The strength of Martin-Löf type theory with a superuniverse. Part I

Michael Rathjen*
School of Mathematics, University of Leeds, Leeds LS2 9JT, UK

Abstract

Universes of types were introduced into constructive type theory by Martin-Löf [12]. The idea of forming universes in type theory is to introduce a universe as a set closed under a certain specified ensemble of set constructors, say $\mathcal{C}$. The universe then "reflects" $\mathcal{C}$.

This is the first part of a paper which addresses the exact logical strength of a particular such universe construction, the so-called *superuniverse* due to Palmgren (cf. [16, 18, 19]).

It is proved that Martin-Löf type theory with a superuniverse, termed **MLS**, is a system whose proof-theoretic ordinal resides strictly above the Feferman-Schütte ordinal Γ_0 but well below the Bachmann-Howard ordinal. Not many theories of strength between Γ_0 and the Bachmann-Howard ordinal have arisen. **MLS** provides a natural example for such a theory.

1 Introduction

Martin-Löf, in 1975 [12] and in his 1984 monograph [13] on an intuitionistic theory of types, gave a framework for a theory of constructive types or sets. The role of universes in this type theory is to allow for the formation of sets of sets which are themselves closed under certain set forming operations that have been introduced at 'earlier' stages, and thereby reflect these earlier stages. In this regard universes in constructive type theory resemble the introduction of reflection cardinals in classical set theory.

Universes have greatly contributed to expanding the realm of constructivism, not least through being an efficacious tool for the development of mathematics in constructive type theory (cf. [15, 19]). An important usage of universes is linked to building sets of propositions and thus defining propositional functions, e.g. for proving the fourth Peano axiom (cf. [13, 27]). In constructive category theory type universes serve as Grothendieck universes of sets in dealing with categories. Another example is Aczel's universe of iterative sets which enabled him to interpret a certain fragment of intuitionistic Zermelo-Fraenkel set theory in constructive type theory (cf. [2, 3, 4]).

In [12, 13] Martin-Löf considered an infinite, externally indexed tower of universes $\mathbf{U}_0 \in \mathbf{U}_1 \in \cdots \in \mathbf{U}_n \in \ldots$ all of which are closed under the same standard ensemble of set forming operations. The next natural step was to implement a *universe operator* into type theory which takes a family of sets and constructs a universe above it. Such a universe operator was formalized by Palmgren while working on a domain-theoretic interpretation of the logical framework with an infinite sequence of universes (cf. [18]). Aiming at extensions of type theory with more powerful axioms, Martin-Löf then suggested finding axioms for a universe $\mathbb{V}$ which itself is closed under the universe operator. The type-theoretic formalization of the pertinent rules appeared first in Griffor and Palmgren [9] and are, in their final form, due to Palmgren [16], where the universe was referred to as a *superuniverse* for intuitionistic type theory.

*Research partially supported by a Heisenberg-Fellowship of the Deutsche Forschungsgemeinschaft. This research was inspired and carried out while I was visiting Uppsala University. I would like to thank the members of the Mathematics Department of Uppsala University for their generous hospitality.

Let $\mathbf{ML}_n$ denote the system with n universes but *without* the $\mathbf{W}$-type. The first indication of the proof-theoretic strength of type theory with universes came with P. Aczel's proof (cf. [1]) that the proof-theoretic ordinal of $\mathbf{ML}_1$ is $\varphi_{\varepsilon_0}(0)$. Feferman [8] then proved *Hancock's Conjecture* about the strength of the theories $\mathbf{ML}_n$ for general n. As a result (independently proved by Aczel), $\mathbf{ML}_{<\omega} := \bigcup_{n<\omega} \mathbf{ML}_n$ has the strength of ramified analysis or the ordinal Γ_0. Later Rathjen [10] and Setzer [25] independently showed that type theory with a single universe closed under $\mathbf{W}$-types exceeds the proof-theoretic strength of **KPi** (the classical theory of one recursively inaccessible ordinal).

Yet stronger universe operators have been studied by Rathjen, Griffor, Palmgren [22] (still predicative) and Setzer [26] (impredicative).

In this paper we will be concerned with the exact logical strength of the *superuniverse*. It is proved that Martin-Löf type theory with a superuniverse, termed **MLS**, is a system whose proof-theoretic ordinal is a landmark residing strictly above the Feferman-Schütte ordinal Γ_0 but well below the Bachmann-Howard ordinal. Not many theories of strength between Γ_0 and the Bachmann-Howard ordinal have arisen. **MLS** is a natural example for such a theory. To define the ordinal of **MLS**, we recall first the Veblen-function φ which is defined by $\varphi_0(\xi) = \omega^\xi$ and, for $\alpha > 0$, φ_α enumerates the common fixed points of the functions $(\varphi_\beta)_{\beta<\alpha}$. Let Γ_α be the α^{th} ordinal $\rho > 0$ such that $\forall \delta, \xi < \rho\ \varphi_\delta(\xi) < \rho$. The function Φ is patterned after φ but the hierarchy starts with $\Phi_0(\xi) = \Gamma_\xi$ and, for $\alpha > 0$, Φ_α enumerates the common fixed points of the functions $(\Phi_\beta)_{\beta<\alpha}$. In terms of the above functions, the proof-theoretic ordinal of **MLS** is $\Phi_{\Gamma_0}(0)$.

It should be emphasized that **MLS** is a type-theory without inductive types (except for the natural numbers), i.e. there are no $\mathbf{W}$-types in **MLS**. Indeed, the technical intricacies of this paper are due to the lack of the $\mathbf{W}$-type. By combining techniques from [10] and [21], it would have been a fairly straightforward undertaking to carry out an ordinal analysis of **MLS** plus $\mathbf{W}$-type.

1.1 Outline of the paper

The paper is organized as follows:

Section 2 gives the type-theoretic rules for the universe operator $\mathbf{U}$ and the superuniverse $\mathbb{V}$ which define the theory **MLS**.

Section 3 is concerned with establishing $\Phi_{\Gamma_0}(0)$ as a lower bound for the proof-theoretic ordinal of **MLS**. To facilitate the presentation and to be able to use results from the literature, we show in a first step that any universe gives rise to a model of a particular fragment of intuitionistic second order arithmetic, dubbed **IAR** for *Intuitionistic Analysis with Replacement*. Apart from Arithmetic Comprehension, **IAR** has a schema that is reminiscent of Replacement in set theory. By iteratively building a tower of universes $\mathbb{V} = \mathbb{V}_0 \in \mathbb{V}_1 \in \ldots \in \mathbb{V}_k \in \mathbb{V}_{k+1} \ldots$ above the superuniverse $\mathbb{V}$, we then show that for any (meta) ordinal $\alpha < \Gamma_0$ and any property $\phi(n) \in \mathbb{V}_k$ $(n \in \mathbb{N})$, transfinite induction holds up to α. In the proof we heavily exploit the fact that the $\mathbb{V}_k$'s are 'models' of **IAR**.

Next, putting to use transfinite induction up to $\alpha < \Gamma_0$, we construct a sequence of sets $\mathfrak{U}_\alpha \in \mathbb{V}$, which are actually unverses when α is a successor. For limit ordinals α, the sets $\mathfrak{U}_\alpha$ are shown to be models of an intuitionistic fragment of second order arithmetic based on *Arithmetical Transfinite Recursion*, $\mathbf{ATR}^i$. The latter fact allows use to draw on well-ordering proofs given in Schütte's book [24] and eventually prove the well-foundedness of any ordinal $< \Phi_{\Gamma_0}(0)$ within **MLS**.

In Section 4 we lay out a classical set theory $\mathbf{T^S}$ and carry out a realizability interpretation of **MLS** in $\mathbf{T^S}$. On the face of it, $\mathbf{T^S}$ is a theory which has fairly strong set existence axioms. With the crucial exception of Foundation, $\mathbf{T^S}$ is a proper extension of Kripke-Platek set theory which postulates the existence of many admissible sets and even the existence of a recursively inaccessible set I, that is, I is admissible and for all $x \in I$ there exists an admissible set $y \in I$ such that $x \in y$. However, as to logical strength, $\mathbf{T^S}$ is much weaker than Kripke-Platek set theory with Infinity axiom. This weakness is due to the lack of Foundation.

To put it roughly, in the realizability interpretation admissible sets are employed to simulate the universe operator and I gets employed for simulating the superuniverse. The interpretation shows that $\mathbf{T^S}$ has at least the proof-theoretic strength of **MLS**. The investigation of the strength of **MLS** will be completed in the second part of this paper, [23], which is devoted to an ordinal analysis of $\mathbf{T^S}$, yielding $\Phi_{\Gamma_0}(0)$ as an upper bound, as well.

One reason for splitting this paper into two parts is its sheer length. But the main reason for the partition is that the two parts may address different audiences. The second part won't require any knowledge of type theory. It will be entirely devoted to the ordinal analysis of a classical theory and contains several new cut elimination theorems for infinitary calculi of ramified set theory. Therefore it should also be of interest to proof theorists, independent of the type-theoretic context.

2 Formalization of type-theoretic universes

The formalisation of universes for intuitionistic type theory we use in this section is that referred to as the *Tarski formulation* in Martin-Löf's monograph [13]. It involves the simultaneous definition of a universe **U** and of a mapping **T** which, given an element of **U**, produces the set *coded* by that element. The codes are built up using codes for ground sets as well as codes for the set constructors. The mapping **T** on codes for ground sets gives them as its values and on codes for constructors applied to other codes produces those constructors applied to corresponding sets. Thus **U** together with **T** give a family of sets.

In what follows we assume familiarity with type theory as presented in Martin-Löf's 1984 monograph [13] or in Beeson's book [7]. In particular, all the system of type theory considered here are based on the **I**-rules of [13] (which are called rules for extensional equality sets in [15]). The only deviation from [13] is that none of the systems has the type of well-founded trees (**W**-type).

2.1 The superuniverse

If $\mathbb{V}$ denotes the superuniverse noted above, then the formation rules for $\mathbb{V}$ are

$$\frac{}{\mathbb{V}\ Set} \qquad \frac{a \in \mathbb{V}}{\mathbb{S}(a)\ Set}\ .$$

The $\mathbb{V}$-introduction rules stating the closure of $\mathbb{V}$ under universe formation have the form:

$$\frac{a \in \mathbb{V} \qquad b \in \mathbb{V}\,[x \in \mathbb{S}(a)]}{\mathbf{u}(a,(x)b) \in \mathbb{V}} \qquad \frac{a \in \mathbb{V} \qquad b \in \mathbb{V}\,[x \in \mathbb{S}(a)]}{\mathbb{S}(\mathbf{u}(a,(x)b)) = \mathbf{U}(\mathbb{S}(a),(x)\mathbb{S}(b))}$$

$$\frac{a \in \mathbb{V} \qquad b \in \mathbb{V}\,[x \in \mathbb{S}(a)] \qquad c \in \mathbb{S}(\mathbf{u}(a,(x)b))}{\mathbf{t}(a,(x)b,c) \in \mathbb{V}}$$

$$\frac{a \in \mathbb{V} \qquad b \in \mathbb{V}\,[x \in \mathbb{S}(a)] \qquad c \in \mathbb{S}(\mathbf{u}(a,(x)b))}{\mathbb{S}(\mathbf{t}(a,(x)b,c)) = \mathbf{T}(\mathbb{S}(a),(x)\mathbb{S}(b),c)}\ .$$

The set $\mathbf{U}(\mathbb{S}(a),(x)\mathbb{S}(b))$ is described separately by a *module* of rules. Given an arbitrary family of sets $(A,(x)B)$ these rules state that $\mathbf{U}(A,(x)B)$ and $(v)\mathbf{T}(A,(x)B,v)$ give the Tarski formulation of a universe reflecting the usual set constructors ($\mathbb{N}$,$\mathbb{N}_0$, $\mathbb{N}_1$, $\mathbf{\Pi}$, $\mathbf{\Sigma}$, $+$, $\mathbf{I}$) while containing codes for the set A and all of the sets $B(x)$ for x in A. However, before describing the rules pertaining to the universe operator **U**, we shall state the rules for reflecting the usual set constructors in $\mathbb{V}$:

$$\frac{}{\hat{\mathbb{N}} \in \mathbb{V}} \qquad \frac{}{\hat{\mathbb{N}}_0 \in \mathbb{V}} \qquad \frac{}{\hat{\mathbb{N}}_1 \in \mathbb{V}}$$

$$\frac{}{\mathbb{S}(\hat{\mathbb{N}}) = \mathbb{N}} \qquad \frac{}{\mathbb{S}(\hat{\mathbb{N}}_0) = \mathbb{N}_0} \qquad \frac{}{\mathbb{S}(\hat{\mathbb{N}}_1) = \mathbb{N}_1}$$

$$\frac{a \in \mathbb{V} \qquad b \in \mathbb{V}\,[x \in \mathbb{S}(a)]}{\hat{\Box}(a,(x)b) \in \mathbb{V}} \qquad \frac{a \in \mathbb{V} \qquad b \in \mathbb{V}\,[x \in \mathbb{S}(a)]}{\mathbb{S}(\hat{\Box}(a,(x)b)) = \Box(\mathbb{S}(a),(x)\mathbb{S}(b))}$$

where $\Box$ is any of the constructors $\mathbf{\Pi}, \mathbf{\Sigma}$,

$$\frac{a \in \mathbb{V} \qquad b \in \mathbb{V}}{a \hat{+} b \in \mathbb{V}} \qquad \frac{a \in \mathbb{V} \qquad b \in \mathbb{V}}{\mathbb{S}(a \hat{+} b) = \mathbb{S}(a) + \mathbb{S}(b)}$$

$$\frac{a \in \mathbb{V} \qquad x \in \mathbb{S}(a) \qquad y \in \mathbb{S}(a)}{\hat{\mathbf{I}}(a,x,y) \in \mathbb{V}} \qquad \frac{a \in \mathbb{V} \qquad x \in \mathbb{S}(a) \qquad y \in \mathbb{S}(a)}{\mathbb{S}(\hat{\mathbf{I}}(a,x,y)) = \mathbf{I}(\mathbb{S}(a),x,y)} \ .$$

2.2 The universe operator

In this subsection we introduce the universe operator $\mathbf{U}$ mentioned at the end of the previous subsection. Given a family of sets

$$A : Set \qquad B(x) : Set \ \ [x \in A], \tag{1}$$

$\mathbf{U}$ and the decoding functional $\mathbf{T}$ produce a universe of sets $\mathbf{U}(A,B)$ whose decoding functional is $\mathbf{T}(A,B)$. In addition to being closed under the standard set formers $\mathbf{\Pi}, \mathbf{\Sigma}, , \ldots,$ the universe $\mathbf{U}(A,B)$ contains codes for the sets $A, B(a)$, where $a \in A$.

2.2.1 U($\mathcal{P}$)-formation

The typing of $\mathbf{U}$ and $\mathbf{T}$ is spelled out in the introduction rules. Let $\mathfrak{A}$ abbreviate the standing assumptions of (1). We shall use the abbreviation $\mathcal{P} := A, B$.

$$\frac{\mathfrak{A}}{\mathbf{U}(\mathcal{P}) : Set} \qquad \frac{\mathfrak{A} \qquad z \in \mathbf{U}(\mathcal{P})}{\mathbf{T}(\mathcal{P}, z) : Set} \ .$$

2.2.2 U($\mathcal{P}$)-introduction

We refrain from repeating the standard introduction rules for universes. That $A, B(a)$ are in $\mathbf{U}(\mathcal{P})$ (though only via codes) is expressed by:

$$\frac{\mathfrak{A}}{\star(\mathcal{P}) \in \mathbf{U}(\mathcal{P})} \qquad \frac{\mathfrak{A}}{\mathbf{T}(\mathcal{P}, \star(\mathcal{P})) = A \ : Set}$$

$$\frac{\mathfrak{A} \qquad a \in A}{\boldsymbol{\ell}(\mathcal{P}, a) \in \mathbf{U}(\mathcal{P})} \qquad \frac{\mathfrak{A} \qquad a \in A}{\mathbf{T}(\mathcal{P}, \boldsymbol{\ell}(\mathcal{P}, a)) = B(a) \ : Set} \ .$$

Thus $\star(\mathcal{P})$ is a code for A in $\mathbf{U}(\mathcal{P})$. $\boldsymbol{\ell}(\mathcal{P}, a)$ (for $a \in A$) provides a code for $B(a)$ in $\mathbf{U}(\mathcal{P})$. Furthermore, $\mathbf{U}(\mathcal{P})$ has the same rules as $\mathbb{V}$ for reflecting the constructors $\mathbb{N}$,$\mathbb{N}_0$, $\mathbb{N}_1$, $\mathbf{\Pi}$, $\mathbf{\Sigma}$, $+$, $\mathbf{I}$.

Definition 2.1 The basis of **MLS** is the type theory exhibited in [13] but without the **W**-type. In addition, **MLS** has the rules for the universe operator **U** and the superuniverse $\mathbb{V}$ given above.

MLU has the same basic type theory and additionally the rules for the universe operator but lacks the superuniverse type.

2.3 The restricted universe operator

We shall also consider a restricted version of the universe operator which only applies to families in $\mathbb{V}$. Formally this means that $\mathbf{U}(\mathcal{P})$-formation is available only in the form

$$\frac{a \in \mathbb{V} \qquad b \in \mathbb{V}\,[x \in \mathbb{S}(a)]}{\mathbf{U}(\mathbb{S}(a),(x)\mathbb{S}(b)) : Set} \quad .$$

The resulting system will be called $\mathbf{MLS}\restriction$.

3 Ordinal functions

Here we introduce the ordinal functions relevant to this paper and gather their characteristic properties.

The Veblen-function φ figures prominently in elementary proof theory (cf. [24], Sec.13).

Definition 3.1 The *Veblen-function* $\varphi\alpha\beta := \varphi_\alpha(\beta)$ is defined by transfinite recursion on α by letting $\varphi_0(\xi) := \omega^\xi$ and, for $\alpha > 0$, φ_α be the function that enumerates the class of ordinals

$$\{\gamma : \forall \xi < \alpha\,[\varphi_\xi(\gamma) = \gamma]\}.$$

Let Γ_α be the α^{th} ordinal $\rho > 0$ such that for all $\beta, \gamma < \rho$, $\varphi\beta\gamma < \rho$

Corollary 3.2 *(i)* $\varphi 0\beta = \omega^\beta$.

(ii) $\xi, \eta < \varphi\alpha\beta \Longrightarrow \xi + \eta < \varphi\alpha\beta$.

(iii) $\xi < \zeta \Longrightarrow \varphi\alpha\xi < \varphi\alpha\zeta$.

(iv) $\alpha < \beta \Longrightarrow \varphi\alpha(\varphi\beta\xi) = \varphi\beta\xi$.

The proof-theoretic ordinal of **MLS** is bigger than Γ_0, and we need another function to obtain a sufficiently large ordinal representation system.

