Proof Theory: From arithmetic to set theory

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1 Cut Elimination: Gentzen's Hauptsatz

In his Inaugural Dissertation from 1935 (published as [41]), Gentzen introduced his sequent calculus and employed the technique of *cut elimination*. As this is a tool of utmost importance in proof theory and ordinal analysis, a rough outline of the underlying ideas will be discussed next.

The most common kind of logical calculi, known as *Hilbert-style* systems, are specified by delineating a collection of schematic logical axioms and some inference rules. The choice of axioms and rules is more or less arbitrary, only subject to the desire to obtain a *complete* system (in the sense of Gödel's completeness theorem). In model theory it is usually enough to know that there is a complete calculus for first order logic as this already entails the compactness theorem.

There are, however, proof calculi without this arbitrariness of axioms and rules. The *natural deduction calculus* and the *sequent calculus* were both invented by *Gentzen*. Both calculi are pretty illustrations of the symmetries of logic. The sequent calculus is a central tool in ordinal analysis and allows for generalizations to so-called infinitary logics. Gentzen's main theorem about the sequent calculus is the *Hauptsatz*, i.e. *the cut elimination theorem*.

A *sequent* is an expression $\Gamma \Rightarrow \Delta$ where Γ and Δ are finite sequences of formulae $A_1, \ldots, A_n$ and $B_1, \ldots, B_m$, respectively. We also allow for the possibility that Γ or Δ (or both) are empty. The empty sequent will be denoted by $\emptyset$. $\Sigma \Rightarrow \Delta$ is read, informally, as Γ yields Δ or, rather, the *conjunction* of the A_i yields the *disjunction* of the B_j. In particular, we have:

- If Γ is empty, the sequent asserts the disjunction of the B_j.
- If Δ is empty, it asserts the negation of the conjunction of the A_i.
- if Γ and Δ are both empty, it asserts the *impossible*, i.e. a *contradiction*.

We use upper case Greek letters $\Gamma, \Delta, \Lambda, \Theta, \Xi \ldots$ to range over finite sequences of formulae. $\Gamma \subseteq \Delta$ means that every formula of Γ is also a formula of Δ.

Next we list the axioms and rules of the sequent calculus.

- *Identity Axiom*

$$A \Rightarrow A$$

  where A is any formula. In point of fact, one could limit this axiom to the case of atomic formulae A.

- *Cut Rule*

$$\frac{\Gamma \Rightarrow \Delta, A \qquad A, \Lambda \Rightarrow \Theta}{\Gamma, \Lambda \Rightarrow \Delta, \Theta}\ \mathrm{Cut}$$

  The formula A is called the *cut formula* of the inference.

- *Structural Rules*

$$\frac{\Gamma \Rightarrow \Delta}{\Gamma' \Rightarrow \Delta'} \qquad \text{if } \Gamma \subseteq \Gamma',\ \Delta \subseteq \Delta'.$$

  A special case of the structural rule, known as *contraction*, occurs when the lower sequent has fewer occurrences of a formula than the upper sequent. For instance, $A, \Gamma \Rightarrow \Delta, B$ follows structurally from $A, A, \Gamma \Rightarrow \Delta, B, B$.

- Rules for Logical Operations

Left | Right

$$\frac{\Gamma \Rightarrow \Delta, A}{\neg A, \Gamma \Rightarrow \Delta} \qquad \frac{B, \Gamma \Rightarrow \Delta}{\Gamma \Rightarrow \Delta, \neg B}$$

$$\frac{\Gamma \Rightarrow \Delta, A \qquad B, \Lambda \Rightarrow \Theta}{A \to B, \Gamma, \Lambda \Rightarrow \Delta, \Theta} \qquad \frac{A, \Gamma \Rightarrow \Delta, B}{\Gamma \Rightarrow \Delta, A \to B}$$

$$\frac{A, \Gamma \Rightarrow \Delta}{A \wedge B, \Gamma \Rightarrow \Delta} \quad \frac{B, \Gamma \Rightarrow \Delta}{A \wedge B, \Gamma \Rightarrow \Delta} \qquad \frac{\Gamma \Rightarrow \Delta, A \qquad \Gamma \Rightarrow \Delta, B}{\Gamma \Rightarrow \Delta, A \wedge B}$$

$$\frac{A, \Gamma \Rightarrow \Delta \qquad B, \Gamma \Rightarrow \Delta}{A \vee B, \Gamma \Rightarrow \Delta} \qquad \frac{\Gamma \Rightarrow \Delta, A}{\Gamma \Rightarrow \Delta, A \vee B} \quad \frac{\Gamma \Rightarrow \Delta, B}{\Gamma \Rightarrow \Delta, A \vee B}$$

$$\frac{F(t), \Gamma \Rightarrow \Delta}{\forall x\, F(x), \Gamma \Rightarrow \Delta}\ \forall \mathrm{L} \qquad \frac{\Gamma \Rightarrow \Delta, F(a)}{\Gamma \Rightarrow \Delta, \forall x\, F(x)}\ \forall \mathrm{R}$$

$$\frac{F(a), \Gamma \Rightarrow \Delta}{\exists x\, F(x), \Gamma \Rightarrow \Delta}\ \exists \mathrm{L} \qquad \frac{\Gamma \Rightarrow \Delta, F(t)}{\Gamma \Rightarrow \Delta, \exists x\, F(x)}\ \exists \mathrm{R}$$

In $\forall L$ and $\exists R$, t is an arbitrary term. The variable a in $\forall R$ and $\exists L$ is an *eigenvariable* of the respective inference, i.e. a is not to occur in the *lower sequent*.

In the rules for logical operations, the formulae highlighted in the premisses are called the *minor formulae* of that inference, while the formula highlighted in the conclusion is the *principal formula* of that inference. The other formulae of an inference are called *side formulae.*

A *proof* (aka *deduction* or *derivation*) $\mathcal{D}$ is a tree of sequents satisfying the following conditions:

- The topmost sequents of $\mathcal{D}$ are identity axioms.
- Every sequent in $\mathcal{D}$ except the lowest one is an upper sequent of an inference whose lower sequent is also in $\mathcal{D}$.

A sequent $\Gamma \Rightarrow \Delta$ is *deducible* if there is a proof having $\Gamma \Rightarrow \Delta$ as its the bottom sequent.

The Cut rule differs from the other rules in an important respect. With the rules for introduction of a connective on the left or the right, one sees that every formula that occurs above the line occurs below the line either directly, or as a subformula of a formula below the line, and that is also true for the structural rules. (Here $A(t)$ is counted as a subformula, in a slightly extended sense, of both $\exists x A(x)$ and $\forall x A(x)$.) But in the case of the Cut rule, the cut formula A vanishes. Gentzen showed that such "vanishing rules" can be eliminated.

Theorem 1.1 (Gentzen's Hauptsatz) *If a sequent $\Gamma \Rightarrow \Delta$ is provable, then it is provable without use of the Cut Rule (called a* cut-free proof*).*

The secret to Gentzen's Hauptsatz is the symmetry of left and right rules for the logical connectives. The proof of the cut elimination theorem is rather intricate as the process of removing cuts interferes with the structural rules. The possibility of contraction accounts for the high cost of eliminating cuts. Let $|\mathcal{D}|$ be the *height* of the deduction $\mathcal{D}$. Also, let $rank(\mathcal{D})$ be *supremum* of the *lengths of cut formulae* occurring in $\mathcal{D}$. Turning $\mathcal{D}$ into a cut-free deduction of the same end sequent results, in the worst case, in a deduction of height $\mathcal{H}(rank(\mathcal{D}), |\mathcal{D}|)$ where $\mathcal{H}(0, n) = n$ and $\mathcal{H}(k+1, n) = 4^{\mathcal{H}(k,n)}$, yielding hyper-exponential growth.

The *Hauptsatz* has an important corollary which explains its crucial role in obtaining consistency proofs.

Corollary 1.2 (The Subformula Property) *If a sequent $\Gamma \Rightarrow \Delta$ is provable, then it has a deduction all of whose formulae are subformulae of the formulae of Γ and Δ.*

Corollary 1.3 *A contradiction, i.e. the empty sequent $\emptyset \Rightarrow \emptyset$, is not deducible.*

Proof: According to the Hauptsatz, if the empty sequent were deducible it would have a deduction without cuts. In a cut-free deduction of the empty sequent only empty sequents can occur. But such a deduction does not exist. □

While mathematics is based on logic, it cannot be developed solely on the basis of *pure logic*. What is needed in addition are *axioms* that assert the *existence* of *mathematical objects* and their properties. Logic plus axioms gives rise to (formal) *theories* such as *first-order arithmetic* or the axioms of *Zermelo-Fraenkel set theory*. What happens when we try to apply the procedure of cut elimination to theories? Well, axioms are detrimental to this procedure. It breaks down because the symmetry of the sequent calculus is lost. In general, one cannot remove cuts from deductions in a theory T when the cut formula is an axiom of T. However, sometimes the axioms of a theory are of *bounded syntactic complexity*. Then the procedure applies partially in that one can remove all cuts that exceed the complexity of the axioms of T. This gives rise *partial cut elimination*. It is a very important tool in proof theory. For example, it works very well if the axioms of a theory can be presented as *atomic intuitionistic sequents* (also called *Horn clauses*), yielding the completeness of *Robinson*'s *resolution method*.

2 Gentzen's consistency proof

Gentzen is perhaps most famous for his consistency proofs of first-order arithmetic of which there are two published ones [42, 45] and an unpublished one [48]

The theory of first-order arithmetic aims at describing the arguably most important structure in mathematics, $\mathfrak{N} = \left(\mathbb{N}; 0^{\mathfrak{N}}, 1^{\mathfrak{N}}, +^{\mathfrak{N}}, \times^{\mathfrak{N}}, E^{\mathfrak{N}}, <^{\mathfrak{N}}\right)$, where $0^{\mathfrak{N}}$ denotes zero, $1^{\mathfrak{N}}$ denotes the number one, $+^{\mathfrak{N}}, \times^{\mathfrak{N}}, E^{\mathfrak{N}}$ denote the successor, addition, multiplication, and exponentiation function, respectively, and $<^{\mathfrak{N}}$ stands for the less-than relation on the natural numbers. In particular, $E^{\mathfrak{N}}(n,m) = n^m$.

Many of the famous theorems and problems of mathematics such as Fermat's and Goldbach's conjecture, the Twin Prime conjecture, and Riemann's hypothesis can be formalized as sentences of the language of $\mathfrak{N}$ and thus concern questions about the structure $\mathfrak{N}$.

Definition 2.1 **First-order number theory** is a theory designed with the intent of axiomatizing the structure $\mathfrak{N}$. It is based on Dedekind's definition of a simply infinite system [23]. These axioms have become universally known as the "Dedekind-Peano axioms" or just the "Peano axioms". Peano presented arithmetic in symbolic notation in [88]. In his honor the formal theory is called **Peano arithmetic**, **PA**. The language of **PA** has the predicate symbols $=, <$, the function symbols $+, \times, E$ (for addition, multiplication,exponentiation) and the constant symbols 0 and 1. The *Axioms of* **PA** comprise the usual equations and laws for addition, multiplication, exponentiation, and the less-than relation. In addition, **PA** has the *Induction Scheme*

$$\textbf{(IND)} \qquad \varphi(0) \wedge \forall x[\varphi(x) \to \varphi(x+1)] \to \forall x \varphi(x)$$

for all formulae φ of the language of **PA**.

Cut elimination fails for first-order arithmetic (i.e. **PA**), not even partial cut elimination is possible since the induction axioms have unbounded complexity. However, one can remove the obstacle to cut elimination in a drastic way by going *infinite*. The so-called *ω-rule* consists of the two types of *infinitary inferences*:

$$\frac{\Gamma \Rightarrow \Delta, F(0);\ \Gamma \Rightarrow \Delta, F(1);\ \ldots\ ; \Gamma \Rightarrow \Delta, F(n);\ \ldots}{\Gamma \Rightarrow \Delta, \forall x\, F(x)}\ \omega\mathrm{R}$$

$$\frac{F(0), \Gamma \Rightarrow \Delta;\ F(1), \Gamma \Rightarrow \Delta;\ \ldots\ ; F(n), \Gamma \Rightarrow \Delta;\ \ldots}{\exists x\, F(x), \Gamma \Rightarrow \Delta}\ \omega\mathrm{L}$$

The price to pay will be that deductions become infinite objects, i.e. *infinite well-founded trees*.

The sequent-style version of Peano arithmetic with the ω-*rule* will be termed $\mathbf{PA}_\omega$. $\mathbf{PA}_\omega$ has no use for free variables. Thus free variables are discarded and all *terms* will be closed. All formulae of this system are therefore closed, too. The *numerals* are the terms $\bar{n}$, where $\bar{0} = 0$ and $\overline{n+1} = S\bar{n}$. We shall identify $\bar{n}$ with the natural number n. All terms t of $\mathbf{PA}_\omega$ evaluate to a numeral $\bar{n}$.

$\mathbf{PA}_\omega$ has all the inference rules of the sequent calculus except for $\forall R$ and $\exists L$. In their stead, $\mathbf{PA}_\omega$ has the ωR and ωL inferences. The *Axioms* of $\mathbf{PA}_\omega$ are the following: (i) $\emptyset \Rightarrow A$ if A is a *true* atomic sentence; (ii) $B \Rightarrow \emptyset$ if B is a *false* atomic sentence; (iii) $F(s_1, \ldots, s_n) \Rightarrow F(t_1, \ldots, t_n)$ if $F(s_1, \ldots, s_n)$ is an atomic sentence and s_i and t_i evaluate to the same numeral.

With the aid of the ω-rule, each instance of the induction scheme becomes logically deducible, albeit the price to pay will be that the proof tree becomes infinite. To describe the cost of cut elimination for $\mathbf{PA}_\omega$, we introduce the measures of *height* and *cut rank* of a $\mathbf{PA}_\omega$ deduction $\mathcal{D}$. We will notate this by

$$\mathcal{D} \,\big|\!\overset{\alpha}{\underset{k}{}}\, \Gamma \Rightarrow \Delta\,.$$

The above relation is defined inductively following the buildup of the deduction $\mathcal{D}$. For the *cut rank* we need the definition of the *length*, $|A|$ of a formula: $|A| = 0$ if A is atomic; $|\neg A_0| = |A_0|+1$; $|A_0 \Box A_1| = \max(|A_0, A_1|) + 1$ where $\Box = \wedge, \vee, \rightarrow$; $|\exists x\, F(x)| = |\forall x\, F(x)| = |F(0)| + 1$.

Now suppose the last inference of $\mathcal{D}$ is of the form

$$\dfrac{\begin{array}{c}\mathcal{D}_0\\ \Gamma_0 \Rightarrow \Delta_0\end{array} \quad \cdots \quad \begin{array}{c}\mathcal{D}_n\\ \Gamma_n \Rightarrow \Delta_n\end{array} \quad \cdots \quad n < \tau}{\Gamma \Rightarrow \Delta}\; I$$

where $\tau = 1, 2, \omega$ and the $\mathcal{D}_n$ are the immediate subdeductions of $\mathcal{D}$. If

$$\mathcal{D}_n \,\big|\!\overset{\alpha_n}{\underset{k}{}}\, \Gamma_n \Rightarrow \Delta_n$$

and $\alpha_n < \alpha$ for all $n < \tau$ then

$$\mathcal{D} \,\big|\!\overset{\alpha}{\underset{k}{}}\, \Gamma \Rightarrow \Delta$$

providing that in the case of I being a *cut* with cut formula A we also have $|A| < k$. We will write $\mathbf{PA}_\omega \,\big|\!\overset{\alpha}{\underset{k}{}}\, \Gamma \Rightarrow \Delta$ to convey that there exists a $\mathbf{PA}_\omega$-deduction $\mathcal{D} \,\big|\!\overset{\alpha}{\underset{k}{}}\, \Gamma \Rightarrow \Delta$. The ordinal analysis of $\mathbf{PA}$ proceeds by first unfolding any $\mathbf{PA}$-deduction into a $\mathbf{PA}_\omega$-deduction:

$$\text{If } \mathbf{PA} \vdash \Gamma \Rightarrow \Delta \text{ then } \mathbf{PA}_\omega \,\big|\!\overset{\omega+m}{\underset{k}{}}\, \Gamma \Rightarrow \Delta \tag{1}$$

for some $m, k < \omega$. The next step is to get rid of the cuts. It turns out that the cost of lowering the cut rank from $k+1$ to k is an exponential with base ω.

Theorem 2.2 (Cut Elimination for $\mathbf{PA}_\omega$) *If* $\mathbf{PA}_\omega \,\big|\!\overset{\alpha}{\underset{k+1}{}}\, \Gamma \Rightarrow \Delta$, *then* $\mathbf{PA}_\omega \,\big|\!\overset{\omega^\alpha}{\underset{k}{}}\, \Gamma \Rightarrow \Delta$.

As a result, if $\mathbf{PA}_\omega \,\big|\!\overset{\alpha}{\underset{n}{}}\, \Gamma \Rightarrow \Delta$, we may apply the previous theorem n times to arrive at a cut-free deduction $\mathbf{PA}_\omega \,\big|\!\overset{\rho}{\underset{0}{}}\, \Gamma \Rightarrow \Delta$ with $\rho = \omega^{\omega^{\cdot^{\cdot^{\cdot^{\omega^\alpha}}}}}$, where the stack has height n. Combining this with the result from (1), it follows that every sequent $\Gamma \Rightarrow \Delta$ deducible in $\mathbf{PA}$ has a cut-free deduction in $\mathbf{PA}_\omega$ of length $< \varepsilon$. Ruminating on the details of how this result was achieved yields a consistency proof for $\mathbf{PA}$ from transfinite induction up to ε_0 for elementary decidable predicates on the basis of finitistic reasoning (as described in (2).

Deductions in $\mathbf{PA}_\omega$ being well-founded infinite trees, they have a natural associated ordinal length, namely: the height of the tree as an ordinal. Thus the passage from finite deductions in $\mathbf{PA}$ to infinite cut-free deductions in $\mathbf{PA}_\omega$ provides an explanation of how the ordinal ε_0 is connected with $\mathbf{PA}$.

