Collapsing functions based on recursively large ordinals: A well–ordering proof for KPM

Michael Rathjen*
Department of Mathematics, Ohio State University,
Columbus, Ohio 43210, USA

Abstract

It is shown how the strong ordinal notation systems that figure in proof theory and have been previously defined by employing large cardinals, can be developed directly on the basis of their recursively large counterparts. Thereby we provide a completely new approach to well–ordering proofs as will be exemplified by determining the proof–theoretic ordinal of the system **KPM** of [R 91].

1 Introduction

The definition procedures of so–called *collapsing functions* in the proof theory of impredicative theories make essential use[1] of uncountable and even large cardinals. Such collapsing functions are then employed in the shape of terms to “name” a countable set of ordinals, and when one succeeds in establishing recursion relations for the ordering between those terms, the set of terms gives rise to a recursive ordinal notation system. Notation systems provided by collapsing functions play a central role in ordinal analysis of fragments of second order arithmetic and weak set theories. However, the well–foundedness of ordinal notation systems arising from collapsing functions on uncountable cardinals can already be proved in appropriate fragments of second order arithmetic[2] and therefore, as regards the notation system, the detour through large cardinals could, in principal, be avoided[3].

As an example, we just mention that the ordinal notations used in the determination of the proof–theoretic ordinals $|\mathsf{ID}_\alpha|$ for theories of α–iterated inductive

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[1] See [Bac 50], [I 70], [S 77], [Bu et al. 81], [Bu 86], [J 84], [P 87], [Bu–S 88], [R 90], [P 91], [Bu 92].

[2] See [S 77], [Bu–P 78], [Bu et al. 81], [Bu–S 88], [S 88]

[3] This does not deny that seizing upon ideas from large cardinals provides a valuable source of inspiration for devising strong ordinal notation systems.

definitions, ID_α, are generated by utilizing a hierarchy of collapsing functions which is contingent upon the uncountable cardinals $\aleph_1, \ldots, \aleph_\alpha$ (Cf. [Bu et al. 81]). As to this issue Feferman remarked (See [F 87], p.436):

> *It has been suggested that, instead, one should be able to interpret the long hierarchies as operating directly on the (Kripke–Platek) admissible number classes τ_α, where $\tau_1 = \omega_1^{rec}$. However, no theory of such classes currently available allows one to "name" higher admissibles in the definition of a function and have a given admissible such as τ_1 closed under it.*

In this paper we show that in the development of proof–theoretic collapsing functions the cardinals can be dispensed with and throughout be replaced with their ordinal recursive analogue. As an example, we treat the strongest ordinal notation system so far available in the literature, $\mathsf{T(M)}$ of [R 90], and verify that the collapsing functions that give rise to $\mathsf{T(M)}$ can be directly developed on the basis of a recursively Mahlo ordinal and the admissible ordinals below it, rather than our earlier approach using a weakly Mahlo cardinal.

Our approach is not confined to the particular notation system $\mathsf{T(M)}$ but translates to other notation systems[4] as well ; $\mathsf{T(K)}$ of [R 93b] being the strongest system for which this claim has been confirmed.

Furthermore, this paper will round off the ordinal analysis of **KPM** begun in [R 91] by showing that the upper bound for the proof–theoretic ordinal of **KPM** obtained there is the proof–theoretic ordinal of **KPM**.

In addition to being a completion of [R 91], the machinery set up for our well-ordering proof of $\mathsf{T(M)}$ furnishes at least two improvements. First, it allows one to replace the use of Π^1_2 comprehension in Schütte's well-ordering proof of Jäger's weaker notation system $T(J)$ (see [J 84],[S 88]) by much weaker principles which are actually the ones that naturally hold in equilibrium with the strength of $T(J)$. Second, but more importantly, it provides a new approach to well–ordering proofs in that the development of the collapsing functions which give rise to the notation system can "almost" be carried out in set theories the proof–theoretic ordinal of which equals the ordertype of the notation system. This contrasts sharply with former detours through large cardinals.

Finally, we should mention a forerunner of this paper, [R 93a], where it was shown that the collapsing functions of [Bu 86] which employed the cardinals $\aleph_1, \ldots, \aleph_\omega$ can also be developed on the basis of the first ω–many admissible ordinals.

As prerequisites for all that is to follow, a knowledge of [R 90] would be desirable. The collapsing functions of [R 90] involve two levels. Those of second level are dealt with in Section 2 and the ones of level 1 will be treated in Section 4. Though not a topic of the present paper, it should also be mentioned that a deeper understanding of collapsing functions cannot be attained without giving attention to their being intimately related to cut elimination procedures for infinitary proof systems.[5]

[4] E.g., to all notation systems of the papers cited in footnote 1.

[5] Cf. [S 77], [Bu et al. 81], [Bu–S 88], [P 91], [R 91], [Bu 92].

We shall assume familiarity with the notion of admissible set and the recursion theory of Σ_1 predicates on admissible sets. We shall use [Bar 75] as our basic reference and will usually stick to the terminology he uses.

In order to fix some notation, let us quickly review some basic notions. Let Ord be the class of *ordinals*. L_α is the collection of *constructible* sets of level $< \alpha$. An ordinal κ is *admissible* if L_κ is an admissible set, i.e. L_κ is a model of **KP**. Let $\mathrm{Ad} = \{\alpha > \omega : \alpha \text{ } \textit{is admissible}\}$. If f is a partial function on the admissible set L_κ, then f is *κ–partial recursive* if the graph of f is definable on L_κ by a set–theoretic Σ_1 formula which may contain parameters from L_κ. f will be called *κ–recursive* if its graph is Δ_1–definable on L_κ. A relation R on L_κ is *κ–recursively enumerable* if R is Σ_1 on L_κ. R is *κ–recursive* if R is Δ_1 on L_κ.

By $\mathsf{fun}(f)$, $\mathsf{dom}(f)$, and $\mathsf{ran}(f)$ we abbreviate that f is a function, the domain of f, and the range of f, respectively. $f \restriction x$ is the restriction of f to x. $f''X$ stands for the set $\{f(z) : z \in X\}$. $\mathsf{TC}(x)$ denotes the transitive closure of x. As usual, $\langle x, y\rangle$ denotes the ordered pair $\{\{x\}, \{x, y\}\}$ and, for $n > 2$, ordered n–tuples are defined by induction on n: $\langle x_1, \cdots, x_n\rangle = \langle x_1, \langle x_2, \cdots, x_n\rangle\rangle$.

For a predicate P, $P(x)$ and $x \in P$ will be used interchangeably.

A binary relation R is said to be *set–like* if, for all sets x, $\{y : R(y, x)\}$ is a set.

If g and h are two partial functions, we write $g(x_1, \cdots, x_n) \simeq h(x_1, \cdots, x_n)$ to mean that either both $g(x_1, \cdots, x_n)$ and $h(x_1, \cdots, x_n)$ are undefined or both are defined and have the same value.

We use lower case Greek letters $\alpha, \beta, \gamma, \ldots$ to range over ordinals.

2 An ordinal recursive analogue of $\alpha \mapsto \chi_\alpha(0)$

In [R 90] we defined a hierarchy of functions $\chi_\alpha : \mathsf{M} \longrightarrow \mathsf{M}$ ($\alpha < \Gamma^{\mathsf{M}}$), where M was assumed to be a weakly Mahlo cardinal. These functions gave rise to a strong ordinal notation system $\mathsf{T}(\mathsf{M})$ which was subsequently used in [R 91] for carrying through the ordinal analysis of the system **KPM**. In this Section we are going to show that similar functions can be developed for recursively Mahlo ordinals, too.

Recall that an admissible ordinal ρ is *recursively Mahlo* if for every ρ–recursive $f : \rho \longrightarrow \rho$ there is an admissible $\kappa < \rho$ that is closed under f, i.e.,
$(\forall \alpha < \kappa)(f(\alpha) < \kappa)$.

We fix a recursively Mahlo ordinal μ_0 throughout this Section. Let ε_{μ_0+1} be the least ordinal $\beta > \mu_0$ such that $\omega^\beta = \beta$. We are going to define a function $\mathsf{Z} : \varepsilon_{\mu_0+1} \longrightarrow \mu_0$ which resembles the function $\alpha \mapsto \chi_\alpha(0)$ of [R 90].

Definition 2.1 $\alpha =_{NF} \omega^\eta + \delta$ means that $\alpha = \omega^\eta + \delta$ and either of the following is true:

(a) $\delta = 0$ and $\eta < \alpha$;

(b) $\delta = \omega^{\delta_1} + \cdots + \omega^{\delta_k}$ with $\eta \geq \delta_1 \geq \ldots \geq \delta_k$ and $k > 0$.

Note that, for any ordinal α which satisfies $0 < \alpha < \omega^\alpha$, there are uniquely determined η and δ such that $\alpha =_{NF} \omega^\eta + \delta$.

Definition 2.2 For $\alpha < \varepsilon_{\mu_0+1}$ and $\beta < \mu_0$ the set $\widetilde{B}(\alpha,\beta)$ and the ordinal $\mathsf{Z}(\alpha)$ are defined by recursion on α as follows.

$$\begin{aligned}
\widetilde{B}^0(\alpha,\beta) &= \beta \cup \{0,\mu_0\};\\
\widetilde{B}^{n+1}(\alpha,\beta) &= \widetilde{B}^n(\alpha,\beta)\\
&\quad \cup \{\xi : \xi =_{NF} \omega^\eta + \delta;\ \eta,\delta \in \widetilde{B}^n(\alpha,\beta);\ \mu_0 < \xi\}\\
&\quad \cup \{\mathsf{Z}(\rho) : \rho < \alpha;\ \rho \in \widetilde{B}^n(\alpha,\beta)\};\\
\widetilde{B}(\alpha,\beta) &= \bigcup\{\widetilde{B}^n(\alpha,\beta) : n < \omega\};\\
\mathsf{Z}(\alpha) &= \text{least } \kappa \in \mathrm{Ad}\,[\,\widetilde{B}(\alpha,\kappa) \cap \mu_0 = \kappa \ \wedge\ \alpha \in \widetilde{B}(\alpha,\kappa)\,].
\end{aligned}$$

In [R 90], exploiting that M is a weakly Mahlo cardinal, it was shown that $\chi_{\mathsf{M}}(\alpha) < \mathsf{M}$ for any $\alpha < \varepsilon_{\mathsf{M}+1}$. Likewise, we intend to utilize that μ_0 is recursively Mahlo for showing that $\mathsf{Z}(\alpha) < \mu_0$ for any $\alpha < \varepsilon_{\mu_0+1}$. However, this is going to be more difficult since simple "cardinality arguments" won't be available any longer. The main idea consists in mimicing the construction of the sets $\widetilde{B}(\alpha,\beta)$ (which takes place outside of L_{μ_0}) in a μ_0–recursive way within L_{μ_0}.

First, we introduce a predicate OR and a binary relation $\sqsubset$ which then will be used for projecting the ordinals $< \varepsilon_{\mu_0+1}$ into L_{μ_0}.

Definition 2.3 (KP) $\widetilde{\mathsf{M}} := \langle 0,1\rangle$ and $\tilde{\omega}^s \oplus t := \langle 2,s,t\rangle$.

$$\begin{aligned}
s \in \mathsf{OR} \quad \text{iff} \quad & \mathsf{Ord}(s) \ \vee\ s = \widetilde{\mathsf{M}}\\
& \vee \exists s_0,s_1 \in \mathsf{OR}[\,s = \tilde{\omega}^{s_0} \oplus s_1 \ \wedge\ \widetilde{\mathsf{M}} \sqsubseteq s_0 \ \wedge\ [(\widetilde{\mathsf{M}} \sqsubset s_0 \wedge s_1 = 0)\\
&\qquad \vee\ (0 \sqsubset s_1 \sqsubseteq \widetilde{\mathsf{M}}) \ \vee\ \exists x \exists y (s_1 = \tilde{\omega}^x \oplus y \wedge x \sqsubseteq s_0)]]
\end{aligned}$$

$$\begin{aligned}
s \sqsubset t \quad \text{iff} \quad & s \in \mathsf{OR} \wedge t \in \mathsf{OR} \wedge\\
& [[\mathsf{Ord}(s) \wedge \mathsf{Ord}(t) \wedge s < t] \ \vee\ [\mathsf{Ord}(s) \wedge \neg\mathsf{Ord}(t)]\\
& \vee [s = \widetilde{\mathsf{M}} \wedge \exists t_0 \exists t_1 (t = \tilde{\omega}^{t_0} \oplus t_1)]\\
& \vee \exists s_0,s_1,t_0,t_1[s = \tilde{\omega}^{s_0} \oplus s_1 \wedge t = \tilde{\omega}^{t_0} \oplus t_1\\
&\qquad \wedge (s_0 \sqsubset t_0 \vee (s_0 = t_0 \wedge s_1 \sqsubset t_1))]]
\end{aligned}$$

Note that the unbounded quantifiers in the above definition of OR and $\sqsubset$ can easily be bounded. $s \in \mathsf{OR}$ and $s \sqsubset t$ are defined (simultaneously) by recursion on the transitive closure of $\{s,t\}$. They are examples of what is called Δ_1 predicates of **KP** in [Bar 75], I.5.1, I.6.6.

Through the use of OR and $\sqsubset$ we intend to simulate Z on L_{μ_0}. Reflection on what is required to make this work leads to the following definition.[6]

[6] Here we draw on ideas from [R 92].

Definition 2.4 (KP)

$$
\begin{aligned}
\widetilde{\mathsf{Z}}(s)=\kappa \quad \text{iff} \quad & s\in\mathsf{OR}\ \wedge\\
& \exists F[\mathsf{fun}(F)\wedge\mathsf{dom}(F)=\omega\times(\kappa+1)\wedge\\
& \forall\delta\le\kappa[F(0,\delta)=\delta\cup\{0,\widetilde{\mathsf{M}}\}\ \wedge\ \forall n<\omega(F(n+1,\delta)=F(n,\delta)\\
& \qquad\qquad \cup\ \{s'\in\mathsf{OR}:\exists s_0,s_1\in F(n,\delta)(s'=\tilde{\omega}^{s_0}\oplus s_1)\}\\
& \qquad\qquad \cup\ \{\widetilde{\mathsf{Z}}(t):\ t\in F(n,\delta)\ \wedge\ t\sqsubset s\})]\ \wedge\\
& \kappa=\text{least }\pi\in\mathrm{Ad}\,[\forall\alpha\in\bigcup_{n<\omega}F(n,\pi)(\alpha<\pi)\ \wedge\ s\in\bigcup_{n<\omega}F(n,\pi)\,]\,].
\end{aligned}
$$

A casual inspection of the definition of $\widetilde{\mathsf{Z}}$ will reveal that the definition is circular, and must thus be a recursion, but the exact nature of this recursion deserves some additional comment. Observe that the definition of $\widetilde{\mathsf{Z}}$ **cannot** be viewed as a definition by recursion on $\sqsubset$ since $\sqsubset$ is not set–like, e.g., $\{s\in\mathsf{OR}:s\sqsubset\widetilde{\mathsf{M}}\}$ is a proper class. For the definition of $\widetilde{\mathsf{Z}}$ we employ the *Second Recursion Theorem* for **KP** as stated in [Bar 75], V.2. We are actually defining a Σ formula $\mathfrak{G}_{\widetilde{\mathsf{Z}}}$ which defines the graph of $\widetilde{\mathsf{Z}}$ by invoking the Second Recursion Theorem. To see this, we have to ensure that $\mathfrak{G}_{\widetilde{\mathsf{Z}}}$ occurs only positively in the right hand side of 2.4. But this can easily be achieved by rewriting the part

$$
\begin{aligned}
\forall n<\omega(F(n+1,\delta)\ &=\ F(n,\delta)\ \cup\ \{s'\in\mathsf{OR}:\exists s_0,s_1\in F(n,\delta)(s'=\tilde{\omega}^{s_0}\oplus s_1)\}\\
&\qquad \cup\{\widetilde{\mathsf{Z}}(t):\ t\in F(n,\delta)\ \wedge\ t\sqsubset s\})
\end{aligned}
$$

as follows

$$
\begin{aligned}
&\forall n<\omega\,\exists g\,(\mathsf{fun}(g)\ \wedge\ \mathsf{dom}(g)=\{t:\ t\in F(n,\delta)\ \wedge\ t\sqsubset s\}\ \wedge\\
&\qquad \forall t\in\mathsf{dom}(g)\mathfrak{G}_{\widetilde{\mathsf{Z}}}(t,g(t))\ \wedge\\
&\qquad F(n+1,\delta)=F(n,\delta)\ \cup\ \{s'\in\mathsf{OR}:\exists s_0,s_1\in F(n,\delta)(s'=\tilde{\omega}^{s_0}\oplus s_1)\}\ \cup\ \mathsf{ran}(g))
\end{aligned}
$$

Note also that the Second Recursion Theorem for **KP** only provides us with a Σ formula defining the partial function $\widetilde{\mathsf{Z}}$ so that the the equivalence of 2.4 is provable in **KP**. However, **KP** does **not** prove $\exists y\,[\widetilde{\mathsf{Z}}(0)=y]$.

