$$

Then $k(\mathbb{J}) \subseteq \mathcal{H}_{\tilde{\alpha_0}+\kappa}[\mathcal{A},\kappa]$. Let $\vec{q}$ be terms of level $< \kappa$. In view of Corollary 9.4, we get

$$\Vdash \vec{t}_\tau \neq \vec{q}, \bigwedge \neg \Gamma[\vec{t}_\tau], \bigvee \Gamma[\vec{q}].$$

Put $B := (\exists \vec{y} \in \mathbb{L}_\kappa)[M^{\vec{s}}_{\mathbb{K}'}(\vec{y}) \,\wedge\, F(\vec{y})]$. Using (43) and (Cut), we get with

$$f(\vec{q},\tau) \quad := \quad no(\vec{t}_\tau \neq \vec{q}, \bigwedge \neg \Gamma[\vec{t}_\tau], \bigvee \Gamma[\vec{q}]),$$

$$\mathcal{H}_{\tilde{\alpha_0}+\tau}[\mathcal{A},\tau,\vec{q}] \,\Big|^{\Psi_{\mathbb{Y}}^{\tilde{\alpha_0}+\tau}\# f(\vec{q},\tau)}_{\Psi_{\mathbb{Y}}^{\tilde{\alpha_0}+\tau}\# f(\vec{q},\tau)}\, \vec{t}_\tau \neq \vec{q}, \bigvee \Gamma[\vec{q}],\, B \tag{55}$$

for all $\tau \in \widetilde{\mathfrak{M}}_{\mathbb{Y}}^{\tilde{\alpha_0}} \cap \kappa$ and $|\vec{q}| < \kappa$. Starting from (55) and applying inferences $(\vee)$ and subsequently an inference $(\neg M_{\mathbb{J}}^{\vec{t}_\kappa})$, we get

$$\mathcal{H}_{\tilde{\alpha_0}+\kappa}[\mathcal{A},\kappa,\vec{q}] \,\Big|^{\Psi_{\mathbb{Y}}^{\tilde{\alpha_0}+\kappa}}_{\Psi_{\mathbb{Y}}^{\tilde{\alpha_0}+\kappa}}\, \neg M_{\mathbb{J}}^{\vec{t}_\kappa}(\vec{q}), \bigvee \Gamma[\vec{q}],\, B \tag{56}$$

for all terms $\vec{q}$ with levels $< \kappa$. Putting to use $(\vee)$ and subsequently $(\forall)$, we arrive at

$$\mathcal{H}_{\tilde{\alpha_0}+\kappa}[\mathcal{A},\kappa] \,\Big|^{\Psi_{\mathbb{Y}}^{\tilde{\alpha_0}+\kappa}+\omega}_{\Psi_{\mathbb{Y}}^{\tilde{\alpha_0}+\kappa}}\, (\forall \vec{v} \in \mathbb{L}_\kappa)\, \big(M_{\mathbb{J}}^{\vec{t}_\kappa}(\vec{v}) \to \bigvee \Gamma[\vec{v}]\big),\, B. \tag{57}$$

Furthermore,

$$\Vdash \Gamma[\vec{t}_\kappa],\, \bigwedge \neg \Gamma[\vec{t}_\kappa]$$

holds by Lemma 9.2(i). As $\bigwedge \neg \Gamma[\vec{t}_\kappa]$ is a $\Sigma_1^{\vee,\wedge}(\mathcal{F}(\mathbb{J}))$-formula we can apply Lemma 10.3 yielding

$$\mathcal{H}_{\tilde{\alpha_0}+\kappa}[\mathcal{A},\kappa] \,\Big|^{\Psi_{\mathbb{X}}^{\tilde{\alpha_0}+\kappa}}_{\Psi_{\mathbb{X}}^{\tilde{\alpha_0}+\kappa}}\, \Gamma[\vec{t}_\kappa], (\exists \vec{v} \in \mathbb{L}_\kappa)\big(M_{\mathbb{J}}^{\vec{t}_\kappa}(\vec{v}) \;\wedge\; \bigwedge \neg \Gamma[\vec{v}]\big)\,. \tag{58}$$

Since $\Psi_{\mathbb{Y}}^{\tilde{\alpha_0}+\kappa} + \omega, \Psi_{\mathbb{X}}^{\tilde{\alpha_0}+\kappa} < \Psi_{\mathbb{X}}^{\tilde{\alpha}+\kappa}$, (Cut) can be applied to (57) and (58). Hence,

$$\mathcal{H}_{\tilde{\alpha}+\kappa}[\mathcal{A},\kappa] \,\Big|\frac{\Psi_{\mathbb{X}}^{\tilde{\alpha}+\kappa}}{\Psi_{\mathbb{X}}^{\tilde{\alpha}+\kappa}}\, \Gamma[\vec{t}_\kappa],\ B\ \ (=\Gamma[\vec{t}_\kappa]). \tag{59}$$

Subcase 5.4: $\mathbb{X}$ is a reflection pattern of the form $(\pi;\delta\text{-}\mathsf{P}_m;\dots)$ where $m>1$. Then there is a reflection pattern $\mathbb{C}$ of the form $\mathbb{C} := (\kappa; \mathsf{M}_{\mathbb{A}}^{<\zeta}\text{-}\mathbf{p}_\kappa(\delta)\text{-}\mathsf{P}_{m-1};\dots)$ with $\tilde{\alpha}\le\zeta$. Further note that $\Gamma[\vec{t}\,]$ is a set of $\mathcal{F}(\mathbb{K})$ formulae, i.e., a set of $\Pi_m(\delta)$-formulae if $\delta>0$ or a set of $\Pi_{m+2}(\delta)$-formulae if $\delta=0$.

Note that $\mathbb{Y}=\mathbb{X}$ in this case. Put

$$\mathbb{J} := (\mathbb{C}, \tilde{\alpha_0}).$$

Then $k(\mathbb{J})\subseteq\mathcal{H}_{\tilde{\alpha_0}+\kappa}[\mathcal{A},\kappa]$. Let $\vec{q}$ be terms of level $<\kappa$. In view of Corollary 9.4, we get

$$\Vdash \vec{t}_\tau\ne\vec{q}, \bigwedge\neg\Gamma[\vec{t}_\tau], \bigvee\Gamma[\vec{q}].$$

Put $B := (\exists\vec{y}\in\mathbb{L}_\kappa)[M_{\mathbb{K}'}^{\vec{s}}(\vec{y})\wedge F(\vec{y})]$. Using (43) and (Cut), we get with

$$f(\vec{q},\tau) \ :=\ no(\vec{t}_\tau\ne\vec{q}, \bigwedge\neg\Gamma[\vec{t}_\tau], \bigvee\Gamma[\vec{q}]),$$

$$\mathcal{H}_{\tilde{\alpha_0}+\tau}[\mathcal{A},\tau,\vec{q}] \,\Big|\frac{\Psi_{\mathbb{Y}}^{\tilde{\alpha_0}+\tau}\#f(\vec{q},\tau)}{\Psi_{\mathbb{Y}}^{\tilde{\alpha_0}+\tau}\#f(\vec{q},\tau)}\, \vec{t}_\tau\ne\vec{q}, \bigvee\Gamma[\vec{q}],\ B \tag{60}$$

for all $\tau\in\widetilde{\mathfrak{M}}_{\mathbb{Y}}^{\tilde{\alpha_0}}\cap\kappa$ and $|\vec{q}|<\kappa$. Starting from (60) and applying inferences $(\vee)$ and subsequently an inference $(\neg M_{\mathbb{J}}^{\vec{t}_\kappa})$, we get

$$\mathcal{H}_{\tilde{\alpha_0}+\kappa}[\mathcal{A},\kappa,\vec{q}] \,\Big|\frac{\Psi_{\mathbb{Y}}^{\tilde{\alpha_0}+\kappa}}{\Psi_{\mathbb{Y}}^{\tilde{\alpha_0}+\kappa}}\, \neg M_{\mathbb{J}}^{\vec{t}_\kappa}(\vec{q}), \bigvee\Gamma[\vec{q}],\ B \tag{61}$$

for all terms $\vec{q}$ with levels $<\kappa$. Putting to use $(\vee)$ and subsequently $(\forall)$, we arrive at

$$\mathcal{H}_{\tilde{\alpha_0}+\kappa}[\mathcal{A},\kappa] \,\Big|\frac{\Psi_{\mathbb{Y}}^{\tilde{\alpha_0}+\kappa}+\omega}{\Psi_{\mathbb{Y}}^{\tilde{\alpha_0}+\kappa}}\, (\forall\vec{v}\in\mathbb{L}_\kappa)\big(M_{\mathbb{J}}^{\vec{t}_\kappa}(\vec{v})\to\bigvee\Gamma[\vec{v}]\big),\ B. \tag{62}$$