Gentzen, though, did not deal explicitly with infinite proof trees in his second published proof of the consistency of $\mathbf{PA}$ [45]. However, in the unpublished first consistency proof [48] he aims at showing that a deduction in the formal system of first-order arithmetic of a sequent gives rise to a a well-founded reduction tree which can be identified with a cut-free deduction tree in in the sequent calculus with the ω-rule.

The infinitary version of **PA** with the ω-rule was explicitly introduced by Schütte in [118]. Incidentally, the ω-rule had already been proposed by Hilbert [62]. Gentzen worked with finite deductions in the sequent calculus version of **PA**, devising an ingenious method of assigning ordinals to purported derivations of the empty sequent (inconsistency). It turns out in recent work by Buchholz [15] that in fact there is a much closer intrinsic connection between the way Gentzen assigned ordinals to deductions in **PA** and the way that ordinals are assigned to infinite deductions in $\mathbf{PA}_\omega$.

In the 1950s infinitary proof theory flourished in the hands of Schütte. He extended his approach to **PA** to systems of ramified analysis and brought this technique to perfection in his monograph "Beweistheorie" [120]. The ordinal representation systems necessary for Schütte's work will be reviewed later.

2.1 Gentzen's result in view of Hilbert's program

Gentzen showed that transfinite induction up to the ordinal

$$\varepsilon_0 \;=\; \sup\{\omega, \omega^\omega, \omega^{\omega^\omega}, \ldots\} \;=\; \text{least } \alpha.\, \omega^\alpha = \alpha$$

suffices to prove the consistency of **PA**. To appreciate Gentzen's result it is pivotal to note that he applied transfinite induction up to ε_0 solely to elementary computable predicates and besides that his proof used only finitistically justified means. Hence, a more precise rendering of Gentzen's result is

$$\mathbf{F} + \text{EC-TI}(\varepsilon_0) \vdash \text{Con}(\mathbf{PA}), \tag{2}$$

where $\mathbf{F}$ signifies a theory that embodies only finitistically acceptable means, EC-TI(ε_0) stands for transfinite induction up to ε_0 for elementary computable predicates, and Con(**PA**) expresses the consistency of **PA**. Gentzen also showed [46] that his result was the best possible in that **PA** proves transfinite induction up to α for arithmetic predicates for any $\alpha < \varepsilon_0$. The compelling picture conjured up by the above is that the non-finitist part of **PA** is encapsulated in EC-TI(ε_0) and therefore "measured" by ε_0, thereby tempting one to adopt the following definition of *proof-theoretic ordinal* of a theory T:

$$|T|_{Con} \;=\; \text{least } \alpha.\, \mathbf{F} + \text{EC-TI}(\alpha) \vdash \text{Con}(T). \tag{3}$$

In the above, many notions were left unexplained. We will now consider them one by one. The *elementary computable functions* are exactly the Kalmar *elementary functions,* i.e. the class of functions which contains the successor, projection, zero, addition, multiplication, and modified subtraction functions and is closed under composition and bounded sums and products. A predicate is elementary computable if its characteristic function is elementary computable.

According to an influential analysis of finitism due to Tait [134], finististic reasoning coincides with a system known as *primitive recursive arithmetic*, **PRA**.[1] For the purposes of ordinal analysis, however, it suffices to identify **F** with an even more restricted theory known as *Elementary Recursive Arithmetic*, **ERA**. **ERA** is a weak subsystem of **PA** having the same defining axioms for $+, \times, E, <$ but with induction restricted to elementary computable predicates.

In order to formalize EC-TI(α) in the language of arithmetic we should first discuss ordinals and the representation of particular ordinals α as relations on $\mathbb{N}$.

Definition 2.3 *A set A equipped with a total ordering $\prec$ (i.e. $\prec$ is transitive, irreflexive, and $\forall x, y \in A\,[x \prec y \lor x = y \lor y \prec x]$) is a* wellordering *if every non-empty subset X of A contains a $\prec$-least element, i.e. $(\exists u \in X)(\forall y \in X)[u \prec y \lor u = y]$.*

An ordinal *is a transitive set wellordered by the elementhood relation $\in$.*

[1] However, there are dissenting voices: "Such people then identify Hilbert's finite standpoint with quantifier-free primitive recursive arithmetic. However, there is reason to believe that Hilbert intended his finite standpoint to encompass Ackermann's function. If so, then his standpoint is beyond quantifier-free primitive recursive arithmetic and lies close to Gentzen's standpoint." [137, p. 365]

Fact 2.4 *Every wellordering* $(A, \prec)$ *is order isomorphic to an ordinal* $(\alpha, \in)$.

Ordinals are traditionally denoted by lower case Greek letters $\alpha, \beta, \gamma, \delta, \ldots$ and the relation $\in$ on ordinals is notated simply by $<$. The operations of addition, multiplication, and exponentiation can be defined on all ordinals, however, addition and multiplication are in general not commutative.

We are interested in representing specific ordinals α as relations on $\mathbb{N}$. In essence Cantor [18] defined the first ordinal representation system in 1897. Natural ordinal representation systems are frequently derived from structures of the form

$$\mathfrak{A} \quad = \quad \langle \alpha, f_1, \ldots, f_n, <_\alpha \rangle \tag{4}$$

where α is an ordinal, $<_\alpha$ is the ordering of ordinals restricted to elements of α and the f_i are functions

$$f_i : \underbrace{\alpha \times \cdots \times \alpha}_{k_i \text{ times}} \longrightarrow \alpha$$

for some natural number k_i.

$$\mathbb{A} \quad = \quad \langle A, g_1, \ldots, g_n, \prec \rangle \tag{5}$$

is a *computable* (or *recursive*) *representation* of $\mathfrak{A}$ if the following conditions hold:

1. $A \subseteq \mathbb{N}$ and A is a computable set.
2. $\prec$ is a computable total ordering on A and the functions g_i are computable.
3. $\mathfrak{A} \cong \mathbb{A}$, i.e. the two structures are isomorphic.

Theorem 2.5 (Cantor, 1897) *For every ordinal* $\beta > 0$ *there exist unique ordinals* $\beta_0 \geq \beta_1 \geq \cdots \geq \beta_n$ *such that*

$$\beta = \omega^{\beta_0} + \ldots + \omega^{\beta_n}. \tag{6}$$

The representation of β in (6) is called the *Cantor normal form.* We shall write $\beta =_{CNF} \omega^{\beta_1} + \cdots \omega^{\beta_n}$ to convey that $\beta_0 \geq \beta_1 \geq \cdots \geq \beta_k$.

ε_0 denotes the least ordinal $\alpha > 0$ such that $(\forall \beta < \alpha)\, \omega^\beta < \alpha$. ε_0 can also be described as the least ordinal α such that $\omega^\alpha = \alpha$.

Ordinals $\beta < \varepsilon_0$ have a Cantor normal form with exponents $\beta_i < \beta$ and these exponents have Cantor normal forms with yet again smaller exponents. As this process must terminate, ordinals $< \varepsilon_0$ can be coded by natural numbers. For instance a coding function

$$\ulcorner . \urcorner : \varepsilon_0 \longrightarrow \mathbb{N}$$

could be defined as follows:

$$\ulcorner \alpha \urcorner = \begin{cases} 0 & \text{if } \alpha = 0 \\ \langle \ulcorner \alpha_1 \urcorner, \ldots, \ulcorner \alpha_n \urcorner \rangle & \text{if } \alpha =_{CNF} \omega^{\alpha_1} + \cdots \omega^{\alpha_n} \end{cases}$$

where $\langle k_1, \cdots, k_n \rangle := 2^{k_1+1} \cdot \ldots \cdot p_n^{k_n+1}$ with p_i being the ith prime number (or any other coding of tuples). Further define:

$$\begin{array}{rclcrclcrcl} A_0 & := & \text{range of } \ulcorner . \urcorner & & \ulcorner \alpha \urcorner \prec \ulcorner \beta \urcorner & :\Leftrightarrow & \alpha < \beta & & & & \\ \ulcorner \alpha \urcorner \hat{+} \ulcorner \beta \urcorner & := & \ulcorner \alpha + \beta \urcorner & & \ulcorner \alpha \urcorner \hat{\cdot} \ulcorner \beta \urcorner & := & \ulcorner \alpha \cdot \beta \urcorner & & \hat{\omega}^{\ulcorner \alpha \urcorner} & := & \ulcorner \omega^\alpha \urcorner. \end{array}$$

Then

$$\langle \varepsilon_0, +, \cdot, \delta \mapsto \omega^\delta, < \rangle \quad \cong \quad \langle A_0, \hat{+}, \hat{\cdot}, x \mapsto \hat{\omega}^x, \prec \rangle.$$

$A_0, \hat{+}, \hat{\cdot}, x \mapsto \hat{\omega}^x, \prec$ are computable (recursive), in point of fact, they are all elementary computable.

Finally, we can spell out the scheme EC-TI(ε_0) in the language of **PA**:

$$\forall x \; [\forall y \; (y \prec x \rightarrow P(y)) \; \rightarrow \; P(x)] \; \rightarrow \; \forall x \, P(x)$$

for all elementary computable predicates P.

3 Extended versions of Hilbert's program

As pointed our before, in the early days of Hilbert's program the difference between finitism and intuitionism had not been clearly realized and they were sometimes thought to be coextensive. The nature of intuitionism was greatly clarified by Kolmogorov [78] who in 1925 gave a formalization of intuitionistic logic and observed the translatability of classical into intuitionistic logic.[2]

His work, though, seems to not have been noticed at the time and it was Heyting's 1930 formalization of intuitionistic predicate logic and intuitionistic arithmetic (later christened Heyting Arithmetic, **HA**) which attained official status. A few years later in 1933, Gentzen and Gödel independently provided translations which illuminated the relationship between classical and intuitionistic arithmetic (still unaware of Kolmogorov's earlier work on logic). Their so-called negative translations showed that, in a sense, Peano Arithmetic is contained in **HA**, and, moreover, that for formulas not containing $\vee$ or $\exists$, provability in **PA** and **HA** amount to the same. Since **HA** comprises just a fragment of Brouwer's intuitionism it became clear that intuitionism has a much wider scope than finitism if the latter is taken to be in the narrow sense of primitive recursive arithmetic. Already in 1933 [56, p.53] Gödel broached the idea of a revised version of Hilbert's program using constructive means that go beyond the limited finitist ones without accepting fully fledged intuitionism which he considered to be problematic, especially on account of the impredicative nature of the intuitionistic implication.

> But there remains the hope that in future one may find other and more satisfactory methods of construction beyond the limits of the system A [capturing finitist methods], which may enable us to found classical arithmetic and analysis upon them. This question promises to be a fruitful field for further investigations.

In his 1938 lecture at Zilsel's [56, p. 94], Gödel explored several routes for extending finitism.

> How then shall we extend? (Extension is necessary.) Three ways are known up to now:
> 1. Higher types of functions (functions of functions of numbers, etc.).
> 2. The modal-logical route (introduction of an absurdity applied to universal sentences and a [notion of] "consequence").
> 3. Transfinite induction, that is, inference by induction is added for certain concretely defined ordinal numbers of the second number class.

The first way, a precursor of the Dialectica interpretation, is inspired by Hilbert's 1926 *Über das Unendliche* which considered a hierarchy of functionals over the natural numbers, not only of finite but also of transfinite type. The third way, of course, is related to Gentzen's consistency proof of **PA** via transfinite inductions while the third way pursues an axiomatic approach to capturing the abstract notion of proof used in intuitionism.

In the aftermath of the incompleteness results, Hilbert made changes to his program and considered a broadened or extended form of finitism. A particulary appealing idea is to pursue Hilbert's program relative to a constructive point of view and determine which parts of classical mathematics are demonstrably consistent relative to that standpoint.[3] As one would aspect, there are differing "schools" of constructivism and different layers of constructivism. Several frameworks for developing mathematics from such a point of view have been proposed. Some we will refer to in this article (arguably the most important) are:

(a) Arithmetical Predicativism.

(b) Theories of higher type functionals.

(c) Takeuti's "Hilbert-Gentzen finitist standpoint".

[2] Kolmogorov already in 1925 drew from this the conclusion that, contrary to Brouwer's views on the matter, a finitary statement proved by using the principle of excluded middle is intuitionistically true ([59, p. 431]).

[3] See [111] for pursuing this with regard to Martin-Löf type theory.

(d) Feferman's explicit mathematics.

(e) Martin-Löf's intuitionistic type theory.

(f) Constructive set theory (Myhill, Friedman, Beeson, Aczel).

At this point we will just give a very rough description of these foundational views. A few more details, especially about their scope on a standard scale of theories and proof-theoretic ordinals, will be provided in subsection 12.1.

(a) Arithmetical Predicativism originated in the writings of Poincaré and Russell in response to the paradoxes. It is characterized by a ban on impredicative definitions. Whilst it accepts the completed infinite set of naturals numbers, all other sets are required to be constructed out of them via an autonomous process of arithmetical definitions. A first systematic attempt at developing mathematics predicatively was made in Weyl's 1918 monograph *Das Kontinuum* [149].

(b) Theories of higher type functionals comprise Gödel's T and Spector's extension of T via functionals defined by bar recursion. These will be discussed in a later chapter as the play an important part in proof theory.

(c) To understand Takeuti's finitist standpoint it is important to pinpoint the place where in a consistency proof à la Gentzen the means of **PRA** are exceeded. Gentzen's proof employs a concrete ordering $\prec$ of type ε_0, it uses an assignment of ordinals to proofs and provides a reduction procedure on proofs such that any alleged proof of an inconsistency is reduced to another proof of an inconsistency which gets assigned a smaller element of the ordering. The ordering, the ordinal assignment and the reduction procedure are actually elementary recursive and the steps described sofar can be carried out in a small fragment of **PRA**. The additional principle needed to infer the consistency of **PA** is the following:

($*$) There are no infinite elementary recursive sequences $\alpha_0, \alpha_1, \alpha_2, \ldots$ such that $\alpha_{n+1} \prec \alpha_n$ holds for all n.

Takeuti refers to ($*$) as the accessibility of $\prec$. Note that this is a weaker property than the wellfoundedness of $\prec$ which refers to arbitrary sequences (Π^0_2 versus Π^1_1). There is nothing special about the case of **PA** since any ordinal analysis of a theory T in the literature can be made to fit this format. Thus espistemologically ($*$) is the fulcrum in any such consistency proof. Takeuti's central idea in [141, 137] is that we can carry out Gedankenexperimente (thought experiments) on concretely given (elementary) sequences to arrive at the insight that ($*$) obtains.[4]

(d) Feferman's explicit mathematics [32, 33] is a theory that describes a realm of concretely and explicitly given objects (a universe U of symbols) equipped with an operation $\bullet$ of application in such a way that given two objects $a, b \in U$, a may be viewed as a program which can be run on input b and may produce an output $a \bullet b \in U$ or never halt (such structures are known as partial combinatory algebras or Schönfinkel algebras). Moreover, some of the objects of U represent sets of elements of U. The construction of new sets out of given sets is either done explicitly by elementary comprehension or by a process of inductive generation. If one also adds principles to the effect that every internal operation (given as $\lambda x.a \bullet x$ for some $a \in U$) which is monotone on sets possesses a least fixed point one arrives at a remarkably strong theory (cf. [104, 106, 107]).

(e) Martin-Löf type theory is an intuitionistic theory of types intended to be a full scale system for formalizing constructive mathematics. Its origins of can be traced to Principia Mathematica, Hilbert's Über das Unendliche, the natural deduction systems of Gentzen, taken in conjunction with Prawitz's reduction procedures, and to Gödel's Dialectica system. It incorporates inductively defined data types which together with the vehicle of internal reflection via universes endow it with considerable consistency strength.

[4] "Because of Gödel's result consistency proofs now require a method that is finite (or constructive) but which is nevertheless very strong when formalized. People think this is impossible or at least unlikely and extremely difficult. The situation is somewhat similar to that of finding a new axiom that carries conviction and decides the continuum hypothesis." [137, p.366]

(f) Constructive set theory (as do the theories under (d) and (e)) sets out to develop a framework for the style of constructive mathematics of Bishop's 1967 *Foundations of constructive analysis* [6] in which he carried out a development of constructive analysis, based on informal notions of constructive function and set, which went substantially further mathematically than anything done before by constructivists. Where Brouwer reveled in differences, Bishop stressed the commonalities with classical mathematics. What was novel about his work was that it could be read as a piece of classical mathematics as well.

The 'manifesto' of constructive set theory was most vividly expressed by Myhill: "*... the argumentation of [6] looks very smooth and seems to follow directly from a certain conception of what sets, functions, etc. are, and we wish to discover a formalism which isolates the principles underlying this conception in the same way that Zermelo-Fraenkel set-theory isolates the principles underlying classical (nonconstructive) mathematics. We want these principles to be such as to make the process of formalization completely trivial, as it is in the classical case.*" ([87, p. 347]

Despite first appearances, there are close connections between the approaches of (d)-(f). Constructive set theory can be interpreted in Martin-Löf type theory (due to Aczel [1]) and explicit mathematics can be interpreted in constructive set theory (see [99]). Perhaps the closest fit between (e) and (f), giving back and forth interpretations, is provided by [114].

4 Consistency beyond PA

It could be said that mathematicians have quite a good intuition about the world of the natural numbers and consequently the consistency of a theory like **PA**, whose axioms appear to be intuitively justified, perhaps by imagining an infinite mind, is not a particularly pressing question. There are, however, parts of mathematics that seem to require more than the means of **PA**. The second of Hilbert's famous problems asks for a consistency proof of analysis. In 1938 Gentzen summarized the situation as follows:

> "*Indeed, it seems not entirely unreasonable to me to suppose that contradictions might possibly be concealed even in classical analysis.*" [47, p. 235]
>
> "*but the most important [consistency] proof of all in practice, that for analysis, is still outstanding.*" [47, p. 236].