Definition 3.3 The function $\Phi\alpha\beta := \Phi_\alpha(\beta)$ is defined by transfinite recursion on α by letting $\Phi_0(\xi) := \Gamma_\xi$ and, for $\alpha > 0$, Φ_α be the function that enumerates the class of ordinals

$$\{\eta : \ \forall \xi < \alpha[\Phi_\xi(\eta) = \eta]\}.$$

Let $\boldsymbol{\Lambda}$ denote the least ordinal $\rho > 0$ such that $\forall \alpha, \beta < \rho$, $\Phi\alpha\beta < \rho$.

Corollary 3.4 *(i)* $\Phi 0\beta = \Gamma_\beta$.

(ii) $\xi, \eta < \Phi\alpha\beta \Longrightarrow \varphi\xi\eta < \Phi\alpha\beta$.

(iii) $\xi < \zeta \Longrightarrow \Phi\alpha\xi < \Phi\alpha\zeta$.

(iv) $\alpha < \beta \Longrightarrow \Phi\alpha(\Phi\beta\xi) = \Phi\beta\xi$.

Definition 3.5 (i) $\alpha =_{NF} \omega^{\alpha_1} + \alpha_2$ if $\alpha = \omega^{\alpha_1} + \alpha_2$ and $\alpha > \alpha_1, \alpha_2$.

(ii) $\alpha =_{NF} \varphi\alpha_1\alpha_2$ if $\alpha = \varphi\alpha_1\alpha_2$ and $\alpha > \alpha_1, \alpha_2$ and $\alpha_1 > 0$.

(iii) $\alpha =_{NF} \Phi\alpha_1\alpha_2$ if $\alpha = \Phi\alpha_1\alpha_2$ and $\alpha > \alpha_1, \alpha_2$.

Lemma 3.6 *For each $0 < \alpha < \boldsymbol{\Lambda}$ there exist unique $\alpha_1, \alpha_2 < \alpha$ such that $\alpha =_{NF} \omega^{\alpha_1} + \alpha_2$ or $\alpha =_{NF} \varphi\alpha_1\alpha_2$ or $\alpha =_{NF} \Phi\alpha_1\alpha_2$. Moreover, the cases are mutually exclusive.*

Proof: This can be proved in almost the same way as for ordinals $< \Gamma_0$ (cf. [24], Ch.V). □

Definition 3.7 There is a primitive recursive ordinal representation system for ordinals $< \boldsymbol{\Lambda}$ as any such ordinal can be uniquely denoted by using the symbols $0, \varphi, \Phi$ and the $<$-comparison between the terms is primitive recursive. This is shown in detail for the ordinal representation of order-type Γ_0 in [24]. By the same token one shows this for $\boldsymbol{\Lambda}$.

Furthermore, a bijection between the ordinals $< \boldsymbol{\Lambda}$ and the set of natural numbers can be arranged. Thereby any natural number codes a unique ordinal $< \boldsymbol{\Lambda}$. To emphasize this shift of perspective, we shall write **OT** instead of $\mathbb{N}$ when we view $\mathbb{N}$ as the set of ordinals $< \boldsymbol{\Lambda}$ rather than the natural numbers.

¿From now on lower case Greek letters $\alpha, \beta, ..$ are supposed to range over ordinals $< \boldsymbol{\Lambda}$ (or more formally **OT**) and λ, λ' range over limit ordinals $< \boldsymbol{\Lambda}$.

Definition 3.8 There is a primitive recursive function $\mathbf{lth} : \mathbf{OT} \to \mathbb{N}$ which assigns a *length* to ordinals as follows:

- $\mathbf{lth}(0) = 0$
- $\mathbf{lth}(\alpha) = \max(\mathbf{lth}(\alpha_1), \mathbf{lth}(\alpha_2)) + 1$ if $\alpha =_{NF} \omega^{\alpha_1} + \alpha_2$ or $\alpha =_{NF} \varphi\alpha_1\alpha_2$ or $\alpha =_{NF} \Phi\alpha_1\alpha_2$.

4 Lower bounds

The main goal of this section is to show that $\Phi\Gamma_0 0$ is a lower bound for the proof-theoretic ordinal of **MLS**. By way of establishing the latter, we show how certain fragments of intuitionistic second order arithmetic can be modelled in **MLU** and **MLS**. These results should also be of independent interest.

4.1 Universes and fragments of second order arithmetic

Each universe in type theory gives rise to a model of a particular fragment of intuitionistic second order arithmetic whose strongest axiom is Replacement. The acronym **IAR** is supposed to stand for *Intuitionistic Analysis with Replacement.*[1]

Definition 4.1 **IAR** is a theory in the language of second order arithmetic with set variables. The language is supposed to have function symbols for all primitive recursive functions. In particular, there are constants and function symbols for 0 (zero) , S (successor), $+$ (plus), $\cdot$ (times). Further, $\langle\ ,\ \rangle$, $(\ )_0$, $(\ )_1$ are function symbols for a bijective primitive recursive pairing function and its inverses. The only predicate symbols are $=$ for equality on the natural numbers and $\in$ for elementhood. The logical rules of **IAR** are those of intuitionistic second order arithmetic. The axioms for arithmetic are the defining axioms for the primitive recursive functions.

Equality for sets will be considered a defined notion, that is to say

$$X = Y \ := \ \forall n[n \in X \leftrightarrow n \in Y].$$

$\exists! X\psi(X)$ stands for $\exists X\, [\psi(X) \wedge \forall Y\, (\psi(Y) \to X = Y)]$.

In addition to the axioms for arithmetic and the usual axioms and rules for intuitionistic second order logic, further axioms of **IAR** axioms are (the universal closures of):

[1]The referee pointed out that Aczel [1] uses a theory **IAAC** in his analysis of $\mathbf{ML}_1$, which is **IAR** without the the uniqueness assumption in the replacement axiom. Aczel attributes **IAAC** it to Martin-Löf.

1. **Induction**:
$$\phi(0) \wedge \forall n[\phi(n) \to \phi(n+1)] \to \forall n \phi(n)$$
for all formulae ϕ.

2. **Arithmetic Comprehension Schema**:
$$\exists X \forall n[n \in X \leftrightarrow \psi(x)]$$
for ψ arithmetical (parameters allowed).

3. **Replacement**:
$$\forall X[\forall n \in X \exists ! Y \phi(n, Y) \to \exists Z \forall n \in X\, \phi(n, (Z)_n)]$$
for all formulas ϕ. Here $\phi(n, (Z)_n)$ arises from $\phi(n, Z)$ by replacing each occurrence $t \in Z$ in the formula by $\langle n, t\rangle \in Z$.

Definition 4.2 In second order arithmetic relations are treated as sets of pairs. Let $k <_R n$ abbreviate $\langle k, n\rangle \in R$. $k \leq_R n$ stands for $k <_R n \ \vee \ k = n$. We say that R is a linear order, abbreviated $\mathbf{LO}(R)$, if $<_R$ is transitive and irreflexive.

The schema of transfinite induction on R, $\mathbf{TI}(R)$, consists of all formulae
$$\forall n \left[\forall k(k <_R n \to \phi(k)) \to \phi(n)\right] \to \forall n \phi(n)$$
where ϕ ranges over all formulae of second order arithmetic.

Proposition 4.3 (**IAR**) *If* $\mathbf{LO}(R)$*,* $\mathbf{TI}(R)$ *and* $\forall n \forall W \exists ! V\, \psi(n, W, V)$*, then there exists* $\bar{Y}$ *such that*
$$\forall n\, \psi(n, \bigcup\{(\bar{Y})_k : k <_R n\}, (\bar{Y})_n).$$

Proof: By induction on n along R one first shows
$$\exists ! X\, \exists Z \left[\forall m <_R n\, \psi(m, \bigcup\{(Z)_k : k <_R m\}, (Z)_m) \ \wedge\ \psi(n, \bigcup\{(Z)_m : m <_R n\}, X)\right]. \tag{2}$$
Assume the latter to be true for all $n <_R n_0$. Using Replacement, there exists Y such that for all $n <_R n_0$,
$$\exists Z \left[\forall m <_R n\, \psi(m, \bigcup\{(Z)_k : k <_R m\}, (Z)_m) \ \wedge\ \psi(n, \bigcup\{(Z)_m : m <_R n\}, (Y)_n)\right]. \tag{3}$$
For any $n <_R n_0$ and any Z satisfying $\forall m <_R n\, \psi(m, \bigcup\{(Z)_k : k <_R m\}, (Z)_m)$ one verifies by induction along R that $\forall m <_R n\ (Z)_m = (Y)_m$. Hence, in view of (3), it follows that for all $n <_R n_0$, $\forall m \leq_R n\, \psi(m, \bigcup\{(Y)_k : k <_R m\}, (Y)_m)$. Further, by assumption there exists an extensionally unique V such that $\psi(n_0, \bigcup\{(Y)_n : n <_R n_0\}, V)$. As a result, (2) holds. From (2) the existence of the desired set $\bar{Y}$ follows by Replacement using the same arguments as above. □

In the following we fix a universe $\mathbf{U}$ which may be of the form $\mathbf{U}(A, B)$ or $\mathbb{V}$. Let $\mathbf{T}_{\mathbf{U}}$ be the associated decoding functional.

Definition 4.4 (Interpretation of **IAR** in a universe)

Terms. Each function symbol for an n-place primitive recursive function is interpreted as a function $f^\diamond \in \mathbb{N} \to^n \mathbb{N}$ in type theory, where $\mathbb{N} \to^0 X := \mathbb{N}$ and $\mathbb{N} \to^{n+1} X := \mathbb{N} \to (\mathbb{N} \to^n X)$. $(\boldsymbol{\lambda} x)t$ is an abbreviation for $\boldsymbol{\lambda}((x)t)$. We define
$$\begin{aligned} s^\diamond &:= (\boldsymbol{\lambda} x)\mathbf{s}_{\mathbb{N}}(x), \\ (P^n_k)^\diamond &:= (\boldsymbol{\lambda} x_1) \cdots (\boldsymbol{\lambda} x_n) x_k, \\ (C^n_k)^\diamond &:= (\boldsymbol{\lambda} x_1) \cdots (\boldsymbol{\lambda} x_n) \mathbf{s}^k_{\mathbb{N}}(0), (\text{ where } \mathbf{s}^0_{\mathbb{N}} := \mathbf{0}; \mathbf{s}^{\mathbf{k+1}}_{\mathbb{N}} := \mathbf{ap}(\mathbf{s}_{\mathbb{N}}, \mathbf{s}^{\mathbf{k}}_{\mathbb{N}})) \end{aligned}$$

$$((\mathrm{Sub}^n_m))^\diamond[g,h_1,\ldots,h_n] \;:=\; (\boldsymbol{\lambda}x_1)\cdots(\boldsymbol{\lambda}x_n)\mathbf{ap}(g^\diamond,\mathbf{ap}(h_1^\diamond,x_1,\ldots,x_n),\ldots,\mathbf{ap}(h_m^\diamond,x_1,\ldots,x_n))$$

$$((\mathrm{Rec}_n[g,h]))^\diamond \;:=\; (\boldsymbol{\lambda}x_1)\cdots(\boldsymbol{\lambda}x_{n+1})\mathbf{R}(x_{n+1},\mathbf{ap}(g^\diamond,x_1,\ldots,x_n),(x,y)\mathbf{ap}(h^\diamond,x_1,\ldots,x_n,x,y)).$$

For terms we let $0^\diamond := \mathbf{0}$, $x^\diamond := x$ if x is a numerical variable, and if t has the form $f(t_1,\ldots,t_n)$ let

$$t^\diamond := f^\diamond(t_1^\diamond,\ldots,t_n^\diamond) := \mathbf{ap}(\ldots\mathbf{ap}(f^\diamond,t_1^\diamond),\ldots,t_n^\diamond).$$

It is obvious that we should interpret the numerical variables of **IAR** in type theory as ranging over $\mathbb{N}$. It might be less obvious how to model the set variables of **IAR**. This is done by defining the *weak power set* of $\mathbb{N}$ with regard to $\mathbf{U}$, $\mathfrak{P}_\mathbf{U}(\mathbb{N})$, as the set of of predicates on $\mathbb{N}$ with truth conditions in the universe $\mathbf{U}$, i.e. $\mathfrak{P}_\mathbf{U}(\mathbb{N}) := \mathbb{N} \to \mathbf{U}$. For $n \in \mathbb{N}$ and $X \in \mathfrak{P}_\mathbf{U}(\mathbb{N})$, membership of n in X is defined by

$$n \,\dot{\in}\, X \;:=\; \mathbf{T}_\mathbf{U}(\mathbf{ap}(X,n)).$$

In dealing with a universe $\mathbf{U}$ as a collection of propositions, we shall often slip back into a Russell-style view of $\mathbf{U}$. This is similar to not distinguishing between a formula and its Gödel number. Thus, for a proposition ψ, rather than saying that there is an $x \in \mathbf{U}$ such that $\mathbf{T}_\mathbf{U}(x) = \psi$ we shall say that ψ is in $\mathbf{U}$ (or $\psi \in \mathbf{U}$).

With the above notion of power set, *comprehension* with respect to a property $\phi(n) \in \mathbf{U}$ $(n \in \mathbb{N})$ is immediate, by letting

$$\{n \in \mathbb{N} : \phi(n)\} \;:=\; (\boldsymbol{\lambda}x)\phi(x) \in \mathfrak{P}_\mathbf{U}(\mathbb{N}). \tag{4}$$

The subset relation ($\dot{\subseteq}$) and extensional equality ($\doteq$) on $\mathfrak{P}_\mathbf{U}(\mathbb{N})$ are defined in the obvious way as follows:

$$X \,\dot{\subseteq}\, Y \;:=\; \forall n \in \mathbb{N}(n \,\dot{\in}\, X \to n \,\dot{\in}\, Y)$$

and

$$X \doteq Y \;:=\; X \,\dot{\subseteq}\, Y \wedge Y \,\dot{\subseteq}\, X.$$

If $X_i \in \mathfrak{P}_\mathbf{U}(\mathbb{N})$ $(\beth \in \mathbb{I})$ is a family of sets with $I \in \mathbf{U}$, let

$$\dot{\bigcup}\{X_i : i \in I\} \;:=\; \{n \in \mathbb{N} : \exists i \in I(n \,\dot{\in}\, X_i)\}.$$

We shall write $s =_\mathbf{N} t$ for $\mathbf{I}(\mathbb{N},s,t)$. Variables n,k,m are supposed to range over $\mathbb{N}$, thus, e.g., $\forall k$ abbreviates $\forall k \in \mathbb{N}$. $\forall n \,\dot{\in}\, X(\ldots)$ is shorthand for $\forall n \in \mathbb{N}(n \,\dot{\in}\, X \to \ldots)$.

Formulae of second order arithmetic are now translated into type expressions as follows:

$$\begin{array}{rcl}
[s = t]^\diamond & := & s^\diamond =_\mathbf{N} t^\diamond, \\
[s \in X]^\diamond & := & s^\diamond \,\dot{\in}\, X, \\
[\phi \circ \psi]^\diamond & := & \phi^\diamond \circ \psi^\diamond \text{ for } \circ \in \{\wedge,\vee,\to\}, \\
\bot^\diamond & := & \mathbf{I}(\mathbb{N},\mathbf{s}_\mathbb{N}(0),0), \\
[Qy\,\phi]^\diamond & := & (Qy \in \mathbb{N})\phi^\diamond \quad \text{for } Q \in \{\forall,\exists\}, \\
[QX\,\phi]^\diamond & := & (QX \in \mathfrak{P}_\mathbf{U}(\mathbb{N}))\phi^\diamond \quad \text{for } \mathbb{Q} \in \{\forall,\exists\}.
\end{array}$$

If $\psi[\vec{x},\vec{Y}]$ is formula of **IAR** (with all free variables shown) and $\vec{n} \in \mathbb{N}$ and $\vec{\mathfrak{Y}} \in \mathfrak{P}_\mathbf{U}(\mathbb{N})$, we shall write

$$\mathbf{U} \models \psi[\vec{n},\vec{\mathfrak{Y}}]$$

to convey that the proposition $\psi^\diamond[\vec{n},\vec{\mathfrak{Y}}]$ holds true.

Definition 4.5 *If F is a family of propositions (i.e. types) over the type A we call F a* species *over A.*

In the following deductions in type theory are presented in an informal style. The following lemma deals with the interpretation of Replacement.

Lemma 4.6 *Suppose* $A \in \mathfrak{P}_{\mathbf{U}}(\mathbb{N})$. *Let* F *be a species over* $\mathbb{N} \times \mathfrak{P}_{\mathbf{U}}(\mathbb{N})$ *such that the following propositions hold true:*

$$\forall n \,\dot{\in}\, A \; \exists X \in \mathfrak{P}_{\mathbf{U}}(\mathbb{N}) \, \mathbb{F}(\ltimes, \mathbb{X}) \tag{5}$$

$$\forall n \,\dot{\in}\, A \; \forall X, Y \in \mathfrak{P}_{\mathbf{U}}(\mathbb{N}) \, [\mathbb{X} \doteq \mathbb{Y} \wedge \mathbb{F}(\ltimes, \mathbb{X}) \;\to\; \mathbb{F}(\ltimes, \mathbb{Y})] \tag{6}$$

$$\forall n \,\dot{\in}\, A \forall X, Y \in \mathfrak{P}_{\mathbf{U}}(\mathbb{N}) \, [\mathbb{F}(\ltimes, \mathbb{X}) \wedge \mathbb{F}(\ltimes, \mathbb{Y}) \;\to \mathbb{X} \doteq \mathbb{Y}]. \tag{7}$$

Then there exists $C \in \mathfrak{P}_{\mathbf{U}}(\mathbb{N})$ *such that, with* $g(n) \;:=\; \{m \in \mathbb{N} : \exists k \,\dot{\in}\, C(\langle n, m\rangle^{\diamond} =_{\mathbf{N}} k)\}$,

$$\forall n \,\dot{\in}\, A \, F(n, g(n)) \text{ holds true.}$$

Proof: From (5) it follows $\forall n[\exists v \in A(n) \;\to\; \exists X \in \mathfrak{P}_{\mathbf{U}}(\mathbb{N}) \, \mathbb{F}(\ltimes, \mathbb{X})]$ true, thus $\forall n \forall v \in A(n) \, \exists X \in \mathfrak{P}_{\mathbf{U}}(\mathbb{N}) \, \mathbb{F}(\ltimes, \mathbb{X})$ true, and hence[2]

$$\forall z \in (\mathbf{\Sigma} v \in \mathbb{N} \, A(v)) \, \exists X \in \mathfrak{P}_{\mathbf{U}}(\mathbb{N}) \, \mathbb{F}(\mathbf{p_0}(\digamma), \mathbb{X}) \text{ true.}$$

As a result, the axiom of choice in type theory provides us with a function

$$h \in (\mathbf{\Sigma} v \in \mathbb{N} \, A(v)) \;\to\; \mathfrak{P}_{\mathbf{U}}(\mathbb{N})$$

so that

$$\forall z \in (\mathbf{\Sigma} v \in \mathbb{N} \, A(v)) \, F(\mathbf{p_0}(z), h(z)) \text{ true.}$$

Now define $g \in \mathbb{N} \to \mathfrak{P}_{\mathbf{U}}(\mathbb{N})$ by

$$g(n) \;:=\; \dot{\bigcup}\{h(z) : z \in (\mathbf{\Sigma} v \in \mathbb{N} \, A(v)) \;\wedge\; \mathbf{p_0}(z) =_{\mathbf{N}} n\}.$$

As a consequence of (7) we have

$$\forall z, z' \in (\mathbf{\Sigma} v \in \mathbb{N} \, A(v)) \, [\mathbf{p_0}(z) =_{\mathbf{N}} \mathbf{p_0}(z') \;\to\; h(z) \doteq h(z')] \text{ true.} \tag{8}$$

Now if $n \,\dot{\in}\, A$ true we may pick $z \in (\mathbf{\Sigma} v \in \mathbb{N} \, A(v))$ such that $\mathbf{p_0}(z) =_{\mathbf{N}} n$ true. Then $g(n) \doteq h(z)$ true by (8) and, as $F(n, h(z))$ true, we obtain $F(n, g(n))$ true using (6). Finally, set

$$C \;:=\; \{z \in \mathbb{N} : \exists v \in \mathbb{N} \, \exists u \,\dot{\in}\, g(v)(\langle v, u\rangle^{\diamond} =_{\mathbf{N}} z)\}.$$

□

Theorem 4.7 *Any theorem of* **IAR** *is true on interpretation in a universe* **U** *(as defined in 4.4). To be more precise, if* ϕ *is a sentence provable in* **IAR**, *then one can show in* **MLU** *as well as in* **MLS** *that for any universe* **U**,

$$\mathbf{U} \models \phi.$$

[2] $\mathbf{p_0}, \mathbf{p_1}$ are defined by $\mathbf{p_0}(c) := \mathbf{E}(c, (x, y).x)$ and $\mathbf{p_1}(c) := \mathbf{E}(c, (x, y).y)$, where $\mathbf{E}$ is the eliminatory constant related to $\mathbf{\Sigma}$.