Furthermore,

$$\Vdash \Gamma[\vec{t}_\kappa],\ \bigwedge\neg\Gamma[\vec{t}_\kappa]$$

holds by Lemma 9.2(i). As $\bigwedge\neg\Gamma[\vec{t}_\kappa]$ is a $\Sigma_1^{\vee,\wedge}(\mathcal{F}(\mathbb{J}))$-formula we can apply Lemma 10.3 yielding

$$\mathcal{H}_{\tilde{\alpha_0}+\kappa}[\mathcal{A},\kappa] \,\Big|\frac{\Psi_{\mathbb{X}}^{\tilde{\alpha_0}+\kappa}}{\Psi_{\mathbb{X}}^{\tilde{\alpha_0}+\kappa}}\, \Gamma[\vec{t}_\kappa], (\exists\vec{v}\in\mathbb{L}_\kappa)\big(M_{\mathbb{J}}^{\vec{t}_\kappa}(\vec{v})\wedge\bigwedge\neg\Gamma[\vec{v}]\big). \tag{63}$$

Since $\Psi_{\mathbb{Y}}^{\tilde{\alpha_0}+\kappa} + \omega, \Psi_{\mathbb{X}}^{\tilde{\alpha_0}+\kappa} < \Psi_{\mathbb{X}}^{\tilde{\alpha}+\kappa}$, (Cut) can be applied to (62) and (63). Hence,

$$\mathcal{H}_{\tilde{\alpha}+\kappa}[\mathcal{A},\kappa] \,\Big|\frac{\Psi_{\mathbb{X}}^{\tilde{\alpha}+\kappa}}{\Psi_{\mathbb{X}}^{\tilde{\alpha}+\kappa}}\, \Gamma[\vec{t}_\kappa],\ B\ \ (=\Gamma[\vec{t}_\kappa]). \tag{64}$$

Subcase 5.5: There are several more cases. However, they can be handled in the same vein as the previous ones.

Case 6: The last inference is (Cut). Then, for some ordinal $\alpha_0 < \alpha$ it holds

$$\mathcal{H}_\gamma[\mathcal{A}] \mid\!\frac{\alpha_0}{\upsilon}\, \Gamma[\vec{t}\,], C(\vec{s}\,)$$

and

$$\mathcal{H}_\gamma[\mathcal{A}] \mid\!\frac{\alpha_0}{\upsilon}\, \Gamma[\vec{t}\,], \neg C(\vec{s}\,)\,,$$

where $rk(C(\vec{s}\,)) < \upsilon$. The assumptions made on the derivation ensure that we can select a projection instance $\mathbb{K}'$ with interval $[\pi, \delta']$ such that $\mathbb{K} \trianglelefteq \mathbb{K}'$ and $C(\vec{s}\,), \neg C(\vec{s}\,) \in \mathcal{F}(\mathbb{K}')$. Let $\mathbb{Y}$ be the projection instance pertaining to $\mathbb{K}'$. By the induction hypothesis , we get that for all $\tau \in \widetilde{\mathfrak{M}}^{\tilde{\alpha_0}}_{\mathbb{Y}}$,

$$\mathcal{H}_{\tilde{\alpha_0}+\tau}[\mathcal{A}, \tau] \mid\!\frac{\Psi^{\tilde{\alpha_0}+\tau}_{\mathbb{Y}}}{\Psi^{\tilde{\alpha_0}+\tau}_{\mathbb{Y}}}\, \Gamma[\vec{t_\tau}\,], C(\vec{s_\tau}\,)$$

and

$$\mathcal{H}_{\tilde{\alpha_0}+\tau}[\mathcal{A}, \tau] \mid\!\frac{\Psi^{\tilde{\alpha_0}+\tau}_{\mathbb{Y}}}{\Psi^{\tilde{\alpha_0}+\tau}_{\mathbb{Y}}}\, \Gamma[\vec{t_\tau}\,], \neg C(\vec{s_\tau}\,)\,,$$

where $\vec{t_\tau} := \mathbf{p}_\tau(\vec{t}\,)$ and $s_\tau := \mathbf{p}_\tau(\vec{s}\,)$. Notice that $rk(C(\vec{s_\tau}\,)) < \Psi^{\tilde{\alpha}+\tau}_{\mathbb{Y}}$. Hence, using (Cut) we get that for all $\tau \in \widetilde{\mathfrak{M}}^{\tilde{\alpha_0}}_{\mathbb{Y}}$,

$$\mathcal{H}_{\tilde{\alpha_0}+\tau}[\mathcal{A}, \tau] \mid\!\frac{\Psi^{\tilde{\alpha_0}+\tau+1}_{\mathbb{Y}}}{\Psi^{\tilde{\alpha_0}+\tau}_{\mathbb{Y}}}\, \Gamma[\vec{t_\tau}\,]\,.$$

One then argues as in Case 5 to conclude that for all $\kappa \in \widetilde{\mathfrak{M}}^{\tilde{\alpha}}_{\mathbb{X}}$,

$$\mathcal{H}_{\tilde{\alpha}+\kappa}[\mathcal{A}, \kappa] \mid\!\frac{\Psi^{\tilde{\alpha}+\kappa}_{\mathbb{X}}}{\Psi^{\tilde{\alpha}+\kappa}_{\mathbb{X}}}\, \Gamma[\mathbf{p}_\kappa(\vec{t}\,)].$$

Case 7: The last inference is $(Ref_{\mathbb{J}})$ with principal formula

$$(\exists \vec{y} \in \mathbb{L}_\tau)[M^{\vec{s}}_{\mathbb{J}}(\vec{y}) \,\wedge\, F(\vec{y})] \in \Gamma[\vec{t}\,]$$

and $F(\vec{s}\,) \in \mathcal{F}(\mathbb{J})$, where $\mathbb{J}$ is a reflection instance with interval $[\nu, \eta]$ and $\nu \neq \pi$. Then there exists $\alpha_0 < \alpha$ such that we have a scenario

$$\mathcal{H}_\gamma[\mathcal{A}] \mid\!\frac{\alpha_0}{\upsilon}\, \Gamma[\vec{t}\,], F(\vec{s}\,)\,. \tag{65}$$

Subcase 7.1: Assume $\nu < \pi$. Then we also have $\eta < \pi$ as was observed at the very beginning of the proof of this theorem. Furthermore, $\mathbf{p}_\kappa(\vec{s}\,) = \vec{s}$ for $\kappa \in \widetilde{\mathfrak{M}}^{\tilde{\alpha_0}}_{\mathbb{X}}$. From the induction hypothesis applied to (65) we then obtain, for all $\kappa \in \widetilde{\mathfrak{M}}^{\tilde{\alpha_0}}_{\mathbb{X}}$,

$$\mathcal{H}_{\tilde{\alpha_0}+\kappa}[\mathcal{A}, \kappa] \mid\!\frac{\Psi^{\tilde{\alpha_0}+\kappa}_{\mathbb{X}}}{\Psi^{\tilde{\alpha_0}+\kappa}_{\mathbb{X}}}\, \Gamma[\vec{t_\kappa}\,], F(\vec{s}\,)\,.$$

As $\kappa \in \widetilde{\mathfrak{M}}^{\tilde{\alpha}}_{\mathbb{X}}$ implies $\kappa \in \widetilde{\mathfrak{M}}^{\tilde{\alpha_0}}_{\mathbb{X}}$ and $\nu < \kappa$, we may apply an inference $(Ref_{\mathbb{J}})$, obtaining

$$\mathcal{H}_{\tilde{\alpha}+\kappa}[\mathcal{A}, \kappa] \mid\!\frac{\Psi^{\tilde{\alpha}+\kappa}_{\mathbb{X}}}{\Psi^{\tilde{\alpha}+\kappa}_{\mathbb{X}}}\, \Gamma[\vec{t_\kappa}\,], (\exists \vec{y} \in \mathbb{L}_\nu)[M^{\vec{s}}_{\mathbb{J}}(\vec{y}) \,\wedge\, F(\vec{y})] \quad (= \Gamma[\mathbf{p}_\kappa(\vec{t}\,)])$$

for all $\kappa \in \widetilde{\mathfrak{M}}^{\tilde{\alpha}}_{\mathbb{X}}$.

Subcase 7.2: Assume $\nu > \pi$. By a standing assumption of the theorem, there exists a reflection instance $\mathbb{K} \trianglelefteq \mathbb{K}'$ with interval $[\pi, \delta^*]$ such that $\delta^* > \eta$ or $\delta^* = \eta$ and $\Pi_3(\eta) \subseteq \mathcal{F}(\mathbb{J})$. $\mathbb{K}'$ can be effectively determined by inspecting $\mathbb{K}$. Let $\mathbb{Y}$ be the projection instance pertaining to $\mathbb{K}'$.