Definition 2.5 Utilizing the normal form of Definition 2.1, an injective projection

$$\wp:\varepsilon_{\mu_0+1}\longrightarrow\mathsf{L}_{\mu_0}$$

is obtained as follows.

$$
\wp(\alpha)=\begin{cases}\alpha & \text{if } \alpha<\mu_0\\ \widetilde{\mathsf{M}} & \text{if } \alpha=\mu_0\\ \tilde{\omega}^{\wp(\alpha_0)}\oplus\wp(\alpha_1) & \text{if } \alpha=_{NF}\omega^{\alpha_0}+\alpha_1>\mu_0\end{cases}
$$

Lemma 2.6 *Let* $\sqsubset^{\mu_0}$ *and* OR^{μ_0} *denote the interpretation of* $\sqsubset$ *and* OR *in* L_{μ_0}*, respectively. Then:*

(i) $\wp''\varepsilon_{\mu_0+1} = \mathsf{OR}^{\mu_0}$.

(ii) $\forall \alpha, \beta < \varepsilon_{\mu_0+1}[\alpha < \beta \leftrightarrow \wp(\alpha) \sqsubset^{\mu_0} \wp(\beta)]$.

Proof. By induction on $\delta < \varepsilon_{\mu_0+1}$ one readily verifies

$$\forall \alpha, \beta < \delta\ (\wp(\alpha), \wp(\beta) \in \mathsf{OR}^{\mu_0}\ \wedge\ [\alpha < \beta \rightarrow \wp(\alpha) \sqsubset^{\mu_0} \wp(\beta)]).$$

This shows $\wp''\varepsilon_{\mu_0+1} \subseteq \mathsf{OR}^{\mu_0}$ and "$\rightarrow$" of (ii). On the other hand,

$$\forall s, t \in \mathsf{OR}^{\mu_0}\ \exists \alpha, \beta < \varepsilon_{\mu_0+1}\ [s = \wp(\alpha)\ \wedge\ t = \wp(\beta)\ \wedge\ (s \sqsubset t \rightarrow \alpha < \beta)]$$

is easily verified by induction on the transitive closure of $\{s, t\}$. □

Lemma 2.7 *If $\widetilde{\mathsf{Z}}^{\mu_0}$ denotes the interpretation of $\widetilde{\mathsf{Z}}$ in L_{μ_0}, then*

$$\forall s \in \mathsf{OR}^{\mu_0}\ \exists \pi < \mu_0\ \widetilde{\mathsf{Z}}^{\mu_0}(s) = \pi.$$

Proof. We proceed by transfinite induction on $\sqsubset^{\mu_0}$. Note that $\sqsubset^{\mu_0}$ is well–founded on account of Lemma 2.6.

Suppose that $\widetilde{\mathsf{Z}}^{\mu_0}$ is defined for all $t \sqsubset^{\mu_0} s$. Then, for all $\delta < \mu_0$, there exists a unique function f_δ satisfying $\mathsf{dom}(f_\delta) = \omega$, $f_\delta(0) = \delta \cup \{0, \widetilde{\mathsf{M}}\}$, and

$$\begin{aligned} \forall n < \omega\, [\, f_\delta(n+1) \quad &= \quad f_\delta(n)\ \cup\ \{s \in \mathsf{OR}^{\mu_0} : \exists s_0, s_1 \in f_\delta(n)(s = \tilde{\omega}^{s_0} \oplus s_1)\} \\ &\qquad \cup \{\widetilde{\mathsf{Z}}^{\mu_0}(r) : r \in f_\delta(n);\ r \sqsubset^{\mu_0} s\}]. \end{aligned}$$

The map $\delta \mapsto f_\delta$ is μ_0–recursive. So, by putting to use the Mahlo recursiveness of μ_0, there is an admissible ordinal $\omega < \kappa < \mu_0$ such that $\forall \delta < \kappa [f_\delta \in \mathsf{L}_\kappa]$ and $s \in \mathsf{L}_\kappa$. Let κ be the least such. $s \in \mathsf{L}_\kappa$ implies $s \in \bigcup_{n<\omega} f_\kappa(n)$ since s is in the closure of $\kappa \cup \{\widetilde{\mathsf{M}}\}$ under $x, y \mapsto \langle 2, x, y\rangle$. Using induction on n, it is readily verified that $f_\kappa(n) \subseteq \bigcup_{\delta<\kappa} f_\delta(n)$ (note that $\delta < \delta'$ implies $f_\delta(n) \subseteq f_{\delta'}(n)$); whence $\bigcup_{n<\omega} f_\kappa(n) \subseteq \mathsf{L}_\kappa$. Letting $F(n, \delta) = f_\delta(n)$ with $\mathsf{dom}(F) = \omega \times (\kappa + 1)$, it follows $\widetilde{\mathsf{Z}}^{\mu_0}(s) = \kappa$ by Definition 2.4. □

Corollary 2.8 *For all $\alpha < \varepsilon_{\mu_0+1}$, $\mathsf{Z}(\alpha) = \widetilde{\mathsf{Z}}^{\mu_0}(\wp(\alpha)) < \mu_0$.*

Proof. We use induction on α. Let $s = \wp(\alpha)$ and $\kappa = \widetilde{\mathsf{Z}}^{\mu_0}(\wp(\alpha)) < \mu_0$. $\widetilde{\mathsf{Z}}^{\mu_0}(\wp(\alpha))$ is defined according to Lemma 2.7. Let $F^{\mu_0} \in \mathsf{L}_{\mu_0}$ be the F of Definition 2.4 within L_{μ_0}. From the induction hypothesis and Lemma 2.6 we then get

$$\forall \delta \leq \kappa\, \forall n < \omega\, [\, \wp'' \widetilde{B}^n(\alpha, \delta) = F^{\mu_0}(n, \delta)].$$

By definition of F^{μ_0}, this implies that κ is the least admissible $> \omega$ such that $\widetilde{B}(\alpha, \kappa) \cap \mu_0 = \kappa$ and $\alpha \in \widetilde{B}(\alpha, \kappa)$. Thence, $\mathsf{Z}(\alpha) = \kappa$. □

For later use we state an absoluteness property of $\widetilde{\mathsf{Z}}$.

Lemma 2.9 *Let $\pi < \tau$ be admissible ordinals $> \omega$. If $s \in \mathsf{OR}^\pi$ and, for some $\kappa < \pi$, $\widetilde{\mathsf{Z}}^\tau(s) = \kappa$, then $\widetilde{\mathsf{Z}}^\pi(s) = \kappa$.*

This means that the graph of $\widetilde{\mathsf{Z}}$ is absolute for admissible sets which contain ω.

Proof. We proceed by induction on $\sqsubset^\pi$ Assume that the assertion is true for all $t \sqsubset^\pi s$. Since $\widetilde{\mathsf{Z}}^\tau(s) = \kappa$, there exists a function $F^\tau \in \mathsf{L}_\tau$ with $\mathsf{dom}(F^\tau) = \omega \times (\kappa+1)$ satisfying $F^\tau(0,\delta) = \delta \cup \{0, \widetilde{\mathsf{M}}\}$,

$$\begin{aligned}\forall n < \omega\,[\,F^\tau(n+1,\delta) \;&=\; F^\tau(n,\delta)\,\cup\,\{s \in \mathsf{OR}^\tau : \exists s_0, s_1 \in F^\tau(n,\delta)(s = \tilde{\omega}^{s_0} \oplus s_1)\}\\ &\quad\cup\{\widetilde{\mathsf{Z}}^\tau(t) : t \sqsubset^\tau s \;\wedge\; t \in F^\tau(n,\delta)\}],\end{aligned}$$

and $\kappa = \text{least}\ \nu \in \text{Ad}\,[\forall \alpha \in \bigcup_{n<\omega} F^\tau(n,\nu)(\alpha < \nu) \;\wedge\; s \in \bigcup_{n<\omega} F^\tau(n,\nu)\,]$.

Observe that $\mathsf{OR}^\pi = \mathsf{OR}^\tau \cap \mathsf{L}_\pi$ and $\sqsubset^\pi = \sqsubset^\tau \cap(\mathsf{L}_\pi \times \mathsf{L}_\pi)$. So, using a subsidiary induction on n and applying the main induction hypothesis, we come to see that for all $\delta \leq \kappa$,

$$\begin{aligned}\forall n < \omega\,[\,F^\tau(n+1,\delta) \;&=\; F^\tau(n,\delta)\,\cup\,v\{s \in \mathsf{OR}^\pi : \exists s_0, s_1 \in F^\tau(n,\delta)(s = \tilde{\omega}^{s_0} \oplus s_1)\}\\ &\quad\cup\{\widetilde{\mathsf{Z}}^\pi(t) : t \sqsubset^\pi s \;\wedge\; t \in F^\tau(n,\delta)\}].\end{aligned}$$

But this means that F^τ arises by Σ–recursion in L_π. Whence, $\widetilde{\mathsf{Z}}^\pi(s) = \widetilde{\mathsf{Z}}^\tau(s)$. □

3 What can be proved in KPM ?

We investigate for which sets s **KPM**[7] proves that $\widetilde{\mathsf{Z}}$ is defined at s. **KPM** does not prove $\forall s \in \mathsf{OR}\,\exists \pi\,(\widetilde{\mathsf{Z}}(z) = \pi)$. Besides means that are available in **KPM**, the proof of Lemma 2.7 made use of transfinite induction along $\sqsubset$, and this is the reason why it cannot be fully formalized in **KPM**. However, **KP** already proves transfinite induction for each of the relations $\sqsubset_n$, where $s \sqsubset_n t$ iff $s \sqsubset t \sqsubset \widetilde{\mathsf{M}}(n)$ with $\widetilde{\mathsf{M}}(0) = \tilde{\omega}^{\widetilde{\mathsf{M}}} \oplus 1$ and $\widetilde{\mathsf{M}}(n+1) = \tilde{\omega}^{\widetilde{\mathsf{M}}(n)} \oplus 0$.

Definition 3.1 For $A(x)$ a set–theoretic formula, let[8]

$$\text{Prog}(\sqsubset, A) \;:=\; \forall s \in \mathsf{OR}\,[\,(\forall t \sqsubset s)A(t) \to A(s)].$$

$\text{TI}(s, \sqsubset)$ denotes the schema

$$\text{Prog}(\sqsubset, A) \to (\forall t \sqsubset s)A(t).$$

Theorem 3.2 *For all (meta) n,* **KP**$\vdash \text{TI}(\widetilde{\mathsf{M}}(n), \sqsubset)$.

[7]For a definition of **KPM** see [R 91].
[8]$(\forall t \sqsubset r)A(t)$ stands for $\forall t(t \sqsubset r \to A(t))$.

Proof. The proof requires some preparations. Ordinal addition, multiplication, and exponentiation can be extended to OR. To wipe out any doubt, we do the boring details. For $s, t \in \mathsf{OR}$, the sum of s and t, $s \tilde{+} t$, is defined by recursion on s as follows. $\alpha \tilde{+} \beta = \alpha + \beta$; $\alpha \tilde{+} t = t$ if $t \notin \mathsf{Ord}$; $s \tilde{+} 0 = s$; $\widetilde{\mathsf{M}} \tilde{+} \alpha = \tilde{\omega}^{\widetilde{\mathsf{M}}} \oplus \alpha$ if $\alpha > 0$; $\widetilde{\mathsf{M}} \tilde{+} \widetilde{\mathsf{M}} = \tilde{\omega}^{\widetilde{\mathsf{M}}} \oplus \widetilde{\mathsf{M}}$; $\widetilde{\mathsf{M}} \tilde{+} (\tilde{\omega}^{\widetilde{\mathsf{M}}} \oplus r) = \tilde{\omega}^{\widetilde{\mathsf{M}}} \oplus (\tilde{\omega}^{\widetilde{\mathsf{M}}} \oplus r)$; $\widetilde{\mathsf{M}} \tilde{+} (\tilde{\omega}^{s_0} \oplus s_1) = \tilde{\omega}^{s_0} \oplus s_1$ if $\widetilde{\mathsf{M}} \sqsubset s_0$; $(\tilde{\omega}^{s_0} \oplus s_1) \tilde{+} \alpha = \tilde{\omega}^{s_0} \oplus (s_1 \tilde{+} \alpha)$; $(\tilde{\omega}^{s_0} \oplus s_1) \tilde{+} \widetilde{\mathsf{M}} = \tilde{\omega}^{s_0} \oplus (s_1 \tilde{+} \widetilde{\mathsf{M}})$; $(\tilde{\omega}^{s_0} \oplus s_1) \tilde{+} (\tilde{\omega}^{t_0} \oplus t_1) = \tilde{\omega}^{s_0} \oplus (s_1 \tilde{+} (\tilde{\omega}^{t_0} \oplus t_1))$ if $t_0 \sqsubseteq s_0$, and $\tilde{\omega}^{t_0} \oplus t_1$ if $s_0 \sqsubset t_0$.

Likewise, we extend $\alpha \mapsto \omega^\alpha$ by letting $\dot{\omega}^\alpha = \omega^\alpha$, $\dot{\omega}^{\widetilde{\mathsf{M}}} = \widetilde{\mathsf{M}}$, and $\dot{\omega}^s = \tilde{\omega}^s \oplus 0$ if $\widetilde{\mathsf{M}} \sqsubset s$.

Let $\dot{\omega}^s 0 = 0$ and $\dot{\omega}^s \alpha = \dot{\omega}^{s \tilde{+} \alpha_1} \dot{+} \cdots \dot{+} \dot{\omega}^{s \tilde{+} \alpha_k}$ if $\alpha = \omega^{\alpha_1} + \cdots + \omega^{\alpha_k}$ with $\alpha_1 \geq \ldots \geq \alpha_k$ and $k > 0$.

Let $A(x)$ be an arbitrary set–theoretic formula and set

$$B(x) := \forall r \in \mathsf{OR}\,[\,(\forall t \sqsubset r) A(t) \to (\forall t \sqsubset r \tilde{+} \dot{\omega}^x) A(t)].$$

We will show

Claim $\quad$ **KP** $\vdash \mathrm{Prog}(\sqsubset, A) \to \mathrm{Prog}(\sqsubset, B)$.

We work informally in **KP**. Assume $\mathrm{Prog}(\sqsubset, A)$, $s \in \mathsf{OR}$, and $(\forall s' \sqsubset s) B(s')$; we want to conclude $B(s)$. By cases we have:

Case 1: s=0. As $B(0)$ means $\forall r \in \mathsf{OR}\,[\,(\forall t \sqsubset r) A(t) \to (\forall t \sqsubset r \tilde{+} 1) A(t)]$, $B(0)$ follows from $\mathrm{Prog}(\sqsubset, A)$.

Case 2: $s = s' \tilde{+} 1$. Suppose $(\forall t \sqsubset r) A(t)$. We want to show $(\forall t \sqsubset r \tilde{+} \dot{\omega}^s) A(t)$. Let $D(\alpha) = (\forall t \sqsubset r \tilde{+} \dot{\omega}^{s'} \alpha) A(t)$. Then $D(0)$. By the use of $B(s')$, $D(\beta)$ implies $D(\beta{+}1)$. If α is a limit and $t' \sqsubset r \tilde{+} \dot{\omega}^{s'} \alpha$, then there is a $\beta < \alpha$ such that $t' \sqsubset r \tilde{+} \dot{\omega}^{s'} \beta$. This shows $\forall \beta < \alpha D(\beta) \ \to \ D(\alpha)$. Hence, applying Foundation, $\forall \alpha\, D(\alpha)$. Since for $t' \sqsubset r \tilde{+} \dot{\omega}^s$ there is an α such that $t' \sqsubset r \tilde{+} \dot{\omega}^{s'} \alpha$, we obtain $(\forall t \sqsubset r \tilde{+} \dot{\omega}^s) A(t)$.

Case 3: s is of the form $s_0 \tilde{+} \dot{\omega}^{s_1}$ with $0 \sqsubset s_1$. For $t \sqsubset r \tilde{+} \dot{\omega}^s$ we then find $s' \sqsubset s$ such that $t \sqsubset r \tilde{+} \dot{\omega}^{s'}$. Therefore $B(s)$ is implied by $(\forall s' \sqsubset s) B(s')$. This completes the proof of Case 3, and hence the proof of the Claim.