Proof: By induction on the length of deductions in **IAR**. A natural deduction formulation of **IAR** is most convenient. That $^\diamond$ validates intuitionistic arithmetic is, for example, proved in [7], XI.17 and [28], Ch. 11, Sect. 4. It is also routine to check that $^\diamond$ validates the (intuitionistic) laws for the second order quantifiers. As to equality, note that

$$\forall u, v \in \mathbb{N} \forall X, Y \in \mathfrak{P}_{\mathbf{U}}(\mathbb{N}) \, [\approxeq =_{\mathbf{N}} \gtrapprox \; \rightarrow \; (\approxeq \dot{\in} \mathbb{X} \leftrightarrow \gtrapprox \dot{\in} \mathbb{Y})]$$

holds true by the very definitions of $=_{\mathbf{N}}$ and $\dot{\in}$. Therefore, if **IA** is obtained from **IAR** by retaining only the part without Arithmetic Comprehension and Replacement, the theorem holds with **IA** in place of **IAR**. We thus only have to check those axioms. As to Arithmetic Comprehension, note that if $\phi(x, \vec{y}, \vec{Z})$ is an arithmetic formula then, for $n, \vec{m} \in \mathbb{N}$ and $\vec{\mathfrak{X}} \in \mathfrak{P}_{\mathbf{U}}(\mathbb{N})$, $\phi(n, \vec{m}, \vec{\mathfrak{X}})^\diamond$ is a small type, that is to say $\phi(n, \vec{m}, \vec{\mathfrak{X}})^\diamond \in \mathbf{U}$; and therefore the validity of Arithmetic Comprehension is confirmed by (4). Replacement is taken care of by Lemma 4.6. $\square$

Let $(\mathbf{U}^+, \mathbf{T}^+)$ be the next universe above $(\mathbf{U}, \mathbf{T}_{\mathbf{U}})$, i.e. $\mathbf{U}^+ = \mathbf{U}(\mathbf{U}, \mathbf{T}_{\mathbf{U}})$ and $\mathbf{T}^+(x) = \mathbf{T}(\mathbf{U}, \mathbf{T}_{\mathbf{U}}, x)$.

Lemma 4.8 (**MLU**, **MLS**) *Let $\phi(n) \in \mathbf{U}^+$ for $n \in \mathbb{N}$. If $R \in \mathfrak{P}_{\mathbf{U}}(\mathbb{N})$ and $\mathbf{U}^+ \models \mathbf{WF}(R)$, then $\mathbf{U} \models \mathbf{TI}(R)$.*

Proof: To prove this we have to consider properties $\phi(n)$ $(n \in \mathbb{N})$, where $\phi(n)$ is the result of translating a second order formula with parameters from $\mathbb{N}$ and $\mathfrak{P}_{\mathbf{U}}(\mathbb{N})$ over $\mathbf{U}$. But then $\phi(n) \in \mathbf{U}^+$ and hence $\{n \in \mathbb{N} : \phi(n)\} \in \mathfrak{P}_{\mathbf{U}^+}(\mathbb{N})$. Due to the latter and $\mathbf{U}^+ \models \mathbf{WF}(R)$, we get $\mathbf{U} \models \mathbf{TI}_\phi(R)$. $\square$

4.2 $\mathbf{ATR}^i$

Definition 4.9 For a set $X \subseteq \mathbb{N}$ and $\beta \in \mathbf{OT}$ let $\beta \subseteq X \; := \; \forall \gamma < \beta \, (\gamma \in X)$.

Further define

$$\begin{aligned} \mathbf{Prog}(A, \alpha) \; &:= \; \forall \beta \leq \alpha \, (\beta \subseteq A \; \rightarrow \; \beta \in A), \\ \mathbf{WF}(\alpha) \; &:= \; \forall X \, [\mathbf{Prog}(X, \alpha) \; \rightarrow \; \forall \xi \leq \alpha \, \xi \in X]. \end{aligned}$$

For a formula $\mathcal{F}(u)$ let $\mathbf{Prog}(\mathcal{F}, \alpha)$ abbreviate

$$\forall \beta \leq \alpha \, (\forall \gamma < \beta \mathcal{F}(\gamma) \; \rightarrow \; \mathcal{F}(\beta)).$$

The schema $\mathbf{TI}(\alpha)$ consists of all formulae

$$\mathbf{Prog}(\mathcal{F}, \alpha) \; \rightarrow \; \forall \xi \leq \alpha \, \mathcal{F}(\xi)$$

for all formulae $\mathcal{F}(u)$ of second order arithmetic.

Definition 4.10 The theory $\mathbf{ATR}_0$ whose central set existence principle is *arithmetical transfinite recursion*, is one of the prominent subsystems of classical second order arithmetic singled out in the Friedman-Simpson program of *Reverse Mathematics*. Let **ATR** be the theory $\mathbf{ATR}_0$ plus the schema of induction (for natural numbers) for all formulae. We shall consider an intuitionistic version $\mathbf{ATR}^i$ whose only difference from **ATR** is that the underlying logic is intuitionistic logic instead of classical logic. The crucial axiom of **ATR** (and thus of $\mathbf{ATR}^i$) is the axiom for arithmetical transfinite recursion. For each arithmetical formula ϕ, which may contain free set and number variables, we have the principle:

$$\mathbf{WO}(R) \; \rightarrow \; \exists X \, \forall y \forall z \big[z \in (X)_y \; \leftrightarrow \; \phi(y, z, \bigcup \{(X)_x : x <_R y\}) \big], \tag{9}$$

where $\mathbf{WO}(R)$ states that R is a linear order of the natural numbers, and that R is well-founded, i.e.

$$\forall X \big[\forall x \, [\forall y \, (\langle y, x \rangle \in R \, \rightarrow \, y \in X) \, \rightarrow \, x \in X] \, \rightarrow \, \forall x \, x \in X \big].$$

For later use, we also isolate the theory $\mathbf{ACA}^i$ which is **IAR** without Replacement. Note that $\mathbf{ATR}^i$ is $\mathbf{ACA}^i$ plus the schema (9).

Lemma 4.11 $\mathbf{ATR}^i \vdash \mathbf{WF}(\alpha) \, \rightarrow \, \mathbf{WF}(\varphi\alpha 0)$.

Proof: Here we will draw on well-ordering proofs given in [24], chapter VIII. Let $\mathcal{R}[X, Y, t]$ be the formula defined in [24], VIII.5. Putting to use arithmetical transfinite recursion, one immediately obtains

$$\mathbf{ATR}^i \vdash \forall \delta \big(\mathbf{WF}(\delta) \, \rightarrow \, \forall X \exists Y \mathcal{R}[X, Y, \delta] \big). \tag{10}$$

Furthermore, [24],VIII, Lemma 10 can be proved in $\mathbf{ACA}^i$ as an inspection of its proof reveals that it only uses arithmetic comprehension, primitive recursive arithmetic and intuitionistic logic; the case distinctions made therein always concern the shape of the ordinal notations, that is decidable properties. The upshot of the preceding is that

$$\mathbf{ACA}^i \vdash \forall \delta \forall \alpha \big(\forall X \exists Y \mathcal{R}[X, Y, \delta] \, \wedge \, \omega^\alpha < \delta \, \wedge \, \mathbf{WF}(\omega^\alpha) \, \rightarrow \, \mathbf{WF}(\varphi\alpha 0) \big). \tag{11}$$

Now, $\mathbf{ACA}^i \vdash \mathbf{WF}(\alpha) \, \rightarrow \, \mathbf{WF}(\omega^\alpha + 1)$ holds by the proof of [24], VIII, Lemma 7c. Thus, letting $\delta := \omega^\alpha + 1$ in (11) and taking into account (10), one obtains $\mathbf{ATR}^i \vdash \mathbf{WF}(\alpha) \, \rightarrow \, \mathbf{WF}(\varphi\alpha 0)$. $\square$

Lemma 4.12 *For any (meta) α,*

$$\mathbf{IAR} + \mathbf{TI}(\omega^{\alpha+1}) \vdash \mathbf{WF}(\varphi\alpha 0).$$

Proof: This is similar to the previous proof. By employing Proposition 4.3, one obtains

$$\mathbf{IAR} + \mathbf{TI}(\omega^{\alpha+1}) \vdash \forall X \exists Y \mathcal{R}[X, Y, \omega^{\alpha+1}]. \tag{12}$$

The rest follows as in Lemma 4.11. $\square$

Lemma 4.13 $\mathbf{ATR}^i \vdash \forall \alpha \big(\mathbf{WF}(\Gamma_\alpha) \, \rightarrow \, \mathbf{WF}(\Gamma_{\alpha+1}) \big)$.

Proof: Assume $\mathbf{WF}(\Gamma_\alpha)$. Let $\beta < \Gamma_{\alpha+1}$. We show $\mathbf{WF}(\beta)$ by induction on the length of β (cf. 3.8); thus we use ordinary induction on the natural numbers. If $\beta =_{NF} \omega^{\beta_1} + \beta_2$, then the induction hypothesis yields $\mathbf{WF}(\beta_1)$, and hence $\mathbf{WF}(\beta_1 + 1)$. By Lemma 4.11, one gets $\mathbf{WF}(\varphi(\beta_1 + 1)0)$. As $\beta < \varphi(\beta_1 + 1)0$, it follows $\mathbf{WF}(\beta)$.

If $\beta =_{NF} \varphi\beta_1\beta_2$, then the inductive assumption provides $\mathbf{WF}(\max(\beta_1, \beta_2))$. Letting $\beta^* := \max(\beta_1, \beta_2) + 1$, it follows $\mathbf{WF}(\beta^*)$. As a result, $\mathbf{WF}(\varphi(\varphi\beta^* 0)0)$ holds by applying Lemma 4.11 twice. As $\beta_1, \beta_2 < \varphi\beta^* 0$, we get $\varphi\beta_1\beta_2 < \varphi(\varphi\beta^* 0)0$. Hence $\mathbf{WF}(\beta)$. If β is not of one of the forms considered thus far, then $\beta \leq \Gamma_\alpha$ and $\mathbf{WF}(\beta)$ holds by assumption.

Finally, from $\forall \beta < \Gamma_{\alpha+1} \, \mathbf{WF}(\beta)$ it ensues $\mathbf{WF}(\Gamma_{\alpha+1})$. $\square$

Lemma 4.14 $\mathbf{ATR}^i \vdash \mathbf{WF}(\Gamma_0)$.

Proof: Arguing in $\mathbf{ATR}^i$, one shows $\mathbf{WF}(\beta)$ for $\beta < \Gamma_0$ by induction on the length of β, similarly as in the previous proof. $\square$

Lemma 4.15 *For any (meta) $\alpha < \varepsilon_0$, $\mathbf{ATR}^i \vdash \mathbf{WF}(\Gamma_\alpha)$.*

Proof: Note for any (meta) $\alpha < \varepsilon_0$, $\mathbf{ATR}^i \vdash \mathbf{TI}(\alpha)$ (by Gentzen's proof). Thus we may use transfinite induction on $\beta \leq \alpha$ to prove $\mathbf{WF}(\Gamma_\beta)$. Inductively assume $\mathbf{WF}(\Gamma_\delta)$ for all $\delta < \beta$. If β is a limit, then the latter implies $\mathbf{WF}(\eta)$ for all $\eta < \Gamma_\beta$, and consequently $\mathbf{WF}(\Gamma_\beta)$. If $\beta = 0$, then the assertion follows from Lemma 4.14. If β is a successor, i.e. $\beta = \beta_0 + 1$ for some $\beta_0 < \beta$, then $\mathbf{WF}(\Gamma_{\beta_0})$ holds by induction hypothesis; thus $\mathbf{WF}(\Gamma_\beta)$ follows by Lemma 4.13. □

Lemma 4.16 *For any β,*

$$\mathbf{ATR}^i + \mathbf{TI}(\beta) \vdash \mathbf{WF}(\Gamma_\beta).$$

Proof: Same as for 4.15. □

4.3 A lower bound for MLU

We fix an ordinal $\alpha < \Gamma_0$ at the meta level. Our objective is to show that **MLU** proves transfinite induction up to α. Put $\rho_0 = \omega$ and $\rho_{n+1} = \varphi\rho_n 0$. Then $\alpha < \rho_n$ for some n.

We shall reason in **MLU**.

Lemma 4.17 (**MLU**) *Suppose $\mathcal{F}(x)$ is a proposition for $x \in \mathbb{N}$. Then*

$$\forall\beta \leq \alpha\,[\forall\xi < \beta\,\mathcal{F}(\xi) \;\rightarrow\; \mathcal{F}(\beta)] \;\rightarrow\; \forall\beta \leq \alpha\,\mathcal{F}(\beta)$$

holds true.

Proof: Let (for external k) $\mathbf{U}_0 = \mathbf{U}(\mathbb{N}, (x)\mathcal{F}(x))$ and $\mathbf{U}_{k+1} = \mathbf{U}_k^+$. We certainly have $\mathbf{U}_n \models \mathbf{WF}(\rho_0)$. Putting to use Lemma 4.8 and Lemma 4.12 n times, we therefore obtain $\mathbf{U}_0 \models \mathbf{WF}(\rho_n)$. As $\mathcal{F}(x) \in \mathbf{U}_0$ for $x \in \mathbb{N}$, we get $\{x \in \mathbb{N} : \mathcal{F}(x)\} \in \mathfrak{P}_{\mathbf{U}_0}(\mathbb{N})$. Thus, using $\mathbf{U}_0 \models \mathbf{WF}(\rho_n)$, it holds $\forall\beta \leq \alpha\,\mathcal{F}(\beta)$. □

Theorem 4.18 Γ_0 *is a lower bound for the proof-theoretic strength of* **MLU**.

Proof: This is immediate by Lemma 4.17. □

4.4 A hierarchy of universes

Again fix a (meta) $\alpha < \Gamma_0$. We will construct a hierarchy

$$(\mathfrak{U}_\xi, \mathfrak{T}_\xi)_{\xi<\alpha}$$

of ever larger sets $\mathfrak{U}_\xi$ with set-valued decoding functions $\mathfrak{T}_\xi$ such that $(\mathfrak{U}_\xi, \mathfrak{T}_\xi)$ is also a universe if ξ is not a limit.

Convention. From now on, we will increasingly write $f(s)$, where $\mathbf{ap}(f, s)$ would have been the correct spelling in type theory. We trust that the context will always make it clear what should have been the correct rendering.

Definition 4.19 For the main construction we consider the type of all codes for families in the superuniverse $\mathbb{V}$,

$$\mathbf{Fam}_{\mathbb{V}} \;:=\; (\boldsymbol{\Sigma} x \in \mathbb{V})[\mathbb{S}(x) \rightarrow \mathbb{V}]. \tag{13}$$

Let $\langle\cdot,\cdot\rangle$ (alias $\mathbf{E}$) denote pairing for the $\mathbf{\Sigma}$-type and $\mathbf{p}_0, \mathbf{p}_1$ be the first and second projection, respectively. Further, define $\mathbf{B}_{\mathbb{V}}(c) := \mathbb{S}(\mathbf{p}_0(c))$ for $c \in \mathbf{Fam}_{\mathbb{V}}$, the base of the family coded by c, and $\mathbf{Fam}_{\mathbb{V}}(c,x) := \mathbb{S}(\mathbf{ap}(\mathbf{p}_1(c),x))$ for $c \in \mathbf{Fam}_{\mathbb{V}}$, $x \in \mathbf{B}_{\mathbb{V}}(c)$, the family of sets over $\mathbf{B}_{\mathbb{V}}(c)$ coded by c. We shall define

$$\hat{\mathbf{u}} \in \mathbf{Fam}_{\mathbb{V}} \to \mathbf{Fam}_{\mathbb{V}}$$

such that (with $\hat{\mathbf{u}}(x) := \mathbf{ap}(\hat{\mathbf{u}}, x)$)

$$\begin{aligned} \mathbf{B}_{\mathbb{V}}(\hat{\mathbf{u}}(c)) &= \mathbf{U}(\mathbf{B}_{\mathbb{V}}(c), (x)\mathbf{Fam}_{\mathbb{V}}(c,x)); \\ \mathbf{Fam}_{\mathbb{V}}(\hat{\mathbf{u}}(c), w) &= \mathbf{T}((\mathbf{B}_{\mathbb{V}}(c), (x)\mathbf{Fam}_{\mathbb{V}}(c,x), w), \end{aligned} \tag{14}$$

by letting

$$\hat{\mathbf{u}} \ := \ (\boldsymbol{\lambda} c)\langle \mathbf{u}(\mathbf{p}_0(c), (x)\mathbf{ap}(\mathbf{p}_1(c), x)), (\boldsymbol{\lambda} w)\mathbf{t}(\mathbf{p}_0(c), (x)\mathbf{ap}(\mathbf{p}_1(c), x), w)\rangle. \tag{15}$$

Hence if c is a code for a universe, then $\hat{\mathbf{u}}(c)$ is a code for a universe above c. Next, we want to iterate the construction transfinitely.