After **PA**, Gentzen worked on a consistency proof for analysis as stenographic notes from 1938, 1943 and 1945 show. Formally, what is called "analysis" can be identified with the theory of second order arithmetic, $\mathbf{Z_2}$. Its language extends that of **PA** with additional variables $X, Y, Z, \ldots$ ranging over sets of numbers, and a binary membership relation $t \in X$. In addition to the axioms of **PA**, $\mathbf{Z_2}$ has the *induction axiom*

$$\forall X(0 \in X \land \forall x(x \in X \to x+1 \in X) \to \forall x(x \in X))$$

and the axiom schema of *comprehension*, (**CA**), which asserts that, for every formula $F(u)$ of $\mathbf{Z_2}$, there is a set $X = \{u \mid F(u)\}$ having exactly those numbers u as members that satisfy $F(u)$, or more formally:

$$(\mathbf{CA}) \qquad \exists X \forall u(u \in X \leftrightarrow F(u)) \tag{7}$$

for all formulae $F(u)$ in which X does not occur.

That $\mathbf{Z_2}$ is often called "analysis" is due to the realization (e.g. Hilbert and Bernays [63]) that, via the coding of real numbers and continuous functions as sets of natural numbers, a good theory of the continuum can be developed from these axioms. Further scrutiny revealed that often fragments of $\mathbf{Z_2}$ suffice. Already in 1918 Hermann Weyl showed that a considerable amount of analysis can be developed by making very lean set-theoretic assumptions. The idea of singling out the minimal fragment of $\mathbf{Z_2}$ required to develop a particular piece of ordinary mathematics led to the research program of *reverse mathematics* (to be reported on later).

Owing to these observations, second order arithmetic and its fragments have been a focal point of proof-theoretic investigations until the late 1970s.

What endows $\mathbf{Z_2}$ with considerable strength is ($\mathbf{CA}$). At first blush it might be a naturally appealing axiom, until one realizes that it involves a certain circularity, called an *impredicative definition*, when the defining formula contains set quantifiers. An impredicative definition of an object refers to a presumed totality of which the object being defined is itself to be a member. For example, to define a set of natural numbers X as

$$X = \{n \mid \forall Y \; F(n, Y)\}$$

is impredicative since it involves the quantified variable 'Y' ranging over arbitrary subsets of the natural numbers $\mathbb{N}$, of which the set X being defined is one member. Determining whether $\forall Y \; F(n, Y)\}$ holds involves an apparent circle since we shall have to know in particular whether $F(n, X)$ holds - but that cannot be settled until X itself is determined. Sweeping uses of impredicativity are present ensue from the core axioms of Zermelo-Fraenkel set theory in the guise of the separation and replacement axioms as well as the powerset axiom. Hermann Weyl in 1946 [150] wrote about an axiom similar to ($\mathbf{CA}$) that it "is a bold, an almost fantastic axiom; there is little justification for it in the real world in which we live, and none at all in the evidence on which our mind bases its constructions".

Since no contradictions have ever been found to ensue from ($\mathbf{CA}$) or much stronger theories such as Zermelo-Fraenkel set theory with the axiom of choice (and even large cardinal axioms) one should perhaps be quite content with this situation and view **ZFC** as a well tested theory.[5] In most sciences this would be the best one could hope for and certainly equate to the stage of corroborating a theory as being true beyond reasonable doubt. Mathematics, however, is unique among the sciences. It doesn't appear to undergo revisions as mathematical proofs establish a statement as true beyond any possible doubt. To be more precise, what is beyond any possible doubt is that mathematical theorems follow from certain first principles such as the axioms of set theory using purely logical inference steps. In practice, the axioms of Zermelo from 1908 and further amendments encapsulated in **ZFC** have stood the test of time and function as a gold standard to which one can in principle resort should a dispute about the validity of a proof arise.[6] One could, however, raise doubts as to how much of **ZFC** has been "tested" as far as main stream mathematics is concerned. Time and again, is has been shown that impredicative existence principles in proofs of statements of ordinary mathematics can be removed, thus leading to a situation as envisioned by Hilbert's program (for example, currently attempts are being made to prove Fermat's last theorem in **PA**). In mathematical logic, we can easily construct arithmetical statements that are not provable in second order arithmetic or **ZFC**, but we hardly find any such statement in mathematical practice. Perhaps the point here is that these arguments are highly unusual, in that they make use of principles of enormous strength that scarcely anyone knows how to use "in real life" mathematics. Naturally we chose a strong system so that it is easy to see that everything can be formalized in the system but the identification of mathematical practice with a certain strong formal system may be be a mirage.

After Gentzen, it was Gaisi Takeuti who worked on a consistency proof for $\mathbf{Z_2}$ in the late 1940s. He conjectured that Gentzen's Hauptsatz not only holds for first order logic but also for higher order logic, also known as simple type theory, **STT**. This came to be known as Takeuti's fundamental conjecture.[7] The particular sequent calculus he introduced was called a generalized logic calculus, GLC [136]. $\mathbf{Z_2}$ can be viewed as a subtheory of GLC. In the setting of GLC the comprehension principle ($\mathbf{CA}$) is encapsulated in the right introduction rule for the existential second order quantifier and the left introduction rule for the universal second order quantifier. In order to display these rules the following notation is convenient. If $F(U)$ and

[5] Takeuti describes the situation as follows: "In Hilbert's day people really worried about contradictions in set theory. But today people have great confidence in set theory. This confidence is based partly on their experience and partly on habit and simply not thinking about the subject." [137, p. 366]

[6] Admittedly, this description is highly idealized. Most mainstream mathematicians would not be able to state the axioms of **ZFC**.

[7] "Having proposed the fundamental conjecture, I concentrated on its proof and spent several years in an anguished struggle trying to resolve the problem day and night." [142, p. 133]

$A(a)$ are formulae then $F(\{v \mid A(v)\})$ arises from $F(U)$ by replacing all subformulae $t \in U$ of $F(U)$ (with U indicated) by $A(t)$. The rules for second order quantifiers can then be stated as follows:[8]

$$\frac{F(\{v \mid A(v)\}), \Gamma \Rightarrow \Delta}{\forall X\, F(X), \Gamma \Rightarrow \Delta}\,(\forall_2\,\mathrm{L}) \qquad \frac{\Gamma \Rightarrow \Delta,\ F(U)}{\Gamma \Rightarrow \Delta,\ \forall X\, F(X)}\,(\forall_2\,\mathrm{R})$$

$$\frac{F(U), \Gamma \Rightarrow \Delta}{\exists X\, F(X), \Gamma \Rightarrow \Delta}\,(\exists_2\,\mathrm{L}) \qquad \frac{\Gamma \Rightarrow \Delta,\ F(\{v \mid A(v)\})}{\Gamma \Rightarrow \Delta,\ \exists X\, F(X)}\,(\exists_2\,\mathrm{R})$$

To deduce an instance $\exists X\, \forall x\, [x \in X \leftrightarrow A(x)]$ of (**CA**) just let $F(U)$ be the formula $\forall x\, [x \in U \leftrightarrow A(x)]$ and observe that $F(\{x \mid A(x)\}) \equiv \forall x\, [A(x) \leftrightarrow A(x)]$, and hence

$$\frac{\begin{array}{c}\vdots\\ \Gamma \Rightarrow \Delta, \forall x\, [A(x) \leftrightarrow A(x)]\end{array}}{\Gamma \Rightarrow \Delta,\ \exists X\, \forall x\, [x \in X \leftrightarrow A(x)]}\,(\exists_2\,\mathrm{R})$$

As the deducibility of the empty sequent is ruled out if cut elimination holds for GLC (or just the fragment GLC_2 corresponding to $\mathbf{Z_2}$), Takeuti's Fundamental Conjecture entails the consistency of $\mathbf{Z_2}$. However note that it does not yield the subformula property as in the first order case since the minor formula $F(\{x \mid A(x)\})$ in $(\exists_2\,\mathrm{R})$ and $(\forall_2\,\mathrm{L})$ may have a much higher (quantifier) complexity than the principal formula $\exists X F(X)$ and $\forall X F(X)$, respectively. Indeed, $\exists X F(X)$ may be a proper subformula of $A(x)$ which clearly exhibits the impredicative nature of these inferences and shows that they are strikingly different from those in predicative analysis where a proper subformula property obtains.

In 1960 Schütte [119] developed a semantic equivalent to the (syntactic) fundamental conjecture using partial or semi-valuations. He employed the method of search trees (or deduction chains) to show that a formula F that cannot be deduced in the cut-free system has a deduction chain without axioms which then gives rise to a partial valuation V assigning the value "false" to F. From the latter he inferred that the completeness of the cut-free system is equivalent to the semantic property that every partial valuation can be extended to a total valuation (basically a Henkin model of **STT**). In 1966 Tait [132] succeeded in proving cut-elimination for second order logic using Schütte's semantic equivalent for that fragment. Around 1967 Takahashi [135] and Prawitz [94] independently proved for full classical simple type that every partial valuation extends to a total one, thereby establishing Takeuti's fundamental conjecture. These results, though, were somewhat disappointing as they were obtained by highly non-constructive methods that provided no concrete method for eliminating cuts in a derivation. Though, Girard showed in 1971 [49] that simple type theory not only allows cut-elimination but that there is also a terminating normalization procedure.[9] These are clearly very interesting result, however, as far as instilling trust in the consistency of $\mathbf{Z_2}$ or **SST** is concerned, the cut elimination or termination proofs are just circular since they blatantly use the very comprehension principles formalized in these theories (and a bit more). To quote Takeuti: "*My fundamental conjecture itself has been resolved in a sense by Motoo Takahashi and Dag Prawitz independently. However, their proofs rely on set theory, and so it cannot be regarded as an execution of Hilbert's Program.*" ([142, p. 133]) Takeuti's work on his conjecture instead focussed on partial results. A major breakthrough that galvanized research in proof theory, especially ordinal-theoretic investigations, was made by him in 1967. In [138] he gave a consistency proof for Π^1_1 comprehension and thereby for the first time obtained an ordinal analysis of an impredicative theory. For this Takeuti vastly extended Gentzen's method of assigning ordinals (ordinal diagrams, to be precise) to purported

[8]Below in $(\forall_2\mathrm{L})$ and $(\exists_2\mathrm{R})$, $A(a)$ is an arbitrary formula. The variable U in $(\forall_2\mathrm{R})$ and $(\exists_2\mathrm{L})$ is an eigenvariable of the respective inference, i.e. U is not to occur in the lower sequent.

[9]In the 1970s Martin-Löf gave a normalization proof for a type theory with a universe that contained itself. The metatheory for this proof was basically a slight extension of the same type theory. Ironically, it turned out that the type theory was inconsistent.

derivations of the empty sequent (inconsistency). It is worth quoting Takeuti's own assessment of his achievements.

"... *the subsystems for which I have been able to prove the fundamental conjecture are the system with* Π^1_1 *comprehension axiom and a slightly stronger system, that is, the one with* Π^1_1 *comprehension axiom together with inductive definitions.[...] Mariko Yasugi and I tried to resolve the fundamental conjecture for the system with the* Δ^1_2 *comprehension axiom within our extended version of the finite standpoint. Ultimately, our success was limited to the system with provably* Δ^1_2 *comprehension axiom. This was my last successful result in this area.*" ([142, p. 133])

5 A brief history of early ordinal representation systems: 1904-1950

Ordinals assigned as lengths to deductions to keep track of the cost of operations such as cut elimination render ordinal analyses of theories particularly transparent. In the case of **PA**, Gentzen could rely on Cantor's normal form for a supply of ordinal representations. For stronger theories, though, segments larger than ε_0 have to be employed. Ordinal representation systems utilized by proof theorists in the 1960s arose in a purely set-theoretic context. This subsection will present some of the underlying ideas as progress in ordinal-theoretic proof theory also hinges on the development of sufficiently strong and transparent ordinal representation systems.

In 1904, Hardy [58] wanted to "construct" a subset of $\mathbb{R}$ of size $\aleph_1$. His method was to represent countable ordinals via increasing sequence of natural numbers and then to correlate a decimal expansion with each such sequence. Hardy used two processes on sequences: (i) Removing the first element to represent the successor; (ii) Diagonalizing at limits. E.g., if the sequence $1, 2, 3, \ldots$ represents the ordinal 1, then $2, 3, 4, \ldots$ represents the ordinal 2 and $3, 4, 5, \ldots$ represents the ordinal 3 etc., while the 'diagonal' $1, 3, 5, \ldots$ provides a representation of ω. In general, if $\lambda = \lim_{n\in\mathbb{N}} \lambda_n$ is a limit ordinal with $b_{n1}, b_{n2}, b_{n3}, \ldots$ representing $\lambda_n < \lambda$, then $b_{11}, b_{22}, b_{33}, \ldots$ represents λ. This representation, however, depends on the sequence chosen with limit λ. A sequence $(\lambda_n)_{n\in\mathbb{N}}$ with $\lambda_n < \lambda$ and $\lim_{n\in\mathbb{N}} \lambda_n = \lambda$ is called a *fundamental sequence* for λ. Hardy's two operations give explicit representations for all ordinals $< \omega^2$.

Veblen [146] extended the initial segment of the countable for which fundamental sequences can be given effectively. The new tools he devised were the operations of *derivation* and *transfinite iteration* applied to *continuous increasing functions* on ordinals.

Definition 5.1 Let ON be the class of ordinals. A (class) function $f : ON \to ON$ is said to be *increasing* if $\alpha < \beta$ implies $f(\alpha) < f(\beta)$ and *continuous* (in the order topology on ON) if

$$f(\lim_{\xi<\lambda} \alpha_\xi) = \lim_{\xi<\lambda} f(\alpha_\xi)$$

holds for every limit ordinal λ and increasing sequence $(\alpha_\xi)_{\xi<\lambda}$. f is called *normal* if it is increasing and continuous.

The function $\beta \mapsto \omega+\beta$ is normal while $\beta \mapsto \beta+\omega$ is not continuous at ω since $\lim_{\xi<\omega}(\xi+\omega) = \omega$ but $(\lim_{\xi<\omega} \xi) + \omega = \omega + \omega$.

Definition 5.2 The *derivative* f' of a function $f : ON \to ON$ is the function which enumerates in increasing order the solutions of the equation $f(\alpha) = \alpha$, also called the *fixed points* of f.

If f is a normal function, $\{\alpha : f(\alpha) = \alpha\}$ is a proper class and f' will be a normal function, too.

Definition 5.3 *Now, given a normal function* $f : ON \to ON$*, define a hierarchy of normal functions as follows:*

$$f_0 = f \qquad f_{\alpha+1} = f'_\alpha$$

$$f_\lambda(\xi) \;=\; \xi^{th} \textit{ element of } \bigcap_{\alpha<\lambda} (\textit{Range of } f_\alpha) \qquad \textit{for } \lambda \textit{ a limit ordinal.}$$

In this way, from the normal function f we get a two-place function, $\varphi_f(\alpha, \beta) := f_\alpha(\beta)$. One usually discusses the hierarchy when $f = \ell$, where $\ell(\alpha) = \omega^\alpha$.

The least ordinal $\gamma > 0$ closed under φ_ℓ, i.e. the least ordinal > 0 satisfying $(\forall \alpha, \beta < \gamma)\ \varphi_\ell(\alpha, \beta) < \gamma$ is called Γ_0. It has a somewhat iconic status, in particular since Feferman [28] and Schütte [121, 122] determined it to be the least ordinal 'unreachable' by certain *predicative* means expressed in terms of autonomous progressions of theories (defined in subsection 7.2).

Veblen extended this idea first to arbitrary finite numbers of arguments, but then also to transfinite numbers of arguments, with the proviso that in, for example $\Phi_f(\alpha_0, \alpha_1, \ldots, \alpha_\eta)$, only a finite number of the arguments α_ν may be non-zero. Finally, Veblen singled out the ordinal $E(0)$, where $E(0)$ is the least ordinal $\delta > 0$ which cannot be named in terms of functions $\Phi_\ell(\alpha_0, \alpha_1, \ldots, \alpha_\eta)$ with $\eta < \delta$, and each $\alpha_\gamma < \delta$.

Though the "great Veblen number" (as $E(0)$ is sometimes called) is quite an impressive ordinal it does not furnish an ordinal representation sufficient for the task of analyzing a theory as strong as Π^1_1 comprehension. Of course, it is possible to go beyond $E(0)$ and initiate a new hierarchy based on the function $\xi \mapsto E(\xi)$ or even consider hierarchies utilizing finite type functionals over the ordinals. Still all these further steps amount to rather modest progress over Veblen's methods. In 1950 Bachmann [4] presented a new kind of operation on ordinals which dwarfs all hierarchies obtained by iterating Veblen's methods. Bachmann builds on Veblen's work but his novel idea was the systematic use of *uncountable ordinals* to keep track of the functions defined by diagonalization. Let Ω be the first uncountable ordinal. Bachmann defines a set of ordinals $\mathfrak{B}$ closed under successor such that with each limit $\lambda \in \mathfrak{B}$ is associated an increasing sequence $\langle \lambda[\xi] : \xi < \tau_\lambda \rangle$ of ordinals $\lambda[\xi] \in \mathfrak{B}$ of length $\tau_\lambda \leq \mathfrak{B}$ and $\lim_{\xi < \tau_\lambda} \lambda[\xi] = \lambda$. A hierarchy of functions $(\overset{\mathfrak{B}}{\varphi}_\alpha)_{\alpha \in \mathfrak{B}}$ is then obtained as follows:

$$\begin{aligned}
&\overset{\mathfrak{B}}{\varphi}_0(\beta) \;=\; 1 + \beta \qquad\qquad \overset{\mathfrak{B}}{\varphi}_{\alpha+1} \;=\; \left(\overset{\mathfrak{B}}{\varphi}_\alpha\right)' \\
&\overset{\mathfrak{B}}{\varphi}_\lambda \;\text{ enumerates }\; \bigcap_{\xi < \tau_\lambda} (\text{Range of } \overset{\mathfrak{B}}{\varphi}_{\lambda[\xi]}) \qquad \text{if } \lambda \text{ is a limit with } \tau_\lambda < \Omega \\
&\overset{\mathfrak{B}}{\varphi}_\lambda \;\text{ enumerates }\; \{\beta < \Omega : \overset{\mathfrak{B}}{\varphi}_{\lambda[\beta]}(0) = \beta\} \qquad \text{if } \lambda \text{ is a limit with } \tau_\lambda = \Omega.
\end{aligned} \tag{8}$$

After the work of Bachmann, the story of ordinal representations becomes very complicated. Significant papers (by Isles, Bridge, Pfeiffer, Schütte, Gerber to mention a few) involve quite horrendous computations to keep track of the fundamental sequences. Also Bachmann's approach was combined with uses of higher type functionals by Aczel and Weyhrauch. Feferman proposed an entirely different method for generating a Bachmann-type hierarchy of normal functions which does not involve fundamental sequences. Buchholz further simplified the systems and proved their computability. For details we recommend the preface to [13].