The Theorem is proved by outer induction on n. Assume $Prog(\sqsubset, A)$. Since $\sqsubset$ is the usual order of the ordinals on $\{s \in \mathsf{OR} : s \sqsubset \widetilde{\mathsf{M}}\}$, we can use Foundation to get $(\forall s \sqsubset \widetilde{\mathsf{M}}) A(s)$. Thus $A(\widetilde{\mathsf{M}})$ by $\mathrm{Prog}(\sqsubset, A)$. This shows $\mathrm{Prog}(\sqsubset, A) \to (\forall t \sqsubset \widetilde{\mathsf{M}}(0)) A(t)$. Whence, **KP** $\vdash \mathrm{TI}(\widetilde{\mathsf{M}}(0), \sqsubset)$.

Now suppose **KP** $\vdash \mathrm{TI}(\widetilde{\mathsf{M}}(n), \sqsubset)$. Assuming $\mathrm{Prog}(\sqsubset, A)$, we want to show $(\forall s \sqsubset \widetilde{\mathsf{M}}(n{+}1)) A(s)$. By the Claim, $\mathrm{Prog}(\sqsubset, B)$; hence $(\forall s \sqsubset \widetilde{\mathsf{M}}(n)) B(s)$ by the induction hypothesis; thus $B(\widetilde{\mathsf{M}}(n))$. Setting $r = 0$ in $B(\widetilde{\mathsf{M}}(n))$, we get $(\forall t \sqsubset \dot{\omega}^{\widetilde{\mathsf{M}}(n)}) A(t)$ which is the same as $(\forall t \sqsubset \widetilde{\mathsf{M}}(n+1)) A(t)$. $\square$

Corollary 3.3 *For all (meta) n,* **KPM** $\vdash \forall s \sqsubset \widetilde{\mathsf{M}}(n)\, \exists \kappa\, (\widetilde{\mathsf{Z}}(s) = \kappa)$.

Proof. Immediate by Theorem 3.2 and the proof of Lemma 2.7. $\square$

4 The collapsing functions $\widetilde{\psi}_\kappa$

In this Section, we define analogues, $\widetilde{\psi}_\kappa$, of the functions ψ_κ of [Bu 86], [J 84], [R 90] by employing admissible ordinals $\kappa > \omega$ instead of uncountable regular cardinals. We continue [R 93a], where it was shown that collapsing functions $\widetilde{\psi}_\kappa$ can be developed on the basis of successor admissibles $> \omega$ instead of uncountable successor cardinals. It will transpire that ordinal recursive analogues for collapsing functions of weakly inaccessible cardinals are much harder to construct.

Definition 4.1 (i) For $s \in \mathsf{OR}$, s^* is defined by induction on s as follows

$$s^* = \begin{cases} s & \text{if } \mathsf{Ord}(s) \\ 0 & \text{if } s = \widetilde{\mathsf{M}} \\ max({s_0}^*, {s_1}^*) & \text{if } s = \tilde{\omega}^{s_0} \oplus s_1. \end{cases}$$

(ii) For $\kappa = \widetilde{\mathsf{Z}}(s)$ with $s \in \mathsf{OR}$ let $\kappa^- = s^*$.

Note that s^* is an ordinal which is in the transitive closure of $\{s\}$.

Since we want to set up everything in such a way that it can easily be formalized in **KPM**, we fix a (meta) natural number $m_0 > 0$ throughout this Section and apply $\widetilde{\mathsf{Z}}$ only to $s \in \mathsf{OR}_{m_0} = \{s \in \mathsf{OR} : s \sqsubset \widetilde{\mathsf{M}}(m_0)\}$. Then **KPM** proves $\mathrm{Prog}(\sqsubset, A) \rightarrow \forall s \in \mathsf{OR}_{m_0} A(s)$ and $\forall s \in \mathsf{OR}_{m_0} \exists \kappa\, [\widetilde{\mathsf{Z}}(s) = \kappa]$. Therefore the following function is also a Δ_1 function of **KPM**.

Definition 4.2 For $s \in \mathsf{OR}_{m_0}$ and $\beta \in \mathsf{Ord}$, let

$$\begin{aligned} \mathfrak{B}^0(s,\beta) &= \beta \cup \{0, \widetilde{\mathsf{M}}\}; \\ \mathfrak{B}^{n+1}(s,\beta) &= \mathfrak{B}^n(s,\beta) \cup \{t \in \mathsf{OR} : t = \tilde{\omega}^{t_0} \oplus t_1;\ t_0, t_1 \in \mathfrak{B}^n(s,\beta)\} \\ &\quad \cup \{\widetilde{\mathsf{Z}}(t) : t \sqsubset s;\ t \in \mathfrak{B}^n(s,\beta)\}; \\ \mathfrak{B}(s,\beta) &= \bigcup \{\mathfrak{B}^n(s,\beta) : n < \omega\} \end{aligned}$$

Using $\mathfrak{B}$, the function $\widetilde{\mathsf{Z}}$ can be characterized as follows.

$$\widetilde{\mathsf{Z}}(s) \quad = \quad \text{least } \kappa \in \mathrm{Ad}\, [\, \mathfrak{B}(s,\kappa) \cap \mathsf{Ord} = \kappa \ \wedge\ s \in \mathfrak{B}(s,\kappa)\,].$$

Next , a criterion will be given that allows one to determine the order of $\widetilde{\mathsf{Z}}(s)$ and $\widetilde{\mathsf{Z}}(t)$ in a recursive way.

Lemma 4.3 *Let $s, t \in \mathsf{OR}_{m_0}$. Then:*

(i) $s^* < \widetilde{\mathsf{Z}}(s)$.

(ii) $\widetilde{\mathsf{Z}}(s) < \widetilde{\mathsf{Z}}(t)$ *iff* $(s \sqsubset t \wedge s^* < \widetilde{\mathsf{Z}}(t)) \vee (t \sqsubset s \wedge \widetilde{\mathsf{Z}}(s) \leq t^*)$.

Proof. (i) By induction on n it is readily verified that $\forall\, t_0 \in \mathfrak{B}^n(s,\delta)\,({t_0}^* \in \mathfrak{B}^n(s,\delta))$. Thus, since $s \in \mathfrak{B}(s, \widetilde{\mathsf{Z}}(s))$ and s^* is an ordinal, $s^* < \widetilde{\mathsf{Z}}(s)$, establishing (i).

(ii) We argue by cases.
CASE 1: $s \sqsubset t$. Then $\widetilde{\mathsf{Z}}(s) < \widetilde{\mathsf{Z}}(t) \rightarrow s^* < \widetilde{\mathsf{Z}}(t)$ by (i).

Suppose $s^* < \widetilde{\mathsf{Z}}(t)$. Then $s^* \in \mathfrak{B}^n(t, \widetilde{\mathsf{Z}}(t))$ and thus $s \in \mathfrak{B}^n(t, \widetilde{\mathsf{Z}}(t))$. Consequently, $\widetilde{\mathsf{Z}}(s) \in \mathfrak{B}^n(t, \widetilde{\mathsf{Z}}(t))$, which yields $\widetilde{\mathsf{Z}}(s) < \widetilde{\mathsf{Z}}(t)$.
CASE 2: $s = t$. Then both sides of the asserted equivalence are false.
CASE 3: $t \sqsubset s$. Then $\widetilde{\mathsf{Z}}(t) < \widetilde{\mathsf{Z}}(s) \leftrightarrow t^* < \widetilde{\mathsf{Z}}(s)$ follows by interchanging the roles of s and t in Case 1. Hence $(*)$ $\widetilde{\mathsf{Z}}(s) \leq \widetilde{\mathsf{Z}}(t) \leftrightarrow \widetilde{\mathsf{Z}}(s) \leq t^*$. But, by (i), $t^* < \widetilde{\mathsf{Z}}(t)$; and therefore $(*)$ implies $\widetilde{\mathsf{Z}}(s) < \widetilde{\mathsf{Z}}(t) \leftrightarrow \widetilde{\mathsf{Z}}(s) \leq t^*$. □

Corollary 4.4 *Let* $\kappa = \widetilde{\mathsf{Z}}(s)$ *with* $s \in \mathsf{OR}_{m_0}$. *Then*

$$\{x \in \mathsf{L}_\kappa \cap \mathsf{OR}_{m_0} : \widetilde{\mathsf{Z}}(x) < \kappa\}$$

is κ*–recursive.*

Proof. We proceed by induction on κ. Note that in view of its definition (cf. 2.3), OR_{m_0} is absolute for admissible sets above ω; thus $\mathsf{L}_\kappa \cap \mathsf{OR}_{m_0}$ is κ–recursive. We argue by cases. If $\kappa = \widetilde{\mathsf{Z}}(0)$ then the set in question is empty and thus κ–recursive. Now assume $\kappa > \widetilde{\mathsf{Z}}(0)$.

CASE 1: There is an $s_0 \in \mathsf{OR}_{m_0}$ such that κ is the next admissible after $\tau := \widetilde{\mathsf{Z}}(s_0)$. As $\widetilde{\mathsf{Z}}(x) < \kappa$ implies $x^* < \widetilde{\mathsf{Z}}(x) \leq \tau$, and thus $x \in \mathsf{L}_\tau$, we get

$$\{x \in \mathsf{L}_\kappa \cap \mathsf{OR}_{m_0} : \widetilde{\mathsf{Z}}(x) < \kappa\} = \{x \in \mathsf{L}_\tau \cap \mathsf{OR}_{m_0} : \widetilde{\mathsf{Z}}(x) < \tau\} \;\cup\; \{s_0\}.$$

So the assertion follows by induction hypothesis.

CASE 2: $\exists \pi < \kappa[\mathrm{Ad}(\pi) \wedge \forall s_0 \in \mathsf{L}_\kappa \cap \mathsf{OR}_{m_0}(\widetilde{\mathsf{Z}}(s_0) < \kappa \rightarrow \widetilde{\mathsf{Z}}(s_0) < \pi)]$. Suppose $x \in \mathsf{L}_\kappa \cap \mathsf{OR}_{m_0}$. If $\widetilde{\mathsf{Z}}(x) < \kappa$ then $\widetilde{\mathsf{Z}}(x) < \pi \wedge x \in \mathsf{L}_\pi \cap \mathsf{OR}_{m_0}$. Hence, by 2.9,

$$x \in \mathsf{L}_\kappa \cap \mathsf{OR}_{m_0} \wedge \widetilde{\mathsf{Z}}(x) < \kappa \;\leftrightarrow\; x \in \mathsf{L}_\pi \cap \mathsf{OR}_{m_0} \wedge \exists \delta < \pi(\widetilde{\mathsf{Z}}^\pi(x) \simeq \delta).$$

Since the formula "$\widetilde{\mathsf{Z}}^\pi(x) \simeq \delta$" is $\Sigma_1(\mathsf{L}_\pi)$, it is $\Delta_0(\mathsf{L}_\kappa)$; confirming κ–recursiveness.

CASE 3: κ is recursively inaccessible. As $x \in \mathsf{L}_\kappa \cap \mathsf{OR}_{m_0}$ entails $x^* < \kappa$, from 4.3(ii) we obtain,

$$(+) \quad \{x \in \mathsf{L}_\kappa \cap \mathsf{OR}_{m_0} : \widetilde{\mathsf{Z}}(x) < \kappa\} = \{x \in \mathsf{L}_\kappa : x \sqsubset s \,\vee\, (s \sqsubset x \wedge \widetilde{\mathsf{Z}}(x) \leq s^*)\}.$$

To see that this yields κ–recursiveness, pick an admissible $\pi > \omega$ such that $s^* < \pi < \kappa$. Then, by 2.9, for $x \in \mathsf{L}_\kappa \cap \mathsf{OR}_{m_0}$ we have

$$(++) \quad \widetilde{\mathsf{Z}}(x) \leq s^* \;\leftrightarrow\; (x \in \mathsf{L}_\pi \wedge \exists \delta < \pi\,[\widetilde{\mathsf{Z}}^\pi(x) \simeq \delta \;\wedge\; \delta \leq s^*]).$$

Therefore κ–recursiveness ensues from $(+)$ and $(++)$. □

During the rest of this section, κ, π, τ will range over $\widetilde{\mathsf{R}}_{m_0} := \{\widetilde{\mathsf{Z}}(s) : s \in \mathsf{OR}_{m_0}\}$.

Definition 4.5 $\widetilde{C}_\kappa(\alpha)$ and $\widetilde{\psi}_\kappa(\alpha)$ are defined by recursion on α as follows.

$$
\begin{array}{rcl}
\widetilde{C}^0_\kappa(\alpha) & = & \kappa^- \cup \{\kappa^-, \widetilde{\mathsf{M}}\}; \\
\widetilde{C}^{n+1}_\kappa(\alpha) & = & \widetilde{C}^n_\kappa(\alpha) \\
& & \cup\, \{s \in \mathsf{OR}_{m_0} : s = \tilde{\omega}^{s_0} \oplus s_1;\ s_0, s_1 \in \widetilde{C}^n_\kappa(\alpha)\} \\
& & \cup\, \{\beta : \beta =_{NF} \omega^{\beta_0} + \beta_1;\ \beta_0, \beta_1 \in \widetilde{C}^n_\kappa(\alpha)\} \\
& & \cup\, \{\widetilde{\mathsf{Z}}(s) : s \in \widetilde{C}^n_\kappa(\alpha)\} \\
& & \cup\, \{\widetilde{\psi}_\pi(\eta) : \pi, \eta \in \widetilde{C}^n_\kappa(\alpha);\ \eta < \alpha;\ \eta \in \widetilde{C}_\pi(\eta)\} \\
& & \cup\, \{\xi : \exists \rho \in \widetilde{C}^n_\kappa(\alpha) \cap \kappa\, (\xi < \rho)\}; \\
\widetilde{C}_\kappa(\alpha) & = & \bigcup_{n<\omega} \widetilde{C}^n_\kappa(\alpha); \\
\widetilde{\psi}_\kappa(\alpha) & = & \text{least ordinal } \beta \text{ such that } \beta \notin \widetilde{C}_\kappa(\alpha).
\end{array}
$$

We will write $\widetilde{\psi}\kappa\alpha$ for $\widetilde{\psi}_\kappa(\alpha)$.

Lemma 4.6 (i) $\widetilde{C}_\kappa(\alpha) \subseteq \mathsf{OR}_{m_0}$

(ii) $\alpha \leq \beta \rightarrow \widetilde{C}_\kappa(\alpha) \subseteq \widetilde{C}_\kappa(\beta)\ \wedge\ \widetilde{\psi}\kappa\alpha \leq \widetilde{\psi}\kappa\beta$.

(iii) $\alpha < \beta\ \wedge\ \alpha \in \widetilde{C}_\kappa(\alpha)\ \rightarrow\ \widetilde{\psi}\kappa\alpha < \widetilde{\psi}\kappa\beta$.

(iv) $\kappa \in \widetilde{C}_\kappa(\alpha)$.

(v) $\omega^{\widetilde{\psi}\kappa\alpha} = \widetilde{\psi}\kappa\alpha\ \wedge\ \widetilde{\psi}\kappa\alpha \notin \widetilde{\mathsf{R}}_{m_0}$.

(vi) $\rho \in \widetilde{C}_\kappa(\alpha) \cap \kappa\ \rightarrow\ \rho \subseteq \widetilde{C}_\kappa(\alpha)$.

(vii) $\pi^- \in \widetilde{C}_\kappa(\alpha) \cap \kappa\ \rightarrow\ \pi \in \widetilde{C}_\kappa(\alpha)$.

Proof. (i) $\widetilde{C}^n_\kappa(\alpha) \subseteq \mathsf{OR}_{m_0}$ follows by induction on n.

(ii) By induction on n one easily establishes $\widetilde{C}^n_\kappa(\alpha) \subseteq \widetilde{C}^n_\kappa(\beta)$.

(iii) The hypothesis implies $\alpha \in \widetilde{C}_\kappa(\beta) \cap \beta$ by (ii). Therefore, $\widetilde{\psi}\kappa\alpha \in \widetilde{C}_\kappa(\beta)$. Since $\widetilde{\psi}\kappa\beta \notin \in \widetilde{C}_\kappa(\beta)$, using (ii), we get $\widetilde{\psi}\kappa\alpha < \widetilde{\psi}\kappa\beta$.

(iv) Let $\kappa = \widetilde{\mathsf{Z}}(t)$. Then $t^* \cup \{t^*\} = \kappa^- \cup \{\kappa^-\} \subseteq \widetilde{C}^0_\kappa(\alpha)$. By the closure properties of $\widetilde{C}_\kappa(\beta)$ it follows $t \in \widetilde{C}_\kappa(\alpha)$. Hence $\kappa = \widetilde{\mathsf{Z}}(t) \in \widetilde{C}_\kappa(\alpha)$.