First note that there is a primitive recursive cut-off function $f : \mathbf{OT} \times \mathbf{OT} \to \mathbf{OT}$ such that

$$f(\xi,\gamma) = \begin{cases} \gamma & \text{if } \gamma < \xi \\ 0 & \text{otherwise} \end{cases}$$

Set $f_\xi(\gamma) := f(\xi,\gamma)$. We will use $s =_{\mathbf{Fam}_{\mathbb{V}}} t$ to abbreviate $s = t \in \mathbf{Fam}_{\mathbb{V}}$.

Let A be a set and $B(v)$ $[v \in A]$ be a family of sets over A such that $A = \mathbb{S}(a)$ and $B(v) = \mathbb{S}(b(v))$, where $a \in \mathbb{V}$ and $b(v) \in \mathbb{V}$ for $v \in \mathbb{S}(a)$. Given $F \in \mathbf{OT} \to \mathbf{Fam}_{\mathbb{V}}$ let

$$\begin{aligned} \mathbf{Hier}_{A,B}(F,\gamma) \ := \ & F(0) =_{\mathbf{Fam}_{\mathbb{V}}} \langle \mathbf{u}(a,(v)b(v)), (\boldsymbol{\lambda} x)\mathbf{t}(a,(v)b(v),x)\rangle \ \wedge \\ & \forall \beta < \gamma \left[F(\beta+1) =_{\mathbf{Fam}_{\mathbb{V}}} \hat{\mathbf{u}}(F(\beta)) \ \wedge \right. \\ & \forall \lambda \le \gamma \left[F(\lambda) =_{\mathbf{Fam}_{\mathbb{V}}} \langle \hat{\mathbf{\Sigma}}(\hat{\mathbb{N}}, (v)\mathbf{p}_0(F(f_\lambda(v)))), (\boldsymbol{\lambda} x)\mathbf{ap}(\mathbf{p}_1(F(f_\lambda(\mathbf{p}_0(x)))), \mathbf{p}_1(x))\rangle \right]. \end{aligned}$$

Note that by Lemma 4.17, we can employ transfinite induction up to α.

By induction on $\gamma \le \alpha$ one readily proves that for $F, H \in \mathbf{OT} \to \mathbf{Fam}_{\mathbb{V}}$,

$$\mathbf{Hier}_{A,B}(F,\gamma) \ \wedge \ \mathbf{Hier}_{A,B}(H,\gamma) \ \to \ \forall \beta \le \gamma \, F(\beta) =_{\mathbf{Fam}_{\mathbb{V}}} H(\beta). \tag{16}$$

Further, by induction on γ, we shall prove

$$\forall \beta \le \gamma \, (\exists F \in \mathbf{OT} \to \mathbf{Fam}_{\mathbb{V}}) \, \mathbf{Hier}_{A,B}(F,\gamma). \tag{17}$$

(17) is obvious for $\gamma = 0$. So assume $\gamma > 0$. Inductively, for all $\beta < \gamma$, there exist $F_\beta \in \mathbf{OT} \to \mathbf{Fam}_{\mathbb{V}}$ such that $\mathbf{Hier}_{A,B}(F_\beta, \beta)$. Using the axiom of choice in type theory, there exists

$$\mathbb{H} \in \mathbf{OT} \to (\mathbf{OT} \to \mathbf{Fam}_{\mathbb{V}})$$

such that, letting $\mathbb{H}_\beta := \mathbb{H}(\beta)$, $\forall \beta < \gamma \, \mathbf{Hier}_{A,B}(\mathbb{H}_\beta, \beta)$. Utilizing that $<$ on $\mathbf{OT}$ is decidable, we may set $\mathbb{H}_\beta := G$ for all $\beta \ge \gamma$, where $G \in \mathbf{OT} \to \mathbf{Fam}_{\mathbb{V}}$ is fixed but arbitrary. Next define

$$\tilde{F}(\beta) := \begin{cases} \mathbb{H}_\beta(\beta) & \text{if } \beta < \gamma \\ G & \text{otherwise;} \end{cases}$$

$$F(\beta) := \begin{cases} \tilde{F}(\beta) & \text{if } \beta \ne \gamma \\ \hat{\mathbf{u}}(\tilde{F}(\gamma_0)) & \text{if } \beta = \gamma = \gamma_0 + 1 \\ \langle \hat{\mathbf{\Sigma}}(\mathbf{n}, (v)\mathbf{p}_0(\tilde{F}(f_\gamma(v)))), (\boldsymbol{\lambda} x)\mathbf{ap}(\mathbf{p}_1(\tilde{F}(f_\gamma(\mathbf{p}_0(x)))), \mathbf{p}_1(x))\rangle & \text{if } \beta = \gamma \text{ is a limit.} \end{cases}$$

Observe that for $0 < \beta < \gamma$ and any ξ we have

$$F(f_\beta(\xi)) =_{\mathbf{Fam}_{\mathbb{V}}} \tilde{F}(f_\beta(\xi)) =_{\mathbf{Fam}_{\mathbb{V}}} \mathbb{H}_{f_\beta(\xi)}(f_\beta(\xi) =_{\mathbf{Fam}_{\mathbb{V}}} \mathbb{H}_\beta(f_\beta(\xi));$$

thus $(\boldsymbol{\lambda} x)F(f_\beta(x)) = (\boldsymbol{\lambda} x)\mathbb{H}_\beta(f_\beta(x)) \in \mathbb{N} \to \mathbf{Fam}_{\mathbb{V}}$, and consequently, if $\beta = \beta_0 + 1$,

$$F(\beta) =_{\mathbf{Fam}_{\mathbb{V}}} \mathbb{H}_\beta(\beta) =_{\mathbf{Fam}_{\mathbb{V}}} \hat{\mathbf{u}}(\mathbb{H}_\beta(\beta_0)) =_{\mathbf{Fam}_{\mathbb{V}}} \hat{\mathbf{u}}(F(\beta_0)),$$

and, if β is a limit,

$$\begin{aligned} F(\beta) =_{\mathbf{Fam}_{\mathbb{V}}} \mathbb{H}_\beta(\beta) \quad &=_{\mathbf{Fam}_{\mathbb{V}}} \quad \langle \hat{\mathbf{\Sigma}}(\hat{\mathbb{N}}, (v)\mathbf{p}_0(\mathbb{H}_\beta(f_\beta(v)))), (\boldsymbol{\lambda} x)\mathbf{ap}(\mathbf{p}_1(\mathbb{H}_\beta(f_\beta(\mathbf{p}_0(x)))), \mathbf{p}_1(x))\rangle \\ &=_{\mathbf{Fam}_{\mathbb{V}}} \quad \langle \hat{\mathbf{\Sigma}}(\hat{\mathbb{N}}, (v)\mathbf{p}_0(F(f_\beta(v)))), (\boldsymbol{\lambda} x)\mathbf{ap}(\mathbf{p}_1(F(f_\beta(\mathbf{p}_0(x)))), \mathbf{p}_1(x))\rangle. \end{aligned}$$

We also have $F(0) =_{\mathbf{Fam}_{\mathbb{V}}} \langle \mathbf{u}(a, (v)b(v)), (\boldsymbol{\lambda} x)\mathbf{t}(a, (v)b(v), x)\rangle$. □

Finally set

$$\begin{aligned} \mathfrak{U}_\gamma \quad &:= \quad \mathbf{B}_{\mathbb{V}}(F(\gamma)), \qquad (18) \\ \mathfrak{T}_\gamma(x) \quad &:= \quad \mathbf{Fam}_{\mathbb{V}}(F(\gamma), x) \quad \text{for } x \in \mathfrak{U}_\gamma. \end{aligned}$$

Note that $(\mathfrak{U}_\gamma, \mathfrak{T}_\gamma)$ is a universe if γ is not a limit.

Corollary 4.20 $\mathfrak{U}_{\delta+1} = \mathbf{U}(\mathfrak{U}_\delta, (x)\mathfrak{T}_\delta(x))$.

4.5 Well ordering proofs in MLS

$\alpha < \Gamma_0$ is kept fixed and all ordinals in this subsection are supposed to be $\leq \alpha$

Definition 4.21 (Models for $\mathbf{ATR}^i$) Let $F \in \mathbb{N} \to \mathbf{Fam}_{\mathbb{V}}$ satisfy $\mathbf{Hier}_{A,B}(\alpha, F)$. Given a limit λ, let

$$\begin{aligned} A_\lambda \quad &:= \quad (\Sigma n \in \mathbb{N})\big(\mathbb{N} \to \mathfrak{U}_{f_\lambda(n)+1}\big), \qquad (19) \\ \mathbb{A}_\lambda \quad &:= \quad \langle \mathbb{N}, 0, \mathbf{suc}, +, \cdot, A_\lambda \rangle. \qquad (20) \end{aligned}$$

We shall show that $\mathbb{A}_\lambda$ gives rise to an interpretation of $\mathbf{ATR}^i$ (in **MLS**). We interpret the language of $\mathbf{ATR}^i$ in $\mathbb{A}_\lambda$ as in Definition 4.4 except that, for $\mathfrak{X} \in A_\lambda$, membership of n in $\mathfrak{X}$ is now defined by

$$n \mathrel{\mathring{\in}} \mathfrak{X} \;:=\; \mathfrak{T}_\beta(\mathbf{ap}(\mathbf{p}_1(\mathfrak{X}), n)),$$

where $\beta := f_\lambda(\mathbf{p}_0(\mathfrak{X})) + 1$, and the set variables are interpreted as ranging over A_λ.

Convention. Given non-limit ordinals ξ, η, $X \in \mathbb{N} \to \mathfrak{U}_\xi$ and $Y \in \mathbb{N} \to \mathfrak{U}_\eta$, we use $X \doteq Y$ to convey that (in the sense of Definition 4.4)

$$\forall n \in \mathbb{N}\,\big[\,\mathfrak{U}_\xi \models n \in X \;\;\leftrightarrow\;\; \mathfrak{U}_\eta \models n \in Y\,\big].$$

Lemma 4.22 *Let $\xi \leq \eta$ be non-limit ordinals. If $X \in \mathbb{N} \to \mathfrak{U}_\xi$, then there exist $Y \in \mathbb{N} \to \mathfrak{U}_\eta$ such that $X \doteq Y$.*

Proof: We proceed by induction on η. If $\eta = 0$ or $\xi = \eta$, then the claim is trivial. So assume $\eta = \eta_0 + 1$ and $\xi < \eta$.

If η_0 is not a limit, then the induction hypothesis provides a $Z \in \mathbb{N} \to \mathfrak{U}_{\eta_0}$ such that $X \doteq Z$. Define $Y \in \mathbb{N} \to \mathfrak{U}_\eta$ by

$$Y(n) \;:=\; \boldsymbol{\ell}(\mathfrak{U}_{\eta_0}, (x)\mathfrak{T}_{\eta_0}(x), Z(n)).$$

Then $\mathfrak{T}_\eta(Y(n)) = \mathfrak{T}_{\eta_0}(Z(n))$, and hence $X \doteq Y$.

Now suppose that η_0 is a limit. Then $\xi < \eta_0$. This time define $Y \in \mathbb{N} \to \mathfrak{U}_\eta$ by

$$Y(n) := \boldsymbol{\ell}(\mathfrak{U}_{\eta_0}, (x)\mathfrak{T}_{\eta_0}(x), \langle \xi, X(n) \rangle).$$

One computes

$$\mathfrak{T}_\eta(Y(n)) = \mathfrak{T}_{\eta_0}(\langle \xi, X(n) \rangle) = \mathfrak{T}_\xi(X(n)),$$

and therefore $X \doteq Y$. □

Remark 4.23 Note first that if $\lambda < \lambda'$, then $\mathbb{A}_\lambda$ can be viewed as a substructure of $\mathbb{A}_{\lambda'}$ as follows: Any $\mathfrak{X} \in A_\lambda$ is of the form $\langle n, X \rangle$ with $X \in \mathbb{N} \to \mathfrak{U}_{\xi+1}$ and $\xi + 1 := f_\lambda(n) + 1$. Then $\mathfrak{X}' := \langle \xi, X \rangle \in A_\lambda$, too, and also $\mathfrak{X}' \in A_{\lambda'}$. Obviously, $\mathbb{A}_\lambda \models \mathfrak{X} = \mathfrak{X}'$. Since the validity of a formula of second order arithmetic is salvaged under substituting $\mathfrak{X}'$ for $\mathfrak{X}$, we see that the mapping $\mathfrak{X} \mapsto \mathfrak{X}'$ embedds $\mathbb{A}_\lambda$ into $\mathbb{A}_{\lambda'}$. Moreover, this embedding preserves the validity of $\mathbf{\Sigma}^1_1$ formulae. By a $\mathbf{\Sigma}^1_1$ formula we mean a formula (possibly with parameters) which starts with a string of existential second order quantifiers followed by a matrix which is arithmetical. Further, both structures satisfy the same arithmetical formulae with parameters from $\mathbb{A}_\lambda$.

In Definition 4.4 we gave an interpretation of **IAR** in an arbitrary universe **U** by letting second order variables range over $\mathfrak{P}_{\mathbf{U}}(\mathbb{N})$. Let $\gamma + 1 < \lambda$, then $\mathfrak{P}_{\mathfrak{U}_{\gamma+1}}(\mathbb{N})$ can be viewed as embedded in A_λ by sending $X \in \mathfrak{P}_{\mathfrak{U}_{\gamma+1}}(\mathbb{N})$ to $\langle \gamma + 1, X \rangle \in A_\lambda$. On account of this embedding, if $\mathcal{F}$ is a $\mathbf{\Sigma}^1_1$ formula with parameters from $\mathfrak{P}_{\mathfrak{U}_{\gamma+1}}(\mathbb{N})$, then $\mathfrak{U}_{\gamma+1} \models \mathcal{F}$ implies $\mathbb{A}_\lambda \models \mathcal{F}$. Likewise, both structures satisfy the same arithmetical formulae with parameters from $\mathfrak{P}_{\mathfrak{U}_{\gamma+1}}(\mathbb{N})$.

In the following we are going to use the above embeddings and the pertaining $\mathbf{\Sigma}^1_1$ persistencies without further ado.

Lemma 4.24 *Let $\lambda < \lambda'$ be limit ordinals. If $\mathbb{A}_{\lambda'} \models \mathbf{WF}(\delta)$, then $\mathbb{A}_\lambda \models \mathbf{TI}(\delta)$.*

Proof: Given a formula of second order arithmetic $\mathcal{F}(v)$ containing parameters from A_λ, one constructs an $X \in \mathbb{N} \to \mathfrak{U}_{\lambda+1}$ such that

$$\forall n \in \mathbb{N}\, [n \dot{\in} X \;\leftrightarrow\; \mathbb{A}_\lambda \models \mathcal{F}(n)] \tag{21}$$

holds true. Set $\mathfrak{X} := \langle \lambda + 1, X \rangle$. Since $\mathbb{A}_{\lambda'} \models \mathbf{Prog}(\mathfrak{X}) \to \forall \xi \leq \delta\, \xi \in \mathfrak{X}$, it follows from (21) that $\mathbb{A}_\lambda \models \mathbf{Prog}(\mathcal{F}) \to \forall \xi \leq \delta\, \mathcal{F}(\xi)$. □

Lemma 4.25 MLS *proves that $\mathbb{A}_\lambda \models \mathbf{ATR}^i$, i.e. $\mathbb{A}_\lambda$ is a model of $\mathbf{ATR}^i$ under the above interpretation.*

Proof: This is similar to the proof that a limit of admissible sets gives rise to a model of $\mathbf{ATR_0}$ (cf. [11], Theorem 4.8). Also Palmgren proved in the first draft of [19] that $\mathbf{ATR}^i$ can be interpreted in type theory with a superuniverse. A similar proof can be used to show that $\mathbb{A}_\lambda \models \mathbf{ATR}^i$. We shall focus on the validity of arithmetical transfinite recursion. Suppose $\mathbb{A}_\lambda \models \mathbf{WO}(\mathfrak{R})$. Let $R := \mathbf{p}_1(\mathfrak{R})$. Further, let $\phi(a, A, \vec{b}, \vec{B})$ be an arithmetical formula and let $\vec{m} \in \mathbb{N}$, $\vec{\mathfrak{X}} = \mathfrak{X}_1, \ldots, \mathfrak{X}_k$, $\vec{\mathfrak{X}} \in A_\lambda$. Since λ is a limit there exists $\delta + 1 < \lambda$ such that $R \in \mathbb{N} \to \mathfrak{U}_\xi$ for some $\xi \leq \delta + 1$ and for all $X_i := \mathbf{p}_1(\mathfrak{X}_i)$ $(1 \leq i \leq k)$ there exists $\gamma \leq \delta + 1$ such that $X_i \in \mathbb{N} \to \mathfrak{U}_\gamma$. By 4.22, we may actually assume that $X_1, \ldots, X_k, R \in \mathbb{N} \to \mathfrak{U}_{\delta+1}$ since for any $Y \in \mathbb{N} \to \mathfrak{U}_{\eta+1}$ with $\eta \leq \delta$ there exists $Y' \in \mathbb{N} \to \mathfrak{U}_{\delta+1}$ such that $Y \doteq Y'$. Set

$$\psi(a, C, D) := \forall z\, [z \in C \;\leftrightarrow\; \phi(a, z, D, \vec{m}, \vec{X})].$$

Since ϕ is arithmetical we get $\mathfrak{U}_{\delta+1} \models \forall n \forall W \exists! V \psi(n, W, V)$ by arithmetic comprehension. ¿From $\mathbb{A}_\lambda \models \mathbf{WO}(\mathfrak{R})$ we also get $\mathfrak{U}_{\delta+1} \models \mathbf{TI}(R)$. Thus, using Proposition 4.3, we obtain $\mathfrak{U}_{\delta+1} \models$

$\exists Y\forall n\psi(n,\bigcup\{(Y)_k : k <_R n\},(Y)_n)$. Since ψ is arithmetic the latter yields (by $\mathbf{\Sigma}^1_1$ persistency upward)

$$\mathbb{A}_\lambda \models \exists Y\forall n\forall x\,[x\in(Y)_n \leftrightarrow \phi(n,x,\bigcup\{(Y)_k : k <_{\mathfrak{R}} n\},\vec{m},\vec{\mathfrak{X}})],$$

showing that arithmetical transfinite induction holds in $\mathbb{A}_\lambda$. □

Lemma 4.26 *Let $\lambda=\sup_{\xi<\mu}\lambda_\xi$ be a limit of limits, i.e. μ is a limit and, for all $\xi<\mu$, λ_ξ is a limit $<\lambda$. If for all $\xi<\lambda$, $\mathbb{A}_{\lambda_\xi}\models\mathbf{WF}(\gamma)$, then $\mathbb{A}_\lambda\models\mathbf{WF}(\gamma)$.*

Proof: Any $\mathfrak{X}$ in A_λ satisfies

$$X\in\mathbb{N}\to\mathfrak{U}_\zeta$$

for some $\zeta<\lambda$, where $X=\mathbf{p}_1(\mathfrak{X})$. Thus we may pick $\xi<\lambda$ such that $\zeta<\lambda_\xi$. Let $\mathfrak{X}':=\langle\zeta,X\rangle$. As a consequence, $\mathfrak{X}'\in A_{\lambda_\xi}$ and hence $A_{\lambda_\xi}\models\mathbf{Prog}(\mathfrak{X}')\to\gamma\in\mathfrak{X}'$. The latter implies $A_\lambda\models\mathbf{Prog}(\mathfrak{X})\to\gamma\in\mathfrak{X}$. This shows $A_\lambda\models\mathbf{WF}(\gamma)$. □

Lemma 4.27 *If $0<\eta$, then*

$$\mathbb{A}_{\kappa+\omega^{2+\gamma}\cdot\eta}\models\forall\beta\,[\,\mathbf{WF}(\beta)\;\to\;\forall\xi\le\gamma\,\mathbf{WF}(\Phi\xi\beta)\,].$$

Proof: We proceed by induction on γ. Suppose $\mathbb{A}_{\kappa+\omega^{2+\gamma}\cdot\eta}\models\mathbf{WF}(\beta)$.