6 Subsystems of second order arithmetic and reverse mathematics

The theory $\mathbf{Z_2}$ of second order arithmetic was already introduced in section 4. Here we will single out some of its more prominent subtheories and introduce the program of reverse mathematics. Using the notations from the previous section we will be able to express the strength of some of them in terms of proof-theoretic ordinals. To this end we will also briefly discuss the notions of proof-theoretic reduction between theories and the assignment of ordinals to theories.

6.1 The language of second order arithmetic

Recall that $\mathbf{Z_2}$ is a two-sorted formal system with one sort of variables $x, y, z, \ldots$ ranging over natural numbers and the other sort $X, Y, Z, \ldots$ ranging over sets of natural numbers. The language $\mathcal{L}_2$ of second-order arithmetic also contains the symbols of **PA**, and in addition has a binary relation symbol $\in$ for elementhood. Formulae are built from the prime formulae $s = t$,

$s < t$, and $s \in X$ (where s, t are numerical terms, i.e. terms of **PA**) by closing off under the connectives $\wedge, \vee, \rightarrow, \neg$, numerical quantifiers $\forall x, \exists x$, and set quantifiers $\forall X, \exists X$.

The basic arithmetical axioms in all theories of second-order arithmetic are the defining axioms for $0, 1, +, \times, E, <$ (as for **PA**) and the *induction axiom*

$$\forall X(0 \in X \wedge \forall x(x \in X \rightarrow x+1 \in X) \rightarrow \forall x(x \in X)).$$

We consider the axiom schema of $\mathcal{C}$*-comprehension* for formula classes $\mathcal{C}$ which is given by

$$\mathcal{C}\text{-}\mathbf{CA} \qquad \exists X \forall u(u \in X \leftrightarrow F(u))$$

for all formulae $F \in \mathcal{C}$ in which X does not occur. Natural formula classes are the *arithmetical formulae*, consisting of all formulae without second order quantifiers $\forall X$ and $\exists X$, and the Π^1_n*-formulae*, where a Π^1_n-formula is a formula of the form $\forall X_1 \ldots QX_n\, A(X_1, \ldots, X_n)$ with $\forall X_1 \ldots QX_n$ being a string of n alternating set quantifiers, commencing with a universal one, followed by an arithmetical formula $A(X_1, \ldots, X_n)$.

Also "mixed" forms of comprehension are of interest, e.g,

$$\Delta^1_n\text{-}\mathbf{CA} \qquad \forall u\,[F(u) \leftrightarrow G(u)] \rightarrow \exists X\, \forall u\,[u \in X \leftrightarrow F(u)]$$

where $F(u)$ is in Π^1_n and $G(u)$ in Σ^1_n.

One also considers Δ^1_n comprehension rules:

$$\Delta^1_n\text{-}\mathbf{CR} \qquad \frac{\forall u\,[F(u) \leftrightarrow G(u)]}{\exists X\, \forall u\,[u \in X \leftrightarrow F(u)]} \qquad \text{if } F(u) \in \Pi^1_n,\ G(u) \in \Sigma^1_n$$

For each axiom scheme $\mathbf{Ax}$ we denote by $(\mathbf{Ax})_0$ the theory consisting of the basic arithmetical axioms plus the scheme $\mathbf{Ax}$. By contrast, $(\mathbf{Ax})$ stands for the theory $(\mathbf{Ax})_0$ augmented by the scheme of induction for all $\mathcal{L}_2$-formulae.

An example for these notations is the theory $(\mathbf{\Pi}^1_1 - \mathbf{CA})_0$ which has the comprehension schema for $\mathbf{\Pi}^1_1$-formulae.

In **PA** one can define an elementary injective pairing function on numbers, e.g $(n, m) := 2^n \times 3^m$. With the help of this function an infinite sequence of sets of natural numbers can be coded as a single set of natural numbers. The n^{th} section of set of natural numbers U is defined by $U_n := \{m : (n, m) \in U\}$. Using this coding, we can formulate a form of the axiom of choice for formulae F in $\mathcal{C}$ by

$$\mathcal{C} - \mathbf{AC} \qquad \forall x \exists Y F(x, Y) \rightarrow \exists Y \forall x F(x, Y_x).$$

6.2 Proof-theoretic reductions

When Hilbert formulated his consistency program many people found it difficult to see what it had to do with his (more understandable) concern with the safety of abstract principles in proofs of real (Π^0_1) statements (e.g. Brouwer).[10] Indeed, a consistency proof for a theory **T** brings about conservativeness for Π^0_1 statements.

Theorem 6.1 *If S and T are theories containing a modicum of arithmetic and F is a Π^0_1 statement then*

$$S \vdash \mathrm{Con}(T) \textit{ and } T \vdash F \textit{ implies } S \vdash F.$$

Moreover, the above theorem is provable in **PRA**.[11]

[10] [7, p. 491] "FOURTH INSIGHT. The recognition of the fact that the (contentual) justification of formalistic mathematics by means of the proof of its consistency contains a vicious circle, since this justification rests upon the (contentual) correctness of the proposition that from the consistency of a proposition the correctness of the proposition follows, that is, upon the (contentual) correctness of the principle of excluded middle."

[11] A clear statement of this point can be found in Hilbert's 1928 Hamburg lecture [61]. There he takes the 'Fermat proposition' F and argues that if we had a proof of it in an 'ideal' system for which we had a finitary consistency proof then we could convert this proof into a finitary proof of F.

Ordinal analyses of theories allow one to compare the strength of theories. This subsection defines the notions of *proof-theoretic reducibility* and *proof-theoretic strength* that will be used henceforth.

All theories T considered in the following are assumed to contain a modicum of arithmetic. For definiteness let this mean that the system **PRA** of Primitive Recursive Arithmetic is contained in T, either directly or by translation.

Definition 6.2 Let T_1, T_2 be a pair of theories with languages $\mathcal{L}_1$ and $\mathcal{L}_2$, respectively, and let Φ be a (primitive recursive) collection of formulae common to both languages. Furthermore, Φ should contain the closed equations of the language of **PRA**.

We then say that T_1 is *proof-theoretically* Φ*-reducible to* T_2, written $T_1 \leq_\Phi T_2$, if there exists a primitive recursive function f such that

$$\mathbf{PRA} \vdash \forall \phi \in \Phi\, \forall x\, [\mathsf{Proof}_{T_1}(x, \phi) \ \rightarrow \ \mathsf{Proof}_{T_2}(f(x), \phi)]. \tag{9}$$

Here Proof_{T_i} is an arithmetized formalization of the provability in T_i. T_1 and T_2 are said to be *proof-theoretically* Φ*-equivalent*, written $T_1 \equiv_\Phi T_2$, if $T_1 \leq_\Phi T_2$ and $T_2 \leq_\Phi T_1$.

The appropriate class Φ is revealed in the process of reduction itself, so that in the statement of theorems we simply say that T_1 is *proof-theoretically reducible* to T_2 (written $T_1 \leq T_2$) and T_1 and T_2 are *proof-theoretically equivalent* (written $T_1 \equiv T_2$), respectively. Alternatively, we shall say that T_1 and T_2 have the *same proof-theoretic strength* when $T_1 \equiv T_2$.

In practice, if $T_1 \equiv T_2$ is shown through an ordinal analysis this always entails that the two theories prove at least the same Π^0_2 sentences (those of the complexity of the twin prime conjecture).

Given a natural ordinal representation system $\langle A, \prec, \ldots \rangle$ of order type τ let $\mathbf{PRA}+\mathrm{TI}_{qf}(<\tau)$ be **PRA** augmented by quantifier-free induction over all initial (externally indexed) segments of $\prec$. This is perhaps best explained via the representation system for ε_0 given at the end of section 2. There one can take the initial segments of $\prec$ to be determined by the Gödel numbers of the ordinals $\omega_0 := 1$ and $\omega_{n+1} := \omega^{\omega_n}$ whose limit is ε_0.

Definition 6.3 We say that a theory T has *proof-theoretic ordinal* τ, written $|T| = \tau$, if T can be proof-theoretically reduced to $\mathbf{PRA} + \mathrm{TI}_{qf}(<\tau)$, i.e.,

$$T \quad \equiv_{\Pi^0_2} \quad \mathbf{PRA} + \mathrm{TI}_{qf}(<\tau).$$

Unsurprisingly, the above notion has certain intensional aspects and hinges on the naturality of the representation system.[12]

6.3 Reverse mathematics

Under the rubric of *Reverse Mathematics* a research programme was initiated by Harvey Friedman and Steve Simpson some thirty years ago. The idea is to ask whether, given a theorem, one can prove its equivalence to some axiomatic system, with the aim of determining what proof-theoretical resources are necessary for the theorems of mathematics. More precisely, the objective of reverse mathematics is to investigate the role of set existence axioms in ordinary mathematics. The main question can be stated as follows:

> *Given a specific theorem τ of ordinary mathematics, which set existence axioms are needed in order to prove τ?*

Central to the above is the reference to what is called 'ordinary mathematics'. This concept, of course, doesn't have a precise definition. Roughly speaking, by ordinary mathematics we mean main-stream, non-set-theoretic mathematics, i.e. the core areas of mathematics which make no essential use of the concepts and methods of set theory and do not essentially depend on the theory of uncountable cardinal numbers.

[12] For a discussion see [105], section 2.

For many mathematical theorems τ, there is a weakest natural subsystem $S(\tau)$ of $\mathbf{Z_2}$ such that $S(\tau)$ proves τ. Very often, if a theorem of ordinary mathematics is proved from the weakest possible set existence axioms, the statement of that theorem will turn out to be provably equivalent to those axioms over a still weaker base theory. This theme is referred to as *Reverse Mathematics*. Moreover, it has turned out that $S(\tau)$ often belongs to a small list of specific subsystems of $\mathbf{Z_2}$ dubbed $\mathbf{RCA}_0$, $\mathbf{WKL_0}$, $\mathbf{ACA}_0$, $\mathbf{ATR}_0$ and $(\mathbf{\Pi}^1_1\mathbf{-CA})_0$, respectively.[13] The systems are enumerated in increasing strength. The main set existence axioms of $\mathbf{RCA}_0$, $\mathbf{WKL_0}$, $\mathbf{ACA}_0$, $\mathbf{ATR}_0$, and $(\mathbf{\Pi}^1_1\mathbf{-CA})_0$ are recursive comprehension, weak König's lemma, arithmetical comprehension, arithmetical transfinite recursion, and $\mathbf{\Pi}^1_1$-comprehension, respectively. For exact definitions of all these systems and their role in reverse mathematics see [130]. The proof-theoretic strength of $\mathbf{RCA}_0$ is weaker than that of $\mathbf{PA}$ while $\mathbf{ACA}_0$ has the same strength as $\mathbf{PA}$. To get a sense of scale, the strengths of the first four theories are best expressed via their proof-theoretic ordinals:

Theorem 6.4 *(i)* $|\mathbf{RCA}_0| = |\mathbf{WKL_0}| = \omega^\omega$.

(ii) $|\mathbf{ACA}_0| = \varepsilon_0$.

(iii) $|\mathbf{ATR}_0| = \Gamma_0$.

$|(\mathbf{\Pi}^1_1\mathbf{-CA})_0|$, however, eludes expression in the ordinal representations introduced so far. This will require the much stronger representation to be introduced in Definition 9.1.

7 Progressions of theories and predicative proof theory

In this section we look at the idea of progressions of theories and how it relates to proof-theoretic work of the early 1960s with its cut elimination techniques for infinitary logics inspired by ramified type theory and a version of Gödel's constructible hierarchy for defining a hierarchy of sets of natural numbers.

Gödel's theorem [51] from 1931 not only showed that any recursively presented theory $\mathbf{T}$ (which is consistent and contains a modicum of arithmetic) is incomplete but also explicitly produced an unprovable yet true statement G_T that can be added to $\mathbf{T}$, making $\mathbf{T} + G_{\mathbf{T}}$ a "less incomplete" theory. This gave rise to various extension procedures $\mathbf{T} \mapsto \mathbf{T}'$ that strengthen a given theory, notably:[14]

(R1) $\mathbf{T}' = \mathbf{T} + \mathrm{Con}(\mathbf{T})$;

(R2) $\mathbf{T}' = \mathbf{T} + \{\mathsf{Proof}_{\mathbf{T}}(\ulcorner\phi\urcorner) \to \phi \mid \phi \text{ closed}\}$.

(R3) $\mathbf{T}' = \mathbf{T} + \{\forall x \mathsf{Proof}_{\mathbf{T}}(\ulcorner\phi(\bar{x})\urcorner) \to \forall x\phi(x) \mid \text{all } \phi(x) \text{ with at most } x \text{ free}\}$.

An obvious idea to redress the incompleteness of $\mathbf{T}$ is to form a sequence of theories $\mathbf{T} = \mathbf{T}_0 \subseteq \mathbf{T}_1 \subseteq \mathbf{T}_2 \subseteq \ldots$ where $\mathbf{T}_{i+1} = \mathbf{T}'_i$ and even to continue this into the transfinite by letting $\mathbf{T}_\lambda = \bigcup_{\alpha<\lambda}\mathbf{T}_\alpha$ for limit ordinals λ and $\mathbf{T}_{\alpha+1} = \mathbf{T}'_\alpha$ for successor ordinals $\alpha+1$. However, since one needs to be able to express the provability predicate for $\mathbf{T}_\lambda$ in the language of $\mathbf{T}_\lambda$ itself one cannot simply use set-theoretic ordinals. Moreover, as one really wants theories whose axioms are effectively presented, i.e., can be enumerated by a recursive function (aka as recursively enumerable) one has to deal with ordinals in an effective way.

[13] Ironically, the counterexamples that come to mind as soon as one begins to think about it come from the main inventor of reverse mathematics.

[14] Below $\mathrm{Con}(\mathbf{T})$ and $\mathsf{Proof}_{\mathbf{T}}$ are arithmetized formalizations of the consistency of $\mathbf{T}$ and provability in $\mathbf{T}$, respectively. $\ulcorner\phi\urcorner$ denotes the Gödel number of a formula ϕ. For every number n, $\bar{n}$ denotes the n^{th} numeral, i.e., the term obtained from 0 by putting n successor function symbols in front of it.

7.1 Recursive progressions

Turing in [145, §7] considers two ways in which the effective representation of ordinals can be achieved. The first way is via the set W of numbers e for recursive well-orderings $\leq_e$, the second is provided by the class of Church-Kleene notations for ordinals [20] that used expressions in the λ-calculus to describe ordinals. The latter approach was then modified in [74] to an equivalent recursion-theoretic definition that uses numerical codes to denote countable ordinals and is known as Kleene's $\mathcal{O}$.

Definition 7.1 Kleene uses $\mathrm{suc}(a) := 2^a$ as notations for successor ordinals and and $\lim(e) := 3 \cdot 5^e$ for limit ordinals.

The class $\mathcal{O}$ of ordinal notations, the partial ordering relation $<_{\mathcal{O}}$ between such notations, and the ordinal $|\,a\,|$ denoted by $a \in \mathcal{O}$ are defined simultaneously as follows:

(i) $0 \in \mathcal{O}$, and $|\,0\,| = 0$.

(ii) If $a \in \mathcal{O}$ then $\mathbf{suc}(a) \in \mathcal{O}$, $a <_{\mathcal{O}} \mathrm{suc}(a)$ and $|\mathrm{suc}(a)| = |a| + 1$.

(iii) If e is an index of a total recursive function and $\{e\}(n) <_{\mathcal{O}} \{e\}(n+1)$ holds for all $n \in \mathbb{N}$, then $\lim(e) \in \mathcal{O}$, and $|\lim(e)| = \sup\{|\{e\}(n)| \mid n \in \mathbb{N}\}$.

(iv) If $a <_{\mathcal{O}} b$ and $b <_{\mathcal{O}} c$ then $a <_{\mathcal{O}} c$.

The first ordinal τ such there is no recursive well-ordering of order type τ is usually denoted by ω^{CK} in honor of Church and Kleene. It can be shown for the above definition of $\mathcal{O}$ that the recursive ordinals are exactly those that have a notation in $\mathcal{O}$.

When it comes to theories T, quite unlike than in other areas of logic (i.e. model theory), the results of this section do not just depend on the set of axioms of T but on the way how the axioms of T are presented. Thus in talking about a theory T we shall always assume that T is given by a Σ^0_1 formula $\psi(v_0)$ such that ϕ is an axiom of T if $\psi(\ulcorner\phi\urcorner)$ holds.

Kleene's $\mathcal{O}$ provides a framework for attaching a transfinite hierarchy of theories to any given theory.

Definition 7.2 A **consistency progression** based on $\mathbf{T}$ is a primitive recursive function $n \mapsto \psi_n$, where $\psi_n(v_0)$ is a Σ^0_1 formula that defines T_n and that $\mathbf{PA}$ proves: (i) $T_0 = \mathbf{T}$; (ii) $T_{\mathrm{suc}(n)} = T'_n$ where $T'_n = T_n + \mathrm{Con}(T_n)$; (iii) $T_{\lim(n)} = \bigcup_x T_{\{n\}(x)}$.

Likewise we define a **local reflection progression** and a **uniform reflection progression** based on $\mathbf{T}$ by requiring that $T_{\mathrm{suc}(n)} = T'_n$ where T'_n is obtained from T_n by the operation (R1)and (R2), respectively.

The existence of primitive recursive functions $n \mapsto \psi_n$ as in the foregoing definition is an easy consequence of the primitive recursive recursion theorem.

If all axioms of $\mathbf{T}$ a true (in the standard model) it can be shown for all $a \in \mathcal{O}$ by transfinite on $|\,a\,|$ that T_a is a true theory (in a sufficiently strong metatheory).