(v) By way of contradiction, assume $\widetilde{\psi}\kappa\alpha < \omega^{\widetilde{\psi}\kappa\alpha}$. Then $\widetilde{\psi}\kappa\alpha =_{NF} \omega^{\delta_0} + \delta_1$ for some $\delta_0, \delta_1 < \widetilde{\psi}\kappa\alpha$. But then $\delta_0, \delta_1 \in \widetilde{C}_\kappa(\alpha)$, which leads to the contradiction $\omega^{\delta_0} + \delta_1 = \widetilde{\psi}\kappa\alpha \in \widetilde{C}_\kappa(\alpha)$.

Next, for the sake of contradiction, suppose $\widetilde{\psi}\kappa\alpha = \widetilde{\mathsf{Z}}(t)$ for some $t \in \mathsf{OR}_{m_0}$. Then $t^* \in \widetilde{C}_\kappa(\alpha)$ since $t^* < \widetilde{\mathsf{Z}}(t)$. But this would imply $\widetilde{\mathsf{Z}}(t) \in \widetilde{C}_\kappa(\alpha)$ and hence the contradiction $\widetilde{\psi}\kappa\alpha \in \widetilde{C}_\kappa(\alpha)$.

(vi) $\xi < \rho \in \widetilde{C}^n_\kappa(\alpha) \cap \kappa$ implies $\xi \in \widetilde{C}^{n+1}_\kappa(\alpha)$.

(vii) All the ordinals it takes to build π via $\widetilde{\mathsf{M}}$ and the operation $s_0, s_1 \mapsto \tilde{\omega}^{s_0} \oplus s_1$ are $\leq \pi^-$. By (vi), they are contained in $\widetilde{C}_\kappa(\alpha)$. □

Lemma 4.7 *(i)* $\tilde{\omega}^{s_0} \oplus s_1 \in \widetilde{C}_\pi(\alpha)\ \rightarrow\ s_0, s_1 \in \widetilde{C}_\pi(\alpha)$.

(ii) $\delta =_{NF} \omega^{\delta_0} + \delta_1 \in \widetilde{C}_\pi(\alpha) \ \rightarrow\ \delta_0, \delta_1 \in \widetilde{C}_\pi(\alpha)$.

(iii) $\widetilde{\mathsf{Z}}(r) \in \widetilde{C}_\pi(\alpha) \ \rightarrow\ r \in \widetilde{C}_\pi(\alpha)$.

(iv) $\delta = \widetilde{\psi}\tau\eta \in \widetilde{C}_\pi(\alpha) \cap \pi \ \rightarrow\ \tau \in \widetilde{C}_\pi(\alpha)$.

Proof. (i) If $\tilde{\omega}^{s_0} \oplus s_1 \in \widetilde{C}_\pi(\alpha)$, then $\tilde{\omega}^{s_0} \oplus s_1 \in \widetilde{C}^{n+1}_\pi(\alpha) \setminus \widetilde{C}^n_\pi(\alpha)$ for some n. But for $\tilde{\omega}^{s_0} \oplus s_1$ the only way to get into $\widetilde{C}^{n+1}_\pi(\alpha)$ is that $s_0, s_1 \in \widetilde{C}^n_\pi(\alpha)$.

(ii) If $\delta < \pi$, then $\delta_0, \delta_1 \in \widetilde{C}_\pi(\alpha)$ by 4.6,(vi).

Suppose $\pi < \delta$. Since δ cannot be of either form $\widetilde{\mathsf{M}}$, $\widetilde{\psi}\tau\eta$ (by 4.6(v)), $\widetilde{\mathsf{Z}}(r)$ (since $\widetilde{\mathsf{Z}}(r) \in \mathrm{Ad}$), or $\tilde{\omega}^{s_0} \oplus s_1$ (not an ordinal), there must be $\xi_0, \xi_1 \in \widetilde{C}_\pi(\alpha)$ such that $\delta =_{NF} \omega^{\xi_0} + \xi_1$. But then $\delta_0 = \xi_0$ and $\delta_1 = \xi_1$.

(iii) If $\widetilde{\mathsf{Z}}(r) < \pi$, then $r^* \in \widetilde{C}_\pi(\alpha) \cap \pi$ by 4.6(vi) since $r^* < \widetilde{\mathsf{Z}}(r)$. As r can be built up from ordinals $\leq r^*$ via $\widetilde{\mathsf{M}}$ and the operation $s_0, s_1 \mapsto \tilde{\omega}^{s_0} \oplus s_1$, $r \in \widetilde{C}_\pi(\alpha)$ and thus $\widetilde{\mathsf{Z}}(r) \in \widetilde{C}_\pi(\alpha)$.

Suppose that $\pi \leq \widetilde{\mathsf{Z}}(r)$ According to 4.6(v), there must be some $r' \in \widetilde{C}_\pi(\alpha)$ such that $\widetilde{\mathsf{Z}}(r) = \widetilde{\mathsf{Z}}(r')$. However, 4.3 ensures that $r = r'$.

(iv) By 4.6(vi), the hypothesis yields $\tau^- \in \widetilde{C}_\pi(\alpha)$; so $\tau \in \widetilde{C}_\pi(\alpha)$ by 4.6(vii). □

The main feature of the functions ψ_κ of [Bu 86] and [R 90] is that $\psi_\kappa(\alpha) < \kappa$. A proof of this fact is trivial since, in the context pertaining to it, κ was assumed to be a regular uncountable cardinal and the set $C_\kappa(\alpha)$ then obviously has cardinality less than κ. When dealing with admissible ordinals, simple cardinal arithmetic does not help. Indeed, the main bulk of this paper will be devoted to establishing $\widetilde{\psi}\kappa\sigma < \kappa$ for σ which satisfy $\sigma \in \widetilde{C}_\kappa(\sigma)$. The proof proceeds by induction on σ.

General Hypotheses: During the rest of this section, we fix ordinals $\bar{\sigma}$ and $\bar{\kappa} = \widetilde{\mathsf{Z}}(\bar{s})$ with $\bar{s} \in \mathsf{OR}_{m_0}$, and we will be assuming the following.

(A1) $\bar{\sigma} \in \widetilde{C}_{\bar{\kappa}}(\bar{\sigma})$

(A2) $\forall \alpha < \bar{\sigma}\, \forall \pi \in \widetilde{\mathsf{R}}_{m_0}\, [\, \alpha \in \widetilde{C}_\pi(\alpha) \rightarrow \widetilde{\psi}\pi\alpha < \pi]$.

From (A2) we can draw the analogues of [R 90], Sec.5 for the functions $\widetilde{\psi}_\pi{\restriction}\bar{\sigma}$.

Lemma 4.8 *Let $\alpha < \bar{\sigma}$ and $\alpha \in \widetilde{C}_\pi(\alpha)$. Then $\widetilde{C}_\pi(\alpha) \cap \pi = \widetilde{\psi}\pi\alpha < \pi$.*

Proof. $\widetilde{\psi}\pi\alpha < \pi$ follows from (A2); thus $\widetilde{\psi}\pi\alpha \subseteq \widetilde{C}_\pi(\alpha) \cap \pi$. On the other hand, if $\xi \in \widetilde{C}_\pi(\alpha) \cap \pi$, then $\xi \subseteq \widetilde{C}_\pi(\alpha)$; hence $\xi < \widetilde{\psi}\pi\alpha$. □

Corollary 4.9 *Let $\alpha < \bar{\sigma}$ and $\alpha \in \widetilde{C}_\kappa(\alpha)$. Then:*

(i) $\pi < \widetilde{\psi}\kappa\alpha \ \leftrightarrow\ \pi < \kappa \wedge \pi^- < \widetilde{\psi}\kappa\alpha$.

(ii) $\widetilde{\psi}\kappa\alpha < \pi \ \leftrightarrow\ \kappa \leq \pi \vee \widetilde{\psi}\kappa\alpha \leq \pi^-$.

Proof. (i): "$\rightarrow$" is immediate by 4.8. For "$\leftarrow$", note that $\pi < \kappa \wedge \pi^- < \widetilde{\psi}\kappa\alpha$ implies $\pi \in \widetilde{C}_\kappa(\alpha) \cap \kappa$; thus $\pi < \widetilde{\psi}\kappa\alpha$.

(ii) is a consequence of (i) and 4.6(v). □

Lemma 4.10 *Suppose $\alpha, \beta < \bar{\sigma}$, $\alpha \in \widetilde{C}_\pi(\alpha)$, and $\beta \in \widetilde{C}_\tau(\beta)$.*

(i) $\widetilde{\psi}\pi\alpha = \widetilde{\psi}\tau\beta \;\rightarrow\; \pi = \tau \wedge \alpha = \beta$.

(ii) $\widetilde{\psi}\pi\alpha < \widetilde{\psi}\tau\beta$ *iff either of the following is true:*

(1) $\pi < \tau \wedge \pi < \widetilde{\psi}\tau\beta$.

(2) $\pi = \tau \wedge \alpha < \beta$.

(3) $\tau < \pi \wedge \widetilde{\psi}\pi\alpha < \tau$.

Proof. (ii) *CASE 1:* $\pi < \tau$. If $\widetilde{\psi}\pi\alpha < \widetilde{\psi}\tau\beta$, then $\pi^- \in \widetilde{C}_\tau(\beta)$ and thus $\pi \in \widetilde{C}_\tau(\beta) \cap \tau$, which yields $\pi < \widetilde{\psi}\tau\beta$ by 4.6(vi) (or 4.8).

From $\pi < \widetilde{\psi}\tau\beta$ it follows $\widetilde{\psi}\pi\alpha < \widetilde{\psi}\tau\beta$, according to 4.8.

CASE 2: $\pi = \tau$. Then $\widetilde{\psi}\pi\alpha < \widetilde{\psi}\tau\beta \leftrightarrow \alpha < \beta$ is a consequence of 4.6(iii).

CASE 3: $\tau < \pi$. If $\widetilde{\psi}\pi\alpha < \widetilde{\psi}\tau\beta$, then $\widetilde{\psi}\pi\alpha < \tau$, since $\widetilde{\psi}\tau\beta < \tau$ by 4.8. Now suppose $\widetilde{\psi}\pi\alpha < \tau$. From $\widetilde{\psi}\tau\beta \leq \widetilde{\psi}\pi\alpha$ it would follow $\tau \in \widetilde{C}_\pi(\alpha) \cap \pi$, and thus the contradiction $\tau < \widetilde{\psi}\pi\alpha$.

(i) is a consequence of (ii) and Lemma 4.6(v). □

From 4.3, 4.6(v), and 4.10, we can glean that any element of $\widetilde{C}_{\bar{\kappa}}(\bar{\sigma}) \setminus \bar{\kappa}$ has a unique representation in terms of the symbols $\widetilde{\mathsf{M}}, +, \omega, \widetilde{\omega}, \oplus, \widetilde{\psi}, \widetilde{\mathsf{Z}}$, and ordinals $< \bar{\kappa}$; this being in a nutshell the idea for coding up $\widetilde{C}_{\bar{\kappa}}(\bar{\sigma})$ κ–recursively on L_κ. Unfortunately, we have to face up with a host of problems. For instance, in order to decide whether $\widetilde{\psi}\pi\beta \in \widetilde{C}_{\bar{\kappa}}(\bar{\sigma})$ we need to determine whether $\beta \in \widetilde{C}_\pi(\beta)$; but if $\bar{\kappa} < \pi$ then, in general, we cannot mimic the whole of $\widetilde{C}_\pi(\beta)$ within $\mathsf{L}_{\bar{\kappa}}$ in a $\bar{\kappa}$–recursive way. It will turn out that in the relevant cases it suffices to inspect a certain subset of $\widetilde{C}_\pi(\beta)$, which can be simulated $\bar{\kappa}$–recursively, and a finite set, $\widetilde{K}_{\pi,\bar{\kappa}}(\beta)$, of ordinals of $\widetilde{C}_{\bar{\kappa}}(\bar{\sigma})$ that traces back how β entered $\widetilde{C}_{\bar{\kappa}}(\bar{\sigma})$.

Definition 4.11 For $\kappa \leq \pi$ define $\widetilde{C}_{\pi,\kappa}(\alpha)$ as follows.

$$
\begin{aligned}
\widetilde{C}^0_{\pi,\kappa}(\alpha) &= \pi^- \cup \{\pi^-, \widetilde{\mathsf{M}}\};\\
\widetilde{C}^{n+1}_{\pi,\kappa}(\alpha) &= \widetilde{C}^n_{\pi,\kappa}(\alpha)\\
&\quad \cup \{s \in \mathsf{OR}_{m_0} : s = \tilde{\omega}^{s_0} \oplus s_1;\; s_0, s_1 \in \widetilde{C}^n_{\pi,\kappa}(\alpha)\}\\
&\quad \cup \{\beta : \beta =_{NF} \omega^{\beta_0} + \beta_1;\; \beta_0, \beta_1 \in \widetilde{C}^n_{\pi,\kappa}(\alpha)\}\\
&\quad \cup \{\widetilde{\mathsf{Z}}(s) : s \in \widetilde{C}^n_{\pi,\kappa}(\alpha)\}\\
&\quad \cup \{\widetilde{\psi}_\tau(\eta) : \tau, \eta \in \widetilde{C}^n_{\pi,\kappa}(\alpha);\; \eta < \alpha;\; \eta \in \widetilde{C}_\tau(\eta)\}\\
&\quad \cup \{\xi : \exists \rho \in \widetilde{C}^n_{\pi,\kappa}(\alpha) \cap \kappa\, (\xi < \rho)\};\\
\widetilde{C}_{\pi,\kappa}(\alpha) &= \bigcup_{n<\omega} \widetilde{C}^n_{\pi,\kappa}(\alpha)
\end{aligned}
$$

Notice that $\widetilde{C}_{\pi,\kappa}(\alpha) \subseteq \widetilde{C}_\pi(\alpha)$. The difference between $\widetilde{C}_\pi(\alpha)$ and $\widetilde{C}_{\pi,\kappa}(\alpha)$ is that $\rho \in \widetilde{C}_\pi(\alpha) \cap \pi$ implies $\rho \subseteq \widetilde{C}_\pi(\alpha)$ whereas in case of $\widetilde{C}_{\pi,\kappa}(\alpha)$ we need to have $\rho \in \widetilde{C}_{\pi,\kappa}(\alpha) \cap \kappa$ in order to be able to conclude $\rho \subseteq \widetilde{C}_{\pi,\kappa}(\alpha)$.

Lemma 4.12 *If $\kappa < \pi$ and*[9] $[\kappa,\pi) \cap \widetilde{C}_{\pi,\kappa}(\alpha) = \emptyset$*, then* $\widetilde{C}_{\pi,\kappa}(\alpha) = \widetilde{C}_{\pi}(\alpha)$ *and* $\widetilde{\psi}\pi\alpha \leq \kappa^{-}$.

Proof. By induction on n prove $\widetilde{C}^{n}_{\pi}(\alpha) = \widetilde{C}^{n}_{\pi,\kappa}(\alpha)$. The assertion is immediate for $n = 0$. Suppose the equality is true for n and we want to verify it for $n+1$. Assume $\delta \in \widetilde{C}^{n}_{\pi,\kappa}(\alpha) \cap \pi$. We have to show that $\delta \subseteq \widetilde{C}^{n+1}_{\pi,\kappa}(\alpha)$. But as $[\kappa,\pi) \cap \widetilde{C}_{\pi,\kappa}(\alpha) = \emptyset$ we must have $\delta \in \widetilde{C}^{n}_{\pi,\kappa}(\alpha) \cap \kappa$, and thus $\delta \in \widetilde{C}^{n+1}_{\pi,\kappa}(\alpha)$.

The remaining cases are immediate.

From $\kappa^{-} < \widetilde{\psi}\pi\alpha$ we would get $\kappa \in \widetilde{C}_{\pi}(\alpha)$, contradicting $\emptyset = [\kappa,\pi) \cap \widetilde{C}_{\pi,\kappa}(\alpha) = [\kappa,\pi) \cap \widetilde{C}_{\pi}(\alpha)$. □

Corollary 4.13 *Let $\kappa < \pi$.*

(i) If $[\kappa,\pi) \cap \widetilde{C}_{\pi,\kappa}(\alpha) = \emptyset$, then $\widetilde{\psi}\pi\alpha \leq \kappa^{-}$.

(ii) If $[\kappa,\pi) \cap \widetilde{C}_{\pi,\kappa}(\alpha) \neq \emptyset$, then $\kappa < \widetilde{\psi}\pi\alpha$.