Case 1: $\gamma=0$. By Lemma 4.24 we obtain $\mathbb{A}_\lambda\models\mathbf{TI}(\beta)$ for all limits $\lambda<\kappa+\omega^2\cdot\eta$. Hence, using Lemma 4.25 plus Lemma 4.16, one gets $\mathbb{A}_\lambda\models\mathbf{WF}(\Gamma_\beta)$. As $\kappa+\omega^2\cdot\eta$ is a limit of limits, 4.26 yields $\mathbb{A}_{\kappa+\omega^2\cdot\eta}\models\mathbf{WF}(\Gamma_\beta)$.

Case 2: $\gamma=\rho+1$. Let $\mu=\pi+\omega^{2+\rho}\cdot\delta<\kappa+\omega^{2+\gamma}\cdot\eta$ with $\delta>0$. Then

$$\mathbb{A}_\mu\models\mathbf{TI}(\beta) \tag{22}$$

and, by induction hypothesis,

$$\mathbb{A}_\mu\models\forall\sigma\,(\mathbf{WF}(\sigma)\;\to\;\forall\xi\le\rho\,\mathbf{WF}(\Phi\xi\sigma)). \tag{23}$$

Let $\mathcal{F}(\nu):=\mathbf{WF}(\Phi\gamma\nu)$. We want to show

$$\mathbb{A}_\mu\models\forall\nu\le\beta\,\mathcal{F}(\nu) \tag{24}$$

by induction on $\nu\le\beta$, i.e. by employing (22). So assume that

$$\mathbb{A}_\mu\models\forall\zeta<\nu\mathcal{F}(\zeta). \tag{25}$$

If $\nu=0$, let $\tau_0:=(\Phi\rho0)+1$ and $\tau_{n+1}:=\Phi\rho\tau_n$. By (23), $\mathbb{A}_\mu\models\mathbf{WF}(\tau_0)$ and, by using (23) iteratively, one gets for all n, $\mathbb{A}_\mu\models\mathbf{WF}(\tau_n)$. As $\Phi\gamma0=\sup_{n<\omega}\tau_n$, the latter yields $\mathbb{A}_\mu\models\mathbf{WF}(\Phi\gamma0)$.

If $\nu=\vartheta+1$, let $\tau_0:=(\Phi\gamma\vartheta)+1$ and $\tau_{n+1}:=\Phi\rho\tau_n$. By (25) and (23) one gets for all n, $\mathbb{A}_\mu\models\mathbf{WF}(\tau_n)$. As $\Phi\gamma\nu=\sup_{n<\omega}\tau_n$, the latter yields $\mathbb{A}_\mu\models\mathbf{WF}(\Phi\gamma\nu)$.

If ν is a limit, then $\Phi\gamma\nu=\sup_{\zeta<\nu}\Phi\gamma\zeta$. By (25), $\mathbb{A}_\mu\models\mathbf{WF}(\Phi\gamma\zeta)$ for all $\zeta<\nu$, and thus $\mathbb{A}_\mu\models\mathbf{WF}(\Phi\gamma\nu)$. This completes the verification of (24).

(24) yields $\mathbb{A}_\mu\models\mathbf{WF}(\Phi\gamma\beta)$. Since $\kappa+\omega^{2+\gamma}\cdot\eta$ is a limit of ordinals of the form $\mu=\pi+\omega^{2+\rho}\cdot\delta$, it follows that $\mathbb{A}_{\pi+\omega^{2+\gamma}\cdot\eta}\models\mathbf{WF}(\Phi\gamma\beta)$, using 4.26.

Case 3: γ is a limit. Here one argues similarly as in the previous case except that in the case $\nu=\vartheta+1$ one employs the fact that $\Phi\gamma\nu=\sup_{\xi<\gamma}\Phi\xi((\Phi\gamma\vartheta)+1)$. □

Corollary 4.28 *For every (meta) $\alpha < \Gamma_0$, if $\mathcal{F}(x)$ is a proposition in $\mathbb{V}$ for every $x \in \mathbb{N}$, then*

$$\forall\beta \leq \Phi\alpha 0\,[\forall\xi < \beta\,\mathcal{F}(\xi) \,\rightarrow\, \mathcal{F}(\beta)] \,\rightarrow\, \forall\beta \leq \Phi\alpha 0\,\mathcal{F}(\beta)$$

holds true.

Proof: By the previous subsection, we get a hierarchy of universes, starting with the family $\mathcal{F}(n)$ over $\mathbb{N}$. Lemma 4.27 yields $\mathbb{A}_\omega \models \mathbf{WF}(\Phi\alpha 0)$. Combining these two results yields the desired result. □

Corollary 4.29 *$\Phi\Gamma_0 0$ is a lower bound for the proof-theoretic ordinal of* **MLS**.

The techniques of this subsection also allow for reading off lower bounds for the theory $\mathbf{MLS}\restriction$ introduced in subsection 2.3. The first observation we need is that for any ordinal $\alpha < \varepsilon_0$ at the meta level we have the following principle:

Lemma 4.30 ($\mathbf{MLS}\restriction$) *Suppose $\mathcal{F}(x)$ is a proposition for $x \in \mathbb{N}$. Then*

$$\forall\beta \leq \alpha\,[\forall\xi < \beta\,\mathcal{F}(\xi) \,\rightarrow\, \mathcal{F}(\beta)] \,\rightarrow\, \forall\beta \leq \alpha\,\mathcal{F}(\beta)$$

holds true.

Proof: This is essentially Gentzen's result. The proof can be carried out in Martin-Löf type theory without any universes. The details can be found in [17]. □

Employing the above induction principle, we can then construct a hierarchy of universes

$$(\mathfrak{U}_\xi, \mathfrak{T}_\xi)_{\xi<\alpha}$$

for any (meta) $\alpha < \varepsilon_0$, arguing on the basis of $\mathbf{MLS}\restriction$. The details are the same as for **MLS**. Putting together the foregoing in the same way as for **MLS**, we get:

Corollary 4.31 *$\Phi\varepsilon_0 0$ is a lower bound for the proof-theoretic ordinal of* $\mathbf{MLS}\restriction$.

5 Modelling MLS in set theory

5.1 A fragment of set theory

The most straightforward interpretation of **MLS** in classical set theory views the type $\Pi(A, B)$ as the set of all functions from A to B. However, such an interpretation would necessitate us to interpret universes as inaccessible sets, i.e. sets of the from V_κ where κ is inaccessible, and therefore already require a framework stronger than **ZFC**. To construct a model of **MLS** that reflects its proof-theoretic strength, we shall resort to realizability models as first developed by Aczel [1] and then extended by Beeson [6]. First we will lay out a set theory $\mathbf{T^S}$ in which the model construction can be formalized. $\mathbf{T^S}$ is a theory which has fairly strong set existence axioms, e.g. $\mathbf{T^S}$ has an axiom which says that any set is an element of an admissible set and thus encapsulates the existence of many admissible sets, furthermore, $\mathbf{T^S}$ postulates the existence of a recursively inaccessible set I, i.e. I is admissible and for all $x \in I$ there exists an admissible set $y \in I$ such that $x \in y$. On the other hand, $\mathbf{T^S}$ is weak with respect to induction principles in that it lacks the foundation axiom. The avoidance of the foundation axiom accounts for its proof-theoretic weakness, indeed $\mathbf{T^S}$ is proof-theoretically weaker than Kripke-Platek set theory with the axiom of infinity. The weakness in proof-theoretic strength of several theories of iterated admissible sets without the foundation axiom was first observed by Jäger. He systematically employed such theories in his monograph [11]. However, the theory $\mathbf{T^S}$ is not one of the systems of [11], in fact $\mathbf{T^S}$ is stronger than all the theories without foundation considered in [11].

Definition 5.1 In addition to the usual language of set theory with equality = and elementhood $\in$, the language of $\mathbf{T^S}$ has a unary predicate symbol $\mathbf{Ad}$ to signify that a set is admissible. For convenience we shall also assume that the language of $\mathbf{T^S}$ has a constant ω for the first infinite ordinal.

We use $A, B, C, ..., F(a), G(a), ..$ as meta-variables for formulae. We shall write $b = \{y \in a : F(y)\}$ for $(\forall y \in b)[y \in a \wedge F(y)] \wedge (\forall y \in a)[F(y) \rightarrow y \in x]$. A formula which contains only bounded quantifiers, i.e. quantifiers of the form $(\forall x \in b), (\exists x \in b)$, is said to be a Δ_0*–formula*.

For a formula A the formula A^b is the result of restricting all unbounded quantifiers in A to b.

The *logical rules* and *axioms* of $\mathbf{T^S}$ are the usual ones for classical predicate logic with equality.

The *set–theoretic axioms* of $\mathbf{T^S}$ are the universal closures of the following formulae:

Extensionality: $\forall x\,(x \in a \leftrightarrow x \in b) \;\rightarrow\; a = b.$

Infinity: $\emptyset \in \omega \;\wedge\; \forall x \in \omega\,\big[x = \emptyset \,\vee\, \exists y \in \omega\,(x = y \cup \{y\})\big].$

ω-Induction: $\forall x \in \omega\,[\forall y \in x\, y \in a \;\rightarrow\; x \in a] \;\rightarrow\; \forall x \in \omega\, x \in a.$

Ad-*Limit:* $\forall x\, \exists y\,(x \in y \;\wedge\; \mathbf{Ad}(y)).$

Ad-*Inaccessible:* $\exists u\,[\mathbf{Ad}(u) \wedge \forall x \in u\, \exists y \in u\,(x \in y \;\wedge\; \mathbf{Ad}(y))].$

Ad-*Linearity:* $\forall u\, \forall v\,[\mathbf{Ad}(u) \;\wedge\; \mathbf{Ad}(v) \;\rightarrow\; u \in v \,\vee\, u = v \,\vee\, v \in u].$

(**Ad**1)*:* $\mathbf{Ad}(a) \;\rightarrow\; \omega \in a \;\wedge\; \forall x \in a\, \forall z \in x\, z \in a.$

(**Ad**2)*:* $\mathbf{Ad}(a) \;\rightarrow\; A^a,$
where the sentence A is a universal closure of one of the following axioms:

- *Pairing:* $\exists x\; (x = \{a, b\}).$
- *Union:* $\exists x\; (x = \bigcup a).$
- Δ_0*–Separation:* $\exists x\; \big(x = \{y \in a : F(y)\}\big)$ for all Δ_0–formulae $F(b)$
- Δ_0*–Collection:* $(\forall x \in a)\exists y G(x, y) \rightarrow \exists z (\forall x \in a)(\exists y \in z) G(x, y)$ for all Δ_0–formulae $G(a, b)$.

Let $\mathbf{T^U}$ be the theory $\mathbf{T^S}$ without the axiom **Ad**-Inaccessible.

Remark 5.2 If we would add the axiom of foundation to $\mathbf{T^S}$ the resulting theory would have an enormous strength. To be more precise, its strength would exceed the one of the fragment of second order arithmetic based on Δ^1_2 comprehension and bar induction. Since the proof-theoretic ordinal of $\mathbf{T^S}$ is way below the Bachmann-Howard ordinal, the latter shows the pivotal role of foundation in connection with the strength of theories of iterated admissibility.

Corollary 5.3 $\mathbf{T^U}$ *proves Pairing, Union, and* Δ_0*-Separation.*

Proof: This is a consequence of **Ad**-Lim and the local validity of the above principles on any admissible set. □

5.2 The next admissible

In order to model the universe operator in set theory one would like to have a way of selecting an admissible set above a given set via a Σ_1-definable operation. Of course, if one had the foundation axiom one would choose the least admissible set above a given set. Fortunately, Gerhard Jäger told me that $\mathbf{T^U}$ proves the existence of the least admissible set, notwithstanding its lack of foundation. The linearity of the admissibles appears to be pivotal for the proof.

Definition 5.4 $a^+ := \bigcap\{b : a \in b \wedge \mathbf{Ad}(b)\}$.

Lemma 5.5 (Jäger) $\mathbf{T^U}$ *proves that* a^+ *is a set and, moreover, that* $\mathbf{Ad}(a^+)$.
The function $a \mapsto a^+$ *is a* Σ_1*-definable function of* $\mathbf{T^U}$.

Proof: We shall reason in $\mathbf{T^U}$. Given a set a pick a set c_0 such that $a \in c_0$ and $\mathbf{Ad}(c_0)$. Using linearity of admissibles, one readily verifies that

$$a^+ := \bigcap\{b \in c_0 \cup \{c_0\} : a \in b \wedge \mathbf{Ad}(b)\}.$$

As a result, a^+ is a set and $a \mapsto a^+$ is Σ_1-definable since

$$x = a^+ \text{ iff } \exists c\,[\mathbf{Ad}(c) \wedge a \in c \wedge x = \bigcap\{b \in c \cup \{c\} : a \in b \wedge \mathbf{Ad}(b)\}].$$

It remains to verify $\mathbf{Ad}(a^+)$. Let $a^{++} := (a^+)^+$. We first show

$$a^+ \neq a^{++}. \tag{26}$$

Aiming at a contradiction, assume $a^+ = a^{++}$. By Δ_0-Separation, $r := \{u \in a^+ : x \notin x\}$ is a set. Furthermore, for any admissible set d with $a^+ \in d$ it holds $r \in d$. Thus $r \in a^{++}$ by definition of a^{++}. Then $r \in a^+$, using $a^+ = a^{++}$. But the latter yields the contradiction

$$r \in r \leftrightarrow r \in a^+ \wedge r \notin r \leftrightarrow r \notin r.$$

On account of (26), there exists a set d such that $\mathbf{Ad}(d)$, $a \in d$, and $a^+ \notin d$. We claim that $d = a^+$. $a^+ \subseteq d$ is obvious. For $d \subseteq a^+$ it suffices to show that

$$\forall b[\mathbf{Ad}(b) \wedge a \in b \rightarrow d \subseteq b].$$

So let $a \in b$ and $\mathbf{Ad}(b)$. Since the admissibles are linearly ordered, we have $d \in b \vee d = b \vee b \in d$. $d \in b \vee d = b$ implies $d \subseteq b$. $b \in d$ is impossible since this would imply $a^+ \in b$ and thus collide with the choice of d. □

5.3 Modelling the universe operator

For this section $\mathbf{T^U}$ suffices as a background theory. We will write $\mathbb{N}$ for ω. To simplify the treatment we will assume that $\mathbf{T^U}$ has names $0, 1, 2, 3, \ldots$ for all (meta) natural numbers. We also assume that $\mathbf{T^S}$ has function symbols for addition and multiplication on $\mathbb{N}$ as well as for a primitive recursive bijective pairing function $\langle,\rangle_{\mathbb{N}} : \mathbb{N}^2 \to \mathbb{N}$ and its primitive recursive inverses $(\,)_0, (\,)_1$, that satisfy $(\langle n,m\rangle_{\mathbb{N}})_0 = n$ and $(\langle n,m\rangle_{\mathbb{N}})_1 = m$. $\mathbf{T^S}$ should also have a symbol T for Kleene's T-predicate and the result extracting function U. Let $[e](n) \simeq k$ be a shorthand for $\exists m[T(e,n,m) \wedge U(m) = k]$. Further, let $\langle n,m,k\rangle_{\mathbb{N}} := \langle\langle n,m\rangle_{\mathbb{N}},k\rangle_{\mathbb{N}}$, $\langle n,m,k,l\rangle_{\mathbb{N}} := \langle\langle n,m,k\rangle_{\mathbb{N}},l\rangle_{\mathbb{N}}$, etc. We use e,d,f,n,m,l,k,s,t,j,i as metavariables for natural numbers. Let

$$\begin{aligned}
\pi(n,m) &= \langle 0, \langle n,m\rangle_{\mathbb{N}}\rangle_{\mathbb{N}} \\
\sigma(n,m) &= \langle 1, \langle n,m\rangle_{\mathbb{N}}\rangle_{\mathbb{N}} \\
\mathrm{pl}(n,m) &= \langle 2, \langle n,m\rangle_{\mathbb{N}}\rangle_{\mathbb{N}} \\
i(n,m,k) &= \langle 3, \langle n,m,k\rangle_{\mathbb{N}}\rangle_{\mathbb{N}} \\
\hat{N} &= \langle 4, 0\rangle_{\mathbb{N}} \\
\hat{N}_k &= \langle 4, k+1\rangle_{\mathbb{N}} \\
\ulcorner\mathbf{u}\urcorner(n,m) &= \langle 6, \langle n,m\rangle_{\mathbb{N}}\rangle_{\mathbb{N}} \\
\ulcorner\mathbf{t}\urcorner(n,m,k) &= \langle 9, \langle n, \langle m,k\rangle_{\mathbb{N}}\rangle_{\mathbb{N}}\rangle_{\mathbb{N}} \\
\ulcorner\star\urcorner &= \langle 11, 0\rangle_{\mathbb{N}} \\
\ulcorner\boldsymbol{\ell}\urcorner(k) &= \langle 12, k\rangle_{\mathbb{N}}
\end{aligned}$$

Remark 5.6 In the following we shall use the so-called *Second Recursion Theorem* (cf. [5],V.2.3) to define relations on admissible sets. Inspection of the proof of the Second Recursion Theorem in [5],V.2.3 reveals that it can be proved in Kripke-Platek set theory if foundation is omitted but the axiom of infinity and set induction over ω are available. Set induction over ω is is needed for formalizing syntax and semantics in set theory. As a consequence, our metatheory $\mathbf{T^U}$ is strong enough to support definitions of relations on admissibles that come into existence via the Second Recursion Theorem.