The above also implies that ν is a successor cardinal and thus $\nu = \eta$. Hence $\mathbb{J}$ is of the form $(\nu; \mathsf{RSC}; \dots)$ and $F(\vec{t}\,)$ is of the form $\forall u^\nu \exists v^\nu G(u, v, \vec{t}\,)$ with matrix in $\Pi_0(\nu)$. From the induction hypothesis applied to (65), we then obtain for all $\tau \in \widetilde{\mathfrak{M}}_{\mathbb{Y}}^{\tilde{\alpha_0}}$,

$$\mathcal{H}_{\tilde{\alpha_0}+\tau}[\mathcal{A}, \tau] \; \frac{\Psi_{\mathbb{Y}}^{\tilde{\alpha_0}+\tau}}{\Psi_{\mathbb{Y}}^{\tilde{\alpha_0}+\tau}} \; \Gamma[\vec{t_\tau}\,], F(\vec{s_\tau}\,)\,, \tag{66}$$

where $\vec{t_\tau} := \mathbf{p}_\tau(\vec{t}\,)$ and $\vec{s_\tau} := \mathbf{p}_\tau(\vec{s}\,)$. Letting $\mathbb{Q}_\tau := ((\mathbf{p}_\tau(\nu); \mathsf{RSC}; \dots)$ be the pertaining reflection instance as defined by clause 1.5 of Definition 3.2, one applies an inference $(Ref_{\mathbb{Q}_\tau})$ to (66), yielding

$$\mathcal{H}_{\tilde{\alpha_0}+\tau}[\mathcal{A}, \tau] \; \frac{\Psi_{\mathbb{Y}}^{\tilde{\alpha_0}+\tau+1}}{\Psi_{\mathbb{Y}}^{\tilde{\alpha_0}+\tau}} \; \Gamma[\vec{t_\tau}\,], (\exists z^\tau)[M_{\mathbb{Q}_\tau}(z) \wedge (\forall u{\in}z)(\exists v{\in}z) G(u, v, \vec{s_\tau})] \tag{67}$$

the latter being the same as

$$\mathcal{H}_{\tilde{\alpha_0}+\tau}[\mathcal{A}, \tau] \; \frac{\Psi_{\mathbb{Y}}^{\tilde{\alpha_0}+\tau+1}}{\Psi_{\mathbb{Y}}^{\tilde{\alpha_0}+\tau}} \; \Gamma[\vec{t_\tau}\,]\,.$$

One then applies the same case distinctions as in case 5 to obtain

$$\mathcal{H}_{\tilde{\alpha}+\kappa}[\mathcal{A}, \kappa] \; \frac{\Psi_{\mathbb{X}}^{\tilde{\alpha}+\kappa}}{\Psi_{\mathbb{X}}^{\tilde{\alpha}+\kappa}} \; \Gamma[\vec{t_\kappa}\,] \tag{68}$$

for all $\kappa \in \widetilde{\mathfrak{M}}_{\mathbb{X}}^{\hat{\alpha}}$. □

Definition 12.2 For $\mu \in Card$, put

$$\underline{\mu} = \begin{cases} \mu + 1 & \text{if } \mu \in Reg \\ \mu & \text{otherwise.} \end{cases}$$

Theorem 12.3 *Let $\mathbb{K}$ be a reflection instance with interval $[\pi, \delta]$. Let $\mathbb{X}$ be the projection instance pertaining to $\mathbb{K}$. Suppose $\mathfrak{A}(\mathcal{A}; \mathbb{K}; \gamma; \mu)$, $NF(\gamma, \omega^{\mu \cdot \alpha})$, and $\Gamma \subseteq \Sigma_1(\pi) \cup \Delta_o(\pi)$. Furthermore, assume that*

$$\mathcal{H}_\gamma[\mathcal{A}] \; \frac{\alpha}{\underline{\mu}} \; \Gamma$$

and that all the reflection inferences $(Ref_{\mathbb{J}})$ of this derivation satisfy $\mathbb{J} \in \mathfrak{R}^\gamma$. Moreover, assume that all reflection inferences $(Ref_{\mathbb{J}'})$ of the derivation whose interval is of the form $[\kappa', \eta']$ with $\kappa' < \pi$ satisfy $\eta' < \pi$ or $\eta' = \pi$ and $\mathcal{F}(\mathbb{J}') \subseteq \mathcal{F}(\mathbb{K})$ (the latter condition will be notated by $\ddagger(\pi, \mathbb{K})$).

Then, for $\hat{\alpha} = \gamma + \omega^{\mu \cdot \alpha}$,

$$\mathcal{H}_{\hat{\alpha}}[\mathcal{A}] \; \frac{\Psi_{\mathbb{X}}^{\hat{\alpha}}}{\Psi_{\mathbb{X}}^{\hat{\alpha}}} \; \Gamma$$

where the reflection inferences $(Ref_{\mathbb{F}})$ appearing in the latter derivation satisfy $\mathbb{F} \in \mathfrak{R}^{\hat{\alpha}}$.

Proof: We proceed by main induction on μ and subsidiary induction on α. Let λ^* be the supremum of all ordinals η such that there is a reflection instance with interval of the form $[\pi, \eta]$. Note that λ^* can be effectively obtained from $\mathbb{K}$ by inspection of $\mathbb{K}$.

Case 1.1: The last inference is $(Ref_{\mathbb{J}})$, where $\mathbb{J} \sim \mathbb{K}$ and $\mathbb{J}$ is of either form $(\pi; \mathsf{RSC}; \dots)$ or $(\pi; \mathsf{RLC}; \dots)$ or $(\pi; \mathsf{M}^{\sigma}_{\mathbb{Z}}\text{-}\pi\text{-}\mathsf{P}_0; \dots)$ or $(\pi; \pi\text{-}\mathsf{P}_0; \lambda\text{-}\mathsf{P}_0; \dots)$. Then

$$\mathcal{H}_\gamma[\mathcal{A}] \mid\!\frac{\alpha_0}{\underline{\mu}}\, \Gamma, A\,,$$

where $\alpha_0 + 1, \pi < \alpha$, $A \equiv \forall x^\pi \exists y^\pi G(x, y) \in \Pi_2(\pi)$, $\mathbb{J} \in \mathcal{H}_\gamma$, $\mathbb{J} \in R^\gamma$, and $\exists z^\pi [M_{\mathbb{J}}(z) \wedge A^{(z,\pi)}] \in \Gamma$. Applying Inversion, i.e. 8.1, we have for all $t \in \mathcal{T}(\pi)$,

$$\mathcal{H}_\gamma[\mathcal{A}, t] \mid\!\frac{\alpha_0}{\underline{\mu}}\, \Gamma, \exists y^\pi G(t, y). \tag{69}$$

For $t \in \mathcal{T}(\pi)$ and $\gamma_t := \gamma + \omega^{\mu \cdot \alpha_0 + |t|}$, by 11.8(iv), it holds $\mathfrak{A}(\mathcal{A} \cup \{t\}; \mathbb{J}; \gamma_t; \mu)$ and also $\gamma_t \in \mathcal{H}_{\gamma_t}[\mathcal{A}, t]$. Therefore we can apply the subsidiary induction hypothesis to (69), so that with $\gamma_t' := \gamma_t + \omega^{\mu \cdot \alpha_0}$ we have for all $t \in \mathcal{T}(\pi)$,

$$\mathcal{H}_{\gamma_t'}[\mathcal{A}, t] \mid\!\frac{\Psi_{\mathbb{J}}^{\gamma_t + \omega^{\mu\cdot\alpha_0}}}{\Psi_{\mathbb{J}}^{\gamma_t + \omega^{\mu\cdot\alpha_0}}}\, \Gamma, \exists y^\pi G(t, y). \tag{70}$$