Proof. (i) follows from 4.12.
As to (ii), suppose $\delta \in [\kappa,\pi) \cap \widetilde{C}_{\pi,\kappa}(\alpha)$. Then $\delta \in \widetilde{C}_{\pi}(\alpha)$ and thus $\delta + 1 \subseteq \widetilde{C}_{\pi}(\alpha)$, which implies $\kappa < \delta + 1 \leq \widetilde{\psi}\pi\alpha$. □

We write $\delta =_{NF} \widetilde{\psi}\tau\eta$ to mean that $\delta = \widetilde{\psi}\tau\eta$ and $\eta \in \widetilde{C}_{\tau}(\eta)$.

Definition 4.14 For $\bar{\kappa} < \pi$ and $s, \pi \in \widetilde{C}_{\bar{\kappa}}(\bar{\sigma})$ we define $\widetilde{K}_{\pi,\bar{\kappa}}(s)$ as follows.

$\widetilde{K}_{\pi,\bar{\kappa}}(s) = \emptyset$ if $s = \widetilde{\mathsf{M}} \vee s \leq \bar{\kappa} \vee s \leq \pi^{-}$.

$\widetilde{K}_{\pi,\bar{\kappa}}(\tilde{\omega}^{s_0} \oplus s_1) = \widetilde{K}_{\pi,\bar{\kappa}}(s_0) \cup \widetilde{K}_{\pi,\bar{\kappa}}(s_1)$.

In the following cases let $\delta \in \widetilde{C}_{\bar{\kappa}}(\bar{\sigma})$ and $\delta > \bar{\kappa}, \pi^{-}$.

$\widetilde{K}_{\pi,\bar{\kappa}}(\delta) = \widetilde{K}_{\pi,\bar{\kappa}}(\delta_0) \cup \widetilde{K}_{\pi,\bar{\kappa}}(\delta_1)$ if $\delta =_{NF} \omega^{\delta_0} + \delta_1$.

$\widetilde{K}_{\pi,\bar{\kappa}}(\delta) = \widetilde{K}_{\pi,\bar{\kappa}}(r)$ if $\delta = \widetilde{\mathsf{Z}}(r)$.

If $\delta =_{NF} \widetilde{\psi}\tau\eta$, then

$$\widetilde{K}_{\pi,\bar{\kappa}}(\delta) = \begin{cases} \widetilde{K}_{\pi,\bar{\kappa}}(\tau) & \text{if } \tau < \pi \\ \{\eta\} \cup \widetilde{K}_{\pi,\bar{\kappa}}(\tau) \cup \widetilde{K}_{\pi,\bar{\kappa}}(\eta) & \text{if } \pi \leq \tau \end{cases}$$

Note that $\widetilde{K}_{\pi,\bar{\kappa}}(s)$ is well defined since the respective representation of s on which the above definition draws is uniquely determined and uses only elements from $\widetilde{C}_{\bar{\kappa}}(\bar{\sigma})$; this follows from 4.3, 4.6, 4.7, and 4.10.

We shall write $\widetilde{K}_{\pi,\bar{\kappa}}(s) < \beta$ to mean $\forall \xi \in \widetilde{K}_{\pi,\kappa}(s)\, [\xi < \beta]$.

Lemma 4.15 *Let $\beta, \pi \in \widetilde{C}_{\bar{\kappa}}(\bar{\sigma})$, $\beta < \bar{\sigma}$, and $\bar{\kappa} < \pi$. If $\widetilde{C}_{\pi,\bar{\kappa}}(\beta) \cap [\bar{\kappa},\pi) \neq \emptyset$, then, for any $s \in \widetilde{C}_{\bar{\kappa}}(\bar{\sigma})$,*

$$(*) \qquad s \in \widetilde{C}_{\pi}(\beta) \iff \widetilde{K}_{\pi,\bar{\kappa}}(s) < \beta.$$

[9] $[\kappa,\pi) := \{\alpha : \kappa \leq \alpha < \pi\}$.

Proof. By induction on $s \in \widetilde{C}^n_{\bar\kappa}(\bar\sigma)$. As $\bar\kappa < \pi$, $\widetilde{C}_{\pi,\bar\kappa}(\beta) \cap [\bar\kappa, \pi) \neq \emptyset$ guarantees $\bar\kappa + 1 \subseteq \widetilde{C}_\pi(\beta)$. Thus, if $\delta \le \bar\kappa$ or $\delta \le \pi^-$, then $\delta \in \widetilde{C}_\pi(\beta)$ and $\widetilde{K}_{\pi,\bar\kappa}(\delta) = \emptyset < \delta$. Also $\widetilde{\mathsf{M}} \in \widetilde{C}_\pi(\beta)$ and $\widetilde{K}_{\pi,\bar\kappa}(\widetilde{\mathsf{M}}) = \emptyset$. So, in particular, $(*)$ holds true for all $x \in \widetilde{C}^0_{\bar\kappa}(\bar\sigma)$.

Now assume that $n > 0$ and $(*)$ being true for all $r \in \widetilde{C}^{n-1}_{\bar\kappa}(\bar\sigma)$. By the above, we may also assume that $s \sqsupset \kappa, \pi^-$. If $s =_{NF} \omega^{\delta_0} + \delta_1$, $s = \tilde\omega^{s_0} \oplus s_1$, or $s = \widetilde{\mathsf{Z}}(s_0)$ with $s_0, s_1 \in \widetilde{C}^{n-1}_{\bar\kappa}(\bar\sigma)$, then $(*)$ is immediate by the inductive assumption.

Let $s =_{NF} \widetilde{\psi}\tau\eta$ with $\tau, \eta \in \widetilde{C}^{n-1}_{\bar\kappa}(\bar\sigma)$ and $\eta < \bar\sigma$. If $\tau < \pi$, then $\widetilde{K}_{\pi,\bar\kappa}(s) = \widetilde{K}_{\pi,\bar\kappa}(\tau)$. Therefore, using 4.7(iv),

$$s \in \widetilde{C}_\pi(\beta) \iff \tau \in \widetilde{C}_\pi(\beta) \iff \widetilde{K}_{\pi,\bar\kappa}(s) < \beta.$$

If $\pi \le \tau$, then

$$\begin{aligned} s \in \widetilde{C}_\pi(\beta) &\iff \tau, \eta \in \widetilde{C}_\pi(\beta) \wedge \eta < \beta \\ &\iff \{\eta\} \cup \widetilde{K}_{\pi,\bar\kappa}(\tau) \cup \widetilde{K}_{\pi,\bar\kappa}(\eta) < \beta \\ &\iff \widetilde{K}_{\pi,\bar\kappa}(s) < \beta. \end{aligned}$$

□

To mimic $\widetilde{C}_{\bar\kappa}(\bar\sigma)$ in $\mathsf{L}_{\bar\kappa}$, we first project $\widetilde{C}_{\bar\kappa}(\bar\sigma)$ into $\mathsf{L}_{\bar\kappa}$ via a function ℓ, and will subsequently verify that $\ell''\widetilde{C}_{\bar\kappa}(\bar\sigma) \in \mathsf{L}_{\bar\kappa}$; thereby establishing $\widetilde{\psi}\bar\kappa\bar\sigma < \bar\kappa$.

Definition 4.16 Set $\mathcal{Z}s = \langle 3, s\rangle$ and $\Upsilon st = \langle 4, s, t\rangle$. Since we also need to "name" ordinals $\delta =_{NF} \omega^{\delta_0} + \delta_1 \in \widetilde{C}_{\bar\kappa}(\bar\sigma)$ with $\delta > \bar\kappa$, we introduce the "sum" $\hat\omega^{s_0} \boxplus s_1 = \langle 5, s_0, s_1\rangle$.

$\ell : \widetilde{C}_{\bar\kappa}(\bar\sigma) \longrightarrow \mathsf{L}_{\bar\kappa}$ is defined as follows.

1. $\ell(\alpha) = \alpha$ if $\alpha < \bar\kappa$.
2. $\ell(\delta) = \hat\omega^{\ell(\delta_0)} \boxplus \ell(\delta_1)$ if $\delta =_{NF} \omega^{\delta_0} + \delta_1 > \bar\kappa$.
3. $\ell(s) = \tilde\omega^{\ell(s_0)} \oplus \ell(s_1)$ if $s = \tilde\omega^{s_0} \oplus s_1$.
4. $\ell(\widetilde{\mathsf{Z}}(r)) = \mathcal{Z}\ell(r)$ if $\widetilde{\mathsf{Z}}(r) \ge \bar\kappa$.
5. $\ell(\widetilde{\mathsf{M}}) = \widetilde{\mathsf{M}}$.
6. $\ell(\delta) = \Upsilon\ell(\pi)\ell(\rho)$ if $\delta =_{NF} \widetilde{\psi}\pi\rho > \bar\kappa$.

Observe that since $\bar s^* < \bar\kappa$, we have $\ell(\bar s) = \bar s$, and hence $\ell(\bar\kappa) = \mathcal{Z}\bar s$.

The next Definition provides a Δ_1 collection of possible notations for elements of $\widetilde{C}_{\bar\kappa}(\bar\sigma)$. Only later on it will be determined which elements of $\mathsf{L}_{\bar\kappa} \cap \mathsf{PN}$ belong to $\ell''\widetilde{C}_{\bar\kappa}(\bar\sigma)$.

Definition 4.17 (**KP**) A Δ_1 predicate PN, a binary relation $\lhd$ and a Δ_1 function $s \mapsto s^\diamond$ are (simultaneously) defined as follows.

1. For all ordinals α, $\alpha \in \mathsf{PN}$ and $\alpha^\diamond = \alpha$.
2. $\widetilde{\mathsf{M}} \in \mathsf{PN}$ and $\widetilde{\mathsf{M}}^\diamond = 0$.

3. If $s_0 \in \mathsf{PN}$ and $\widetilde{\mathsf{M}} \lhd s_0$, then $s = \tilde{\omega}^{s_0} \oplus 0 \in \mathsf{PN}$ and $s^\diamond = s_0^\diamond$.
4. If $s_0, s_1 \in \mathsf{PN}$, $\widetilde{\mathsf{M}} \unlhd s_0$ and either $0 \lhd s_1 \unlhd \widetilde{\mathsf{M}}$ or $s_1 = \tilde{\omega}^x \oplus y$ with $x \unlhd s_0$, then $\tilde{\omega}^{s_0} \oplus s_1 \in \mathsf{PN}$ and $(\tilde{\omega}^{s_0} \oplus s_1)^\diamond$ is the maximum of $s_0^\diamond$ and $s_1^\diamond$ with respect to $\lhd$.
5. If $s_0 \in \mathsf{PN}$, $\neg\mathsf{Ord}(s_0)$, $s_0 \lhd \widetilde{\mathsf{M}}$, and s_0 is of neither form $\mathcal{Z}t$ or Υrt, then $s = \hat{\omega}^{s_0} \boxplus 0 \in \mathsf{PN}$ and $s^\diamond = s$.
6. If $s_0, s_1 \in \mathsf{PN}$, $\neg\mathsf{Ord}(s_0)$, $s_0 \lhd \widetilde{\mathsf{M}}$, and either $0 \lhd s_1 \unlhd s_0$ or $s_1 = \hat{\omega}^x \boxplus y$ with $x \unlhd s_0$, then $s = \hat{\omega}^{s_0} \boxplus s_1 \in \mathsf{PN}$ and $s^\diamond = s$.
7. If $s \in \mathsf{PN}$ then $\mathcal{Z}s \in \mathsf{PN}$ and $(\mathcal{Z}s)^\diamond = \mathcal{Z}s$.
8. If $\mathcal{Z}s, t \in \mathsf{PN}$ and $t \lhd \widetilde{\mathsf{M}}$, then $\Upsilon(\mathcal{Z}s)t \in \mathsf{PN}$ and $(\Upsilon(\mathcal{Z}s)t)^\diamond = \Upsilon(\mathcal{Z}s)t$.

   In the following definition of $\lhd$ we make the tacit assumption that all sets are from PN.
9. If $\mathsf{Ord}(s)$ and $\neg\mathsf{Ord}(t)$, then $s \lhd t$.
10. If s is of either form $\hat{\omega}^{s_0} \boxplus s_1$, $\mathcal{Z}s_0$ or $\Upsilon s_0 s_1$, then $s \lhd \widetilde{\mathsf{M}}$.
11. If $t = \tilde{\omega}^{t_0} \oplus t_1$ and s is of either form $\hat{\omega}^{s_0} \boxplus s_1$, $\widetilde{\mathsf{M}}$, $\mathcal{Z}s_0$ or $\Upsilon s_0 s_1$, then $s \lhd t$.
12. If t is of either form $\mathcal{Z}t_0$ or $\Upsilon t_0 t_1$, then
$$\hat{\omega}^{s_0} \boxplus s_1 \lhd t \iff s_0 \lhd t$$
    and
$$t \lhd \hat{\omega}^{s_0} \boxplus s_1 \iff t \unlhd s_0.$$
13. $\hat{\omega}^{s_0} \boxplus s_1 \lhd \hat{\omega}^{t_0} \boxplus t_1 \iff (s_0 \lhd t_0 \vee (s_0 = t_0 \wedge s_1 \lhd t_1))$.
14. $\tilde{\omega}^{s_0} \oplus s_1 \lhd \tilde{\omega}^{t_0} \oplus t_1 \iff (s_0 \lhd t_0 \vee (s_0 = t_0 \wedge s_1 \lhd t_1))$.
15. $\mathcal{Z}s \lhd \Upsilon t_0 t_1 \iff s^\diamond \lhd \Upsilon t_0 t_1 \wedge \mathcal{Z}s \lhd t_0$.
16. $\Upsilon t_0 t_1 \lhd \mathcal{Z}s \iff t_0 \unlhd \mathcal{Z}s \vee \Upsilon t_0 t_1 \unlhd s^\diamond$.
17. $\mathcal{Z}s \lhd \mathcal{Z}t \iff (s \lhd t \wedge s^\diamond \lhd \mathcal{Z}t) \vee (t \lhd s \wedge \mathcal{Z}s \unlhd t^\diamond)$.
18. $$\begin{aligned} \Upsilon s_0 s_1 \lhd \Upsilon t_0 t_1 \iff & (s_0 \lhd t_0 \wedge s_0 \lhd \Upsilon t_0 t_1) \\ & \vee(s_0 = t_0 \wedge s_1 \lhd t_1) \\ & \vee(t_0 \lhd s_0 \wedge \Upsilon s_0 s_1 \lhd t_0). \end{aligned}$$

For $t \in \mathsf{PN}$ of the form $\mathcal{Z}r$, we set $t^- = r^\diamond$.

The above definition proceeds by recursion on the (provably in **KP**) well–founded set–like relation $\prec$ which is defined by

$$\begin{aligned} \langle x, y\rangle \prec \langle x', y'\rangle \iff & (x = x' \wedge y \in \mathsf{TC}(y')) \vee (y = y' \wedge x \in \mathsf{TC}(x')) \\ & \vee(x \in \mathsf{TC}(x') \wedge y \in \mathsf{TC}(y')). \end{aligned}$$

It is then obvious that in each case (1) through (14) the induction is pushed back; note that always $s^\diamond \in \mathsf{TC}(\{s\})$. The definition of $s^\diamond$ in (4) assumes that $s_0^\diamond$ and $s_1^\diamond$ are ordered with regard to $\lhd$. This is true, but formally we could have set $s^\diamond = 0$ if this was not the case.

Lemma 4.18 *(i)* $\mathsf{OR} \subseteq \mathsf{PN}$ *and* $\sqsubset = \lhd \cap (\mathsf{OR} \times \mathsf{OR})$.

(ii) If $s \in \mathsf{OR}$ *then* $s^* = s^\diamond$.

(iii) $\lhd$ *is not a well–ordering since, e.g., with* $r = \mathcal{Z}0$*, using 4.17(16),(18), we get*

$$\Upsilon rr \rhd \Upsilon r(\Upsilon rr) \rhd \Upsilon r(\Upsilon r(\Upsilon rr)) \rhd \cdots .$$

Proof. (i) is immediate by induction on $\prec$. (ii) follows by induction on $\mathsf{TC}(s)$. □

Definition 4.19 Recall that $\bar\kappa = \widetilde{\mathsf{Z}}(\bar s)$. In order to simulate the function $\widetilde{K}_{\pi,\bar\kappa}$ on PN we define a $\bar\kappa$–recursive function $\mathfrak{K}_t$ by recursion on $\mathsf{TC}(r)$ as follows. Let $r, t \in \mathsf{PN}$, $t = \mathcal{Z}t_0$, and $\mathcal{Z}\bar s \lhd t$.

(1) If $r \unlhd \mathcal{Z}\bar s$ or $r \unlhd t^-$, then $\mathfrak{K}_t(r) = \emptyset$.