For numbers a outside $\mathcal{O}$ the theories T_a bear no interest. They may actually be inconsistent.[15]

Theorem 7.3 (Turing's completeness result) *For any true Π^0_1 sentence ϕ a number $a_\phi \in \mathcal{O}$ with $|\,a\,| = \omega + 1$ can be constructed such that $T_a \vdash \phi$. Moreover, the function $\phi \mapsto a_\phi$ is given by a primitive recursive function.*

At first glance Turing's theorem seems to provide some insight into the nature of true Π^0_1 statements. That this is an illusion is revealed by its simple proof which is just based on the

[15]For example, the recursion theorem ensures that there exists an e such that $\{e\}(0) = \mathrm{suc}(\lim(e))$. As $T_{\{e\}(0)} \subseteq T_{\lim(e)}$ and $T_{\{e\}(0)}$ proves the consistency of $T_{\lim(e)}$, both theories are inconsistent.

trick of coding the truth of ϕ as a member of $\mathcal{O}$.[16] The proof also shows that the infinitely many iterated consistency axioms $\mathrm{Con}(T_0), \mathrm{Con}(T_1), \ldots$ of $T_{\mathbf{suc}(\lim(e))}$ are totally irrelevant for proving θ. As it turns out, the reason why one has to go to stage $\omega+1$ is simply that only at stage ω a non-standard definition of the axioms of $\bigcup_{n<\omega} T_n$ can be introduced. And actually a non-standard definition of the axioms of T_0 would serve the same purpose. Setting $\vartheta(v_0) :\Leftrightarrow \psi_0(v_0) \vee (\exists x\, \neg\psi(x) \wedge v_0 = \ulcorner 0 = 1 \urcorner)$, the theory T_0' defined by the Σ^0_1 formula ϑ_0 has the same axioms as T_0, but the difference is that the consistency of T_0' implies θ (provably so in **PA**).

Also note that epistemologically recognizing that $\mathbf{suc}(\lim(e))$ is in $\mathcal{O}$ hinges on us knowing that θ is true, and hence nothing is gained by further knowing that $T_{\mathbf{suc}(\lim(e))} \vdash \theta$.

The proof of the Theorem 7.3 works with any consistency progression. Turing actually considered slightly stronger progressions in that he used a special version of local reflection progressions, where (R2) is restricted to Π^0_2 sentences, i.e., sentences of $\forall\exists$ form with primitive recursive matrix. He took at one of his main aims to show that these progressions are complete for Π^0_2 sentences. However, it is easy to show that this is not the case.

Theorem 7.4 *Let $(T_a)_{a\in\mathcal{O}}$ be a progression based on the local reflection principle. Then the following hold:*[17]

(i) $\bigcup_{a\in\mathcal{O}} T_a \subseteq T_0 +$ *all true* Π^0_1 *sentences.*

(ii) There is a true Π^0_2 sentence that is not provable in $\bigcup_{a\in\mathcal{O}} T_a$.

The problem left open after Turing's thesis, namely whether any stronger progressions can be complete for Π^0_2 statements, was addressed by Feferman [27] with the amazing result that progressions based on the uniform reflection principle were not only complete with respect to Π^0_2 sentences but for all arithmetical sentences.

Theorem 7.5 (Feferman's completeness theorem 1962) *Let $(T_a)_{a\in\mathcal{O}}$ be a progression based on the uniform reflection principle with $T_0 = \mathbf{PA}$.*

For any true arithmetical sentence θ there exists $a \in \mathcal{O}$ such that $T_a \vdash \theta$. Moreover, $a \in \mathcal{O}$ can be chosen such that $|\,a\,| \leq \omega^{\omega^2+1}$.

In contrast to Turing's 7.3, the proof of this theorem is rather difficult and involved and also utilizes Shoenfield's theorem [127] that Peano arithmetic with the recursive ω-rule is complete for arithmetical statements. Accessible accounts can be found in Torkel Franzén's book [36] and his paper [37]. Again, however, as far as mathematical knowledge is concerned, the same circularity as in Turing's completeness theorem obtains in Theorem 7.5 in that recognizing an $a \in \mathcal{O}$ with $T_a \vdash \psi$ is at least as hard as recognizing that ψ is true. The starting point for

[16]Let θ be $\forall x\, \psi(x)$ with ψ primitive recursive. Define e by the recursion theorem such that provably in **PA**

$$\{e\}(n) = \begin{cases} n_\Omega & \text{if } \psi(\bar{k}) \text{ is true for every } k \leq n \\ \mathbf{suc}(\lim(e)) & \text{otherwise.} \end{cases}$$

Note that $\{e\}$ is clearly total (by induction on n). On account of θ being true, we have $\{e\}(n) = n_\Omega$ for all n. Thus $\lim(e)$ belongs to Ω and $|\lim(e)| = \omega$. We claim that the consistency of $T_{\lim(e)}$ entails that θ is true. For if θ were false we would have $\neg\psi(\bar{m})$ for some m and thus $T_{\{e\}(n)} = T_{\mathbf{suc}(\lim(e))}$ for all $n \geq m$. But, by design, $T_{\mathbf{suc}(\lim(e))}$ proves the consistency of $T_{\lim(e)}$ and $T_{\{e\}(n)}$ is a subtheory of $T_{\lim(e)}$ for all n. Thus $T_{\lim(e)}$ would prove its own consistency, rendering it inconsistent (by Gödel's second incompleteness theorem). The foregoing reasoning can be formalized in **PA** and a fortiori in $T_{\mathbf{suc}(\lim(e))}$. As a result, $T_{\mathbf{suc}(\lim(e))} \vdash \theta$. $\square$

[17]**Proof**: Let $T^* := T_0 +$ all true Π^0_1 sentences.

(i): We show by induction on $a \in \mathcal{O}$ that $T_a \subseteq T^*$. Only the successor step needs to be looked at. So suppose $a = \mathrm{suc}(b)$ and $T_b \subseteq T^*$. It suffices to show that $T^* \vdash \mathsf{Proof}_{T_b}(\ulcorner\phi\urcorner) \to \phi$ holds for every sentence ϕ. There are two cases to consider. If $\mathsf{Proof}_{T_b}(\ulcorner\phi\urcorner)$ is false then $\neg\mathsf{Proof}_{T_b}(\ulcorner\phi\urcorner)$ is a true Π^0_1 sentence and hence $T^* \vdash \neg\mathsf{Proof}_{T_b}(\ulcorner\phi\urcorner)$ which entails $T^* \vdash \mathsf{Proof}_{T_b}(\ulcorner\phi\urcorner) \to \phi$. If $\mathsf{Proof}_{T_b}(\ulcorner\phi\urcorner)$ is true the $T_b \vdash \phi$ and, by the induction hypothesis, $T^* \vdash \phi$ which also yields $T^* \vdash \mathsf{Proof}_{T_b}(\ulcorner\phi\urcorner) \to \phi$.

(ii) The provability predicate for T^* is of complexity Σ^0_2. If $\bigcup_{a\in\mathcal{O}} T_a$ proved all true Π^0_2 sentences then T^* would be Π^0_2 complete. But that is absurd on account of the arithmetical hierarchy theorem. $\square$

constructing such an $a \in \mathcal{O}$ therefore is the truth of ψ and as in Turing's theorem one proceeds to cook up a via application of the primitive recursion theorem, albeit this time a very intricate one.

7.2 Autonomous progressions

The problem of gaining insight into which expressions are actually notations for ordinals led to the concept of **autonomous progressions of theories**, where one is allowed to ascend to a theory T_a only if one already has shown in a previously accepted theory T_b that $a \in \mathcal{O}$. This idea of generating a hierarchy of theories via a boot-strapping process appeared for the first time in Kreisel 1958 [82] where it was proposed as a way of characterizing finitism and predicativism in mathematically precise way. In more formal terms, the starting point is a theory $\mathbf{T}_0$ which is accepted as correct and an extension procedure $\mathbf{T} \mapsto \mathbf{T}'$ which is viewed as leading from a correct theory $\mathbf{T}$ to a more encompassing correct theory $\mathbf{T}'$. Moreover, the language of these theories is supposed to contain a formula $\mathrm{Acc}(x)$ such that provability of $\mathrm{Acc}(\bar{a})$ in a correct theory entails that $a \in \mathcal{O}$.[18] Kreisel singled out two autonomous progressions of theories $\{\mathbf{F}_a\}$ and $\{\mathbf{R}_a\}$ for finitism and predicativity, respectively, and determined the least upper bound of the $|a|$ appearing in the first hierarchy to be the ordinal ε_0 which is also the proof-theoretic ordinal of **PA**. The determination of the least upper bound for the predicative hierarchy $\{\mathbf{R}_a\}$ was achieved independently by Feferman [28] and Schütte [121, 122]. It turned out that this ordinal can be expressed in a notation system developed by Veblen and was shown to be Γ_0, the least non-zero ordinal closed under the Veblen function $\alpha, \beta \mapsto \varphi_\alpha(\beta)$. This was a genuine proof-theoretic result with the tools coming ready-made from Schütte's 1960 monograph [120] Schütte had calculated the proof-theoretic ordinals of the $\mathbf{R}_a$ as a function of $|a|$, using cut elimination techniques for logics with infinitary rules (dubbed "semi-formal systems"). If $|a| = \omega^\alpha$ then $|\mathbf{R}_a| = \varphi_\alpha(0)$. This work will be discussed next.

7.3 Proof theory of the ramified analytic hierarchy

A major stumbling block for proving Takeuti's fundamental conjecture is that in $(\forall_2\mathrm{L})$ and $(\exists_2\mathrm{R})$ inferences the minor formula $F(\{v \mid A(v)\})$ can have a much higher complexity than the principal (inferred) formula $QX\,F(X)$. If, instead, one allowed these inferences only in cases where the 'abstraction' term $\{v \mid A(v)\}$ had (in some sense) a lower complexity than $QX\,F(X)$, cut elimination could be restored. To implement this idea, one introduces a hierarchy of sets (formally represented by abstraction terms) whose complexity is stratified by ordinal levels α, and a pertaining hierarchy of quantifiers $\forall X^\beta$ and $\exists X^\beta$ conceived to range over sets of levels $< \beta$.

The semi-formal system $\mathbf{RA}^*$ of Ramified Analysis "Semi-formal" is a terminology employed by Schütte and refers to the fact that this proof system has rules with infinitely many premisses, similar to the ω rule.

Definition 7.6 In the following we assume that the ordinals come from some representation system. The language of $\mathbf{RA}^*$ is an extension of that of first order arithmetic. For each ordinal α and $\beta > 0$ it has free set variables $U_0^\alpha, U_1^\alpha, U_2^\alpha \ldots$ of level α and bound set variables of level β. The level, $\mathrm{lev}(A)$, of a formula A of $\mathbf{RA}^*$ is defined to be the maximum of the levels of set variables that occur in A. Expressions of the form $\{x \mid A(x)\}$ with $A(u)$ a formula will be called *abstraction terms*, their level being the same as that of the formula $A(u)$.

The inference rules of $\mathbf{RA}^*$ comprise those of the sequent calculus with the exception of $(\forall R)$ and $(\exists L)$ which are replaced by those for the ω-rule: ωR and ωL. Below $\mathfrak{P}_\beta$ stands for the

[18] This is straightforward for languages allowing for quantifiers over sets of natural numbers, but for theories like **PA** one would have to add a new predicate symbol to the language.

set of all abstraction terms with levels $< \beta$. The rules for the set quantifiers are as follows:

$$\frac{F(P), \Gamma \Rightarrow \Delta,}{\forall X^\beta F(X^\beta), \Gamma \Rightarrow \Delta}\, \forall_\beta \mathrm{L} \qquad \frac{\Gamma \Rightarrow \Delta,\ F(P) \ \text{ all } P \in \mathfrak{P}_\beta}{\Gamma \Rightarrow \Delta,\ \forall X^\beta F(X^\beta)}\, \forall_\beta \mathrm{R}$$

$$\frac{F(P), \Gamma \Rightarrow \Delta \quad \text{all } P \in \mathfrak{P}_\beta}{\exists X^\beta F(X^\beta), \Gamma \Rightarrow \Delta}\, \exists_\beta \mathrm{L} \qquad \frac{\Gamma \Rightarrow \Delta,\ F(P)}{\Gamma \Rightarrow \Delta,\ \exists X^\beta F(X^\beta)}\, \exists_\beta \mathrm{R}$$

where in $\forall_\beta$L and $\exists_\beta$R, P is an abstraction term of level $< \beta$.

As per usual, the price one has to pay for rules with infinitely many premisses is that derivations become infinite (well-founded) trees. The length of a derivation can then be measured by the ordinal rank associated with the tree. One also wants to keep track of the complexity of cuts in the derivation. The length we assign to a formula A, $|A|$, measures its complexity. It is an ordinal of the form $\omega \cdot \alpha + n$ where α is the level of A and $n < \omega$. One then defines a notion of derivability in **RA**,

$$\mathbf{RA}^* \mid\!\frac{\alpha}{\rho}\, \Gamma \Rightarrow \Delta$$

where α majorizes the transfinite length of the derivation and ρ conveys that all cut formulae in the derivation have length $< \rho$.

Cut elimination works smoothly for $\mathbf{RA}^*$, however, the prize one has to pay can only be measured in terms of Veblen's φ function. The optimal result is the so-called second cut elimination theorem.

Theorem 7.7 (Second Cut Elimination Theorem)
If $\mathbf{RA}^* \mid\!\frac{\alpha}{\rho+\omega^\nu}\, \Gamma \Rightarrow \Delta$ *then* $\mathbf{RA}^* \mid\!\frac{\varphi_\nu(\alpha)}{\rho}\, \Gamma \Rightarrow \Delta$.

It entails of course the special case that $\mathbf{RA}^* \mid\!\frac{\alpha}{\omega^\nu}\, \Gamma \Rightarrow \Delta$ yields $\mathbf{RA}^* \mid\!\frac{\varphi_\nu(\alpha)}{0}\, \Gamma \Rightarrow \Delta$, and thus, as the latter deduction is cut-free, all cuts can be removed. Several subtheories of $\mathbf{Z_2}$ can be interpreted in $\mathbf{RA}^*$, yielding upper bounds for their proof-theoretic ordinals via Theorem 7.7. Here is selection:

Theorem 7.8 *(i)* $|\mathbf{ACA}_0| = \varepsilon_0$.

(ii) $|\mathbf{ACA}| = \varphi_2(0) = \varepsilon_{\varepsilon_0}$.

(iii) $|(\Delta^1_1\text{-}\mathbf{CR})| = \varphi_\omega(0)$.

(iv) $|(\Delta^1_1\text{-}\mathbf{CA})| = |(\Sigma^1_1\text{-}\mathbf{AC})| = \varphi_{\varepsilon_0}(0)$.

To obtain the results about theories in (iii) and (iv) it is somewhat easier to first reduce them to systems of the form $(\Pi^0_1\text{-}\mathbf{CA})_{<\rho}$ as the latter have a straightforward interpretation in $\mathbf{RA}^*$. Reductions of $(\Delta^1_1\text{-}\mathbf{CR})$ to $(\Pi^0_1\text{-}\mathbf{CA})_{<\omega^\omega}$ and of $(\Sigma^1_1\text{-}\mathbf{AC})$ to $(\Pi^0_1\text{-}\mathbf{CA})_{<\varepsilon_0}$ are due to Feferman [28] and Friedman [38], respectively.

8 The Dialectica interpretation: Gödel and Spector

Among the proposals for extending finitary methods put forward in his 1938 lecture at Zilsel's, Gödel appears to have favored the route via higher type functions. Details of what came to be known as the Dialectica interpretation were not published until 1958 [53] but the D-interpretation itself was arrived at by 1941. Gödel's system T axiomatizes a class of functions that he called the *primitive recursive functionals of finite type*. T is a largely equational theory whose axioms are equations involving terms for higher type functionals with just a layer of propositional logic on top of that. In this way the quantifiers, problematic for finists and irksome to intuitionists, are avoided. To explain the benefits of the D-interpretation we need to have a closer look at the syntax of T.

Definition 8.1 T has a many-sorted language in that each terms is assigned a type. Type (symbols) are generated from 0 by the rule: If σ and τ are types then so is $\sigma \to \tau$. Intuitively the ground type 0 is the type of natural numbers. If σ and τ are types that are already understood then $\sigma \to \tau$ is a type whose objects are considered to be functions from objects of type σ to objects of type τ. In addition to variables $x^\tau, y^\tau, z^\tau, \ldots$ for each type τ, the language of T has special constants 0, Suc, $\mathrm{K}_{\sigma,\tau}$, $\mathrm{S}_{\rho,\sigma,\tau}$, and R_σ for all types ρ, σ, τ. The meaning of these constants is explained by their defining equations. $\mathrm{K}_{\sigma,\tau}$ and $\mathrm{S}_{\rho,\sigma,\tau}$ are familiar from combinatory logic which was introduced by Schönfinkel in 1924 [117] and became more widely known through Church's work [19]. 0 plays the role of the first natural number while Suc embodies the successor function on objects of type 0. The constants R_σ, called *recursors*, provide the main vehicle for defining functionals by recursion on $\mathbb{N}$. Term formation starts with constants and variables, and if s and t are terms of type $\sigma \to \tau$ and σ, respectively, then $s(t)$ is a term of type τ. To increase readability we shall write $t(r, s)$ instead of $(t(r))(s)$ and $t(r, s, q)$ instead of $(t(r, s))(q)$ etc. Also $\mathrm{Suc}(t)$ will be shortened to t'. The defining axioms for the constants are the following:[19]

1. $\neg t' = 0$
2. $t' = r' \to t = r$
3. $\mathrm{K}_{\sigma,\tau}(s, t) = s$
4. $\mathrm{S}_{\rho,\sigma,\tau}(r, s, t) = (r(t))(s(t))$
5. $\mathrm{R}_\sigma(f, g, 0) = f$
6. $\mathrm{R}_\sigma(f, g, n') = g(n, \mathrm{R}_\sigma(f, g, n))$.

The axioms of T consist of the above defining axioms, equality axioms and axioms for propositional logic. Inference rules are modus ponens and the induction rule

$$\text{from } A(0) \text{ and } A(x) \to A(x') \text{ conclude } A(t)$$

for t of type 0 and x not in $A(0)$.