Definition 5.7 Let $A \subseteq \mathbb{N}$, $=_A$ be an equivalence relation on A and let $(\langle B_i, =_{B_i}\rangle)_{i\in A}$ be a family such that for all $i \in A$, B_i is a subset of $\mathbb{N}$ together with a equivalence relation $=_{B_i}$ on B_i. In what follows, $\mathcal{P}$ stands for the tuple $A, =_A, (\langle B_i, =_{B_i}\rangle)_{i\in A}$. Thanks to Lemma 5.5, we can pick the least admissible set $\mathfrak{A}_\mathcal{P}$ such that $\mathcal{P} \in \mathfrak{A}_\mathcal{P}$. Using the Second Recursion Theorem (cf. [5],V.2.3) on $\mathfrak{A}_\mathcal{P}$ we define simultaneously seven relations $R_1^\mathcal{P}, \dots, R_6^\mathcal{P}$ on $\mathbb{N}$, uniformly in $\mathcal{P}$. To increase intelligibility, we shall write

$$\begin{array}{lll}
\mathbb{U}^\mathcal{P} \models^w n \ \mathbf{set} & \text{for} & R_1^\mathcal{P}(n) \\
\mathbb{U}^\mathcal{P} \models^w n \in m & \text{for} & R_2^\mathcal{P}(n.m) \\
\mathbb{U}^\mathcal{P} \models^w n \notin m & \text{for} & R_3^\mathcal{P}(n,m) \\
\mathbb{U}^\mathcal{P} \models^w n = m \in k & \text{for} & R_4^\mathcal{P}(n,m,k) \\
\mathbb{U}^\mathcal{P} \models^w n = m \notin k & \text{for} & R_5^\mathcal{P}(n,m,k) \\
\mathbb{U}^\mathcal{P} \models^w n = m & \text{for} & R_6^\mathcal{P}(n,m) \\
\mathbb{U}^\mathcal{P} \models^w \mathbf{Fam}(k,f) & \text{for} & R_7^\mathcal{P}(f,k),
\end{array}$$

where $\mathbf{Fam}(k,f)$ is spelled out as *"f is a family of types over k"*.

The clauses in the definition are as follows:

(i)

$$\begin{array}{lll}
\mathbb{U}^\mathcal{P} \models^w \ulcorner\star\urcorner \ \mathbf{set} & & \\
\mathbb{U}^\mathcal{P} \models^w m \in \ulcorner\star\urcorner & \text{if} & m \in A \\
\mathbb{U}^\mathcal{P} \models^w m \notin \ulcorner\star\urcorner & \text{if} & m \notin A \\
\mathbb{U}^\mathcal{P} \models^w m = n \in \ulcorner\star\urcorner & \text{if} & m,n \in A \ \wedge \ n =_A m \\
\mathbb{U}^\mathcal{P} \models^w m \neq n \in \ulcorner\star\urcorner & \text{if} & m,n \notin A \ \vee \ n \neq_A m
\end{array}$$

$$\begin{array}{lll}
\mathbb{U}^\mathcal{P} \models^w \ulcorner\boldsymbol{\ell}\urcorner(k) \ \mathbf{set} & \text{if} & k \in A \\
\mathbb{U}^\mathcal{P} \models^w m \in \ulcorner\boldsymbol{\ell}\urcorner(k) & \text{if} & k \in A \ \wedge \ m \in B_k \\
\mathbb{U}^\mathcal{P} \models^w m \notin \ulcorner\boldsymbol{\ell}\urcorner(k) & \text{if} & k \notin A \ \vee \ m \notin B_k \\
\mathbb{U}^\mathcal{P} \models^w m = n \in \ulcorner\boldsymbol{\ell}\urcorner(k) & \text{if} & k \in A \ \wedge \ m,n \in B_k \ \wedge \ m =_{B_k} n \\
\mathbb{U}^\mathcal{P} \models^w m = n \notin \ulcorner\boldsymbol{\ell}\urcorner(k) & \text{if} & k \notin A \ \vee \ m \notin B_k \ \vee \ n \notin B_k \ \vee \ m \neq_{B_k} n
\end{array}$$

$$\begin{array}{lll}
\mathbb{U}^\mathcal{P} \models^w \langle 4,j\rangle \ \mathbf{set} & \text{if} & j \in \mathbb{N} \\
\mathbb{U}^\mathcal{P} \models^w m \in \langle 4,j\rangle & \text{if} & j = 0 \ \vee \ m+1 < j \\
\mathbb{U}^\mathcal{P} \models^w m \notin \langle 4,j\rangle & \text{if} & j \neq 0 \ \wedge \ m+1 \geq j \\
\mathbb{U}^\mathcal{P} \models^w n = m \in \langle 4,j\rangle & \text{if} & n = m \ \wedge \ (j = 0 \ \vee \ m+1 < j) \\
\mathbb{U}^\mathcal{P} \models^w n = m \notin \langle 4,j\rangle & \text{if} & n \neq m \ \vee \ (j \neq 0 \ \wedge \ m+1 \geq j).
\end{array}$$

(ii) If $\mathbb{U}^\mathcal{P} \models^w k$ **set**, $\forall j[\mathbb{U}^\mathcal{P} \models^w j \notin k \vee \ \mathbb{U}^\mathcal{P} \models^w [e](j)\ \mathbf{set}]$ and $\forall j,i[\mathbb{U}^\mathcal{P} \models^w i = j \notin k \vee \ \mathbb{U}^\mathcal{P} \models^w [e](i) = [e](j)]$, then

$$\mathbb{U}^\mathcal{P} \models^w \mathbf{Fam}(k,e).$$

(iii) If $\mathbb{U}^{\mathcal{P}} \models^w \mathbf{Fam}(k,e)$, then $\mathbb{U}^{\mathcal{P}} \models^w \pi(k,e)$ **set** and $\mathbb{U}^{\mathcal{P}} \models^w \sigma(k,e)$ **set** and

$$
\begin{array}{lll}
\mathbb{U}^{\mathcal{P}} \models^w n \in \pi(k,e) & \text{if} & \forall i\,\big(\mathbb{U}^{\mathcal{P}} \models^w i \notin k \ \vee \ \ \mathbb{U}^{\mathcal{P}} \models^w [n](i) \in [e](i)\big) \text{ and} \\
 & & \forall i,j\,\big[\mathbb{U}^{\mathcal{P}} \models^w i = j \notin k \ \vee \ \ \mathbb{U}^{\mathcal{P}} \models^w [n](i) = [n](j) \in [e](i)\big] \\
\mathbb{U}^{\mathcal{P}} \models^w n \notin \pi(k,e) & \text{if} & \exists i\,\big(\mathbb{U}^{\mathcal{P}} \models^w i \in k \ \wedge \ \ \mathbb{U}^{\mathcal{P}} \models^w [n](i) \notin [e](i)\big) \text{ or} \\
 & & \exists i,j\,\big[\mathbb{U}^{\mathcal{P}} \models^w i = j \in k \wedge \mathbb{U}^{\mathcal{P}} \models^w [n](i) = [n](j) \notin [e](i)\big] \\
\mathbb{U}^{\mathcal{P}} \models^w n = m \in \pi(k,e) & \text{if} & \mathbb{U}^{\mathcal{P}} \models^w n \in \pi(k,e) \text{ and } \mathbb{U}^{\mathcal{P}} \models^w m \in \pi(k,e) \text{ and} \\
 & & \forall j\,\big[\mathbb{U}^{\mathcal{P}} \models^w j \notin k \ \vee \ \ \mathbb{U}^{\mathcal{P}} \models^w [n](j) = [m](j) \in [e](j)\big] \\
\mathbb{U}^{\mathcal{P}} \models^w n = m \notin \pi(k,e) & \text{if} & \mathbb{U}^{\mathcal{P}} \models^w n \notin \pi(k,e) \text{ or } \mathbb{U}^{\mathcal{P}} \models^w m \notin \pi(k,e) \text{ or} \\
 & & \exists j\,\big[\mathbb{U}^{\mathcal{P}} \models^w j \in k \ \wedge \ \ \mathbb{U}^{\mathcal{P}} \models^w [n](j) = [m](j) \notin [e](j)\big] \\
\mathbb{U}^{\mathcal{P}} \models^w n \in \sigma(k,e) & \text{if} & \mathbb{U}^{\mathcal{P}} \models^w (n)_0 \in k \text{ and } \mathbb{U}^{\mathcal{P}} \models^w (n)_1 \in [e]((n)_0) \\
\mathbb{U}^{\mathcal{P}} \models^w n \notin \sigma(k,e) & \text{if} & \mathbb{U}^{\mathcal{P}} \models^w (n)_0 \notin k \text{ or } \mathbb{U}^{\mathcal{P}} \models^w (n)_1 \notin [e]((n)_0) \\
\mathbb{U}^{\mathcal{P}} \models^w n = m \in \sigma(k,e) & \text{if} & \mathbb{U}^{\mathcal{P}} \models^w n \in \sigma(k,e) \text{ and } \mathbb{U}^{\mathcal{P}} \models^w m \in \sigma(k,e) \text{ and} \\
 & & \mathbb{U}^{\mathcal{P}} \models^w (n)_0 = (m)_0 \in k \text{ and } \mathbb{U}^{\mathcal{P}} \models^w (n)_1 = (m)_1 \in [e]((n)_0) \\
\mathbb{U}^{\mathcal{P}} \models^w n = m \notin \sigma(k,e) & \text{if} & \mathbb{U}^{\mathcal{P}} \models^w n \notin \sigma(k,e) \text{ or } \mathbb{U}^{\mathcal{P}} \models^w m \notin \sigma(k,e) \text{ or} \\
 & & \mathbb{U}^{\mathcal{P}} \models^w (n)_0 = (m)_0 \notin k \text{ or } \mathbb{U}^{\mathcal{P}} \models^w (n)_1 = (m)_1 \notin [e]((n)_0).
\end{array}
$$

(iv) If $\mathbb{U}^{\mathcal{P}} \models^w n$ **set** and $\mathbb{U}^{\mathcal{P}} \models^w m$ **set**, then $\mathbb{U}^{\mathcal{P}} \models^w \mathrm{pl}(n,m)$ **set** and

$$
\begin{array}{lll}
\mathbb{U}^{\mathcal{P}} \models^w i \in \mathrm{pl}(n,m) & \text{if} & [(i)_0 = 0 \text{ and } \mathbb{U}^{\mathcal{P}} \models^w (i)_1 \in n] \text{ or} \\
 & & [(i)_0 = 1 \text{ and } \mathbb{U}^{\mathcal{P}} \models^w (i)_1 \in m] \\
\mathbb{U}^{\mathcal{P}} \models^w i \notin \mathrm{pl}(n,m) & \text{if} & [(i)_0 \neq 0 \text{ and } (i)_0 \neq 1] \text{ or } [(i)_0 = 0 \text{ and } \mathbb{U}^{\mathcal{P}} \models^w (i)_1 \notin n] \text{ or} \\
 & & [(i)_0 = 1 \text{ and } \mathbb{U}^{\mathcal{P}} \models^w (i)_1 \notin m] \\
\mathbb{U}^{\mathcal{P}} \models^w i = j \in \mathrm{pl}(n,m) & \text{if} & [(i)_0 = (j)_0 = 0 \text{ and } \mathbb{U}^{\mathcal{P}} \models^w (i)_1 = (j)_1 \in n] \text{ or} \\
 & & [(i)_0 = (j)_0 = 1 \text{ and } \mathbb{U}^{\mathcal{P}} \models^w (i)_1 = (j)_1 \in m] \\
\mathbb{U}^{\mathcal{P}} \models^w i = j \notin \mathrm{pl}(n,m) & \text{if} & [(i)_0 = (j)_0 \neq 0 \text{ and } (i)_0 = (j)_0 \neq 1] \text{ or} \\
 & & [(i)_0 = (j)_0 = 0 \text{ and } \mathbb{U}^{\mathcal{P}} \models^w (i)_1 = (j)_1 \notin n] \text{ or} \\
 & & [(i)_0 = (j)_0 = 1 \text{ and } \mathbb{U}^{\mathcal{P}} \models^w (i)_1 = (j)_1 \notin m].
\end{array}
$$

(v) If $\mathbb{U}^{\mathcal{P}} \models^w n$ **set**, then $\mathbb{U}^{\mathcal{P}} \models^w i(n,m,k)$ **set** and

$$
\begin{array}{lll}
\mathbb{U}^{\mathcal{P}} \models^w s \in i(n,m,k) & \text{if} & s = 0 \text{ and } \mathbb{U}^{\mathcal{P}} \models^w m = k \in n, \\
\mathbb{U}^{\mathcal{P}} \models^w s \notin i(n,m,k) & \text{if} & s \neq 0 \text{ or } \mathbb{U}^{\mathcal{P}} \models^w m = k \notin n, \\
\mathbb{U}^{\mathcal{P}} \models^w s = s' \in i(n,m,k) & \text{if} & s = s' = 0 \text{ and } \mathbb{U}^{\mathcal{P}} \models^w m = k \in n \\
\mathbb{U}^{\mathcal{P}} \models^w s = s' \notin i(n,m,k) & \text{if} & s = s' \neq 0 \text{ or } \mathbb{U}^{\mathcal{P}} \models^w m = k \notin n.
\end{array}
$$

(vi)

$$\begin{aligned}\mathbb{U}^{\mathcal{P}} \models^w e = f \quad \text{if} \quad & \mathbb{U}^{\mathcal{P}} \models^w e \ \ \mathbf{set} \text{ and } \mathbb{U}^{\mathcal{P}} \models^w f \ \ \mathbf{set} \text{ and} \\ & \forall s(\mathbb{U}^{\mathcal{P}} \models^w s \notin e \ \vee \ \ \mathbb{U}^{\mathcal{P}} \models^w s \in f) \text{ and} \\ & \forall s(\mathbb{U}^{\mathcal{P}} \models^w s \in e \ \vee \ \ \mathbb{U}^{\mathcal{P}} \models^w s \notin f) \text{ and} \\ & \forall s,t(\mathbb{U}^{\mathcal{P}} \models^w s = t \notin e \ \vee \ \ \mathbb{U}^{\mathcal{P}} \models^w s = t \in f) \text{ and} \\ & \forall s,t(\mathbb{U}^{\mathcal{P}} \models^w s = t \in e \ \vee \ \ \mathbb{U}^{\mathcal{P}} \models^w s = t \notin f).\end{aligned}$$

Now we come to the second step of this definition, namely using the above relations to define membership and equality for $\mathbb{U}^{\mathcal{P}}$. We have to overcome the following difficulty: for k such that $\mathbb{U}^{\mathcal{P}} \models^w k$ **set** we cannot prove that $\mathbb{U}^{\mathcal{P}} \models^w n \in k$ and $\mathbb{U}^{\mathcal{P}} \models^w n \notin k$ are complementary relations, since that seems to require an inductive definition of the relations. The solution (due to Aczel in another context) is this: define

$$\begin{array}{lll} \mathbb{U}^{\mathcal{P}} \models k \ \mathbf{set} & \text{is} & \mathbb{U}^{\mathcal{P}} \models^w k \ \ \mathbf{set} \text{ and } \forall n \left(\mathbb{U}^{\mathcal{P}} \models^w n \in k \text{ iff } \mathbb{U}^{\mathcal{P}} \not\models^w n \notin k \right) \\ & & \text{and } \forall n,m \left(\mathbb{U}^{\mathcal{P}} \models^w n = m \in k \text{ iff } \mathbb{U}^{\mathcal{P}} \not\models^w n = m \notin k \right) \\ \mathbb{U}^{\mathcal{P}} \models n \in k & \text{is} & \mathbb{U}^{\mathcal{P}} \models k \ \mathbf{set} \text{ and } \mathbb{U}^{\mathcal{P}} \models^w n \in k \\ \mathbb{U}^{\mathcal{P}} \models n = m \in k & \text{is} & \mathbb{U}^{\mathcal{P}} \models k \ \mathbf{set} \text{ and } \mathbb{U}^{\mathcal{P}} \models^w n = m \in k \\ \mathbb{U}^{\mathcal{P}} \models k = k' & \text{is} & \mathbb{U}^{\mathcal{P}} \models k \ \mathbf{set} \text{ and } \mathbb{U}^{\mathcal{P}} \models k' \ \mathbf{set} \text{ and} \\ & & \forall n \left(\mathbb{U}^{\mathcal{P}} \models^w n \in k \text{ iff } \mathbb{U}^{\mathcal{P}} \models^w n \in k' \right). \end{array} \tag{27}$$

Corollary 5.8 *The mapping which assigns to $\mathcal{P}$ the relations of (27) is Σ_1-definable in $\mathbf{T^U}$.*

Proof: The second Recursion Theorem provides formulae which define these relations on the least admissible containing $\mathcal{P}$. □

5.4 Modelling the superuniverse

Here $\mathbf{T^S}$ will be our metatheory. On account of **Ad**-Inaccessible, let $\mathbb{B}_0$ be an admissible set which is also a limit of admissibles, i.e. $\forall x \in \mathbb{B}_0 \, \exists y \in \mathbb{B}_0 \, (x \in y \wedge \mathbf{Ad}(y))$. To model the superuniverse in $\mathbf{T^S}$ we shall define three relations:

$$\begin{array}{l} \mathbb{V} \models k \ \mathbf{set} \\ \mathbb{V} \models n \in k \\ \mathbb{V} \models n = m \in k \\ \mathbb{V} \models k = k'. \end{array} \tag{28}$$

As in the case of the universe operator, the first step toward defining the relations of (28) consists in defining several auxiliary relations via the Second Recursion Theorem on the admissible $\mathbb{B}_0$, namely the following:

$$\begin{array}{l} \mathbb{V} \models^w k \ \mathbf{set} \\ \mathbb{V} \models^w n \in k \\ \mathbb{V} \models^w n \notin k \\ \mathbb{V} \models^w n = m \in k \\ \mathbb{V} \models^w n = m \notin k \\ \mathbb{V} \models^w n = m \\ \mathbb{V} \models^w \mathbf{Fam}(k, f). \end{array} \tag{29}$$

The clauses in the definition are as follows:

(i)

$$\begin{array}{lll}
\mathbb{V} \models^w \langle 4, j\rangle \ \textbf{set} & \text{if} & j \in \mathbb{N} \\
\mathbb{V} \models^w m \in \langle 4, j\rangle & \text{if} & j = 0 \ \vee \ m+1 < j \\
\mathbb{V} \models^w m \notin \langle 4, j\rangle & \text{if} & j \neq 0 \ \wedge \ m+1 \geq j \\
\mathbb{V} \models^w n = m \in \langle 4, j\rangle & \text{if} & n = m \ \wedge \ (j = 0 \ \vee \ m+1 < j) \\
\mathbb{V} \models^w n = m \notin \langle 4, j\rangle & \text{if} & n \neq m \ \vee \ (j \neq 0 \ \wedge \ m+1 \geq j).
\end{array}$$