Set $\delta_t := \Psi_{\mathbb{J}}^{\gamma_t + \omega^{\mu\cdot\alpha_0}}$, $\gamma^* := \gamma + \omega^{\mu\cdot\alpha_0+\pi}$ and let $\eta := \Psi_{\mathbb{J}}^{\gamma + \omega^{\mu\cdot\alpha_0+\pi}}$. With the aid of the Bounding Lemma 8.7 we then obtain from (70),

$$\mathcal{H}_{\gamma^*}[\mathcal{A}, t] \mid\!\frac{\delta_t}{\delta_t}\, \Gamma, \exists y^\eta G(t, y) \tag{71}$$

for all $t \in \mathcal{T}(\pi)$ which satisfy $\delta_t \le \eta$. Due to $\mathfrak{A}(\mathcal{A}; \mathbb{K}; \gamma; \mu)$ and $NF(\gamma, \omega^{\mu\cdot\alpha_0+\pi})$, it follows that $\mathbb{J}, \gamma + \omega^{\mu\cdot\alpha_0+\pi} \in C(\gamma + \omega^{\mu\cdot\alpha_0+\pi}, \pi)$. From this we gather that (as $\eta = \Psi_{\mathbb{J}}^{\gamma+\omega^{\mu\cdot\alpha_0+\pi}}$) $\mathbb{L}_\eta \in \hat{\mathcal{M}}_{\mathbb{J}}$ Whence,

$$\Vdash M_{\mathbb{J}}(\mathbb{L}_\eta). \tag{72}$$

Furthermore, one computes that if $t \in \mathcal{T}(\eta)$, then $\delta_t < \eta$. Therefore

$$\mathcal{H}_{\hat{\alpha}}[\mathcal{A}] \mid\!\frac{\eta}{\eta}\, \Gamma, \forall x^\eta \exists y^\eta G(x, y) \tag{73}$$

follows from (71). Since $\eta < \pi$,

$$\mathcal{H}_{\hat{\alpha}}[\mathcal{A}] \mid\!\frac{\omega^{\eta+\omega}}{\eta}\, \Gamma, \exists z^\pi [\hat{\mathcal{M}}_{\mathbb{J}}(z) \wedge A^{(z,\pi)}] \quad (= \Gamma) \tag{74}$$

follows from (73) and (72). Finally, it remains to verify $\eta < \Psi_{\mathbb{K}}^{\hat{\alpha}}$. We have $\gamma + \omega^{\mu\cdot\alpha_0+\pi} < \gamma + \omega^{\mu\cdot\alpha} = \hat{\alpha}$ as $\alpha_0 + 1, \pi < \alpha$ and $\pi \le \mu$. From $NF(\gamma, \omega^{\mu\cdot\alpha})$ it follows $\gamma, \mu, \mathbb{J} \in C(\hat{\alpha}, \Psi_{\mathbb{K}}^{\hat{\alpha}})$; so $\gamma + \omega^{\mu\cdot\alpha_0+\pi} \in C(\hat{\alpha}, \Psi_{\mathbb{K}}^{\hat{\alpha}}) \cap \hat{\alpha}$, hence $\eta < \Psi_{\mathbb{K}}^{\hat{\alpha}}$. Therefore we arrive at

$$\mathcal{H}_{\hat{\alpha}}[\mathcal{A}] \mid\!\frac{\Psi_{\mathbb{X}}^{\hat{\alpha}}}{\Psi_{\mathbb{X}}^{\hat{\alpha}}}\, \Gamma$$

by (74).

Case 1.2: The last inference is $(Ref_{\mathbb{J}})$, where $\mathbb{J} \sim \mathbb{K}$ and $\mathbb{J}$ is not of the form $(\pi; \mathsf{RSC}; \dots)$

or $(\pi;\mathsf{RLC};\dots)$ or $(\pi;\mathsf{M}^{\sigma}_{\mathbb{Z}}\text{-}\pi\text{-}\mathsf{P}_0;\dots)$ or $(\pi;\pi\text{-}\mathsf{P}_0;\lambda\text{-}\mathsf{P}_0;\dots)$. Let $[\pi,\delta^*]$ be the interval of $\mathbb{J}$ and let $\mathbb{Y}$ be the projection instance pertaining to $\mathbb{J}$. Then

$$\mathcal{H}_{\gamma}[\mathcal{A}] \mathrel{\big|\!\!\frac{\alpha_0}{\underline{\mu}}} \Gamma, F(\vec{t}\,), \tag{75}$$

where $\alpha_0+1,\pi<\alpha$, $F(\vec{t}\,)\in\mathcal{F}(\mathbb{J})$, and $(\exists\vec{y}\in\mathbb{L}_\pi)[M^{\vec{t}}_{\mathbb{J}}(\vec{y})\wedge F(\vec{y}\,)]\in\Gamma$.

Subcase 1.2.1: Firstly, assume that $\lambda^*\geq\underline{\mu}$ or $\lambda^*=\mu$ and that there is a reflection instance $\mathbb{P}$ with interval $[\pi,\nu]$, where $\nu>\mu$ or $\nu=\mu$ and $\Pi_3(\mu)\subseteq\mathcal{F}(\mathbb{P})$. One can also, then, arrange that $\mathbb{P}\in\Re^{\gamma}$ and $\mathbb{P}\in\mathcal{H}_\gamma[\mathcal{A}]$ as $\mathbb{P}$ is effectively constructible from $\mathbb{J}$ and μ. In this situation Theorem 12.1 is directly applicable. Set $\pi_0:=\Psi^{\tilde{\alpha_0}}_{\mathbb{Y}}$. Then $\pi_0\in\widetilde{\mathfrak{M}}^{\hat{\alpha_0}}_{\mathbb{Z}}$. Since $\mathfrak{B}(\mathcal{A};\mathbb{J};\gamma)$ holds, we may employ Theorem 12.1 to obtain

$$\mathcal{H}_{\tilde{\alpha_0}+\pi_0}[\mathcal{A},\pi_0] \mathrel{\big|\!\!\frac{\Psi^{\tilde{\alpha_0}+\pi_0}_{\mathbb{Y}}}{\Psi^{\tilde{\alpha_0}+\pi_0}_{\mathbb{Y}}}} \Gamma^{(\pi_0,\pi)}, F(\mathbf{p}_{\pi_0}(\vec{t}\,)). \tag{76}$$

It also holds

$$\Vdash M^{\vec{t}}_{\mathbb{J}}(\mathbf{p}_{\pi_0}(\vec{t}\,));$$

and hence

$$\mathcal{H}_{\tilde{\alpha_0}+\pi_0}[\mathcal{A},\pi_0] \mathrel{\big|\!\!\frac{\Psi^{\tilde{\alpha_0}+\pi_0+\omega}_{\mathbb{Y}}}{\Psi^{\tilde{\alpha_0}+\pi_0}_{\mathbb{Y}}}} \Gamma^{(\pi_0,\pi)}, (\exists\vec{y}\in\mathbb{L}_\pi)[M^{\vec{t}}_{\mathbb{J}}(\vec{y})\wedge F(\vec{y},z)]. \tag{77}$$

The latter yields

$$\mathcal{H}_{\tilde{\alpha_0}+\pi_0}[\mathcal{A},\pi_0] \mathrel{\big|\!\!\frac{\Psi^{\tilde{\alpha_0}+\pi_0}_{\mathbb{Y}}+\omega}{\Psi^{\tilde{\alpha_0}+\pi_0}_{\mathbb{Y}}}} \Gamma \tag{78}$$

by Lemma 7.17,(iv). To arrive at the desired conclusion note that $\pi_0\in\mathcal{H}_{\tilde{\alpha}}[\mathcal{A}]$ and $\Psi^{\tilde{\alpha_0}+\pi_0}_{\mathbb{Y}}+\omega<\Psi^{\hat{\alpha}}_{\mathbb{X}}$. Thus from (78) we get

$$\mathcal{H}_{\hat{\alpha}}[\mathcal{A}] \mathrel{\big|\!\!\frac{\Psi^{\hat{\alpha}}_{\mathbb{X}}}{\Psi^{\hat{\alpha}}_{\mathbb{X}}}} \Gamma\,.$$

Subcase 1.2.2: Suppose $\lambda^*<\mu$. Set $\tau:=(\lambda^*)^+$. Thus τ is a successor cardinal. Let $\mathbb{F}:=(\tau;\mathsf{RSC};\dots)$ be the uniquely determined reflection instance pertaining to τ. Then $\mathbb{F}\in\,\in\mathcal{H}_\gamma[\mathcal{A}]$. Moreover, we have $\tau\leq\mu$, $\mathfrak{A}(\mathcal{A};\mathbb{F};\gamma;\mu)$, and $\Gamma\cup\{F(\vec{t}\,)\}\subseteq\Delta_0(\tau)$ since $\lambda^*<\tau$. Note also that τ satisfies $\ddagger(\tau)$. Using the subsidiary induction hypothesis we thus get

$$\mathcal{H}_{\hat{\alpha_0}}[\mathcal{A}] \mathrel{\big|\!\!\frac{\Psi^{\hat{\alpha_0}}_{\mathbb{F}}}{\Psi^{\hat{\alpha_0}}_{\mathbb{F}}}} \Gamma, F(\vec{t}\,)\,.$$

Employing predicative cut elimination 8.5, we obtain

$$\mathcal{H}_{\hat{\alpha_0}}[\mathcal{A}] \mathrel{\big|\!\!\frac{\varphi\eta(\eta+1)}{\upsilon}} \Gamma, F(\vec{t}\,) \tag{79}$$

with $\eta:=\Psi^{\hat{\alpha_0}}_{\mathbb{F}}$, where $\upsilon=\lambda^*$ if $\lambda^*>\pi$ and $\upsilon=\pi+1$ if $\lambda^*=\pi$.