Henceforth, let $r \rhd \mathcal{Z}\bar s, t^-$.

(2) $\mathfrak{K}_t(\widetilde{\mathsf{M}}) = \emptyset$.

(3) $\mathfrak{K}_t(\hat\omega^{r_0} \boxplus r_1) = \mathfrak{K}_t(r_0) \cup \mathfrak{K}_t(r_1)$.

(4) $\mathfrak{K}_t(\tilde\omega^{r_0} \oplus r_1) = \mathfrak{K}_t(r_0) \cup \mathfrak{K}_t(r_1)$.

(5) $\mathfrak{K}_t(\mathcal{Z}r_0) = \mathfrak{K}_t(r_0)$.

(6)

$$\mathfrak{K}_t(\Upsilon r_0 r_1) = \begin{cases} \mathfrak{K}_t(r_0) & \text{if } r_0 \lhd t \\ \{r_1\} \cup \mathfrak{K}_t(r_0) \cup \mathfrak{K}_t(r_1) & \text{if } t \unlhd r_0. \end{cases}$$

Lemma 4.20 *(i)* $\forall t, r \in \widetilde{C}_{\bar\kappa}(\bar\sigma)[t \sqsubset r \iff \ell(t) \lhd \ell(r)]$;

(ii) $\forall t, \pi \in \widetilde{C}_{\bar\kappa}(\bar\sigma)[\ell''\widetilde{K}_{\pi,\bar\kappa}(t) = \mathfrak{K}_{\ell(\pi)}(\ell(t))]$.

(iii) $\forall t \in \mathsf{OR}^{\bar\kappa} \cap \widetilde{C}_{\bar\kappa}(\bar\sigma)[\ell(t) = t]$, *(where* $\mathsf{OR}^{\bar\kappa} := \mathsf{OR} \cap \mathsf{L}^{\bar\kappa}$*).*

Proof. Proceed by induction on $\prec$. Then (i) follows immediately from the characterizations of $\sqsubset$ given in 4.3, 4.9, 4.10, and the definitions of $\lhd$ and ℓ.

(ii) follows from (i) using the definitions of $\widetilde{K}_{\pi,\bar\kappa}$ and $\mathfrak{K}_{\ell(\pi)}$.

(iii) is readily shown by induction on $\mathsf{TC}(t)$. □

Definition 4.21 Using the Second Recursion Theorem we define simultaneously two $\bar\kappa$–partial recursive functions $\mathcal{F}$ and $\mathfrak{C}$. We will write $\mathfrak{C}_s^n(r)$ instead of $\mathfrak{C}(s, r, n)$, $\mathfrak{C}_s(r)$ instead of $\bigcup_{n<\omega} \mathfrak{C}_s^n(r)$, and $\mathcal{F}_s(r)$ instead of $\mathcal{F}(s, r)$. *The definition is supposed to take place within* $\mathsf{L}_{\bar\kappa}$.[10]

[10] For $t, t' \in \mathsf{PN}$, $[t, t') := \{x \in \mathsf{PN} : t \unlhd x \lhd t'\}$. $\mathfrak{K}_{\mathcal{Z}x}(y) \lhd y \;:\iff\; \forall u \in \mathfrak{K}_{\mathcal{Z}x}(y)\,[u \lhd y]$.

If $s = \mathcal{Z}s_0 \in \mathsf{PN}$ with $s^- < \bar{\kappa}$, and $r \in \mathsf{PN}$, then

$$
\begin{aligned}
\mathfrak{C}^0_s(r) &\;\backsimeq\; s^- \cup \{s^-, \widetilde{\mathsf{M}}\} && (1)\\
\mathfrak{C}^{n+1}_s(r) &\;\backsimeq\; \mathfrak{C}^n_s(r) \cup \{\delta : \delta =_{NF} \omega^{\delta_0} + \delta_1;\ \delta_0, \delta_1 \in \mathfrak{C}^n_s(r)\} && (2)\\
&\qquad \cup \{\tilde{\omega}^y \oplus z \in \mathsf{PN} :\ y \lhd \widetilde{\mathsf{M}}(m_0 - 1);\ y, z \in \mathfrak{C}^n_s(r)\} && (3)\\
&\qquad \cup \{\hat{\omega}^y \boxplus z \in \mathsf{PN} :\ y, z \in \mathfrak{C}^n_s(r)\} && (4)\\
&\qquad \cup \{\widetilde{\mathsf{Z}}^{\bar{\kappa}}(x) : x \in \mathsf{OR}^{\bar{\kappa}}_{m_0};\ \widetilde{\mathsf{Z}}(x) < \bar{\kappa};\ x \in \mathfrak{C}^n_s(r)\} && (5)\\
&\qquad \cup \{\mathcal{Z}x : x \notin \mathsf{OR}^{\bar{\kappa}}_{m_0}\ \vee\ (x \in \mathsf{OR}^{\bar{\kappa}}_{m_0} \wedge \bar{\kappa} \le \widetilde{\mathsf{Z}}(x));\ x \in \mathfrak{C}^n_s(r)\} && (6)\\
&\qquad \cup \{\mathcal{F}_{\mathcal{Z}\bar{s}}(y) :\ y \lhd r;\ y \in \mathfrak{C}_{\mathcal{Z}\bar{s}}(y);\ y, \mathcal{Z}\bar{s} \in \mathfrak{C}^n_s(r)\} && (7)\\
&\qquad \cup \{\mathcal{F}_{\mathcal{Z}x}(y) : x \in \mathsf{OR}^{\bar{\kappa}}_{m_0};\ \bar{\kappa} < \widetilde{\mathsf{Z}}(x);\ y \lhd r;\ y \in \mathfrak{C}_{\mathcal{Z}x}(y); && (8)\\
&\qquad\qquad\qquad \mathfrak{C}_{\mathcal{Z}x}(y) \cap [\mathcal{Z}\bar{s}, \mathcal{Z}x) = \emptyset;\ \mathcal{Z}x, y \in \mathfrak{C}^n_s(r)\}\\
&\qquad \cup \{\Upsilon(\mathcal{Z}x)y : x \in \mathsf{OR}^{\bar{\kappa}}_{m_0};\ \bar{\kappa} < \widetilde{\mathsf{Z}}(x);\ y \lhd r;\ \mathfrak{K}_{\mathcal{Z}x}(y) \lhd y && (9)\\
&\qquad\qquad\qquad \mathfrak{C}_{\mathcal{Z}x}(y) \cap [\mathcal{Z}\bar{s}, \mathcal{Z}x) \ne \emptyset;\ \mathcal{Z}x, y \in \mathfrak{C}^n_s(r)\}\\
&\qquad \cup \{\Upsilon(\mathcal{Z}x)y : x \notin \mathsf{OR}^{\bar{\kappa}}_{m_0};\ y \lhd r;\ \mathfrak{K}_{\mathcal{Z}x}(y) \lhd y;\ \mathcal{Z}x, y \in \mathfrak{C}^n_s(r)\} && (10)\\
&\qquad \cup \{\xi : \exists \alpha \in \mathfrak{C}^n_s(r)(\xi < \alpha)\} && (11)\\
\mathcal{F}_s(r) &\;\backsimeq\; \min\{\xi : \xi \notin \mathfrak{C}^n_s(r)\}. && (12)
\end{aligned}
$$

To ensure that this definition falls under the Second Recursion Theorem, we have to draw on 4.4, where it was shown that $\{x \in \mathsf{OR}^{\bar{\kappa}}_{m_0} : \widetilde{\mathsf{Z}}(x) < \bar{\kappa}\}$ is $\bar{\kappa}$–recursive (thus $\{x \in \mathsf{OR}^{\bar{\kappa}}_{m_0} : \bar{\kappa} \le \widetilde{\mathsf{Z}}(x)\}$, too).

The idea behind the foregoing definition is that if s codes π and r codes α and $\pi^- < \bar{\kappa} \le \pi$, then $\mathfrak{C}^n_s(r)$ simulates $\widetilde{C}^n_{\pi,\bar{\kappa}}(\alpha)$ and $\mathcal{F}_s(r) = \min\{\xi : \xi \notin \widetilde{C}_{\pi,\bar{\kappa}}(\alpha)\}$.

Proposition 4.22 *Let $\alpha \in \widetilde{C}_{\bar{\kappa}}(\bar{\sigma})$ and $\alpha \le \bar{\sigma}$. If $\pi \in \widetilde{C}_{\bar{\kappa}}(\bar{\sigma})$ and $\pi^- < \bar{\kappa} \le \pi$, then $\ell''\widetilde{C}^n_{\pi,\bar{\kappa}}(\alpha) = \mathfrak{C}^n_{\ell(\pi)}(\ell(\alpha))$ and $\mathcal{F}_{\ell(\pi)}(\ell(\alpha)) = \min\{\xi : \xi \notin \widetilde{C}_{\pi,\bar{\kappa}}(\alpha)\}$; in particular, $\mathfrak{C}^n_{\ell(\pi)}(\ell(\alpha))$ and $\mathcal{F}_{\ell(\pi)}(\ell(\alpha))$ are defined.*

Proof. Notice that $\widetilde{C}_{\pi,\bar{\kappa}}(\alpha) \subseteq \widetilde{C}_{\bar{\kappa}}(\bar{\sigma})$. Also observe that, since $\bar{s}^* < \bar{\kappa}$, we have $\ell(\bar{s}) = \bar{s}$, and hence $\ell(\bar{\kappa}) = \mathcal{Z}\bar{s}$.

We proceed by main induction on α and subsidiary induction on n. Let $\pi = \widetilde{\mathsf{Z}}(r)$. From $r^* = \pi^- < \bar{\kappa}$ it follows $r \in \mathsf{OR}^{\bar{\kappa}}_{m_0}$; thus $\ell(r) = r$ by 4.20(iii), yielding $\ell(\pi) = \mathcal{Z}\ell(r) = \mathcal{Z}r$.

For $n = 0$, we then have (note that $\pi^- = r^\diamond$ by 4.18(ii)),

$$\ell''\widetilde{C}^0_{\pi,\bar{\kappa}}(\alpha) = \ell''(\pi^- \cup \{\pi^-, \widetilde{\mathsf{M}}\}) = \pi^- \cup \{\pi^-, \widetilde{\mathsf{M}}\} = r^\diamond \cup \{r^\diamond, \widetilde{\mathsf{M}}\} = \mathfrak{C}^0_{\ell(\pi)}(\ell(\alpha)).$$

Now assume $n > 0$ and $\ell''\widetilde{C}^{n-1}_{\pi,\bar{\kappa}}(\alpha) = \mathfrak{C}^{n-1}_{\ell(\pi)}(\ell(\alpha))$. We want to show

$$(+) \quad \ell''\widetilde{C}^n_{\pi,\bar{\kappa}}(\alpha) = \mathfrak{C}^n_{\ell(\pi)}(\ell(\alpha)).$$

First, we will be concerned with "$\subseteq$".

CASE 1: $\delta =_{NF} \omega^{\delta_0}+\delta_1 \in \widetilde{C}^n_{\pi,\bar\kappa}(\alpha)$; $\delta_0,\delta_1 \in \widetilde{C}^{n-1}_{\pi,\bar\kappa}(\alpha)$. Then $\ell(\delta_0),\ell(\delta_1) \in \mathfrak{C}^{n-1}_{\ell(\pi)}(\ell(\alpha))$.

If $\delta_0 < \bar\kappa$, then $\ell(\delta) = \delta \in \mathfrak{C}^n_{\ell(\pi)}(\ell(\alpha))$ by 4.21(2).

If $\delta_0 \geq \bar\kappa$, then $\neg\mathsf{Ord}(\ell(\delta_0))$. Since also $\ell(\delta_0) \lhd \widetilde{\mathsf{M}}$, we get $\hat\omega^{\ell(\delta_0)} \boxplus \ell(\delta_1) \in \mathsf{PN}$. Hence, $\ell(\delta) = \hat\omega^{\ell(\delta_0)} \boxplus \ell(\delta_1) \in \mathfrak{C}^n_{\ell(\pi)}(\ell(\alpha))$ by 4.21(4).

CASE 2: $s = \tilde\omega^{s_0} \oplus s_1 \in \widetilde{C}^n_{\pi,\bar\kappa}(\alpha)$; $s_0,s_1 \in \widetilde{C}^{n-1}_{\pi,\bar\kappa}(\alpha)$. Then $\ell(s_0),\ell(s_1) \in \mathfrak{C}^{n-1}_{\ell(\pi)}(\ell(\alpha))$. From $s \in \mathsf{OR}_{m_0}$ (and 4.20) it follows $\tilde\omega^{\ell(s_0)} \oplus \ell(s_1) \in \mathsf{PN}$ and $\ell(s_0) \lhd \widetilde{\mathsf{M}}(m_0-1)$. Thus $\ell(s) = \tilde\omega^{\ell(s_0)} \oplus \ell(s_1) \in \mathfrak{C}^n_{\ell(\pi)}(\ell(\alpha))$ by 4.21(3).

CASE 3: $\tau = \widetilde{\mathsf{Z}}(t) \in \widetilde{C}^n_{\pi,\bar\kappa}(\alpha)$ with $t \in \widetilde{C}^{n-1}_{\pi,\bar\kappa}(\alpha)$. Then $\ell(t) \in \mathfrak{C}^{n-1}_{\ell(\pi)}(\ell(\alpha))$.

Suppose $\bar\kappa \leq \tau$. If $t \in \mathsf{OR}^{\bar\kappa}_{m_0}$, then $t = \ell(t)$; hence $\ell(\tau) = \mathcal{Z}t \in \mathfrak{C}^n_{\ell(\pi)}(\ell(\alpha))$ by 4.21(6). If $t \notin \mathsf{OR}^{\bar\kappa}_{m_0}$, then $\ell(t) \notin \mathsf{OR}^{\bar\kappa}_{m_0}$; thus $\ell(\tau) = \mathcal{Z}\ell(\tau) \in \mathfrak{C}^n_{\ell(\pi)}(\ell(\alpha))$, again by 4.21(6).

Now let $\tau < \bar\kappa$. Then $t \in \mathsf{OR}^{\bar\kappa}_{m_0}$. So $\ell(t) = t$. Hence $t \in \mathfrak{C}^{n-1}_{\ell(\pi)}(\ell(\alpha))$. By 2.9, $\tau = \widetilde{\mathsf{Z}}^{\bar\kappa}(t)$. Hence $\ell(\tau) = \widetilde{\mathsf{Z}}^{\bar\kappa}(t) \in \mathfrak{C}^n_{\ell(\pi)}(\ell(\alpha))$ by 4.21(5).

CASE 4: $\widetilde\psi\tau\beta \in \widetilde{C}^n_{\pi,\bar\kappa}(\alpha)$; $\tau,\beta \in \widetilde{C}^{n-1}_{\pi,\bar\kappa}(\alpha)$; $\beta < \alpha$; $\beta \in \widetilde{C}_\tau(\beta)$; $\bar\kappa < \tau$; $\bar\kappa < \widetilde\psi\tau\beta$. Let $\tau = \widetilde{\mathsf{Z}}(r)$. Then, by 4.13, $\widetilde{C}_{\tau,\bar\kappa}(\beta) \cap [\bar\kappa,\tau) \neq \emptyset$. $\ell(\beta) \lhd \ell(\alpha)$ holds by 4.20(i). From $\beta \in \widetilde{C}_\tau(\beta)$ and $\widetilde{C}_{\tau,\bar\kappa}(\beta) \cap [\bar\kappa,\tau) \neq \emptyset$ we get $\widetilde{K}_{\tau,\bar\kappa}(\beta) < \beta$ by 4.15. Thus $\mathfrak{K}_{\ell(\tau)}(\ell(\beta)) \lhd \ell(\beta)$ by 4.20.

Suppose $r \in \mathsf{OR}^{\bar\kappa}_{m_0}$. Then $\tau^- = r^* < \bar\kappa$. So we can apply the main induction hypothesis to τ and β, giving $\mathfrak{C}_{\ell(\tau)}(\ell(\beta)) \cap [\mathcal{Z}\bar s,\ell(\tau)) \neq \emptyset$. Thus, by 4.21(9), $\Upsilon(\mathcal{Z}r)\ell(\beta) \in \mathfrak{C}^n_{\ell(\pi)}(\ell(\alpha))$. Since $\ell(\tau) = \mathcal{Z}r$ follows from $r \in \mathsf{OR}^{\bar\kappa}_{m_0}$, we get $\ell(\widetilde\psi\tau\beta) = \Upsilon(\mathcal{Z}r)\ell(\beta) \in \mathfrak{C}^n_{\ell(\pi)}(\ell(\alpha))$.