(ii) Suppose $\mathbb{V} \models^w \mathbf{Fam}(k, e)$ and $A, R \in \mathbb{B}_0$ with $A \subseteq \mathbb{N}$, $R \subseteq \mathbb{N} \times \mathbb{N}$, and f, g are functions in $\mathbb{B}_0$ with domain A such that $\forall s \in A\, [f(s) \subseteq \mathbb{N} \ \wedge \ g(s) \subseteq \mathbb{N} \times \mathbb{N}]$. Let

$$\begin{array}{rll}
s =_A t & \text{iff} & s, t \in A \ \wedge \ \langle s, t\rangle \in R, \\
B_s & := & f(s), \\
r =_{B_s} r' & \text{iff} & r, r' \in B_s \ \wedge \ \langle r, r'\rangle \in g(s).
\end{array}$$

Further, suppose

$$\begin{array}{l}
\forall n \left[(n \in A \ \vee \ \mathbb{V} \models^w n \notin k) \ \wedge \ (n \in A \ \rightarrow \ \mathbb{V} \models^w n \in k)\right] \ \wedge \\
\forall n, m \left[(B_n = B_m \ \vee \ \mathbb{V} \models^w n \neq m \in k) \ \wedge \ (B_n = B_m \ \rightarrow \ \mathbb{V} \models^w n = m \in k)\right] \ \wedge \\
\forall n \in A \, \forall m \left[(m \in B_{[e](n)} \ \vee \ \mathbb{V} \models^w m \notin [e](n)) \ \wedge \ (m \in B_{[e](n)} \ \rightarrow \ \mathbb{V} \models^w m \in [e](n))\right].
\end{array}$$

Define $\mathcal{P} := A, =_A, \langle (B_s, =_{B_s})\rangle_{s \in A}$. Then

$$\begin{array}{lll}
\mathbb{V} \models^w \ulcorner \mathbf{u} \urcorner(k, e) \ \textbf{set} & & \\
\mathbb{V} \models^w m \in \ulcorner \mathbf{u} \urcorner(k, e) & \text{if} & \mathbb{U}^{\mathcal{P}} \models m \ \textbf{set} \\
\mathbb{V} \models^w m \notin \ulcorner \mathbf{u} \urcorner(k, e) & \text{if} & \mathbb{U}^{\mathcal{P}} \not\models m \ \textbf{set} \\
\mathbb{V} \models^w m = n \in \ulcorner \mathbf{u} \urcorner(k, e) & \text{if} & \mathbb{U}^{\mathcal{P}} \models m = n \\
\mathbb{V} \models^w m \neq n \in \ulcorner \mathbf{u} \urcorner(k, e) & \text{if} & \mathbb{U}^{\mathcal{P}} \not\models m = n \\
\mathbb{V} \models^w \ulcorner \mathbf{t} \urcorner(k, e, l) \ \textbf{set} & \text{if} & \mathbb{U}^{\mathcal{P}} \models l \ \textbf{set} \\
\mathbb{V} \models^w m \in \ulcorner \mathbf{t} \urcorner(k, e, l) & \text{if} & \mathbb{U}^{\mathcal{P}} \models m \in l \\
\mathbb{V} \models^w m \notin \ulcorner \mathbf{t} \urcorner(k, e, l) & \text{if} & \mathbb{U}^{\mathcal{P}} \not\models m \in l \\
\mathbb{V} \models^w m = n \in \ulcorner \mathbf{t} \urcorner(k, e, l) & \text{if} & \mathbb{U}^{\mathcal{P}} \models m = n \in l \\
\mathbb{V} \models^w m \neq n \in \ulcorner \mathbf{t} \urcorner(k, e, l) & \text{if} & \mathbb{U}^{\mathcal{P}} \not\models m = n \in l.
\end{array}$$

(iii) If $\mathbb{V} \models^w k \ \ \textbf{set}$, $\forall j [\mathbb{V} \models^w j \notin k \vee \ \mathbb{V} \models^w [e](j) \ \textbf{set}]$ and $\forall j, i [\mathbb{V} \models^w i = j \notin k \vee \ \mathbb{V} \models^w [e](i) = [e](j)]$, then

$$\mathbb{V} \models^w \mathbf{Fam}(k, e).$$

(iv) If $\mathbb{V} \models^w \mathbf{Fam}(k, e)$, then $\mathbb{V} \models^w \pi(k, e)$ **set** and

$$\begin{array}{lll}
\mathbb{V} \models^w n \in \pi(k, e) & \text{if} & \forall i \, (\mathbb{V} \models^w i \notin k \ \vee \ \mathbb{V} \models^w [n](i) \in [e](i)) \ \text{and} \\
 & & \forall i, j \, [\mathbb{V} \models^w i = j \notin k \ \vee \ \mathbb{V} \models^w [n](i) = [n](j) \in [e](i)] \\
\mathbb{V} \models^w n = m \in \pi(k, e) & \text{if} & \mathbb{V} \models^w n \in \pi(k, e) \text{ and } \mathbb{V} \models^w m \in \pi(k, e) \text{ and} \\
 & & \forall j \, [\mathbb{V} \models^w j \notin k \ \vee \ \mathbb{V} \models^w [n](j) = [m](j) \in [e](j).]
\end{array}$$

If $\mathbb{V} \models^w \mathbf{Fam}(k, e)$, then $\mathbb{V} \models^w \sigma(k, e)$ **set** and

$$
\begin{array}{lll}
\mathbb{V} \models^w n \in \sigma(k,e) & \text{if} & \mathbb{V} \models^w (n)_0 \in k \text{ and } \mathbb{V} \models^w (n)_1 \in [e]((n)_0). \\
\mathbb{V} \models^w n = m \in \sigma(k,e) & \text{if} & \mathbb{V} \models^w n \in \sigma(k,e) \text{ and } \mathbb{V} \models^w m \in \sigma(k,e) \text{ and} \\
 & & \mathbb{V} \models^w (n)_0 = (m)_0 \in k \text{ and } \mathbb{V} \models^w (n)_1 = (m)_1 \in [e]((n)_0).
\end{array}
$$

(v) If $\mathbb{V} \models^w n$ **set** and $\mathbb{V} \models^w m$ **set**, then $\mathbb{V} \models^w \mathrm{pl}(n, m)$ **set** and

$$
\begin{array}{lll}
\mathbb{V} \models^w i \in \mathrm{pl}(n,m) & \text{if} & [(i)_0 = 0 \text{ and } \mathbb{V} \models^w (i)_1 \in n] \text{ or} \\
 & & [(i)_0 = 1 \text{ and } \mathbb{V} \models^w (i)_1 \in m]. \\
\mathbb{V} \models^w i = j \in \mathrm{pl}(n,m) & \text{if} & [(i)_0 = (j)_0 = 0 \text{ and } \mathbb{V} \models^w (i)_1 = (j)_1 \in n] \text{ or} \\
 & & [(i)_0 = (j)_0 = 1 \text{ and } \mathbb{V} \models^w (i)_1 = (j)_1 \in m].
\end{array}
$$

(vi) If $\mathbb{V} \models^w n$ **set**, then $\mathbb{V} \models^w i(n, m, k)$ **set** and

$$
\begin{array}{lll}
\mathbb{V} \models^w s \in i(n,m,k) & \text{if} & s = 0 \text{ and } \mathbb{V} \models^w m = k \in n, \\
\mathbb{V} \models^w s = s' \in i(n,m,k) & \text{if} & s = s' = 0 \text{ and } \mathbb{V} \models^w m = k \in n.
\end{array}
$$

(vii)

$$
\begin{array}{lll}
\mathbb{V} \models^w e = f & \text{if} & \mathbb{V} \models^w e \ \mathbf{set} \text{ and } \mathbb{V} \models^w f \ \mathbf{set} \text{ and} \\
 & & \forall s (\mathbb{V} \models^w s \notin e \ \vee \ \mathbb{V} \models^w s \in f) \text{ and} \\
 & & \forall s (\mathbb{V} \models^w s \in e \ \vee \ \mathbb{V} \models^w s \notin f) \text{ and} \\
 & & \forall s, t (\mathbb{V} \models^w s = t \notin e \ \vee \ \mathbb{V} \models^w s = t \in f) \text{ and} \\
 & & \forall s, t (\mathbb{V} \models^w s = t \in e \ \vee \ \mathbb{V} \models^w s = t \notin f).
\end{array}
$$

Now we come to the second step of this definition, namely using the above relations to define the relations of (28):

$$
\begin{array}{lll}
\mathbb{V} \models k \ \mathbf{set} & \text{iff} & \mathbb{V} \models^w k \ \mathbf{set} \text{ and } \forall n \big(\mathbb{V} \models^w n \in k \text{ iff } \mathbb{V} \not\models^w n \notin k\big) \\
 & & \text{and } \forall n, m \big(\mathbb{V} \models^w n = m \in k \text{ iff } \mathbb{V} \not\models^w n = m \notin k\big) \\
\mathbb{V} \models n \in k & \text{iff} & \mathbb{V} \models k \ \mathbf{set} \text{ and } \mathbb{V} \models^w n \in k \\
\mathbb{V} \models n = m \in k & \text{iff} & \mathbb{V} \models k \ \mathbf{set} \text{ and } \mathbb{V} \models^w n = m \in k \\
\mathbb{V} \models k = k' & \text{iff} & \mathbb{V} \models k \ \mathbf{set} \text{ and } \mathbb{V} \models k' \ \mathbf{set} \text{ and} \\
 & & \forall n \big(\mathbb{V} \models^w n \in k \text{ iff } \mathbb{V} \models^w n \in k'\big) \\
\mathbb{V} \models \mathbf{Fam}(k, f) & \text{iff} & \mathbb{V} \models k \ \mathbf{set} \\
 & & \text{and } \forall n (\mathbb{V} \models n \in k \text{ then } \mathbb{V} \models [f](n) \ \mathbf{set}) \text{ and} \\
 & & \forall n, m (\mathbb{V} \models n = m \in k \text{ then } \mathbb{V} \models [f](n) = [f](m).
\end{array}
$$

Lemma 5.9 *If* $\mathbb{V} \models \mathbf{Fam}(k, f)$*, then* $\mathbb{V} \models \mathbf{u}(k, f)$ **set**.

Proof: First note that the relations of (29) are Σ_1 on $\mathbb{B}_0$. Define $\mathcal{P} := A, =_A, \langle (B_s, =_{B_s} \rangle_{s \in A}$ by:

$$
\begin{array}{rcl}
A & := & \{s \in \mathbb{N} : \mathbb{V} \models^w s \in k\}, \\
s =_A t & \text{iff} & s, t \in A \wedge \mathbb{V} \models^w s = t \in k, \\
B_s & := & \{r \in \mathbb{N} : \mathbb{V} \models^w r \in [e](s)\}, \\
r =_{B_s} r' & \text{iff} & \mathbb{V} \models^w r = r' \in [e](s).
\end{array}
$$

The assumption $\mathbb{V} \models \mathbf{Fam}(k, f)$ implies that

$$\begin{array}{rcl} A & := & \{s \in \mathbb{N} : \mathbb{V} \not\models^w s \notin k\}, \\ s =_A t & \text{iff} & s, t \in A \wedge \mathbb{V} \not\models^w s \neq t \in k, \\ B_s & := & \{r \in \mathbb{N} : \mathbb{V} \not\models^w r \notin [e](s)\}, \\ r =_{B_s} r' & \text{iff} & \mathbb{V} \not\models^w r \neq r' \in [e](s). \end{array}$$

Therefore, using Δ_1-comprehension and Σ-replacement in $\mathbb{B}_0$, one gets $A, R, f, g \in \mathbb{B}_0$, where $R := \{\langle n, m\rangle : n =_A m\}$, f is the function $(s \mapsto B_s)_{s\in A}$ and g is the function $(s \mapsto =_{B_s})_{s\in A}$. Hence, by the inductive clause (ii), it follows $\mathbb{V} \models^w \mathbf{u}(k, f)$ **set**. Note that if A', R', f', g' would also satisfy the conditions of clause (ii), then $A = A'$, $R = R'$, $f = f'$, and $g = g'$. This ensures that $\mathbb{V} \models \mathbf{u}(k, f)$ **set**. □

5.5 The interpretation

In order to define its interpretation in $\mathbf{T^S}$, we need a more detailed account of the syntax of **MLS**. Here we will follow [7], Ch.XI; however, for the readers convenience, we shall recall most of the definitions.

If B is any expression, and $x_1, \ldots, x_n$ are variables, we form the expression $(x_1, \ldots, x_n)B$. The symbol $\stackrel{\triangle}{\equiv}$ will be used for the relation on expressions satisfying

$$((x_1, \ldots, x_n)B)(x_1, \ldots, x_n) \stackrel{\triangle}{\equiv} B$$

and $A \stackrel{\triangle}{\equiv} C$ for expressions A and C which differ only in the renaming of bound variables (cf. [7], XI6).

Definition 5.10 (cf. [7], XI.20.3) The *constants* of **MLS** are: $\mathbf{\Pi}, \mathbf{\Sigma}, \mathbf{I}, +, \mathbb{N}, \mathbb{N}_0, \mathbb{N}_1, \mathbf{U}, \mathbb{V}, \mathbf{T}, \mathbb{S}, \hat{\mathbf{\Pi}}$, $\hat{\mathbf{\Sigma}}, \hat{\mathbf{I}}, \hat{+}, \hat{\mathbb{N}}, \hat{\mathbb{N}}_0, \hat{\mathbb{N}}_1$, $\mathbf{0}, \mathbf{s}_{\mathbb{N}}, \mathbf{r}$, $\boldsymbol{\lambda}$, $\mathbf{ap}$, $\mathbf{E}$, $\mathbf{i}, \mathbf{j}, \mathbf{D}$, $\mathbf{J}, \mathbf{R}, \mathbf{R}_0 \mathbf{R}_1, \mathbf{u}, \mathbf{t}, \star, \boldsymbol{\ell}$. The *terms* are generated by:

1. Every constant and variable is a term;
2. If t and s are terms, then $t(s)$ and (t, s) are terms;
3. If t is a term, then $(x_1, \ldots, x_n)t$ is a term, where the x_i are variables.

To ensure that all terms occurring in derivations of **MLS** are terms in the above sense, it should be said that for a constant $\mathbf{c}$ of **MLS**, $\mathbf{c}(t_1, t_2)$, $\mathbf{c}(t_1, t_2, t_3)$, $\mathbf{c}(t_1, t_2, t_3, t_4)$ etc. are to be read as $\mathbf{c}((t_1, t_2))$, $\mathbf{c}(((t_1, t_2), t_3))$, $\mathbf{c}((((t_1, t_2), t_3), t_4))$ etc., respectively.

Free and bound occurrences of variables in terms are defined as usual, letting abstraction, i.e. the formation of $(x_1, \ldots, x_n)t$ bind the variables $x_1, \ldots, x_n$. We now would like to assign to every term t of **MLS** a corresponding term $\hat{t}$ of the theory $\mathbf{T^S}$ by replacing the abstract application of **MLS** with recursive application. However, the technicality here is that recursive application cannot be expressed on the level of terms within $\mathbf{T^S}$. Therefore, we first assign to every term t of **MLS** an application term t^* of the theory **EON** of [7], VI.2 or equivalently of the theory **APP** as described in [28], Ch.9, Sect.3. It is then a straightforward matter to translate a formula of the form $t^* \in X$ into a legitimate formula of $\mathbf{T^S}$.

Definition 5.11 The theory **EON** is defined in [7], VI.2.

EON has two important devices for defining functions, the Abstraction Theorem (cf. [7], VI.2.2) and the Recursion Theorem (cf. [7], VI.2.7) which we shall employ in the next definition.

Definition 5.12 We now assign to each term t of **MLS** an application term t^* of the theory **EON**. Occurrences of λ in the definition of t^* denote the λ-operator introduced in [7], VI.2.2. We shall write (x, y) for $\mathbf{p}xy$ and, inductively, $(x_1, \ldots, x_{k+1})$ for $\mathbf{p}(x_1, \ldots, x_k)x_{k+1}$. For constants $\mathbf{c}$ we define $\boldsymbol{c^*}$ by:

$$
\begin{array}{rl}
\mathbf{0}^* & \text{is } \mathbf{0} \\
\mathbf{\Pi}^* & \text{is } \lambda x.\lambda y.(0, x, y) \\
\hat{\mathbf{\Pi}}^* & \text{is } \lambda x.\lambda y.(0, x, y) \\
\mathbf{\Sigma}^* & \text{is } \lambda x.\lambda y.(1, x, y) \\
\hat{\mathbf{\Sigma}}^* & \text{is } \lambda x.\lambda y.(1, x, y) \\
+^* & \text{is } \lambda x.\lambda y.(2, x, y) \\
\hat{+}^* & \text{is } \lambda x.\lambda y.(2, x, y) \\
\mathbf{I}^* & \text{is } \lambda x.\lambda y.(3, x, y, z) \\
\hat{\mathbf{I}}^* & \text{is } \lambda x.\lambda y.(3, x, y, z) \\
\mathbb{N}^* & \text{is } (4, 0) \\
\hat{\mathbb{N}}^* & \text{is } (4, 0) \\
\mathbb{N}_{\mathbf{k}}^* & \text{is } (4, k+1) \\
\hat{\mathbb{N}}_{\mathbf{k}}^* & \text{is } (4, k+1) \\
\mathbf{U}^* & \text{is } \lambda x.\lambda y.(5, x, y, z) \\
\mathbf{u}^* & \text{is } \lambda x.\lambda y.(6, x, y, z) \\
\mathbb{V}^* & \text{is } (7, 0) \\
\mathbf{s}_{\mathbf{N}}^* & \text{is } \mathbf{s}_{\mathbb{N}} \\
\mathbf{r}^* & \text{is } \mathbf{0} \\
\boldsymbol{\lambda}^* & \text{is } \lambda x.x \quad \text{(i.e. } \mathbf{skk}) \\
\mathbf{ap}^* & \text{is } \lambda x.\lambda y.yx \\
\mathbf{E}^* & \text{is } \lambda x.\lambda y.y(\mathbf{p_0}x, \mathbf{p_1}x) \\
\mathbf{i}^* & \text{is } \lambda x.(0, x) \\
\mathbf{j}^* & \text{is } \lambda x.(1, x) \\
\mathbf{D}^* & \text{is } \lambda x.\lambda y.\lambda z(0, \mathbf{p_0}x, y(\mathbf{p_0}x), z(\mathbf{p_1}x)) \\
\mathbf{J}^* & \text{is } \lambda x.\lambda y.y \\
\mathbf{T}^* & \text{is } \lambda x.\lambda y.\lambda z.z \\
\mathbf{t}^* & \text{is } \lambda x.\lambda y.\lambda z.z \\
\mathbb{S}^* & \text{is } \lambda x.x \\
\star^* & \text{is } \lambda x.\lambda y.(11, 0) \\
\boldsymbol{\ell}^* & \text{is } \lambda x.\lambda y.\lambda z.(12, z)
\end{array}
$$