Let $\sigma:=\varphi\eta(\eta+1)$. One easily verifies that $\mathfrak{B}(\mathcal{A};\mathbb{J};\tau)$ and $NF(\tau,\omega^{\pi\cdot\sigma})$. Set $\xi:=\tau+\omega^{\pi\cdot\sigma}$ and let $\pi_1:=\Psi^{\xi}_{\mathbb{Y}}$. Then $\pi_1\in\widetilde{\mathfrak{M}}^{\xi}_{\mathbb{Y}}$. We are now in a situation where Theorem 12.1 is applicable. Thus we get

$$\mathcal{H}_{\xi+\pi_1}[\mathcal{A},\pi_1] \mathrel{\big|\!\!\frac{\Psi^{\xi+\pi_1}_{\mathbb{Y}}}{\Psi^{\xi+\pi_1}_{\mathbb{Y}}}} \Gamma^{(\pi_1,\pi)}, F(\mathbf{p}_{\pi_1}(\vec{t}\,)). \tag{80}$$

We also have

$$\Vdash M^{\vec{t}}_{\mathbb{K}}(\mathbf{p}_{\pi_1}(\vec{t})),$$

and hence we can conclude that

$$\mathcal{H}_{\xi+\pi_1}[\mathcal{A},\pi_1] \,\big|^{\Psi^{\xi+\pi_1}_{\mathbb{Y}}+\omega}_{\Psi^{\xi+\pi_1}_{\mathbb{Y}}}\, \Gamma^{(\pi_1,\pi)}, \exists z^{\pi}(\exists \vec{y}\in z)[M^{\vec{t}}_{\mathbb{K}}(\vec{y}) \wedge F(\vec{y})]. \tag{81}$$

The latter yields

$$\mathcal{H}_{\xi+\pi_1}[\mathcal{A},\pi_1] \,\big|^{\Psi^{\xi+\pi_1}_{\mathbb{Y}}+\omega}_{\Psi^{\xi+\pi_1}_{\mathbb{Y}}}\, \Gamma \tag{82}$$

by Lemma 7.17,(iv). To arrive at the desired conclusion note that $\xi < \hat{\alpha}$ and $\xi \in C(\hat{\alpha}, \Psi^{\hat{\alpha}}_{\mathbb{X}})$. Hence $\pi_1 \in C(\hat{\alpha}, \Psi^{\hat{\alpha}}_{\mathbb{X}})$ and $\xi + \pi_1 < \hat{\alpha}$, yielding $\Psi^{\xi+\pi_1}_{\mathbb{Y}} + \omega < \Psi^{\hat{\alpha}}_{\mathbb{X}}$. As a result we get from (82) that

$$\mathcal{H}_{\hat{\alpha}}[\mathcal{A}] \,\big|^{\Psi^{\hat{\alpha}}_{\mathbb{X}}}_{\Psi^{\hat{\alpha}}_{\mathbb{X}}}\, \Gamma\,.$$

Subcase 1.2.3: Suppose $\lambda^* = \mu$ and $\mu > \pi$. If μ is not regular then $\mu = \underline{\mu} = \lambda$ and this case has been dealt with in subcase 1.2.1. Likewise we may exclude that there exists a reflection instance $\mathbb{P}$ with interval $[\pi, \mu]$ and $\Pi_3(\mu) \subseteq \mathcal{F}(\mathbb{P})$ since that case has been dealt with in subcase 1.2.1, as well. Consequently we may assume that μ is regular. Moreover, μ must be a successor cardinal or $\mu = \mathbf{I}$. Let $\mathbb{F} := (\mu; \mathsf{RSC}; \dots)$ or $\mathbb{F} := (\mu; \mathsf{RLC}; \dots)$ be the respective uniquely determined reflection instance pertaining to μ. Then $\mathbb{F} \in\, \in \mathcal{H}_{\gamma}[\mathcal{A}]$ and $\mathfrak{A}(\mathcal{A}; \mathbb{F}; \gamma; \mu)$. Note also that μ satisfies $\ddagger(\mu)$, that is, every reflection inference $\mathbb{I}$ in the derivation with interval $[\kappa', \delta']$ and $\kappa' < \mu$ satisfies $\delta' < \mu$ or $\delta' = \mu$ and $\mathcal{F}(\mathbb{I}) \subseteq \mathcal{F}(\mathbb{F})$.

The highest complexity that the formula $F(\vec{t})$ can assume under the constraints of this subcase is $\Pi_2(\mu)$. So let us assume that $F(\vec{t}) \equiv \forall x^{\mu} \exists y^{\mu} G(x, y, \vec{t}) \in \Pi_2(\mu)$. Applying Inversion, i.e. 8.1, we have for all $s \in \mathcal{T}(\mu)$,

$$\mathcal{H}_{\gamma}[\mathcal{A}, s] \,\big|^{\alpha_0}_{\underline{\mu}}\, \Gamma, \exists y^{\mu} G(s, y, \vec{t}). \tag{83}$$

For $s \in \mathcal{T}(\mu)$ and $\gamma_s := \gamma + \omega^{\mu\cdot\alpha_0 + |s|}$, by 11.8(iv), it holds $\mathfrak{A}(\mathcal{A} \cup \{s\}; \mathbb{F}; \gamma_s; \mu)$ and also $\gamma_s \in \mathcal{H}_{\gamma_s}[\mathcal{A}, s]$. Therefore we can apply the subsidiary induction hypothesis to (83), so that with $\gamma'_s := \gamma_s + \omega^{\mu\cdot\alpha_0}$, for all $s \in \mathcal{T}(\mu)$,

$$\mathcal{H}_{\gamma'_s}[\mathcal{A}, s] \,\big|^{\Psi^{\gamma_s+\omega^{\mu\cdot\alpha_0}}_{\mathbb{F}}}_{\Psi^{\gamma_s+\omega^{\mu\cdot\alpha_0}}_{\mathbb{F}}}\, \Gamma, \exists y^{\mu} G(s, y, \vec{t}). \tag{84}$$

Employing $(\forall)$ to (84), we obtain

$$\mathcal{H}_{\gamma^*}[\mathcal{A}] \,\big|^{\mu}_{\mu}\, \Gamma, F(\vec{t}), \tag{85}$$

where $\gamma^* = \gamma + \omega^{\mu\cdot(\alpha_0+1)}$. Note that (84) satisfies the requirements of Theorem 12.1 as all cut formulae of the derivation belong to $\mathcal{F}(\mathbb{J})$ and there are no reflection inferences with interval $[\mu, \mu]$ in this derivation. One easily verifies that $\mathfrak{B}(\mathcal{A}; \mathbb{J}; \gamma^*)$ and $NF(\gamma^*, \omega^{\pi\cdot\mu})$. Set $\xi^* := \gamma^* + \omega^{\pi\cdot\mu}$ and let $\pi_1 := \Psi^{\xi^*}_{\mathbb{Y}}$. Then $\pi_1 \in \widetilde{\mathfrak{M}}^{\xi^*}_{\mathbb{Y}}$. We are now in a situation where Theorem 12.1 is applicable. Thus we get

$$\mathcal{H}_{\xi^*+\pi_1}[\mathcal{A}, \pi_1] \,\big|^{\Psi^{\xi^*+\pi_1}_{\mathbb{Y}}}_{\Psi^{\xi^*+\pi_1}_{\mathbb{Y}}}\, \Gamma^{(\pi_1,\pi)}, F(\mathbf{p}_{\pi_1}(\vec{t})). \tag{86}$$

We also have

$$\Vdash M_{\mathbb{K}}^{\vec{t}}(\mathbf{p}_{\pi_1}(\vec{t})),$$

and hence we can conclude

$$\mathcal{H}_{\xi^*+\pi_1}[\mathcal{A},\pi_1] \,\Big|\frac{\Psi_{\mathbb{Y}}^{\xi^*+\pi_1}+\omega}{\Psi_{\mathbb{Y}}^{\xi^*+\pi_1}}\, \Gamma^{(\pi_1,\pi)}, \exists z^{\pi}(\exists \vec{y} \in z)[M_{\mathbb{K}}^{\vec{t}}(\vec{y}) \wedge F(\vec{y})]. \tag{87}$$