If $r \notin \mathsf{OR}^{\bar\kappa}_{m_0}$, then $\ell(r) \notin \mathsf{OR}^{\bar\kappa}_{m_0}$; hence $\ell(\widetilde\psi\tau\beta) = \Upsilon(\mathcal{Z}\ell(r))\ell(\beta) \in \mathfrak{C}^n_{\ell(\pi)}(\ell(\alpha))$ by 4.21(10).

CASE 5: $\widetilde\psi\tau\beta \in \widetilde{C}^n_{\pi,\bar\kappa}(\alpha)$; $\tau,\beta \in \widetilde{C}^{n-1}_{\pi,\bar\kappa}(\alpha)$; $\beta < \alpha$; $\beta \in \widetilde{C}_\tau(\beta)$; $\widetilde\psi\tau\beta < \bar\kappa < \tau$. Then $\ell(\tau),\ell(\beta) \in \mathfrak{C}^{n-1}_{\ell(\pi)}(\ell(\alpha))$.

Let $\tau = \widetilde{\mathsf{Z}}(x)$. As $\tau^- < \bar\kappa$, we get $x \in \mathsf{OR}^{\bar\kappa}_{m_0}$. Hence $\ell(\tau) = \mathcal{Z}x \in \mathfrak{C}^{n-1}_{\ell(\pi)}(\ell(\alpha))$. From $\widetilde\psi\tau\beta < \bar\kappa \leq \tau$ it follows $\widetilde{C}_{\tau,\bar\kappa}(\beta) \cap [\bar\kappa,\tau) = \emptyset$ and $\widetilde{C}_{\tau,\bar\kappa}(\beta) = \widetilde{C}_\tau(\beta)$ by 4.12. As $\beta < \alpha$, the main induction hypothesis yields $\ell''(\widetilde{C}_{\tau,\bar\kappa}(\beta) \cap [\bar\kappa,\tau)) = \mathfrak{C}_{\mathcal{Z}x}(\ell(\beta)) \cap [\mathcal{Z}\bar s,\mathcal{Z}x) = \emptyset$; so $\mathcal{F}_{\mathcal{Z}x}(\ell(\beta)) = \min\{\xi : \xi \notin \widetilde{C}_{\tau,\bar\kappa}(\beta)\} = \min\{\xi : \xi \notin \widetilde{C}_\tau(\beta)\} = \widetilde\psi\tau\beta$. As $\beta \in \widetilde{C}_\tau(\beta)$ implies $\beta \in \widetilde{C}_{\tau,\bar\kappa}(\beta)$, the main induction hypothesis also yields $\ell(\beta) \in \mathfrak{C}_{\mathcal{Z}x}(\ell(\beta))$. Gathering all the above information, $\widetilde\psi\tau\beta \in \mathfrak{C}^n_{\ell(\pi)}(\ell(\alpha))$ holds true by 4.21(8).

CASE 6: $\widetilde\psi\bar\kappa\beta \in \widetilde{C}^n_{\pi,\bar\kappa}(\alpha)$; $\bar\kappa,\beta \in \widetilde{C}^{n-1}_{\pi,\bar\kappa}(\alpha)$; $\beta < \alpha$; $\beta \in \widetilde{C}_{\bar\kappa}(\beta)$. Thence $\mathcal{Z}\bar s,\ell(\beta) \in \mathfrak{C}^{n-1}_{\ell(\pi)}(\ell(\alpha))$ as $\ell(\bar\kappa) = \mathcal{Z}\bar s$. Also, $\ell(\beta) \lhd \ell(\alpha)$. So, by the main induction hypothesis, $\ell(\beta) \in \mathfrak{C}_{\mathcal{Z}\bar s}(\ell(\beta))$ and $\mathcal{F}_{\mathcal{Z}\bar s}(\ell(\beta)) = \widetilde\psi\bar\kappa\beta$. Whence, $\ell(\widetilde\psi\bar\kappa\beta) = \widetilde\psi\bar\kappa\beta \in \mathfrak{C}^n_{\ell(\pi)}(\ell(\alpha))$ by 4.21(7).

CASE 7: $\widetilde{\psi}\tau\beta\in\widetilde{C}^n_{\pi,\bar\kappa}(\alpha)$; $\tau,\beta\in\widetilde{C}^{n-1}_{\pi,\bar\kappa}(\alpha)$; $\beta<\alpha$; $\beta\in\widetilde{C}_\tau(\beta)$; $\tau<\bar\kappa$. Then $\ell(\tau)=\tau\in\mathfrak{C}^{n-1}_{\ell(\pi)}(\ell(\alpha))$. As $\widetilde{\psi}\tau\beta<\tau$, $\ell(\widetilde{\psi}\tau\beta)=\widetilde{\psi}\tau\beta\in\mathfrak{C}^n_{\ell(\pi)}(\ell(\alpha))$ by 4.21(11).

CASE 8: $\xi\in\widetilde{C}^n_{\pi,\bar\kappa}(\alpha)$ and $\xi<\delta$ for some $\delta\in\widetilde{C}^{n-1}_{\pi,\bar\kappa}(\alpha)\cap\bar\kappa$. Then $\ell(\delta)=\delta\in\mathfrak{C}^{n-1}_{\ell(\pi)}(\ell(\alpha))$; thus $\ell(\xi)=\xi\in\mathfrak{C}^n_{\ell(\pi)}(\ell(\alpha))$ by 4.21(11).

This concludes the proof of "$\subseteq$". Next, we will be establishing "$\supseteq$".

CASE 9: $\delta=_{NF}\omega^{\delta_0}+\delta_1\in\mathfrak{C}^n_{\ell(\pi)}(\ell(\alpha))$; $\delta_0,\delta_1\in\mathfrak{C}^{n-1}_{\ell(\pi)}(\ell(\alpha))$. As only ordinals get sent to ordinals via ℓ, we must have $\delta_0,\delta_1\in\widetilde{C}^{n-1}_{\pi,\bar\kappa}(\alpha)$; thus $\delta\in\widetilde{C}^{n-1}_{\pi,\bar\kappa}(\alpha)$. Since $\delta<\bar\kappa$ (note that $\mathfrak{C}^{n-1}_{\ell(\pi)}(\ell(\alpha))\in\mathsf{L}_{\bar\kappa}$), we get $\ell(\delta)=\delta$.

CASE 10: $s=\tilde\omega^{s_0}\oplus s_1\in\mathfrak{C}^n_{\ell(\pi)}(\ell(\alpha))$; $s\in\mathsf{PN}$; $s_0\lhd\widetilde{\mathsf{M}}(m_0-1)$; $s_0,s_1\in\mathfrak{C}^{n-1}_{\ell(\pi)}(\ell(\alpha))$. Then $s_0=\ell(x_0)$ and $s_1=\ell(x_1)$ for some $x_0,x_1\in\widetilde{C}^{n-1}_{\pi,\bar\kappa}(\alpha)$. From 4.20 we then glean $\tilde\omega^{x_0}\oplus x_1\in\widetilde{C}^n_{\pi,\bar\kappa}(\alpha)$. Furthermore, $\ell(\tilde\omega^{x_0}\oplus x_1)=\tilde\omega^{s_0}\oplus s_1$.

CASE 11: $s=\hat\omega^{s_0}\boxplus s_1\in\mathfrak{C}^n_{\ell(\pi)}(\ell(\alpha))$; $s\in\mathsf{PN}$; $s_0,s_1\in\mathfrak{C}^{n-1}_{\ell(\pi)}(\ell(\alpha))$. $s\in\mathsf{PN}$ comprises $\neg\mathsf{Ord}(s_0)$ and $s_0\lhd\widetilde{\mathsf{M}}$. So, by 4.20, $s_0=\ell(\delta_0)$ and $s_1=\ell(\delta_1)$ for ordinals $\delta_0,\delta_1\in\widetilde{C}^{n-1}_{\pi,\bar\kappa}(\alpha)$ with $\bar\kappa\leq\delta_0$. Let $\delta=\omega^{\delta_0}+\delta_1$. Then $\delta=_{NF}\omega^{\delta_0}+\delta_1\in\widetilde{C}^n_{\pi,\bar\kappa}(\alpha)$ and $\ell(\delta)=s$.

CASE 12: $\widetilde{\mathsf{Z}}^{\bar\kappa}(x)\in\mathfrak{C}^n_{\ell(\pi)}(\ell(\alpha))$, $x\in\mathfrak{C}^{n-1}_{\ell(\pi)}(\ell(\alpha))$, $x\in\mathsf{OR}^{\bar\kappa}_{m_0}$, and $\widetilde{\mathsf{Z}}(x)<\bar\kappa$. Then $\ell(r)=x$ for some $r\in\widetilde{C}^{n-1}_{\pi,\bar\kappa}(\alpha)$. But $x\in\mathsf{OR}^{\bar\kappa}_{m_0}$ forces $r=x$. Hence $\widetilde{\mathsf{Z}}(x)\in\widetilde{C}^n_{\pi,\bar\kappa}(\alpha)$. From $\widetilde{\mathsf{Z}}(x)<\bar\kappa$ we conclude $\ell(\widetilde{\mathsf{Z}}(x))=\widetilde{\mathsf{Z}}(x)=\widetilde{\mathsf{Z}}^{\bar\kappa}(x)$.

CASE 13: $\mathcal{Z}x\in\mathfrak{C}^n_{\ell(\pi)}(\ell(\alpha))$, $x\in\mathfrak{C}^{n-1}_{\ell(\pi)}(\ell(\alpha))$, and $x\notin\mathsf{OR}^{\bar\kappa}_{m_0}\vee\bar\kappa\leq\widetilde{\mathsf{Z}}(x)$. Choose $r\in\widetilde{C}^{n-1}_{\pi,\bar\kappa}(\alpha)$ such that $\ell(r)=x$. Then $\widetilde{\mathsf{Z}}(r)\in\widetilde{C}^n_{\pi,\bar\kappa}(\alpha)$.

If $x\notin\mathsf{OR}^{\bar\kappa}_{m_0}$, then $r\notin\mathsf{OR}^{\bar\kappa}_{m_0}$, and therefore $\bar\kappa\leq r^*$, yielding $\bar\kappa<\widetilde{\mathsf{Z}}(r)$. Hence $\ell(\widetilde{\mathsf{Z}}(r))=\mathcal{Z}\ell(r)=\mathcal{Z}x$.

If $x\in\mathsf{OR}^{\bar\kappa}_{m_0}$, then $\bar\kappa\leq\widetilde{\mathsf{Z}}(x)$ and $r=x$; thus $\ell(\widetilde{\mathsf{Z}}(r))=\ell(\widetilde{\mathsf{Z}}(x))=\mathcal{Z}x$.

CASE 14: $\mathcal{F}_{\mathcal{Z}\bar s}(y)\in\mathfrak{C}^n_{\ell(\pi)}(\ell(\alpha))$; $y\lhd\ell(\alpha)$; $y\in\mathfrak{C}_{\mathcal{Z}\bar s}(y)$; $y,\mathcal{Z}\bar s\in\mathfrak{C}^{n-1}_{\ell(\pi)}(\ell(\alpha))$. Then $y=\ell(\beta)$ for some $\beta\in\widetilde{C}^{n-1}_{\pi,\bar\kappa}(\alpha)$. $\bar\kappa\in\widetilde{C}^{n-1}_{\pi,\bar\kappa}(\alpha)$ follows from $\ell(\bar\kappa)=\mathcal{Z}\bar s$. $\ell(\beta)\lhd\ell(\alpha)$ implies $\beta<\alpha$. So, by employing the main induction hypothesis, $\mathcal{F}_{\mathcal{Z}\bar s}(\ell(\beta))=\widetilde{\psi}\bar\kappa\beta$ and $\beta\in\widetilde{C}_{\bar\kappa}(\beta)$ (since $\ell(\beta)\in\mathfrak{C}_{\mathcal{Z}\bar s}(\ell(\beta))$). Hence $\widetilde{\psi}\bar\kappa\beta\in\widetilde{C}^n_{\pi,\bar\kappa}(\alpha)$.

CASE 15: $\mathcal{F}_{\mathcal{Z}x}(y)\in\mathfrak{C}^n_{\ell(\pi)}(\ell(\alpha))$; $x\in\mathsf{OR}^{\bar\kappa}_{m_0}$; $\bar\kappa<\widetilde{\mathsf{Z}}(x)$; $y\lhd\ell(\alpha)$; $y\in\mathfrak{C}_{\mathcal{Z}x}(y)$; $\mathfrak{C}_{\mathcal{Z}x}(y)\cap[\mathcal{Z}\bar s,\mathcal{Z}x)=\emptyset$; $\mathcal{Z}x,y\in\mathfrak{C}^{n-1}_{\ell(\pi)}(\ell(\alpha))$. Then $\ell(\tau)=\mathcal{Z}x$ and $\ell(\beta)=y$ for some $\tau,\beta\in\widetilde{C}^{n-1}_{\pi,\bar\kappa}(\alpha)$. However, since $x\in\mathsf{OR}^{\bar\kappa}_{m_0}$, we must have $\tau=\widetilde{\mathsf{Z}}(x)$. $\ell(\beta)\lhd\ell(\alpha)$ implies $\beta<\alpha$.

¿From $x\in\mathsf{OR}^{\bar\kappa}_{m_0}$ it follows $\tau^-<\bar\kappa$. Since $\tau^-<\bar\kappa<\tau$ and $\beta<\alpha$, the main induction hypothesis applies, providing us with $\mathcal{F}_{\mathcal{Z}x}(y)=\min\{\xi:\xi\notin\widetilde{C}_{\tau,\bar\kappa}(\beta)\}$ and $\widetilde{C}_{\tau,\bar\kappa}(\beta)\cap[\bar\kappa,\tau)=\emptyset$. Applying 4.12, we then get $\widetilde{C}_\tau(\beta)=\widetilde{C}_{\tau,\bar\kappa}(\beta)$, showing $\widetilde{\psi}\tau\beta=\mathcal{F}_{\mathcal{Z}x}(y)$.

From $\ell(\beta)\in\mathfrak{C}_{\ell(\tau)}(\ell(\beta))$ we obtain $\beta\in\widetilde{C}_{\tau,\bar\kappa}(\beta)$ by the main induction hypothesis; whence $\beta\in\widetilde{C}_{\tau}(\beta)$. This establishes $\widetilde{\psi}\tau\beta\in\widetilde{C}^n_{\pi,\bar\kappa}(\alpha)$. Furthermore, $\ell(\widetilde{\psi}\tau\beta)=\widetilde{\psi}\tau\beta$ as $\widetilde{\psi}\tau\beta<\bar\kappa$ by 4.13(i).

CASE 16: $\Upsilon(\mathcal{Z}x)y\in\mathfrak{C}^n_{\ell(\pi)}(\ell(\alpha))$; $x\in\mathsf{OR}^{\bar\kappa}$; $\bar\kappa<\widetilde{\mathsf{Z}}(x)$; $y\lhd\ell(\alpha)$; $\mathfrak{K}_{\mathcal{Z}x}(y)\lhd y$; $\mathfrak{C}_{\mathcal{Z}x}(y)\cap[\mathcal{Z}\bar s,\mathcal{Z}x)\neq\emptyset$; $\mathcal{Z}x,y\in\mathfrak{C}^{n-1}_{\ell(\pi)}(\ell(\alpha))$. Let τ and β be defined as in the previous case. We now get $\widetilde{C}_{\tau,\bar\kappa}(\beta)\cap[\bar\kappa,\tau)\neq\emptyset$ from the main induction hypothesis, yielding $\widetilde{\psi}\tau\beta>\bar\kappa$ by 4.13(ii). ¿From $\mathfrak{K}_{\mathcal{Z}x}(y)\lhd y$ we obtain $\widetilde{K}_{\tau,\bar\kappa}(\beta)<\beta$ (using 4.20(ii)), and hence $\beta\in\widetilde{C}_{\tau}(\beta)$ by 4.15. This shows $\widetilde{\psi}\tau\beta\in\widetilde{C}^n_{\pi,\bar\kappa}(\alpha)$. Finally, $\ell(\widetilde{\psi}\tau\beta)=\Upsilon\ell(\tau)\ell(\beta)=\Upsilon(\mathcal{Z}x)y$.