The first step towards the D-interpretation of Heyting arithmetic in T consists of associating to each formula A of arithmetic a syntactic translation A^D which is of the form

$$A^D \quad \equiv \quad \exists x^{\sigma_1} \ldots \exists x^{\sigma_n} \forall y^{\tau_1} \ldots \forall y^{\tau_m}\, A_D(\vec{x}, \vec{y}\,)$$

with $A_D(\vec{x}, \vec{y}\,)$ being quantifier free. Thus A^D is not a formula of T but of its augmentation via quantifiers $\forall x^\tau$ and $\exists y^\tau$ for all types τ. The translation proceeds by induction on the buildup of A. The cases where the outermost logical symbol of A is among $\wedge, \vee, \exists x, \forall x$ are rather straightforward. The crucial case occurs when A is an implication $B \to C$. To increase readability we shall suppress the typing of variables. Let $B^D \equiv \exists \vec{x}\, \forall \vec{y}\, B_D(\vec{x}, \vec{y}\,)$ and $C^D \equiv \exists \vec{u}\, \forall \vec{v}\, C_D(\vec{u}, \vec{v}\,)$. Then one uses a series of judicious equivalences to bring the quantifiers in $B^D \to C^D$ to the front and finally employs skolemization of existential variables as follows:

$$\begin{array}{rl}
(i) & \exists \vec{x}\, \forall \vec{y}\, B_D(\vec{x}, \vec{y}\,) \to \exists \vec{u}\, \forall \vec{v}\, C_D(\vec{u}, \vec{v}\,) \\
(ii) & \forall \vec{x}\, [\forall \vec{y}\, B_D(\vec{x}, \vec{y}\,) \to \exists \vec{u}\, \forall \vec{v}\, C_D(\vec{u}, \vec{v}\,)] \\
(iii) & \forall \vec{x}\, \exists \vec{u}\, [\forall \vec{y}\, B_D(\vec{x}, \vec{y}\,) \to \forall \vec{v}\, C_D(\vec{u}, \vec{v}\,)] \\
(iv) & \forall \vec{x}\, \exists \vec{u}\, \forall \vec{v}\, [\forall \vec{y}\, B_D(\vec{x}, \vec{y}\,) \to C_D(\vec{u}, \vec{v}\,)] \\
(v) & \forall \vec{x}\, \exists \vec{u}\, \forall \vec{v}\, \exists \vec{y}\, [B_D(\vec{x}, \vec{y}\,) \to C_D(\vec{u}, \vec{v}\,)] \\
(vi) & \forall \vec{x}\, \exists \vec{u}\, \exists Y\, \forall \vec{v}\, [B_D(\vec{x}, Y(\vec{y}\,)) \to C_D(\vec{u}, \vec{v}\,)] \\
(vii) & \exists U\, \exists Z\, \forall \vec{x}\, \forall \vec{v}\, [B_D(\vec{x}, Z(\vec{x}, \vec{v}\,)) \to C_D(U(\vec{x}\,), \vec{v}\,)].
\end{array}$$

[19] Terms have to be of appropriate type, e.g. in (2) below, t and r have to be of type 0 while in (3) s and t have to be of type σ and τ, respectively. The required typing should always be clear from the context.

A^D is then defined to be the formula in (vii). Note, however, that these equivalences are not necessarily justified constructively. Only (i) and (ii) hold constructively whereas (v) and (vi) are justified constructively only if one also accepts the axiom of choice for all finite types (AC'). Equivalences (ii) and (iv) use a certain amount of classical logic known as the *principle of independence of premise* (IP') and *Markov's principle* (MP') for all finite types, respectively. At this point $A \mapsto A^D$ is just a syntactic translation. But amazingly it gives rise to a meaningful interpretation of **HA** in T.

Theorem 8.2 (Gödel 1958) *Suppose $\mathcal{D}$ is a proof of A in* **HA**. *Then one can effectively construct a sequence of terms $\vec{t}$ (from $\mathcal{D}$) such that T proves $A_D(\vec{t}, \vec{y})$.*

If one combines the D-interpretation with the Kolmogorov-Gentzen-Gödel negative translation of **PA** into **HA** one also arrives at an interpretation of **PA** in T. Some interesting consequences of the latter are that the consistency of **PA** follows finitistically from the consistency of T and that every total recursive function of **PA** is denoted by a term of T.

The three principles (AC'), (IP') and (MP') which figured in the D-translation actually characterize the D-translation in the sense that over the quantifier extension of T with intutionistic logic, called $\mathbf{HA}^\omega$, they are equivalent to the schema

$$A \leftrightarrow A^D$$

for all formulae A of that theory. Principles similar to the three above are also often validated in another type of computational interpretation of intuitionistic theories known as *realizability*. Thus it appears that they intrinsically related to computational interpretations of such theories.

A further pleasing aspect of Gödel's interpretation is that it can be extended to stronger systems such as higher order systems and even to set theory ([17, 25]). Moreover, it sometimes allows one to extract computational information even from proofs of specific classical theorems (see e.g. [77]). It behaves nicely with respect to modus ponens and thus works well for ordinary proofs that are usually structured via a series of lemmata. This is in contrast to cut elimination which often requires a computationally costly transformation of proofs.

Spector's [131] extended Gödel's functional interpretation, engineering an interpretation of $\mathbf{Z_2}$ into T augmented via a scheme of transfinite recursion on higher type orderings. This type of recursion, called *bar recursion*, is conceptually related to Brouwer's bar induction principle.

8.1 Bar induction

The Bar Theorem is a theorem about trees. It occupies a prominent place in Brouwer's development of intuitionistic mathematics and has also played a central role in proof theory in the 1960s and 1970s. Here we will give a brief account of it since it is essential to Spector's functional interpretation of second order arithmetic. Its proof-theoretic analysis provides a nice demonstration of Buchholz' Ω-rule.

Definition 8.3 Let $\mathbb{N}^*$ be the set of all finite sequences of natural numbers which includes the empty sequence $\langle\rangle$. $\mathbb{N}^*$ can be viewed as an infinite tree which grows from the root $\langle\rangle$ upwards. If $s \in \mathbb{N}^*$, $m \in \mathbb{N}$ and $s = \langle s_0, \ldots s_k\rangle$ then the immediate successor nodes or children of s are of the form $s * \langle m\rangle$ defined as $\langle s_0, \ldots, s_k, m\rangle$. t is a node above s if t is of the form $s * \langle k_0\rangle * \ldots \langle k_r\rangle$.

A bar of $\mathbb{N}^*$ is a subclass B of $\mathbb{N}^*$ such that every infinite path through $\mathbb{N}^*$ goes through B; in Brouwer's terminology, every infinite path is "barred" by B. More formally this is defined as follows. For a function $f : \mathbb{N} \to \mathbb{N}$ and $n \in \mathbb{N}$, $\bar{f}n$ denotes the sequence $\langle f(0), \ldots, f(n-1)\rangle$ ($\bar{f}(0) = \langle\rangle$). A **bar** B for $\mathbb{N}^*$ is a subclass of $\mathbb{N}^*$ such that for all $f : \mathbb{N} \to \mathbb{N}$ there exists $n \in \mathbb{N}$ such that $\bar{f}n \in B$.

Bar induction is the following principle:

$$\mathbf{BI}_{gen} \qquad \text{Hyp } 1 \wedge \text{Hyp } 2 \wedge \text{Hyp } 3 \wedge \text{Hyp } 4 \to Q(\langle\rangle)$$

where

(Hyp 1)	B is a bar
(Hyp 2)	$\forall s \in \mathbb{N}^* \, \forall n \in \mathbb{N} \, (s \in B \to s * \langle n \rangle \in B)$
(Hyp 3)	$\forall s \in \mathbb{N}^* \, (s \in B \to s \in Q)$
(Hyp 4)	$\forall s \in \mathbb{N}^* \, [(\forall k \in \mathbb{N} \, s * \langle k \rangle \in Q) \to s \in Q)$.

Clause (Hyp 4) asserts that the property of being in Q propagates to a node s if all its children belong to Q. Since all nodes in the bar belong to Q by (Hyp 3), the intuitive idea behind this principle is that the clauses (Hyp 1-4) guarantee that membership in Q "percolates" from the bar all the way down to the root.[20]

Brouwer's justification for the Bar theorem [7], that is, of the general validity of Bar Induction, rests on the idea that any canonical proof of (Hyp 1) in infinitary logic has a particular structure which allows one, when supplied with proofs of (Hyp 2) - (Hyp 4), to transform it into a proof of $Q(\langle\rangle)$. With hindsight, one could say that Brouwer is assuming that a canonical proof is something like a cut free proof in ω-logic.

The notions of Definition 8.3 can be easily formalized in the language of second order arithmetic. If one doesn't impose any further restrictions on the complexity of the bar B and the predicate Q (i.e. allowing them to be expressed by any formula of the language), then $\mathbf{BI}_{gen}$ is surprisingly strong when classical logic is assumed.

Theorem 8.4 $\mathbf{BI}_{gen}$ *implies full second order comprehension* $\mathbf{CA}$ *(actually* $\mathbf{AC}$*) and is conservative over* $\mathbf{Z_2}$ *for* Π^1_4 *sentences.*[21] *(The proof is not difficult but will be omitted.)*

On the other hand, when the ambient logic is intuitionistic logic, $\mathbf{BI}_{gen}$ is much weaker than $\mathbf{Z_2}$. To obtain an intuitionistic theory of that strength based on the idea of bar induction one needs to consider bar induction on higher types.

8.2 An outline of Spector's interpretation

In 1960, Clifford Spector [131] gave a consistency proof of $\mathbf{Z_2}$ by means of a functional interpretation. To find a class of functionals sufficient unto the task of lifting the D-interpretation to $\mathbf{Z_2}$, he defined the so-called bar recursive functionals. The crucial step in the interpretation is to furnish a functional interpretation of the negative translation of $\mathbf{BI}_{gen}$, which by Theorem 8.4 gives rise to an interpretation of $\mathbf{Z_2}$. For this he extended intuitionistic $\mathbf{BI}_{gen}$ to all finite types. Bar induction for type σ, $\mathbf{BI}_\sigma$, is formulated similar to $\mathbf{BI}_{gen}$, the difference being that instead of just looking at the tree of all finite sequences of natural numbers $\mathbb{N}^*$, one takes the full tree of finite sequences $\langle F_1, \ldots, F_r\rangle$ of objects F_i of type σ, $\mathcal{T}_\sigma$. A bar of the latter is defined completely analogous to a bar of $\mathbb{N}^*$.

Instead of $\mathbf{BI}_\sigma$, Spector's extension of T has a scheme, $\mathbf{BR}_\sigma$ for defining functionals by *bar recursion* on the tree $\mathcal{T}_\sigma$ (we omit the details). The first step is to interpret the theory $\mathbf{HA}^\sharp + \mathbf{BI}_\sigma$ in Spector's $T + \mathbf{BR}_\sigma$, where $\mathbf{HA}^\sharp$ is the theory $\mathbf{HA}^\omega$ augmented by the axioms

$$A \leftrightarrow A^D \tag{10}$$

with A^D being the Gödel dialectica interpretation of A. It is easy to give a functional interpretation of $A \leftrightarrow A^D$: merely observe that $(A \leftrightarrow A^D)^D$ is identical to $(A \leftrightarrow A)^D$. With this it is not too difficult to see that $\mathbf{HA}^\sharp + \mathbf{BI}_\sigma$ has a functional interpretation in $\mathbf{T} + \mathrm{BR}_\sigma$.

[20] Brouwer did not include (Hyp 2) among the hypothesis. It expresses the "mononotonicity' of the bar. Classically it's rather superflous but intuitionistically it is essential as Kleene observed who showed that $\mathbf{BI}_{gen}$ formulated without (Hyp 2) yields instances of excluded middle that are incompatible with Brouwer's continuity principles [76, 7.14].

[21] A bar B gives rise to an wellfounded ordering $\prec$ on $\mathbb{N}^*$ where $t \prec s$ means that t is a node above s and $t \notin B$. Viewed in this way, $\mathbf{BI}_{gen}$ is a principle of transfinite induction along $\prec$. For the implication $\mathbf{BI}_{gen} \to \mathbf{CA}$ to hold it is important that B can be of arbitrary complexity. Thus the moral drawable from this is that impredicative comprehension can be deduced from transfinite induction on impredicatively defined orderings.

The next step, which furnishes the interpretation of classical $\mathbf{BI}_{gen}$ (and thereby of full $\mathbf{CA}$) is to look at the negative interpretation of some instances of $\mathbf{BI}_\sigma$ in $\mathbf{HA}^\sharp + \mathbf{BI}_\sigma$. The main step is now to verify the negative interpretation of (Hyp 1) of special forms of $\mathbf{BI}_\sigma$,

$$\forall f \exists n\, P(\bar{f}n) \tag{11}$$

with the predicate $P(c)$ being of the form $\exists Z\, B(Z,c)$ where $B(Z,c)$ is quantifier free and c is of type σ (see [64, Lemma 4D]). The negative translation of (11) is

$$\forall f\, \neg\neg\exists n\, P(\bar{f}n)^N;\ \text{i.e.,}\ \forall f\, \neg\neg\exists n\, \neg\neg\exists Z\, B(Z, \bar{f}n)$$

since $B(Z,c)$ is quantifier free. The D-translation of the latter formula is the same as that of $\forall f \exists n \neg\neg\exists Z\, B(Z, \bar{f}n)$. Note that a generalization of Markov's principle is a consequence of (10). As a result, $\forall f \neg\neg\exists n\, P(\bar{f}n)^N$ is equivalent in $\mathbf{HA}^\sharp$ to $\forall f \exists n\, P(\bar{f}n)^N$, so (Hyp 1) has been restored.

Spector was rather cautious not to claim that his theory $T + \mathbf{BR}$ gives a constructive interpretation of $\mathbf{Z_2}$.

> "*The author believes that the bar theorem is itself questionable, and that until the bar theorem can be given a suitable foundation, the question of whether bar induction is intuitionistic is premature.*"

The question of constructivity of bar recursion was also answered in the negative by Kreisel. On closer inspection of the proof, one is left with the impression that logically complex comprehensions are traded in for inductions on highly complex higher type relations.[22] That notwithstanding, Spector's result is quite remarkable and his interpretation has been applied to extract computational information from classical proofs (cf. [77]).

8.3 Bar induction and Buchholz' Ω-rule

Classically $\mathbf{BI}_{gen}$ is very strong but if the background logic is assumed to be just intuitionistic it is a lot weaker, namely of the same strength as the theory of non-iterated inductive definitions $\mathbf{ID}_1$ (see section 10) (the Bachmann-Howard ordinal being its proof-theoretic ordinal; see section 9). So $\mathbf{BI}_{gen}$ is an example of a theory where the law of excluded middle makes an enormous difference. A classical theory of the same strength as intuitionistic $\mathbf{BI}_{gen}$ is obtained by requiring the bar B to be a set. This theory is usually denoted by $\mathbf{BI}$. An equivalent formalization of $\mathbf{BI}$ is given by the schema of quantifier elimination (see [31]):

$$(\forall^2\text{-E}) \qquad \forall X\, A(X) \to A(F) \tag{12}$$

for any arithmetical formula $A(X)$ and arbitrary L_2-formula $F(u)$, where $A(F)$ arises from $A(X)$ by replacing all occurrences of the form $t \in X$ in the formula by $F(t)$.

Ever since the great successes with the ω-rule, which restored cut elimination in arithmetic, proof theorists were looking for stronger forms of infinitary proof rules that could bring about cut elimination for genuinely impredicative theories. Buchholz was the first who succeeded in finding such a rule. He introduced the Ω-rule in [9] and extended versions of it are the central tool in [14]. A sequent calculus version of it was used in [100] to give a proof-theoretic bound on Kruskal's theorem and this is the version we will briefly discuss. According to the intuitionistic interpretation of an implication (called the Brouwer-Heyting-Kolmogoroff (BHK) interpretation) the truth of an implication $C \to D$ is explained in terms of a construction that transforms any proof of C into a proof of D. This idea may serve as a first approach to the Ω-rule:

($\tilde{\Omega}$) If for every cut free proof[23] $\mathcal{D}$ of $\forall X A(X)$ we have $\mathcal{T}(\mathcal{D}) \vdash \Theta \Rightarrow \Xi$, then $\mathcal{T}$ is considered to be a proof of $\Theta, \forall X F(X) \Rightarrow \Xi$.

[22] Girard is also very skeptical about the intuitionistic setting of Spector's interpretation: "all these topics have *nothing to do* with intuitionism" ([50, p. 479])

[23] As matter of clarification, cut free proofs are not allowed to contain other instances of $\tilde{\Omega}$.

Since any cut free proof of $\forall X A(X)$ can be transformed into a proof of $A(F)$, just by the operation $\mathcal{T}$ of substituting $F(t)$ for $t \in X$ in $A(X)$, this rule allows one to prove ($\forall^2$-E). However, ($\tilde{\Omega}$) is just too naive an approach since this rule does not behave well with respect to cut elimination, particularly since side formulae (assumptions) are not taken into account. So the actual Ω-rule takes a rather more involved form:

Definition 8.5

(Ω) If for all finite sets of Σ^1_1-formulae Γ and Π^1_1-formulae Δ, every cut free proof $\mathcal{D}$ of $\Gamma \Rightarrow \Delta, \forall X A(X)$ can be transformed into a proof $\mathcal{T}(\mathcal{D})$ of $\Gamma, \Theta \Rightarrow \Delta, \Xi$, then $\mathcal{T}$ is considered to be a proof of $\Theta, \forall X F(X) \Rightarrow \Xi$.

With the help of the Ω-rule one obtains an ordinal analysis of (**BI**) (see [14, 100]). In view of Brouwer's speculative justification of bar induction, it is a very pleasing outcome that a proof-theoretic analysis can be obtained via a rule that embodies transformations on infinite canonical proof trees.

9 A glimpse at ordinal representation systems beyond 1975

Bachmann's bold move of using large ordinals to generate names for small ordinal ordinals was a very important idea. To obtain ordinal analyses of ever stronger theories one has to find new ways of defining ordinal representation systems that can encapsulate their strength. The later goes hand in hand with the development of new cut elimination techniques that are capable of removing cuts in (infinitary) proof systems with strong reflection rules. Ordinal representations, however, appear to pose a considerable barrier to understanding books and articles in this research area. Nonetheless we think that they are the best way to express the proof-theoretic strength of a theory as they provide a scale by means of which one can get a grasp of how much stronger a theory S_1 is than another theory S_2 (rather than the bland statement that S_1 is stronger than S_2).