The latter yields

$$\mathcal{H}_{\xi^*+\pi_1}[\mathcal{A},\pi_1] \,\Big|\frac{\Psi_{\mathbb{Y}}^{\xi^*+\pi_1}+\omega}{\Psi_{\mathbb{Y}}^{\xi^*+\pi_1}}\, \Gamma \tag{88}$$

by Lemma 7.17,(iv). To arrive at the desired conclusion note that $\xi^* < \hat{\alpha}$ since $\alpha_0 + 1 < \alpha$. Also note that $\xi^* \in C(\hat{\alpha}, \Psi_{\mathbb{X}}^{\hat{\alpha}})$. Hence $\pi_1 \in C(\hat{\alpha}, \Psi_{\mathbb{X}}^{\hat{\alpha}})$ and $\xi^* + \pi_1 < \hat{\alpha}$, yielding $\Psi_{\mathbb{Y}}^{\xi^*+\pi_1} + \omega < \Psi_{\mathbb{X}}^{\hat{\alpha}}$. As a result we get from (88) that

$$\mathcal{H}_{\hat{\alpha}}[\mathcal{A}] \,\Big|\frac{\Psi_{\mathbb{X}}^{\hat{\alpha}}}{\Psi_{\mathbb{X}}^{\hat{\alpha}}}\, \Gamma\,.$$

Case 2: The last inference is of the form $(\mathit{Ref}_{\mathbb{J}})$, where $\mathbb{J}$ has an interval $[\kappa, \zeta]$ with $\kappa < \pi$. Then

$$\mathcal{H}_{\gamma}[\mathcal{A}] \,\Big|\frac{\alpha_0}{\underline{\mu}}\, \Gamma\,, \tag{89}$$

where $\alpha_0 + 1, \kappa < \alpha$, where A is the minor formula of that inference and the principal formula of the inference belongs to Γ. Since in this case A is a $\Delta_0(\pi)$ formula we can apply the subsidiary induction hypothesis directly to (89), yielding

$$\mathcal{H}_{\hat{\alpha_0}}[\mathfrak{X}] \,\Big|\frac{\Psi_{\mathbb{X}}^{\hat{\alpha_0}}}{\Psi_{\mathbb{X}}^{\hat{\alpha_0}}}\, \Gamma, A\,.$$

Due to $\Psi_{\mathbb{X}}^{\hat{\alpha_0}} + \kappa < \Psi_{\mathbb{X}}^{\hat{\alpha}}$, the same inference $(\mathit{Ref}_{\mathbb{J}})$ leads to

$$\mathcal{H}_{\hat{\alpha}}[\mathcal{A}] \,\Big|\frac{\Psi_{\mathbb{X}}^{\hat{\alpha}}}{\Psi_{\mathbb{X}}^{\hat{\alpha}}}\, \Gamma\,.$$

Case 3: The last inference is $(\bigvee)$ with principal formula $C \cong \bigvee(C_\iota)_{\iota \in J} \in \Gamma$. Then

$$\mathcal{H}_{\gamma}[\mathcal{A}] \,\Big|\frac{\alpha_0}{\underline{\mu}}\, \Gamma, C_{\iota_0}$$

for some $\alpha_0 < \alpha$ and $\iota_0 \in J \restriction \alpha$. By the subsidiary induction hypothesis , we obtain

$$\mathcal{H}_{\hat{\alpha_0}}[\mathfrak{X}] \,\Big|\frac{\Psi_{\mathbb{X}}^{\hat{\alpha_0}}}{\Psi_{\mathbb{X}}^{\hat{\alpha_0}}}\, \Gamma, C_{\iota_0}\,,$$

whence,

$$\mathcal{H}_{\hat{\alpha}}[\mathcal{A}] \,\Big|\frac{\Psi_{\mathbb{X}}^{\hat{\alpha}}}{\Psi_{\mathbb{X}}^{\hat{\alpha}}}\, \Gamma, C \quad (= \Gamma)$$

via $(\bigvee)$.

Case 4: The last inference is $(\bigwedge)$ with principal formula $C \cong \bigwedge(C_\iota)_{\iota\in J} \in \Gamma$. This means

$$\mathcal{H}_\gamma[\mathcal{A}, \iota] \; \overset{\alpha_\iota}{\underset{\mu}{\vdash}} \; \Gamma, C_\iota$$

and $|\iota| \leq \alpha_\iota < \alpha$ for $\iota \in J$. The conditions on Γ enforce that $C \in \Delta_0(\pi)$. Due to $k(C) \subseteq \mathcal{H}_\gamma[\mathcal{A}](\emptyset) \cap \pi$, we must have $|\iota| < \Psi_{\mathbb{X}}^{\gamma+1}$ for all $\iota \in J$. Let $\gamma_\iota := \gamma + \omega^{\mu\cdot\alpha_\iota + |\iota|}$. From $NF(\gamma_\iota, \omega^{\mu\cdot\alpha_\iota})$ it follows $\mathfrak{A}(\mathcal{A} \cup \{\gamma_\iota\}; \mathbb{K}; \gamma_\iota; \mu)$ for all $\iota \in J$. The subsidiary induction hypothesis then yields

$$\mathcal{H}_{\delta_\iota}[\mathcal{A}, \iota] \; \overset{\Psi_{\mathbb{K}}^{\delta_\iota}}{\underset{\Psi_{\mathbb{K}}^{\delta_\iota}}{\vdash}} \; \Gamma, C_\iota$$

for all $\iota \in J$, where $\delta_\iota := \gamma_\iota + \omega^{\mu\cdot\alpha_\iota} \in C(\hat{\alpha}, \Psi_{\mathbb{X}}^{\hat{\alpha}})$. $|\iota| \leq \alpha_\iota < \alpha$ implies $\delta_\iota < \hat{\alpha}$; thus $\Psi_{\mathbb{X}}^{\delta_\iota} < \Psi_{\mathbb{X}}^{\hat{\alpha}}$. So, using $(\bigwedge)$, we conclude

$$\mathcal{H}_{\hat{\alpha}}[\mathcal{A}] \; \overset{\Psi_{\mathbb{X}}^{\hat{\alpha}}}{\underset{\Psi_{\mathbb{X}}^{\hat{\alpha}}}{\vdash}} \; \Gamma\,.$$

Case 5: The last inference is (Cut). Then there exist $\alpha_0 < \alpha$ and an $RS(\mathcal{OT})$–formula A with $rk(A) < \underline{\mu}$, so that

$$\mathcal{H}_\gamma[\mathcal{A}] \; \overset{\alpha_0}{\underset{\mu}{\vdash}} \; \Gamma, A \tag{90}$$

and

$$\mathcal{H}_\gamma[\mathcal{A}] \; \overset{\alpha_0}{\underset{\mu}{\vdash}} \; \Gamma, \neg A. \tag{91}$$

Subcase 5.1: $rk(A) < \pi$.

Then $rk(A) < \Psi_{\mathbb{X}}^{\hat{\alpha_0}}$ and $A \in \Delta_0(\pi)$, hence $\neg A \in \Delta_0(\pi)$. Therefore, applying the subsidiary induction hypothesis to (90) and (91), we get

$$\mathcal{H}_{\hat{\alpha_0}}[\mathfrak{X}] \; \overset{\Psi_{\mathbb{X}}^{\hat{\alpha_0}}}{\underset{\Psi_{\mathbb{X}}^{\hat{\alpha_0}}}{\vdash}} \; \Gamma, A \quad \text{and} \quad \mathcal{H}_{\hat{\alpha_0}}[\mathfrak{X}] \; \overset{\Psi_{\mathbb{X}}^{\hat{\alpha_0}}}{\underset{\Psi_{\mathbb{X}}^{\hat{\alpha_0}}}{\vdash}} \; \Gamma, \neg A\,;$$

whence

$$\mathcal{H}_{\hat{\alpha}}[\mathcal{A}] \; \overset{\Psi_{\mathbb{X}}^{\hat{\alpha}}}{\underset{\Psi_{\mathbb{X}}^{\hat{\alpha}}}{\vdash}} \; \Gamma$$

by means of (Cut) since $\Psi_{\mathbb{X}}^{\hat{\alpha_0}} < \Psi_{\mathbb{X}}^{\hat{\alpha}}$.

Subcase 5.2: $\pi \leq rk(A) < \mu$.

We can select $\sigma \in \mathcal{H}_\gamma[\mathcal{A}]$ so that $\sigma \in Card$ and

$$\sigma \leq rk(A) < \sigma^+.$$

Set $\tau := \sigma^+$. Let $\mathbb{F} := (\tau; \mathsf{RSC}; \dots)$ be the unique reflection instance pertaining to τ. Then $\mathbb{F} \in\; \in \mathcal{H}_\gamma[\mathcal{A}]$. Moreover, $\pi < \tau \leq \mu$.