$\mathbf{R}_{\mathbf{k}}^*$ is $\lambda m.\lambda x_0 \cdots \lambda x_{k-1}.e_k(m, x_0, \ldots, x_k - 1)$, where e_k is chosen so that **EON** proves

$$e_k(m, x_0, \ldots, x_k - 1) \simeq \begin{cases} x_m & \text{if } m < k \\ \Omega & \text{otherwise,} \end{cases}$$

where $\Omega := \lambda x.xx(\lambda x.xx)$ (signifying undefinedness). $\mathbf{R}^*$ is an application term of **EON** introduced by the Recursion Theorem to satisfy (provably in **EON**) $\mathbf{R}^*(a, b, 0) = a$ and $\mathbf{R}^*(a, b, \mathbf{s_N x}) \simeq \mathbf{b}(\mathbf{x}, \mathbf{R}^*(\mathbf{a}, \mathbf{b}, \mathbf{x}))$. For complex terms of **MLS** we define:

$$
\begin{array}{c}
((x_1, \ldots, x_n)t)^* \text{ is } \lambda x_1 \cdots \lambda x_n.t^*; \\
(t(s))^* \text{ is } t^*s^*; \\
(t, s)^* \text{ is } \mathbf{p}t^*s^*.
\end{array}
$$

Definition 5.13 Rather than translating application terms of **EON** into the set-theoretic language of $\mathbf{T^S}$, we define the translation of expressions of the form $t \simeq u$, where t is an application term of

EON and u is a variable. First, we select indices $\bar{\mathbf{k}}, \bar{\mathbf{s}}, \bar{\mathbf{p}}, \bar{\mathbf{p}}_{\mathbf{0}}, \bar{\mathbf{p}}_{\mathbf{1}}$, $\bar{\mathbf{s}}_{\mathbf{N}}$, $\bar{\mathbf{p}}_{\mathbf{N}}$ and $\bar{\mathbf{d}}$ for partial recursive functions so that:

$$\begin{array}{rcl}
[[\bar{\mathbf{k}}](n)](m) & \simeq & n \\
[[[\bar{\mathbf{s}}](n)](m)](l) & \simeq & [[n](l)]([m](l)) \\
[[\bar{\mathbf{p}}](n)](m) & \simeq & 2^n \cdot 3^m = \langle n, m \rangle_{\mathbb{N}} \\
[\bar{\mathbf{p}}_{\mathbf{0}}](n) & \simeq & (n)_0 \\
[\bar{\mathbf{p}}_{\mathbf{1}}](n) & \simeq & (n)_1 \\
[\bar{\mathbf{s}}_{\mathbf{N}}](n) & \simeq & n+1 \\
[\bar{\mathbf{p}}_{\mathbf{N}}](0) & \simeq & 0 \\
[\bar{\mathbf{p}}_{\mathbf{N}}](n+1) & \simeq & n \\
[[[[\bar{\mathbf{d}}](n)](m)](x)](y) & \simeq & \begin{cases} x & \text{if } n = m \\ y & \text{if } n \neq m \end{cases}
\end{array}$$

The definition of $(t \simeq u)^\wedge$ proceeds along the way that t was built up.

$$\begin{array}{lll}
(v \simeq u)^\wedge & \text{is} & v = u \wedge v \in \omega \ \text{ if } v \text{ is a variable;} \\
(0 \simeq u)^\wedge & \text{is} & 0 = u; \\
(\boldsymbol{c} \simeq u)^\wedge & \text{is} & \bar{\boldsymbol{c}} = u \text{ if } \boldsymbol{c} \text{ is one of the constants } \mathbf{k}, \mathbf{s}, \mathbf{p}, \mathbf{p_0}, \mathbf{p_1}, \mathbf{s}_{\mathbb{N}}, \mathbf{p_N}, \mathbf{d}.
\end{array}$$

A complex application term of **EON** has the form ts. Let

$$(ts \simeq u)^\wedge := \exists x, y \in \omega[(t \simeq x)^\wedge \ \wedge \ (s \simeq y)^\wedge \ \wedge \ [x](y) \simeq u].$$

EON is interpreted in $\mathbf{T^S}$ by setting

$$\begin{array}{rcl}
(\mathrm{Ap}(s,t,r))^\diamond & := & \exists u \in \omega\, [(st \simeq u)^\wedge \ \wedge \ (r \simeq u)^\wedge] \\
(s = t)^\diamond & := & \exists u \in \omega\, [(s \simeq u)^\wedge \ \wedge \ (t \simeq u)^\wedge] \\
(s \in \mathbf{N})^\diamond & := & \exists u \in \omega\, [(s \simeq u)^\wedge] \\
(\phi \Box \psi)^\diamond & := & \phi^\diamond \Box \psi^\diamond \quad (\Box \in \{\wedge, \vee, \to\}) \\
(\forall x \phi(x))^\diamond & := & \forall x \in \omega\, \phi(x)^\diamond.
\end{array}$$

Definition 5.14 The *set terms* of **MLS** are defined inductively by

1. $\mathbb{N}$, $\mathbf{N_k}$ (for $k = 0, 1$) and $\mathbb{V}$ are set terms;

2. If A and B are set terms, so is $(A+B)$;

3. If $B(x)$ and A are set terms, and x is not free in A or in B, and t is a term, then $\mathbf{\Pi}(A, B)$, $\mathbf{\Sigma}(A, B)$, $\mathbf{U}(A, B)$, and $\mathbf{T}(A, B, t)$ are set terms;

4. If A is a set term and t, s are any terms of **MLS**, then $\mathbf{I}(A, s, t)$ is a set term;

5. If t is a term, then $\mathbb{S}(t)$ is a set term;

6. If A is a set term and $B \stackrel{\triangle}{\equiv} A$, then B is a set term.

Definition 5.15 (Interpretation of **MLS** in $\mathbf{T^S}$) By induction on the complexity of the set term A we shall assign to each judgement Φ of **MLS** of the form $u \in A$ or $u = v \in A$ (u, v variables) a formula $(\Phi)^\wedge$ of $\mathbf{T^S}$ with the same free variables. $(ux \in A)^\wedge$ and $(ux = uy \in A)^\wedge$ will be used as shorthand for $\exists z \in \omega[[u](x) \simeq z \ \wedge \ (z \in A)^\wedge]$ and $\exists z \in \omega[[u](x) \simeq z \ \wedge \ [u](y) \simeq z \ \wedge \ (z \in A)^\wedge]$, respectively. Likewise, $(u \in A(vx))^\wedge$ abbreviates $\exists z \in \omega[[v](x) \simeq z \wedge (u \in A(z))^\wedge]$, etc. The clauses in the definition are as follows:

$$
\begin{array}{rcl}
(u \in \mathbf{\Pi}(A,B))^\wedge & \text{is} & \forall x \in \omega[(x \in A)^\wedge \to (ux \in B(x))^\wedge] \wedge \\
& & \forall x,y \in \omega[(x = y \in A)^\wedge \to (ux = uy \in B(x))^\wedge] \\
(u = v \in \mathbf{\Pi}(A,B))^\wedge & \text{is} & \forall x \in \omega[(x \in A)^\wedge \to (ux = vx \in B(x))^\wedge] \wedge \\
& & (u \in \mathbf{\Pi}(A,B))^\wedge \wedge (v \in \mathbf{\Pi}(A,B))^\wedge \\
(u \in \mathbf{\Sigma}(A,B))^\wedge & \text{is} & (\bar{\mathbf{p}}_\mathbf{0} u \in A)^\wedge \wedge (\bar{\mathbf{p}}_\mathbf{1} u \in B(\bar{\mathbf{p}}_\mathbf{0} u))^\wedge \\
(u = v \in \mathbf{\Sigma}(A,B))^\wedge & \text{is} & (\bar{\mathbf{p}}_\mathbf{0} u = \bar{\mathbf{p}}_\mathbf{0} v \in A)^\wedge \wedge (\bar{\mathbf{p}}_\mathbf{1} u = \bar{\mathbf{p}}_\mathbf{1} v \in B(\bar{\mathbf{p}}_\mathbf{0} u))^\wedge \\
(u \in (A+B))^\wedge & \text{is} & [\bar{\mathbf{p}}_\mathbf{0} u = 0 \wedge (\bar{\mathbf{p}}_\mathbf{1} u \in A)^\wedge] \vee [\bar{\mathbf{p}}_\mathbf{0} u = 1 \wedge (\bar{\mathbf{p}}_\mathbf{1} u \in B)^\wedge] \\
(u = v \in (A+B))^\wedge & \text{is} & [\bar{\mathbf{p}}_\mathbf{0} u = 0 \wedge \bar{\mathbf{p}}_\mathbf{0} v = 0 \wedge (\bar{\mathbf{p}}_\mathbf{1} u = \bar{\mathbf{p}}_\mathbf{1} v \in A)^\wedge] \vee \\
& & [\bar{\mathbf{p}}_\mathbf{0} u = 1 \wedge \bar{\mathbf{p}}_\mathbf{0} v = 1 \wedge (\bar{\mathbf{p}}_\mathbf{1} u = \bar{\mathbf{p}}_\mathbf{1} v \in B)^\wedge] \\
(u \in \mathbf{I}(A,b,c))^\wedge & \text{is} & u = 0 \wedge (b = c \in A)^\wedge \\
(u = v \in \mathbf{I}(A,b,c))^\wedge & \text{is} & u = 0 \wedge v = 0 \wedge (b = c \in A)^\wedge \\
(u \in \mathbb{N})^\wedge & \text{is} & u \in \omega \\
(u = v \in \mathbb{N})^\wedge & \text{is} & u = v \wedge u \in \omega \\
(u \in \mathbb{N}_\mathbf{k})^\wedge & \text{is} & u \in \omega \wedge u \in \bar{k} \\
(u = v \in \mathbb{N}_\mathbf{k})^\wedge & \text{is} & u \in \omega \wedge u = v \wedge u \in \bar{k} \\
(u \in \mathbb{V})^\wedge & \text{is} & \mathbb{V} \models u \ \ \mathbf{set} \\
(u = v \in \mathbb{V})^\wedge & \text{is} & \mathbb{V} \models u = v \\
(u \in \mathbb{S}(w))^\wedge & \text{is} & \mathbb{V} \models w \ \ \mathbf{set} \wedge \mathbb{V} \models u \in w \\
(u = v \in \mathbb{S}(w))^\wedge & \text{is} & \mathbb{V} \models u = v \in w \\
(u \in \mathbf{U}(A,B))^\wedge & \text{is} & \mathbb{U}^\mathcal{P} \models u \ \ \mathbf{set} \\
(u = v \in \mathbf{U}(A,B))^\wedge & \text{is} & \mathbb{U}^\mathcal{P} \models u = v \\
(u \in \mathbf{T}(A,B,w))^\wedge & \text{is} & \mathbb{U}^\mathcal{P} \models u \in w \\
(u = v \in \mathbf{T}(A,B,w))^\wedge & \text{is} & \mathbb{U}^\mathcal{P} \models u = v \in w,
\end{array}
$$

where $\mathcal{P} := \tilde{A}, =_{\tilde{A}}, \langle \tilde{B}_i, =_{\tilde{B}_i} \rangle, k_0, e_0$ is determined by: $\tilde{A} := \{x \in \mathbb{N} : (x \in A)^\wedge\}$, $x =_{\tilde{A}} y$ iff $(x = y \in A)^\wedge$, $\tilde{B}_i := \{x : (x \in B(i))^\wedge\}$, $x =_{\tilde{B}_i} y$ iff $(x = y \in B(i))^\wedge$.

If s and t are arbitrary terms of **MLS** and A is a set term of **MLS**, we set:

$$
\begin{array}{rcl}
(t \in A)^\wedge & \text{iff} & \exists u \in \omega[(t^* \simeq u)^\wedge \wedge (u \in A)^\wedge], \\
(s = t \in A)^\wedge & \text{iff} & \exists u, v \in \omega[(s^* \simeq u)^\wedge \wedge (t^* \simeq v)^\wedge \wedge (u = v \in A)^\wedge].
\end{array}
$$

For set terms A and B we define $(A = B)^\wedge$ by

$$
\forall u \in \omega[(u \in A)^\wedge \leftrightarrow (u \in B)^\wedge] \wedge \forall u, v \in \omega[(u = v \in A)^\wedge \leftrightarrow (u = v \in B)^\wedge].
$$

Theorem 5.16 (Soundness of the Interpretation of **MLS** in $\mathbf{T^S}$.) *If Φ is a judgement of* **MLS** *not of the form "A set," then* $\mathbf{T^S} \vdash (\Phi)^\wedge$.

Proof: First note that if an expression of the form A set, $s \in A$, $s = t \in A$, or $A = B$ appears in a derivation of **MLS**, then A is a set term in the sense of Definition 5.14, as is readily seen by induction on derivations in **MLS**.[3] This ensures that any judgement of **MLS** gets translated under $^\wedge$. Secondly, it should be clear that the above interpretation replaces the abstract application of **MLS** by recursive application in a faithful way, i.e. the equations which the rules of **MLS** prescribe

[3] Incidentally, there are more set terms than those that appear in such derivations.

for the constants of **MLS** are satisfied by their translations. Since **MLS** derivations may involve hypothetical judgements, 5.16 has to be stated in a more general form so as to be able to carry out the proof by induction on derivations in **MLS**. For the details we refer to [7], XI.20.3.1, where this is done for the interpretation of $\mathbf{ML_0}$ in **EON**.

Theorem 5.17 (Soundness of the Interpretation of **MLU** in $\mathbf{T^U}$.) *If* Φ *is a judgement of* **MLU** *not of the form "A set," then* $\mathbf{T^U} \vdash (\Phi)^\wedge$.

Proof: Same as for the previous theorem except that one doesn't have to handle the superuniverse. □

References

[1] P. Aczel: *The strength of Martin–Löf's intuitionistic type theory with one universe*, in: S. Miettinen, S. Väänänen (eds.): Proceedings of Symposia in Mathematical Logic, Oulu, 1974, and Helsinki, 1975, Report No. 2 (University of Helsinki, Department of Philosophy, 1977) 1–32.

[2] P. Aczel: *The Type Theoretic Interpretation of Constructive Set Theory*, in: MacIntyre, A. Pacholski, L., Paris, J. (eds.), *Logic Colloquium '77* (North-Holland, Amsterdam, 1978).

[3] P. Aczel: *The Type Theoretic Interpretation of Constructive Set Theory: Choice Principles*, in: Troelstra, A. S., van Dalen, D. (eds.), *The L.E.J. Brouwer Centenary Symposium*, (North–Holland, Amsterdam, 1982).

[4] P. Aczel: *The Type Theoretic Interpretation of Constructive Set Theory: Inductive Definitions*, in: Marcus, R. B. et al. (eds), *Logic, Methodology, and Philosophy of Science VII*, (North–Holland, Amsterdam 1986).

[5] J Barwise: *Admissible Sets and Structures*, Springer, Berlin 1975.

[6] M. Beeson: *Recursive Models for Constructive Set Theories*, Annals of Math. Logic 23, 126–178 (1982).

[7] M. Beeson: *Foundations of Constructive Mathematics*, (Springer Verlag, Berlin, 1985)

[8] S. Feferman: *Iterated inductive fixed-point theories: Application to Hancock's conjecture*, in: G. Metakides (ed.): *Patras Logic Symposium* (North-Holland, Amsterdam, 1982) 171–196.

[9] E. Griffor and E. Palmgren: *A Intuitionistic Theory of Transfinite Types* (preprint 1990) 22 pages.

[10] E. Griffor and M. Rathjen: *The strength of some Martin–Löf type theories.* Archive for Mathematical Logic 33 (1994) 347–385.

[11] Jäger, G.: *Theories for admissible sets: a unifying approach to proof theory* (Bibliopolis, Naples, 1986).

[12] P. Martin-Löf: *An intuitionistic theory of types: predicative part*, in: H.E. Rose and J. Shepherdson (eds.): *Logic Colloquium '73* (North-Holland, Amsterdam, 1975) 73–118.

[13] P. Martin-Löf: *Intuitionistic Type Theory*, (Bibliopolis, Naples, 1984).

[14] J. Myhill: *Constructive Set Theory*, JSL 40 (1975) 347–382.

[15] B. Nordström, K. Petersson and J.M. Smith: *Programming in Martin–Löf's Type Theory*, (Clarendon Press, Oxford, 1990).

[16] E. Palmgren: *Transfinite hierarchies of universes*, Department of Mathematics Report 1991:7 (Uppsala University, 1991), 16 pages.

[17] E. Palmgren: *Type-Theoretic Interpretations of Iterated, Strictly Positive Inductive Definitions*, Arch. Math. Logic 32, 75–99 (1993).

[18] E. Palmgren: *An information system interpretation of Martin-Löf's partial type theory with universes*, Information and Computation 106 (1993) 26–60.

[19] E. Palmgren: *On universes in type theory.* To appear in: *Proceedings of "Twenty-five years of type theory"*, Venice, 1995 (Oxford University Press).

[20] Aarne Ranta: *Type-theoretical grammar* (Clarendon Press, Oxford, 1994).

[21] M. Rathjen: *Untersuchungen zu Teilsystemen der Zahlentheorie zweiter Stufe und der Mengenlehre mit einer zwischen* Δ^1_2-CA *und* $\Delta^1_2-CA+BI$ *liegenden Beweisstärke.* Publication of the Institute for Mathematical Logic and Foundational Research of the University of Münster (1989). *Revision of 1988 Münster University doctoral thesis.* MR 91m#03062 .

[22] M. Rathjen, E. Griffor, E. Palmgren: *Inaccessibility in constructive set theory and type theory.* Annals of Pure and Applied Logic 94 (1998) 181–200.

[23] M. Rathjen: *The strength of Martin-Löf type theory with a superuniverse. Part II.* 23 pages. To appear in: Archive for Mathematical Logic.

[24] K. Schütte: *Proof Theory* (Springer, Berlin, 1977).

[25] A. Setzer: *A well-ordering proof for the proof theoretical strength of Martin-Löf type theory.* Annals of Pure and Applied Logic 92 (1998) 113–159.

[26] A. Setzer: *Extending Martin-Löf type theory by one Mahlo-universe*, to appear in: archive for Mathematical Logic.

[27] J.M. Smith: *The independence of Peano's fourth axiom from Martin-Löf's type theory without universes.* Journal of Symbolic Logic 53 (1989).

[28] A. S. Troelstra and D. van Dalen: *Constructivism in Mathematics: an Introduction*, volume II, North–Holland, Amsterdam 1988.