As an example we will introduce an ordinal representation system which characterizes the theory (Π^1_1-**CA**) + **BI**, following [10]. It is based on certain ordinal functions ψ_{Ω_n} which are often called *collapsing functions*. The definition of these functions, that is of the value $\psi_{\Omega_n}(\alpha)$ at α, proceeds by recursion on α and gets intertwined with the definition of sets of ordinals $C^{\Omega_\omega}(\alpha, \beta)$, dubbed "Skolem hulls" since they are defined as the smallest structures closed under certain functions specified below.

Let $\mathbb{N}^+$ be the natural numbers without 0. Below we shall assume that Ω_n ($n \in \mathbb{N}^+$) is a "large" ordinal and that $\omega < \Omega_n < \Omega_{n+1}$. Their limit, $\sup_{n\in\mathbb{N}^+} \Omega_n$, will be denoted by Ω_ω.

Definition 9.1 By recursion on α we define:

$$C^{\Omega_\omega}(\alpha, \beta) = \begin{cases} \textit{closure of } \beta \cup \{0, \Omega_\omega\} \cup \{\Omega_n \mid n \in \mathbb{N}^+\} \\ \textit{under:} \\ +, (\xi \mapsto \omega^\xi) \\ (\xi \longmapsto \psi_{\Omega_n}(\xi))_{\xi<\alpha} \text{ for } n \in \mathbb{N}^+ \end{cases}$$

$$\psi_{\Omega_n}(\alpha) = \min\{\rho < \Omega_n \mid C^{\mathbf{I}}(\alpha, \rho) \cap \Omega_n = \rho\}.$$

At this point it is not clear whether $\psi_{\Omega_n}(\alpha)$ will actually be defined for all α since there might not exist a $\rho < \Omega_n$ such that

$$C^{\mathbf{I}}(\alpha, \rho) \cap \Omega_n = \rho.$$

This is where the "largeness" of Ω_n comes into play. One (easy) way of guaranteeing this consists in letting Ω_n be the n^{th} uncountable regular cardinal, that is $\Omega_n := \aleph_n$. However, such strong set-theoretic assumptions can be avoided. For instance, it suffices to let Ω_n be the n^{th} recursively regular ordinal (which is a countable ordinal) (see [98]).

To get a better feel for what ψ_{Ω_n} is doing, note that if $\rho = \psi_{\Omega_n}(\alpha)$, then $\rho < \Omega_n$ and with $[\rho, \Omega_n)$ being the interval consisting of ordinals $\rho \leq \alpha < \Omega_n$ one has

$$[\rho, \Omega_n) \cap C^{\Omega_\omega}(\alpha, \rho) = \emptyset$$

thus the order-type of the ordinals below Ω_n which belong to the "Skolem hull" $C^{\Omega_\omega}(\alpha,\rho)$ is ρ. In more pictorial terms, ρ is said to be the α^{th} *collapse* of Ω_n since the order-type of Ω_n viewed from within the structure $C^{\Omega_\omega}(\alpha,\rho)$ is actually ρ.

The ordinal representation system we are after is provided by the set

$$C^{\Omega_\omega}(\varepsilon_{\Omega_\omega+1},0)$$

where $\varepsilon_{\Omega_\omega+1}$ is the least epsilon number after Ω_ω, i.e., the least ordinal $\eta > \Omega_\omega$ such that $\omega^\eta = \eta$. The proof-theoretic ordinal of $(\Pi^1_1\text{-}\mathbf{CA}) + \mathbf{BI}$ is $\psi_{\Omega_1}(\varepsilon_{\Omega_\omega+1})$. Although the definition of the set $C^{\Omega_\omega}(\varepsilon_{\Omega_\omega+1},0)$ and its ordering is set-theoretic, it turns that it also has a purely elementary recursive definition which can be given in a fragment of **PRA**. Thus the set-theoretic presentation mainly serves the purpose of a "visualization" of an elementary well-ordering.

The pattern of definition exhibited in Definition 9.1 continues for stronger systems, albeit only as a basic template since for theories beyond the level of $(\Delta^1_2\text{-}\mathbf{CA}) + \mathbf{BI}$ substantially new ideas are required. Analogies between large set-theoretic ordinals (cardinals) and recursively large ordinals on the one hand and ordinal representation systems on the other hand can be a fruitful source of inspiration for devising new representation systems. More often than not, hierarchies and structural properties that have been investigated in set theory and recursion theory on ordinals turn out to have proof-theoretic counterparts.

10 Investigations of theories of inductive definitions

Spector's [131] functional interpretation of $\mathbf{Z_2}$ via bar recursive functionals was of great interest to proof theory. However, it was not clear whether there was a constructive foundation of these functionals along the lines of hereditarily continuous functionals that can be represented by computable functions (akin to [75], [80]) which would make them acceptable on intuitionistic grounds. In 1963 Kreisel conducted a seminar the expressed aim of which was to assay the constructivity of Spector's interpretation (see [81]). Specifically he asked whether an intuitionistic theory of monotonic inductive definitions, $\mathbf{ID}^i_1(\text{mon})$, could model bar recursion, or even more specifically, formally capture a class of indices of representing functions of these functionals. In a subsequent report the seminar's conclusion was later summarized by Kreisel:

> *... the answer is negative by a wide margin, since not even bar recursion of type 2 can be proved consistent [from intuitionistically accepted principles].*

[81] not only introduced theories of one inductive definition but also of ν-times transfinitely iterated inductive definitions, $\mathbf{ID}_\nu$. Albeit it soon became clear that even the theories $\mathbf{ID}_\nu$ couldn't reach the strength of $\mathbf{Z_2}$ (in point of fact, such theories are much weaker than the fragment of $\mathbf{Z_2}$ based on Π^1_2-comprehension) they became the subject of proof-theoretical investigation in their own right and occupied the attention of proof theorists for at least another 15 years. One reason for this interest was surely that the intuitionistic versions corresponding to the accessible (i.e., well-founded) part of a primitive recursive ordering are immediately constructively appealing and a further reason was that they were thought to be more amenable to direct proof-theoretic treatments than fragments of $\mathbf{Z_2}$ or set theories.

We shall not give a detailed account of the formalization of these theories, but focus on the non-iterated case $\mathbf{ID}_1$ and its intuitionistic version $\mathbf{ID}^i_1$ to convey the idea. A monotone operator on $\mathbb{N}$ is a map Γ that sends a set $X \subseteq \mathbb{N}$ to a subset $\Gamma(X)$ of $\mathbb{N}$ and is monotone, i.e. $X \subseteq Y \subseteq \mathbb{N}$ implies $\Gamma(X) \subseteq \Gamma(Y)$. Owing to monotonicity, the operator Γ will have a least fixed point $I_\Gamma \subseteq \mathbb{N}$, i.e. $\Gamma(I_\Gamma) = I_\Gamma$ and for every other fixed point X it holds $I_\Gamma \subseteq X$. Set-theoretically I_Γ is obtained by iterating Γ through the ordinals,

$$\Gamma^0 = \emptyset, \quad \Gamma^1 = \Gamma(\Gamma^0), \quad \Gamma^\alpha = \Gamma(\bigcup_{\xi<\alpha} \Gamma^\xi).$$

Monotonicity ensures (in set theory) that one finds an ordinal τ such that $\Gamma(\Gamma^\tau) = \Gamma^\tau$, and the set Γ^τ will be the least fixed point. If one adds a new 1-place predicate symbol P to the language of arithmetic, one can describe the so-called positive arithmetical operators. They are of the form

$$\Gamma_A(X) = \{n \in \mathbb{N} \mid A(n, X)\}$$

where $A(x, P^+)$ is a formula of the language of **PA** augmented by P in which the predicate P occurs only positively (indicated by the superscript +). The syntactic condition of positivity then ensures that the operator Γ_A is monotone. The language of $\mathbf{ID}_1$ is an extension of that of **PA**. It contains a unary predicate symbol I_A for each positive arithmetical operator Γ_A and the axioms

$$(Id1) \quad \forall x\, (A(x, I_A) \leftrightarrow I_A(x)) \tag{13}$$

$$(Id2) \quad \forall x\, [A(x, F) \rightarrow F(x)] \rightarrow \forall x\, [I_A(x) \rightarrow F(x)] \tag{14}$$

where in $(Id2)$ $F(x)$ is an arbitrary formula of $\mathbf{ID}_1$ and $A(x, F)$ arises from $A(x, P)$ by replacing every occurrence of $P(t)$ in the formula by $F(t)$. Collectively these axioms assert that I_A is the least fixed point of Γ_A, or more accurately the least among all sets of naturals definable in the language of $\mathbf{ID}_1$.

$\mathbf{ID}_1^i$ will be used to denote the intuitionistic version. Its subtheory $\mathbf{ID}_1^i(\mathcal{O})$ is obtained by just adding the predicate symbol I_A and the pertaining axioms $(Id1)$ and $(Id2)$, where Γ_A is the operator that defines Kleene's $\mathcal{O}$ (cf. Definition 7.1).

By a complicated passage through formal theories for choice sequences it was known that the theory $\mathbf{ID}_1$ can be reduced to $\mathbf{ID}_1^i(\mathcal{O})$. The first ordinal analysis for the theory $\mathbf{ID}_1^i(\mathcal{O})$ was obtained by Howard [65]. Via the known proof-theoretical reductions this entailed also an ordinal analysis for $\mathbf{ID}_1$. The proof-theoretic ordinal of $\mathbf{ID}_1$ is the *Bachmann-Howard ordinal*, which is denoted by $\psi_{\Omega_1}(\varepsilon_{\Omega_1+1})$ in the system of Definition 9.1.

As inductively defined sets can be the starting point of another inductive definition, the procedure of inductively defining predicates can be iterated along any wellordering ν in a uniform way. This leads to the theories $\mathbf{ID}_\nu$ which allow one to formalize ν-times iterated inductive definitions, where ν stands for a primitive recursive well-ordering. If ν is a wellordering on constructive grounds then also the ν-times iterated version of Kleene's $\mathcal{O}$ has a clear constructive meaning. As a result the formal theories $\mathbf{ID}_\nu^i(\mathcal{O})$ that embody this process are constructively justified. The topic of theories of iterated inductive definitions was flourishing at the 1968 conference on Intuitionism and Proof Theory in Buffalo (see [72]). One of the main proof-theoretic goals was to find a reduction of the classical theories $\mathbf{ID}_\nu$ to their intuitionistic counterparts $\mathbf{ID}_\nu^i(\mathcal{O})$. This was all the more desirable because of known reductions of important fragments of second order arithmetic to theories of the former kind. Friedman [38] had shown that the second order system with the Σ_2^1-axiom of choice can be interpreted in the system $(\Pi_1^1\text{-CA})_{<\varepsilon_0}$ of less than ε_0-fold iterated Π_1^1 comprehensions and Feferman [30] had shown that less than ν-fold iterated Π_1^1 comprehensions could be interpreted in the system

$$\mathbf{ID}_{<\nu} := \bigcup_{\alpha<\nu} \mathbf{ID}_\alpha$$

for $\nu = \omega^\gamma$ with γ limit. However, Zucker [151] showed that there are definitive obstacles to a straight forward reduction of the theories ID_ν for $\nu > 1$ to their intuitionistic cousins. Ordinal analyses for theories for theories of finitely iterated inductive definitions were first obtained by Pohlers [90] and then also for transfinitely iterated inductive definitions [91], using Takeuti's reduction procedure for Π_1^1-comprehension. Working independently, Buchholz [9] used a new type of rules, dubbed $\Omega_{\mu+1}$-rules to recapture these results without use of Takeuti's methods. These rules are an extension of the Ω-rule described in Definition 8.5. Meanwhile, Sieg [128] attacked the problem by a method adapted from Tait [133] who had used cut elimination for an infinitary propositional logic with formulae indexed over constructive number classes to obtain a consistency proof for second order arithmetic theory with the schema of Σ_2^1 dependent choices. Sieg achieved a reduction of $\mathbf{ID}_{<\nu}$ to $\mathbf{ID}_{<\nu}^i(\mathcal{O})$ for limit ν by carrying out the proof theory for a system of $\mathbf{PL}_\alpha$ of propositional logic with infinitely long conjunctions and disjunctions indexed over the constructive number classes $\mathcal{O}_\alpha$ for $\alpha < \nu$ inside $\mathbf{ID}^i_{\alpha+1}(\mathcal{O})$. As $\mathbf{ID}_\alpha$ can be reduced to $\mathbf{PL}_\alpha$ this brought about the reduction.

Using an extended version of the representation system from Definition 9.1 if $\nu > \omega$,[24] the outcome of all these efforts can be summarized by the following theorem.[25]

Theorem 10.1 *For recursive ν,*

$$|\mathbf{ID}_\nu| = |\mathbf{ID}^i_\nu(\mathcal{O})| = \psi_{\Omega_\nu}(\varepsilon_{\Omega_\nu+1}).$$

A generalized treatment of theories of iterated inductive definitions for arbitrary wellorderings and of autonomous iteration was carried out in [95, 112]. These theories are stronger than Δ^1_2-$\mathbf{CA}$.

Theorem 10.1 played an important role in determining the exact strength of some fragments of $\mathbf{Z_2}$. The major ordinal-theoretic results pertaining to subsystems of $\mathbf{Z_2}$ of the pre 1980 area given in the next theorem.

Theorem 10.2

(i) $|\Pi^1_1\text{-}\mathbf{CA}_0| = |\mathbf{ID}_{<\omega}| = \psi_{\Omega_1}(\Omega_\omega)$

(ii) $|\Pi^1_1\text{-}\mathbf{CA}| = \psi_{\Omega_1}(\Omega_\omega \cdot \varepsilon_0)$

(iii) $|\Pi^1_1\text{-}\mathbf{CA} + \mathbf{BI}| = |\mathbf{ID}_\omega| = \psi_{\Omega_1}(\varepsilon_{\Omega_\omega+1})$

(iv) $|\Delta^1_2\text{-}\mathbf{CR}| = \psi_{\Omega_1}(\Omega_{\omega^\omega})$

(v) $|\Delta^1_2\text{-}\mathbf{CA}| = \psi_{\Omega_1}(\Omega_{\varepsilon_0})$

Upper bounds for (i) and (iii) are due to Takeuti [138] while upper bounds for (iv) and (v) are owed to Takeuti and Yasugi[139]. The exacts bound in (iii) and (ii) are due to Pohlers [91] and Buchholz [9], respectively. Other exact bounds involve the work of several people.

The next challenge after Δ^1_2-$\mathbf{CA}$ was posed by the theory Δ^1_2-$\mathbf{CA} + \mathbf{BI}$. Its treatment not only required a considerably stronger ordinal representation system but also coincided with a shift away from $\mathcal{L}_2$ theories and theories of iterated inductive definitions to a direct proof-theoretic treatment of set theories. Pioneering work on the proof theory of set theories is mainly due to Jäger [66, 67]. The analysis of Δ^1_2-$\mathbf{CA} + \mathbf{BI}$ is joint work of Jäger and Pohlers [68] and provides a particularly fine showcase for the universality of Pohlers' *method of local predicativity.*

Theorem 10.3 $|\Delta^1_2\text{-}\mathbf{CA} + \mathbf{BI}| = \psi_{\Omega_1}(\varepsilon_{I+1})$

The "I" in the foregoing notation is supposed to be indicative of "*inaccessible cardinal*". Indeed, the easiest way to build an extended ordinal representation system sufficient unto this task (modeled on Definition 9.1) is to add an inaccessible I, close the Skolem hulls under $\xi \mapsto \Omega_\xi$ for $\xi < I$ and introduce collapsing functions ψ_π for all π of either form I or $\Omega_{\xi+1}$.

The goal of giving an ordinal analysis of full second order arithmetic has not been attained yet. For many years Π^1_2-comprehension posed a formidable challenge and the quest for its ordinal analysis attained something of a holy grail status (cf. [35]). At first blush it might be difficult to see why the latter comprehension is so much more powerful than Δ^1_2-comprehension (plus $\mathbf{BI}$). To get a sense for the difference, it is advisable to work in (admissible) set theory and consider a hierarchy of recursively large ordinal notions wherein these comprehension schemes correspond to the bottom and the top end of the scale, respectively. Thus we turn to set theories in the next section.

[24]The extension then has ordinals Ω_α for all $\alpha \leq \nu$ and each ordinal of the form $\Omega_{\beta+1}$ gets furnished with its own collapsing function $\psi_{\Omega_{\beta+1}}$.

[25]For a detailed account of the history of the proof theory of iterated inductive definitions see [13].

11 Proof theory of set theories

With the work of Jäger and Pohlers (see [67, 68]) the forum of ordinal analysis switched from the realm of second-order arithmetic to set theory, shaping what is now called *admissible proof theory*, after the models of *Kripke-Platek set theory*, **KP**. Their work culminated in the analysis of the system $(\Delta^1_2\text{-}\mathbf{CA}) + \mathbf{BI}$.

By and large, ordinal analyses for set theories are more uniform and transparent than for subsystems of $\mathbf{Z_2}$. The axiom systems for set theories considered in this paper are formulated in the usual language of set theory (called $\mathcal{L}_\in$ hereafter) containing $\in$ as the only non-logical symbol besides $=$. Formulae are built from prime formulae $a \in b$ and $a = b$ by use of propositional connectives and quantifiers $\forall x, \exists x$. Quantifiers of the forms $\forall x \in a$, $\exists x \in a$ are called *bounded*. *Bounded* or Δ_0*-formulae* are the formulae wherein all quantifiers are bounded; Σ_1-formulae are those of the form $\exists x \varphi(x)$ where $\varphi(a)$ is a Δ_0-formula. For $n > 0$, Π_n-formulae (Σ_n-formulae) are the formulae with a prefix of n alternating unbounded quantifiers starting with a universal (existential) one followed by a Δ_0-formula. The class of Σ-formulae is the smallest class of formulae containing the Δ_0-formulae which is closed under $\wedge$, $\vee$, bounded quantification and unbounded existential quantification.