Subcase 5.2.1: There is no reflection instance $\mathbb{P}$ with interval $[\kappa, \eta]$ such that $\pi \leq \kappa < \tau$ and $\eta > \tau$, or $\eta \geq \tau$ and $\Pi_3(\tau) \subseteq \mathcal{F}(\mathbb{P})$. We then have $\mathfrak{A}(\mathcal{A}; \mathbb{F}; \gamma; \mu)$ and $\Gamma \cup \{A, \neg A\} \subseteq \Delta_0(\tau)$. Using the subsidiary induction hypothesis we get

$$\mathcal{H}_{\hat{\alpha_0}}[\mathcal{A}] \; \overset{\Psi_{\mathbb{F}}^{\hat{\alpha_0}}}{\underset{\Psi_{\mathbb{F}}^{\hat{\alpha_0}}}{\vdash}} \; \Gamma, A \quad \text{and} \quad \mathcal{H}_{\hat{\alpha_0}}[\mathcal{A}] \; \overset{\Psi_{\mathbb{F}}^{\hat{\alpha_0}}}{\underset{\Psi_{\mathbb{F}}^{\hat{\alpha_0}}}{\vdash}} \; \Gamma, \neg A\,,$$

whence,

$$\mathcal{H}_{\hat{\alpha_0}}[\mathcal{A}] \,\Big|\frac{\Psi_{\mathbb{F}}^{\hat{\alpha_0}+1}}{\Psi_{\mathbb{F}}^{\hat{\alpha_0}}}\, \Gamma\,, \tag{92}$$

as $rk(A) < \Psi_{\mathbb{F}}^{\hat{\alpha_0}}$. Employing predicative cut elimination, 8.5, we obtain

$$\mathcal{H}_{\hat{\alpha_0}}[\mathcal{A}] \,\Big|\frac{\varphi\eta(\eta+1)}{\underline{\sigma}}\, \Gamma \tag{93}$$

with $\eta := \Psi_{\mathbb{F}}^{\hat{\alpha_0}}$. Note that $\pi \leq \sigma$. Furthermore, $\mathfrak{A}(\mathcal{A};\mathbb{K}\,;\hat{\alpha_0};\sigma)$ and $NF(\hat{\alpha_0}, \omega^{\nu\cdot\varphi\eta(\eta+1)})$. Also $\sigma < \mu$. Therefore, letting $\zeta := \hat{\alpha_0} + \omega^{\sigma\cdot\varphi\eta(\eta+1)}$, we can use the main induction hypothesis on (93) to conclude

$$\mathcal{H}_{\zeta}[\mathcal{A}] \,\Big|\frac{\Psi_{\mathbb{X}}^{\zeta}}{\Psi_{\mathbb{X}}^{\zeta}}\, \Gamma\,.$$

Noting that $\zeta < \hat{\alpha}$ and $\Psi_{\mathbb{X}}^{\zeta} < \Psi_{\mathbb{X}}^{\hat{\alpha}}$, this implies

$$\mathcal{H}_{\hat{\alpha}}[\mathcal{A}] \,\Big|\frac{\Psi_{\mathbb{X}}^{\hat{\alpha}}}{\Psi_{\mathbb{X}}^{\hat{\alpha}}}\, \Gamma\,.$$

Subcase 5.2.2: There is a reflection instance $\mathbb{P}$ with interval $[\kappa, \eta]$ such that $\pi \leq \kappa < \tau$ and $\eta > \tau$, or $\eta = \tau$ and $\Pi_3(\tau) \subseteq \mathcal{F}(\mathbb{P})$. Then such a reflection instance can be found in $\mathcal{H}_\gamma[\mathcal{A}]$ and $\mathfrak{R}^\gamma$. Let $\mathbb{Z}$ be the projection instance pertaining to $\mathbb{P}$. We also have $\mathfrak{B}(\mathcal{A};\mathbb{P};\gamma)$ and $NF(\gamma, \omega^{\kappa\cdot\alpha_0})$. Let $\kappa_1 = \Psi_{\mathbb{Z}}^{\widetilde{\alpha_0}}$, where $\widetilde{\alpha_0} = \gamma + \omega^{\kappa\cdot\alpha_0}$. Then $\kappa_1 \in \widetilde{\mathfrak{M}_{\mathbb{Z}}^{\widetilde{\alpha_0}}}$. We are now in a situation where Theorem 12.1 is applicable. Thus, letting $A \equiv B(\vec{t})$, we get

$$\mathcal{H}_{\widetilde{\alpha_0}+\kappa_1}[\mathcal{A}, \kappa_1] \,\Big|\frac{\Psi_{\mathbb{Z}}^{\widetilde{\alpha_0}+\kappa_1}}{\Psi_{\mathbb{Z}}^{\widetilde{\alpha_0}+\kappa_1}}\, \Gamma, B(\mathbf{p}_{\kappa_1}(\vec{t})) \tag{94}$$

and

$$\mathcal{H}_{\widetilde{\alpha_0}+\kappa_1}[\mathcal{A}, \kappa_1] \,\Big|\frac{\Psi_{\mathbb{Z}}^{\widetilde{\alpha_0}+\kappa_1}}{\Psi_{\mathbb{Z}}^{\widetilde{\alpha_0}+\kappa_1}}\, \Gamma, \neg B(\mathbf{p}_{\kappa_1}(\vec{t})). \tag{95}$$

Applying (Cut) to the latter derivations we arrive at

$$\mathcal{H}_{\widetilde{\alpha_0}+\kappa_1}[\mathcal{A}, \kappa_1] \,\Big|\frac{\kappa}{\kappa}\, \Gamma. \tag{96}$$

Letting $\gamma^* = \gamma + \omega^{\alpha_0+1}$ we get

$$\mathcal{H}_{\gamma^*}[\mathcal{A}] \,\Big|\frac{\kappa}{\kappa}\, \Gamma \tag{97}$$

from (96). Next one verifies $\mathfrak{A}(\mathcal{A};\mathbb{K}\,;\gamma^*;\kappa)$ and $NF(\gamma^*, \omega^{\kappa\cdot\kappa})$. Therefore from (97) one obtains by the main induction hypothesis ($\kappa < \mu$),

$$\mathcal{H}_{\zeta}[\mathcal{A}] \,\Big|\frac{\Psi_{\mathbb{X}}^{\zeta}}{\Psi_{\mathbb{X}}^{\zeta}}\, \Gamma\,,$$

where $\zeta = \gamma^* + \omega^{\kappa\cdot\kappa}$. Noting that $\zeta < \hat{\alpha}$ and $\Psi_{\mathbb{X}}^{\zeta} < \Psi_{\mathbb{X}}^{\hat{\alpha}}$, this implies

$$\mathcal{H}_{\hat{\alpha}}[\mathcal{A}] \,\Big|\frac{\Psi_{\mathbb{X}}^{\hat{\alpha}}}{\Psi_{\mathbb{X}}^{\hat{\alpha}}}\, \Gamma\,.$$

Subcase 5.3: $rk(A) = \mu$ and $\mu \in Reg$.
Then, either A or $\neg A$ is of the form $\exists x^\mu F(x)$ with $F(\mathbb{L}_0) \in \Delta_0(\mu)$. If $\alpha_0 < \mu$, then $\neg A$ never gets used as a principal formula of an inference in $\mathcal{H}_\gamma[\mathcal{A}] \,\frac{|\alpha_0}{\underline{\mu}}\, \Gamma, \neg A$, and therefore, $\mathcal{H}_\gamma[\mathcal{A}] \,\frac{|\alpha_0}{\underline{\mu}}\, \Gamma$. Thus, by the subsidiary induction hypothesis we get $\mathcal{H}_{\hat{\alpha_0}}[\mathfrak{X}] \,\frac{|\Psi_{\mathbb{X}}^{\hat{\alpha_0}}}{\Psi_{\mathbb{X}}^{\hat{\alpha_0}}}\, \Gamma$, whence $\mathcal{H}_{\hat{\alpha}}[\mathcal{A}] \,\frac{|\Psi_{\mathbb{X}}^{\hat{\alpha}}}{\Psi_{\mathbb{X}}^{\hat{\alpha}}}\, \Gamma$ since $\Psi_{\mathbb{X}}^{\hat{\alpha_0}} < \Psi_{\mathbb{X}}^{\hat{\alpha}}$.