CASE 17: $\Upsilon(\mathcal{Z}x)y\in\mathfrak{C}^n_{\ell(\pi)}(\ell(\alpha))$; $x\notin\mathsf{OR}^{\bar\kappa}$; $y\lhd\ell(\alpha)$; $\mathfrak{K}_{\mathcal{Z}x}(y)\lhd y$; $\mathcal{Z}x,y\in\mathfrak{C}^{n-1}_{\ell(\pi)}(\ell(\alpha))$. Then $\mathcal{Z}x=\ell(\tau)$ and $y=\ell(\beta)$ for some $\tau,\beta\in\widetilde{C}^{n-1}_{\pi,\bar\kappa}(\alpha)$. As $x\notin\mathsf{OR}^{\bar\kappa}_{m_0}$ forces $\delta>\tau^-\geq\bar\kappa$, we get $\widetilde{C}_{\tau,\bar\kappa}(\beta)\cap[\bar\kappa,\tau)\neq\emptyset$. Therefore reasoning like in Case 16 confirms that $\widetilde{\psi}\tau\beta\in\widetilde{C}^n_{\pi,\bar\kappa}(\alpha)$ and $\ell(\delta)=\Upsilon(\mathcal{Z}x)y$.

CASE 18: $\zeta\in\mathfrak{C}^n_{\ell(\pi)}(\ell(\alpha))$ and $\zeta<\eta$ for some $\eta\in\mathfrak{C}^{n-1}_{\ell(\pi)}(\ell(\alpha))$. Then $\eta=\ell(t)$ for some $t\in\widetilde{C}^{n-1}_{\pi,\bar\kappa}(\alpha)$. However, by definition of ℓ, this requires $\eta=t$. Since $\eta<\bar\kappa$, $\eta\in\widetilde{C}^{n-1}_{\pi,\bar\kappa}(\alpha)\cap\bar\kappa$; thence $\ell(\zeta)=\zeta\in\widetilde{C}^n_{\pi,\bar\kappa}(\alpha)$.

This concludes the proof of (+). Using induction on n, we then get $\ell''\widetilde{C}_{\pi,\bar\kappa}(\alpha)=\mathfrak{C}_{\ell(\pi)}(\ell(\alpha))\in\mathsf{L}_{\bar\kappa}$, confirming $\mathcal{F}_{\ell(\pi)}(\ell(\alpha))=\min\{\xi:\xi\notin\widetilde{C}_{\pi,\bar\kappa}(\alpha)\}$ because of $\ell''(\widetilde{C}_{\pi,\bar\kappa}(\alpha)\cap\bar\kappa)=\widetilde{C}_{\pi,\bar\kappa}(\alpha)$. □

Corollary 4.23 $\widetilde{\psi}\bar\kappa\bar\sigma<\bar\kappa$.

Proof. By assumption (A1), we have $\bar\sigma\in\widetilde{C}_{\bar\kappa}(\bar\sigma)$. Hence $\widetilde{\psi}\bar\kappa\bar\sigma=\mathcal{F}_{\ell(\bar\kappa)}(\ell(\bar\sigma))\in\mathsf{L}_{\bar\kappa}$ by 4.22. □

Theorem 4.24 **KPM** $\vdash\ \forall\sigma\,\forall\kappa\in\widetilde{\mathsf{R}}_{m_0}\,(\sigma\in\widetilde{C}_{\kappa}(\sigma)\ \rightarrow\ \widetilde{\psi}\kappa\sigma<\kappa)$.

Proof. In this Section, arguing in **KPM** , it was shown that for all ordinals σ,

$$\forall\kappa\in\widetilde{\mathsf{R}}_{m_0}\,[\forall\alpha<\sigma\,\forall\pi\in\widetilde{\mathsf{R}}_{m_0}\,(\,\alpha\in\widetilde{C}_{\pi}(\alpha)\rightarrow\widetilde{\psi}\pi\alpha<\pi)\ \wedge\ \sigma\in\widetilde{C}_{\kappa}(\sigma)\ \rightarrow\ \widetilde{\psi}\kappa\sigma<\kappa\,].$$

So the assertion follows by transfinite induction on σ. □

Corollary 4.25 *Lemma 4.8 and Lemma 4.10 are true for all α and β.*

5 Lower bounds for |KPM|

Let $\mathsf{T}_m(\mathsf{M})$ arise from the ordinal notation system $\mathsf{T}(\mathsf{M})$ of [R 90], 6.1 by requiring in clause (T4) that $\gamma < \mathsf{M}(m)$, where $\mathsf{M}(0) = \mathsf{M} + 1$ and $\mathsf{M}(k+1) = \omega^{\mathsf{M}(k)}$. We want to show that, for $m < m_0$, $\mathsf{T}_m(\mathsf{M})$ can be embedded into $\widetilde{C}_{\widetilde{\mathsf{Z}}(0)}(\widetilde{\mathsf{Z}}(\widetilde{\mathsf{M}}(m)))$. To this end we will need some auxiliary functions which take care of the functions φ, Φ, and χ of $\mathsf{T}(\mathsf{M})$.

We shall assume that $m \geq 2$. We also adopt the notations from [R 90].

Definition 5.1 $\tilde{+}$ and $\dot{\omega}^s$ were defined at the beginning of the proof of 3.2. For $n > 0$ set $\widetilde{\mathsf{M}}{\cdot}n = \widetilde{\mathsf{M}} \tilde{+} \cdots \tilde{+} \widetilde{\mathsf{M}}$, where the sum is taken n times. Put $\widetilde{\mathsf{M}}^n = \dot{\omega}^{\widetilde{\mathsf{M}}\cdot n}$.

For $s \in \mathsf{OR}$, $\widetilde{\mathsf{M}}^n \!\cdot s$ is defined by recursion on s as follows. $\widetilde{\mathsf{M}}^n \!\cdot \alpha = \dot{\omega}^{\widetilde{\mathsf{M}}\cdot n}\alpha$ (cf. proof of 3.2); $\widetilde{\mathsf{M}}^n \!\cdot \widetilde{\mathsf{M}} = \widetilde{\mathsf{M}}^{n+1}$; $\widetilde{\mathsf{M}}^n \!\cdot (\tilde{\omega}^{s_0} \oplus s_1) = \dot{\omega}^{\widetilde{\mathsf{M}}\cdot n \tilde{+} s_0} \tilde{+} \widetilde{\mathsf{M}}^n \!\cdot s_1$.

1. $\widetilde{\varphi}0\beta = \omega^\beta$. For $\alpha > 0$, let $\widetilde{\varphi}\alpha\beta = \widetilde{\mathsf{Z}}(\widetilde{\mathsf{M}}{\cdot}\alpha \tilde{+} \beta)$.
2. For $\alpha > 0$ set $\widetilde{\Phi}\alpha\beta = \widetilde{\mathsf{Z}}(\widetilde{\mathsf{M}}^3 \!\cdot \alpha \tilde{+} \beta)$.
3. $\widetilde{\chi}0\alpha = \widetilde{\mathsf{Z}}(\widetilde{\mathsf{M}}^2 \!\cdot (1+\alpha))$. For $s \in \mathsf{OR}_{m_0}$ with $s \neq 0$, set $\widetilde{\chi}s\alpha = \widetilde{\mathsf{Z}}(\widetilde{\mathsf{M}}^4 \!\cdot s \tilde{+} \alpha)$.

Lemma 5.2 $\widetilde{C}_{\widetilde{\mathsf{Z}}(0)}(\widetilde{\mathsf{Z}}(\widetilde{\mathsf{M}}(m)))$ *is closed under* $\widetilde{\varphi}, \widetilde{\Phi}$, *and* $\widetilde{\chi}$.

Proof. If $s \sqsubset \widetilde{\mathsf{M}}(m)$ then $\widetilde{\mathsf{M}}^n \!\cdot s \tilde{+} \alpha \sqsubset \widetilde{\mathsf{M}}(m)$ since $m \geq 2$. □

Lemma 5.3 (i) If $f = \widetilde{\varphi}$ or $f = \widetilde{\Phi}$, then

$$\begin{aligned} f\alpha\beta < f\alpha'\beta' \quad \iff \quad & (\alpha < \alpha' \wedge \beta < f\alpha'\beta') \\ & \vee (\alpha = \alpha' \wedge \beta < \beta') \\ & \vee (\alpha' < \alpha \wedge f\alpha\beta \leq \beta'). \end{aligned}$$

(ii)

$$\begin{aligned} \widetilde{\chi}s\alpha < \widetilde{\chi}s'\alpha' \quad \iff \quad & (s \sqsubset s' \wedge \alpha < \widetilde{\chi}s'\alpha' \wedge s^* < \widetilde{\chi}s'\alpha') \\ & \vee (s = s' \wedge \alpha < \alpha') \\ & \vee (s' \sqsubset s \wedge (\widetilde{\chi}s\alpha \leq \alpha' \vee \widetilde{\chi}s\alpha \leq (s')^*)). \end{aligned}$$

(iii) $\gamma, \delta < \widetilde{\chi}0\beta \implies \widetilde{\varphi}\gamma\delta < \widetilde{\chi}0\beta$.

(iv) $\alpha > 0 \wedge \gamma, \delta < \widetilde{\Phi}\alpha\beta \implies \widetilde{\varphi}\gamma\delta < \widetilde{\Phi}\alpha\beta \wedge \widetilde{\chi}0\delta < \widetilde{\Phi}\alpha\beta$.

(v) $s \neq 0 \wedge s \in \mathsf{OR}_{m_0} \wedge \gamma, \delta < \widetilde{\chi}s\alpha \implies \widetilde{\varphi}\gamma\delta < \widetilde{\chi}s\alpha \wedge \widetilde{\Phi}\gamma\delta < \widetilde{\chi}s\alpha$.

Proof. Notice that $(\widetilde{\mathsf{M}}^n \!\cdot s \tilde{+} \alpha)^* = \max(s^*, \alpha)$. Therefore (i) through (v) issue from 4.3. □

In the next definition we shall deviate from our convention that lower case Greek letters always range over ordinals in that we also use those to range over elements of the notation system $\mathsf{T}(\mathsf{M})$. But no confusion is to be feared.

Definition 5.4 $\jmath : \mathsf{T}_m(\mathsf{M}) \longrightarrow \widetilde{C}_{\widetilde{\mathsf{Z}}(0)}(\widetilde{\mathsf{Z}}(\widetilde{\mathsf{M}}(m)))$ is defined as follows.[11]

1. $\jmath(0) = 0$ and $\jmath(\mathsf{M}) = \widetilde{\mathsf{M}}$.
2. If $\gamma =_{NF} \omega^{\gamma_0} + \gamma_1$, set $\jmath(\gamma) = \omega^{\jmath(\gamma_0)} + \jmath(\gamma_1)$ if $\gamma < \mathsf{M}$, and $\jmath(\gamma) = \tilde{\omega}^{\jmath(\gamma_0)} \oplus \jmath(\gamma_1)$ if $\gamma > \mathsf{M}$.
3. If $\gamma =_{NF} \varphi\alpha\beta$, then $\jmath(\gamma) = \widetilde{\varphi}\jmath(\gamma)\jmath(\beta)$.
4. If $\gamma =_{NF} \Phi\alpha\beta$, then $\jmath(\gamma) = \widetilde{\Phi}\jmath(\alpha)\jmath(\beta)$.
5. If $\gamma =_{NF} \chi\alpha\beta$, then $\jmath(\gamma) = \widetilde{\chi}\jmath(\alpha)\jmath(\beta)$.
6. If $\gamma =_{NF} \psi\kappa\alpha$, then $\jmath(\gamma) = \widetilde{\psi}\jmath(\kappa)\jmath(\alpha)$.

Lemma 5.5 *$\jmath$ preserves the respective orderings.*

Proof. By induction on $\mathrm{G}\alpha + \mathrm{G}\beta$ one verifies

$$(*) \quad \alpha, \beta \in \mathsf{T}_m(\mathsf{M}) \wedge \alpha < \beta \implies \jmath(\alpha) \sqsubset \jmath(\beta).$$

The punch–line of 4.25 and 5.3 is that $\widetilde{\psi}$, $\widetilde{\varphi}$, $\widetilde{\Phi}$, and $\widetilde{\chi}$ obey the same recursive order relations as the functions ψ, φ, Φ, and χ of [R 90], respectively. This establishes $(*)$. However, the checking of the many cases is best done on scratch paper. □

Corollary 5.6 *For all (meta) m,* **KPM** $\vdash$ "$\langle \mathsf{T}_m(\mathsf{M}), < \rangle$ is well–founded."

Proof. For $m < m_0$ this follows from 5.5. Though m_0 was kept fixed throughout the paper, it is yet an arbitrary (meta) natural number, confirming the statement. □

Lemma 5.7 $\{\alpha \in \mathsf{T}(\mathsf{M}) : \alpha < \psi(\chi 00)(\chi\mathsf{M}(m)0)\} \subseteq \mathsf{T}_{m+1}(\mathsf{M})$.

Proof. Set $\Omega_1 = \chi 00$. If $\alpha < \psi\Omega_1(\chi\mathsf{M}(m)0)$, then $\mathrm{K}_{\Omega_1}\alpha < \chi\mathsf{M}(m)0$ by [R 90], 7.2. So the Lemma will follow from

$$(*) \quad \{\beta\} \cup \mathrm{K}_{\Omega_1}\beta < \chi\mathsf{M}(m)0 \implies \beta \in \mathsf{T}_{m+1}(\mathsf{M}).$$

The proof of $(*)$ is by induction on $\mathrm{G}\beta$. Suppose $\{\beta\} \cup \mathrm{K}_{\Omega_1}\beta < \chi\mathsf{M}(m)0$.

CASE 1: $\beta =_{NF} \chi\beta_0\beta_1$. Then $\beta_0 < \mathsf{M}(m)$. Furthermore, $\mathrm{SC}_{\mathsf{M}}(\beta_0) \cup \{\beta_1\} < \chi\mathsf{M}(m)0$ and $\bigcup\{\mathrm{K}_{\Omega_1}\delta : \delta \in \mathrm{SC}_{\mathsf{M}}(\beta_0)\} \cup \mathrm{K}_{\Omega_1}\beta_1 < \chi\mathsf{M}(m)0$ by definition of K_{Ω_1} and SC_{M}. Also, for all $\delta \in \mathrm{SC}_{\mathsf{M}}(\beta_0)\}$, $\mathrm{G}\delta < \mathrm{G}\beta$. Hence the induction hypothesis yields $\mathrm{SC}_{\mathsf{M}}(\beta_0) \cup \{\beta_1\} \subseteq \mathsf{T}_{m+1}(\mathsf{M})$. This implies $\beta_0, \beta_1 \in \mathsf{T}_{m+1}(\mathsf{M})$; whence $\beta \in \mathsf{T}_{m+1}(\mathsf{M})$ as $\beta_0 < \mathsf{M}(m)$.

CASE 2: $\beta =_{NF} \psi\pi\eta$. Then $\mathrm{K}_{\Omega_1}\beta = \mathrm{K}_{\Omega_1}\pi \cup \mathrm{K}_{\Omega_1}\eta \cup \{\eta\} < \chi\mathsf{M}(m)0$. Thus $\eta \in \mathsf{T}_{m+1}(\mathsf{M})$ by induction hypothesis.

CASE 2.1: $\pi < \chi\mathsf{M}(m)0$. Then the induction hypothesis yields $\pi < \chi\mathsf{M}(m)0$. Hence $\beta \in \mathsf{T}_{m+1}(\mathsf{M})$.

[11] $=_{NF}$ with regard to terms of $\mathsf{T}(\mathsf{M})$ is defined in [R 90].

CASE 2.2: $\chi\mathsf{M}(m)0 \leq \pi$. $\chi\mathsf{M}(m)0 < \pi$ would yield the contradiction $\chi\mathsf{M}(m)0 < \beta$. Thus $\chi\mathsf{M}(m)0 = \pi$. Since $\chi\mathsf{M}(m)0 \in \mathsf{T}_{m+1}(\mathsf{M})$, clearly $\beta \in \mathsf{T}_{m+1}(\mathsf{M})$.

The remaining Cases follow immediately by employing the induction hypothesis. □

Corollary 5.8 $|\mathbf{KPM}| = \psi(\chi 00)(\psi(\chi\varepsilon_{\mathsf{M}+1}0)0)$.

Proof. We have $\sup_{n<\omega}\chi\mathsf{M}(n)0 = \psi(\chi\varepsilon_{\mathsf{M}+1}0)0$; and therefore

$$\psi(\chi 00)(\psi(\chi\varepsilon_{\mathsf{M}+1}0)0) = \sup_{n<\omega}\psi(\chi 00)(\chi\mathsf{M}(n)0).$$

Now, by 5.6, $\mathbf{KPM} \vdash$ "$\langle \mathsf{T}_{m+1}(\mathsf{M}), <\rangle$ *is well–founded*" for all m. Therefore, by 5.7, **KPM** proves that the initial segment of $\mathsf{T}(\mathsf{M})$ determined by $\psi(\chi 00)(\chi\mathsf{M}(m)0)$ is well–founded. This gives "$\geq$". "$\leq$" follows from [R 91], Theorem 7.15. □

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