Now assume $\mu \le \alpha_0$. Since $\mu \in \mathcal{H}_\gamma[\mathcal{A}]$ we have $\mu \in C(\gamma+1, \Psi_{\mathbb{X}}^{\gamma+1})$ and thus $o(\mu) < \gamma+1$. Therefore we can select a reflection instance $\mathbb{J} \in \Re^\gamma$ with $i(\mathbb{J}) = \mu$ and $k(\mathbb{J}) \subseteq \mathcal{H}_\gamma[\mathcal{A}]$. Let $\mathbb{Z}$ be the projection instance pertaining to $\mathbb{J}$. Then $\mathfrak{A}(\mathcal{A};\mathbb{J};\gamma;\mu)$ and $\Gamma, A \subseteq \Delta_0(\mu) \cup \Sigma_1(\mu)$. Applying the subsidiary induction hypothesis to (90) and using the Bounding Lemma 8.7, we obtain

$$\mathcal{H}_{\hat{\alpha_0}}[\mathcal{A}] \,\frac{|\Psi_{\mathbb{J}}^{\hat{\alpha_0}}}{\Psi_{\mathbb{J}}^{\hat{\alpha_0}}}\, \Gamma, A^{(\Psi_{\mathbb{J}}^{\hat{\alpha_0}},\mu)}\,. \tag{98}$$

From (91), by employing 7.17(iii) and the fact that $\Psi_{\mathbb{J}}^{\hat{\alpha_0}} \in \mathcal{H}_{\hat{\alpha_0}}[\mathcal{A}]$ we get

$$\mathcal{H}_{\hat{\alpha_0}}[\mathcal{A}] \,\frac{|\alpha_0}{\underline{\mu}}\, \Gamma, \neg A^{(\Psi_{\mathbb{Z}}^{\hat{\alpha_0}},\mu)}. \tag{99}$$

Since $\mathfrak{A}(\mathcal{A};\mathbb{J};\hat{\alpha_0};\mu)$ and $NF(\hat{\alpha_0}, \omega^{\mu\cdot\alpha_0})$, the subsidiary induction hypothesis can be used on (99), furnishing

$$\mathcal{H}_\delta[\mathcal{A}] \,\frac{|\Psi_{\mathbb{Z}}^{\delta}}{\Psi_{\mathbb{Z}}^{\delta}}\, \Gamma, \neg A^{(\Psi_{\mathbb{Z}}^{\hat{\alpha_0}},\mu)}\,, \tag{100}$$

where $\delta := \hat{\alpha_0} + \omega^{\mu\cdot\alpha_0}$. Using (Cut) applied to (98) and (99), we obtain

$$\mathcal{H}_\delta[\mathcal{A}] \,\frac{|\Psi_{\mathbb{Z}}^{\hat{\alpha_0}}+1}{\Psi_{\mathbb{Z}}^{\hat{\alpha_0}}}\, \Gamma\,. \tag{101}$$

If $\mu = \pi$, then (101) implies

$$\mathcal{H}_{\hat{\alpha}}[\mathcal{A}] \,\frac{|\Psi_{\mathbb{X}}^{\hat{\alpha}}}{\Psi_{\mathbb{X}}^{\hat{\alpha}}}\, \Gamma\,,$$

noting that $\Psi_{\mathbb{Z}}^{\hat{\alpha_0}} < \Psi_{\mathbb{X}}^{\hat{\alpha}}$.

From now on, let $\pi < \mu$. Then we can select $\sigma \in Card$ such that $\sigma \in \mathcal{H}_\gamma[\mathcal{A}]$ and $\sigma \le \Psi_{\mathbb{Z}}^{\delta} < \sigma^+ \le \mu$. Through the use of predicative cut elimination, (101) yields

$$\mathcal{H}_\delta[\mathcal{A}] \,\frac{|\eta}{\underline{\sigma}}\, \Gamma\,, \tag{102}$$

where we put $\eta := \varphi(\Psi_{\mathbb{Z}}^{\delta})((\Psi_{\mathbb{Z}}^{\delta})+1)$. Set $\gamma' := \delta + \omega^{\mu\cdot\alpha_0}$. Then $\delta < \gamma'$ and $NF(\gamma', \omega^{\nu\cdot\eta})$ since $\sigma < \mu$ as well as $\eta < \sigma \le \alpha_0$. Since $\pi < \mu$ and $\pi \in C(\gamma+1, \Psi_{\mathbb{Z}}^{\gamma+1})$, we get $\pi < \Psi_{\mathbb{Z}}^{\delta}$; thence $\pi \le \sigma$. Note that $\mathfrak{A}(\mathcal{A};\mathbb{K};\sigma)$. Since $\sigma < \mu$, we can use the main induction hypothesis on (102), so that with $\rho := \gamma' + \omega^{\sigma\cdot\eta}$,

$$\mathcal{H}_\rho[\mathcal{A}] \,\frac{|\Psi_{\mathbb{X}}^{\rho}}{\Psi_{\mathbb{X}}^{\rho}}\, \Gamma\,. \tag{103}$$

One readily verifies $\rho < \hat{\alpha}$ and $\rho \in C(\hat{\alpha}, \Psi_{\mathbb{X}}^{\hat{\alpha}})$. Therefore, by (103),

$$\mathcal{H}_{\hat{\alpha}}[\mathcal{A}] \,\frac{|\Psi_{\mathbb{X}}^{\hat{\alpha}}}{\Psi_{\mathbb{X}}^{\hat{\alpha}}}\, \Gamma\,.$$

□

Theorem 12.4 *Let $\rho_0 := 1$ and $\rho_{n+1} := \mathbf{I}^{\rho_n}$. The property of being an admissible set above ω can be expressed by a Δ_0–formula. (For definiteness, let this be the formula displayed in [21].) If B is a Σ_1–sentence and*

$$\mathbf{KPi} + \exists M\,(M \text{ is transitive } \wedge\ M \prec_1 V) \vdash \forall x[Ad(x) \to B^x],$$

then there is a $k < \omega$ such that

$$\mathcal{H}_{\rho_k} \overset{\Psi_{\mathbb{K}}^{\rho_k}}{\underset{0}{\vdash}} B^{\mathbb{L}_{\Omega_1}}\,,$$

where $\mathbb{K} = (\omega^+; \mathsf{RSC}; \emptyset; 0)$.

Proof: Let $\Omega_1 := \omega^+$. According to Theorem 9.16, there is an $m < \omega$ satisfying

$$\mathcal{H}_0 \overset{\mathbf{I}\cdot\omega^m}{\underset{\mathbf{I}+m}{\vdash}} \neg Ad(\mathbb{L}_{\Omega_1}), B^{\mathbb{L}_{\Omega_1}}\,.$$

Moreover, the latter derivation contains only reflection inferences $(Ref_{\mathbb{J}})$ with $\mathbb{J} \in \Re^0$. Applying Corollary 8.6 several times, we get

$$\mathcal{H}_0 \overset{\rho_{m+2}}{\underset{\mathbf{I}+1}{\vdash}} \neg Ad(\mathbb{L}_{\Omega_1}), B^{\mathbb{L}_{\Omega_1}}\,.$$

Since

$$\mathcal{H}_0 \overset{\Omega_1\cdot\omega^\omega}{\underset{\Omega_1+\omega}{\vdash}} Ad(\mathbb{L}_{\Omega_1})\,,$$

it follows that

$$\mathcal{H}_0 \overset{\rho_{m+2}+1}{\underset{\mathbf{I}+1}{\vdash}} B^{\mathbb{L}_{\Omega_1}}\,.$$

Letting $\gamma := \rho_{m+4}$, we have $NF(\gamma, \mathbf{I}^{\rho_{m+2}})$ and $\mathfrak{B}(\emptyset; \mathbb{K}\,; \gamma; \mathbf{I})$. So we can apply Theorem 12.3 to get

$$\mathcal{H}_{\rho_n} \overset{\Psi_{\mathbb{K}}^{\rho_n}}{\underset{\Psi_{\mathbb{K}}^{\rho_n}}{\vdash}} B^{\mathbb{L}_{\Omega_1}}$$

provided that $n > m + 4$. Using predicative cut elimination, Theorem 8.5, this leads to

$$\mathcal{H}_{\hat{\alpha}} \overset{\varphi\delta\delta}{\underset{0}{\vdash}} B^{\mathbb{L}_{\Omega_1}}\,,$$

where $\delta := \Psi_{\mathbb{K}}^{\rho_n}$. For $k := n + 3$, one easily verifies $\rho_n < \rho_k$ and $\varphi\delta\delta < \Psi_{\mathbb{K}}^{\rho_k}$. Hence,

$$\mathcal{H}_{\rho_k} \overset{\Psi_{\mathbb{K}}^{\rho_k}}{\underset{0}{\vdash}} B^{\mathbb{L}_{\Omega_1}}\,.$$

□

Corollary 12.5

$$|\mathbf{KPi} + \exists M\,(M \text{ is transitive } \wedge\ M \prec_1 V)| \leq \Psi_{\mathbb{K}}^{\varepsilon_{\mathbf{I}+1}}.$$

($|\mathbf{KPi} + \exists M\,(M \text{ is transitive } \wedge\ M \prec_1 V)|$ denotes the proof–theoretic ordinal of $\mathbf{KPi} + \exists M\,(M \text{ is transitive } \wedge\ M \prec_1 V)$).

□

Remark 12.6 The bound given in 12.5 is indeed sharp as can be shown by methods developed in [15] which consist in re-interpreting the ordinal representation system on the basis of recursively large counterparts of ν-reducible cardinals. But we will not give a proof for that in this paper.

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