One of the set theories which is amenable to ordinal analysis is Kripke-Platek set theory, **KP**. Its standard models are called *admissible sets*. One of the reasons that this is an important theory is that a great deal of set theory requires only the axioms of **KP**. An even more important reason is that admissible sets have been a major source of interaction between model theory, recursion theory and set theory (cf. [5]). **KP** arises from **ZF** by completely omitting the power set axiom and restricting separation and collection to bounded formulae. These alterations are suggested by the informal notion of 'predicative'. To be more precise, the axioms of **KP** consist of *Extensionality, Pair, Union, Infinity, Bounded Separation*

$$\exists x \, \forall u \, [u \in x \leftrightarrow (u \in a \, \wedge \, F(u))]$$

for all bounded formulae $F(u)$, *Bounded Collection*

$$\forall x \in a \, \exists y \, G(x,y) \;\rightarrow\; \exists z \, \forall x \in a \, \exists y \in z \, G(x,y)$$

for all bounded formulae $G(x,y)$, and *Set Induction*

$$\forall x \; [(\forall y \in x \, H(y)) \rightarrow H(x)] \, , \rightarrow \; \forall x \, H(x)$$

for all formulae $H(x)$.

A transitive set A such that $(A, \in)$ is a model of **KP** is called an *admissible set*. Of particular interest are the models of **KP** formed by segments of Gödel's *constructible hierarchy* **L**. The constructible hierarchy is obtained by iterating the definable powerset operation through the ordinals

$$\begin{aligned} \mathbf{L}_0 &= \emptyset, \\ \mathbf{L}_\lambda &= \bigcup \{\mathbf{L}_\beta : \beta < \lambda\} \; \lambda \text{ limit} \\ \mathbf{L}_{\beta+1} &= \big\{X : \; X \subseteq \mathbf{L}_\beta; \; X \text{ definable over } \; \langle \mathbf{L}_\beta, \in \rangle \big\}. \end{aligned}$$

So any element of **L** of level α is definable from elements of **L** with levels $< \alpha$ and the parameter $\mathbf{L}_\alpha$. An ordinal α is *admissible* if the structure $(\mathbf{L}_\alpha, \in)$ is a model of **KP**.

Formulae of $\mathcal{L}_2$ can be easily translated into the language of set theory. Some of the subtheories of $\mathbf{Z_2}$ considered above have set-theoretic counterparts, characterized by extensions of **KP**. **KPi** is an extension of **KP** via the axiom

$$(Lim) \qquad \forall x \exists y [x {\in} y \, \wedge \, y \textit{ is an admissible set}].$$

KPl denotes the system **KPi** without Bounded Collection. It turns out that $(\Delta^1_2\text{-}\mathbf{CA}) + \mathbf{BI}$ proves the same $\mathcal{L}_2$-formulae as **KPi**, while $(\mathbf{\Pi}^1_1{-}\mathbf{CA})$ proves the same $\mathcal{L}_2$-formulae as **KPl** (see [70]).

11.1 Admissible proof theory.

KP is the weakest in a line of theories that were analyzed by proof theorists of the Munich school in the late 1970s and 1980s. It can be viewed as a set-theoretic version of other well known theories.

Theorem 11.1 $|\mathbf{KP}| = |\mathbf{ID}_1| = |(\mathbf{BI})| = \psi_{\Omega_1}(\varepsilon_{\Omega_1+1})$.

In many respects, **KP** is a very special case. Several fascinating aspects of ordinal analysis do not yet exhibit themselves at the level of **KP**.

Recall that **KPl** is the set-theoretic version of $(\mathbf{\Pi}^1_1\mathbf{-CA})+\mathbf{BI}$, while **KPi** is the set-theoretic counterpart to $(\Delta^1_2\text{-}\mathbf{CA})+\mathbf{BI}$. The main axiom of **KPl** says that every set is contained in an admissible set (one also says that the admissible sets are cofinal in the universe) without requiring that the universe is also admissible, too. To get a sense of scale for comparing **KP**, **KPl**, and **KPi** it is perhaps best to relate the large cardinal assumptions that give rise to the pertaining ordinal representation systems. In the case of **KPl** the assumptions is that there are infinitely many large ordinals $\Omega_1, \Omega_2, \Omega_3, \ldots$ (where Ω_n can be taken to be $\aleph_n$) each equipped with their own 'collapsing' function $\alpha \mapsto \psi_{\Omega_n}(\alpha)$ as we saw in section 9. The ordinal system sufficient for **KPi** is built using the much bolder assumption that there is an inaccessible cardinal I.

As the above set theories are based on the notion of admissible set it is suitable to call the proof theory concerned with them 'admissible proof theory'. The salient feature of admissible sets is that they are models of Bounded Collection and that that principle is equivalent to Σ Reflection on the basis of the other axioms of **KP** (see [5]). Furthermore, admissible sets of the form $\mathbf{L}_\kappa$ also satisfy Π_2 reflection, i.e., if $\mathbf{L}_\kappa \models \forall x\, \exists y\, C(x,y,\vec{a})$ with $C(x,y)$ bounded and $\vec{a} \in \mathbf{L}_\kappa$, then there exists $\rho < \kappa$ such that $\vec{a} \in \mathbf{L}_\rho$ and $\mathbf{L}_\rho \models \forall x\, \exists y\, C(x,y,\vec{a})$.

In essence, admissible proof theory is a gathering of cut-elimination and collapsing techniques that can handle infinitary calculi of set theory with Σ and/or Π_2 reflection rules, and thus lends itself to ordinal analyses of theories of the form **KP**+ *"there are x many admissibles"* or **KP**+ *"there are many admissibles"*.

A theory on the verge of admissible proof theory is **KPM**, designed to axiomatize essential features of a recursively Mahlo universe of sets. An admissible ordinal κ is said to be recursively Mahlo if it satisfies Π_2-reflection in the above sense but with the extra condition that the reflecting set $\mathbf{L}_\rho$ be admissible as well. The ordinal representation [96] for **KPM** is built on the assumption that there exists a (weakly) Mahlo cardinal. The novel feature of over previous work is that there are two layers of collapsing functions. In all of the ordinal representation systems for admissible proof theory, collapsed ordinals $\psi_\pi(\alpha)$ are intrinsically singular, i.e., they can be approached from below by a definable sequence $(\beta_\xi)_{\xi<\lambda}$ of ordinals β_ξ with $\xi < \lambda < \psi_\pi(\alpha)$. In the representation system for **KPM** this is no longer the case. One needs a collapsing function ψ_M whose values $\psi_M(\delta)$ are regular ordinals themselves, meaning that they are furnished with their own collapsing function $\xi \mapsto \psi_{\psi_M(\delta)}(\xi)$. The ordinal analysis for **KPM** was carried out in [97]. A different approach to **KPM** using ordinal diagrams is due to Arai [2].

11.2 Beyond admissible proof theory

Gentzen fostered hopes that with sufficiently large constructive ordinals one could establish the consistency of analysis, i.e., $\mathbf{Z_2}$. The purpose of this section is to report on the next major step in analyzing fragments of $\mathbf{Z_2}$. This is obviously the ordinal analysis of the system $(\mathbf{\Pi}^1_2\text{–}\mathbf{CA})$.[26] The strength of $(\mathbf{\Pi}^1_2\text{–}\mathbf{CA})$ dwarfs that of $(\Delta^1_2\text{-}\mathbf{CA})+\mathbf{BI}$. The treatment of Π^1_2 comprehension posed formidable technical challenges (see [103, 108, 109]. Other approaches to ordinal analysis of systems above $\mathbf{\Pi}^1_1\text{–}\mathbf{AC}$ are due to Arai (see [2, 3]) who uses ordinal diagrams and finite deductions.

The means of admissible proof theory are certainly too weak to deal with the next level of reflection having three alternations of quantifiers, i.e. Π_3-reflection.

[26]For more background information see [140],p.259, [35],p.362, [93],p.374.

Definition 11.2 $\alpha > 0$ is said to be Π_n*–reflecting* if $\mathbf{L}_\alpha \models \Pi_n$–reflection. By Π_n–reflection we mean the scheme $C \to \exists z[Tran(z) \wedge z \neq \emptyset \wedge C^z]$, where C is Π_n, $Tran(z)$ expresses that z is a transitive set and C^z is the formula resulting from C by restricting all unbounded quantifiers in C to z.

An ordinal representation, $\mathcal{T}(\mathcal{K})$, for dealing with Π_3-reflection was developed in [102], utilizing the notion of a weakly compact cardinal. That such a cardinal notion played a role is not a mere accident. Indeed, in [115] the recursively large analogue of a weakly compact cardinal was equated with a Π_3-reflecting ordinal. The levels of collapsing functions in $\mathcal{T}(\mathcal{K})$ now become transfinite. They mirror a transfinite hierarchy of Mahloness. Moreover, the proof-theoretic treatment of **KP** + Π_3-Reflection features a new technique for collapsing families of proofs, called "stationary collapsing".

Climbing up in the hierarchy of Π_n-reflection, stronger cardinal notions are required to develop the pertaining representation systems. Another description of a weakly compact cardinal is that it is Π^1_1-indescribable. As a rule of thumb, one can develop a representation system sufficient for analyzing Π_{n+3}-reflection by making use of a Π^1_n-indescribable cardinals (see [108]). Already at that level things become very involved. At this point the reader might ask whether the theories with Π_n-reflection carry us anywhere near the level of Π^1_2-comprehension. The answer is, unfortunately, "no" by a wide margin. To make this more visible, we need a few more preparations. on the set–theoretic side, Π^1_2 comprehension corresponds to Σ_1 separation, i.e. the scheme of axioms

$$\exists z(z = \{x \in a : \phi(x)\})$$

for all Σ_1 formulas ϕ. The precise relationship is as follows:

Theorem 11.3 $\mathbf{KP} + \Sigma_1$ *separation and* $(\Pi^1_2 - \mathbf{CA}) + \mathbf{BI}$ *prove the same sentences of second order arithmetic.*

The ordinals κ such that $\mathbf{L}_\kappa \models \mathbf{KP} + \Sigma_1$-*Separation* are familiar from ordinal recursion theory.

Definition 11.4 An admissible ordinal κ is said to be *nonprojectible* if there is no total κ–recursive function mapping κ one–one into some $\beta < \kappa$, where a function $g : \mathbf{L}_\kappa \to \mathbf{L}_\kappa$ is called κ*–recursive* if it is Σ definable in $\mathbf{L}_\kappa$.

The key to the 'largeness' properties of nonprojectible ordinals is that for any nonprojectible ordinal κ, $\mathbf{L}_\kappa$ is a limit of Σ_1–elementary substructures, i.e. for every $\beta < \kappa$ there exists a $\beta < \rho < \kappa$ such that $\mathbf{L}_\rho$ is a Σ_1–elementary substructure of $\mathbf{L}_\kappa$, written $\mathbf{L}_\rho \prec_1 \mathbf{L}_\kappa$.

Such ordinals satisfying $\mathbf{L}_\rho \prec_1 \mathbf{L}_\kappa$ have strong reflecting properties. For instance, if $\mathbf{L}_\rho \models C$ for some set–theoretic sentence C (containing parameters from $\mathbf{L}_\rho$), then there exists a $\gamma < \rho$ such that $\mathbf{L}_\gamma \models C$. This is because $\mathbf{L}_\rho \models C$ implies $\mathbf{L}_\kappa \models \exists\gamma\, C^{\mathbf{L}_\gamma}$, hence $\mathbf{L}_\rho \models \exists\gamma\, C^{\mathbf{L}_\gamma}$ using $\mathbf{L}_\rho \prec_1 \mathbf{L}_\kappa$.

The last result makes it clear that an ordinal analysis of Π^1_2 comprehension would necessarily involve a proof–theoretic treatment of reflections beyond those surfacing in admissible proof theory. The notion of stability will be instrumental.

Definition 11.5 α is δ*–stable* if $\mathbf{L}_\alpha \prec_1 \mathbf{L}_{\alpha+\delta}$.

For our purposes we need refinements of this notion, the simplest being provided by:

Π_n–reflection for all n suffices to express one step in the $\prec_1$ relation.

Lemma 11.6 (cf. [115], 1.18) $\mathbf{L}_\kappa \prec_1 \mathbf{L}_{\kappa+1}$ *iff* κ *is* Π_n*–reflecting for all* n.

In the following, we will gradually slice Π^1_2 comprehension into degrees of reflection to achieve a sense of scale. A further refinement of the notion of δ-stability will be addressed next.

Definition 11.7 κ is said to be δ-Π_n-reflecting if whenever $C(u, \vec{x})$ is a set–theoretic Π_n formula, $a_1, \ldots, a_r \in \mathbf{L}_\kappa$ and $\mathbf{L}_{\kappa+\delta} \models C[\kappa, a_1, \ldots, a_n]$, then there exists $\kappa_0, \delta_0 < \kappa$ such that $a_1, \ldots, a_r \in \mathbf{L}_{\kappa_0}$ and $\mathbf{L}_{\kappa_0+\delta_0} \models C[\kappa_0, a_1, \ldots, a_n]$.

Putting the previous definition to work, one gets:

Corollary 11.8 *If κ is $\delta+1$-Σ_1-reflecting, then, for all n, κ is δ-Σ_n-reflecting.*

At this point let us return to proof theory to explain the need for even further refinements of the preceding notions. Recall that the first nonprojectible ordinal ρ is a limit of smaller ordinals ρ_n such that $\mathbf{L}_{\rho_n} \prec_1 \mathbf{L}_\rho$. In the ordinal representation system for $\Pi^1_2 - \mathbf{CA}$, there will be symbols $\mathfrak{E}_n$ and $\mathfrak{E}_\omega$ for ρ_n and ρ, respectively. They are proof-theoretic analogues of cardinals with very high degrees of indescribability. They were called "reducible cardinals" in [109].

The associated infinitary proof system will have rules

$$(Ref_{\Sigma(\mathbb{L}_{\mathfrak{E}_n+\delta})}) \quad \frac{\Gamma \Rightarrow \Delta, C(\vec{s})^{\mathbb{L}_{\mathfrak{E}_n+\delta}}}{\Gamma \Rightarrow \Delta, (\exists z \in \mathbb{L}_{\mathfrak{E}_n})(\exists \vec{x} \in \mathbb{L}_{\mathfrak{E}_n})[Tran(z) \wedge C(\vec{x})^z]},$$

where $C(\vec{x})$ is a Σ formula, $\vec{s}$ are set terms of levels $< \mathfrak{E}_n + \delta$, and $\delta < \mathfrak{E}_\omega$. These rules suffice to bring about the embedding $\mathbf{KP} + \Sigma_1$-Separation into the infinitary proof system, but other reflection rules galore will be needed to carry out cut-elimination. For example, there will be "many" ordinals $\pi, \delta \in \mathcal{OR}$ that play the role of δ-Π_{n+1}-reflecting ordinals by virtue of corresponding reflection rules in the infinitary calculus. The corresponding collapsing functions also have new features. Instead of collapsing a single ordinal they will have to collapse intervals. In that way they are reminiscent of inverses of elementary embeddings, with the latter being associated with very large cardinals in classical set theory.

12 Benefits of ordinal-theoretic proof theory

Results that have been achieved through ordinal analysis mainly fall into four groups: (1) Consistency of subsystems of classical second order arithmetic and set theory relative to constructive theories, (2) reductions of theories formulated as conservation theorems, (3) combinatorial independence results, and (4) classifications of provable functions and ordinals. Below we shall just provide a few examples (for more details see [105]).

12.1 Reduction to constructive frameworks

The reductions we have in mind, underlies a broadened view of "constructivity". Constructive theories of functions and sets that relate to Bishop's constructive mathematics as theories like **ZFC** relate to Cantorian set theory have been proposed by Myhill, Martin–Löf, Feferman and Aczel. Among those are Feferman's constructive theory of operations and classes, $\mathbf{T_0}$ ([32, 33]), Martin-Löf's intuitionistic type theory [85] and constructive set theory, especially Constructive Zermelo-Fraenkel Set Theory, **CZF**, the latter also combined with the regular extension axiom, **REA**. By employing an ordinal analysis for **KPi** it has been shown that **KPi** and consequently $\Delta^1_2 - \mathbf{CA}$+**BI** can be reduced to both of these theories.

Theorem 12.1 (Feferman [32], Jäger [69], Jäger and Pohlers [68], Rathjen [99]) $\Delta^1_2 - \mathbf{CA}$+**BI**, **KPi**, $\mathbf{T_0}$ *and* **CZF**+**REA** *are proof-theoretically equivalent. In particular, these theories prove the same theorems in the negative arithmetic fragment.*

Theorem 12.2 (Rathjen [99]; Setzer [125]) *The soundness of the negative arithmetic fragment of* $\Delta^1_2 - \mathbf{CA}$+**BI** *and* **KPi** *is provable in Martin-Löf's 1984 type theory.*

A detailed account of these results has been given in [105], section 3.

12.2 Combinatorial independence results and new combinatorial principles

Gödel's Incompleteness Theorems raised the question of whether there is a strictly mathematical example of an incompleteness in first-order Peano arithmetic and stronger systems, one which is mathematically simple and interesting and does not require the numerical coding of metamathematical notions. The first such examples were found by Gentzen and Goodman. Recall from section 2 the ordinal representation for ε_0 based on Cantor's normal form with its ordering $\prec$. Let $\mathrm{PRWO}(\varepsilon_0)$ be the statement that there are no infinite primitive recursive $\prec$-descending sequences.

Theorem 12.3 (Gentzen 1938)

(i) The theory of primitive recursive arithmetic, **PRA***, proves that* $\mathrm{PRWO}(\varepsilon_0)$ *implies the consistency of* **PA**.

(ii) Assuming that **PA** *is consistent,* **PA** *does not prove* $\mathrm{PRWO}(\varepsilon_0)$.

12.3 is not explicitly stated in [45] but it is an immediate consequence of his consistency proof of **PA** (cf. [113]). Goodstein, upon studying Gentzen's [42], established a connection between descending sequences of ordinals below ε_0 and certain sequences of natural numbers. He realized that given two ordinals $\alpha, \beta < \varepsilon_0$ one could replace the base ω in their complete Cantor normal forms by a sufficiently large number b and the resulting natural numbers $\hat{\mathrm{T}}^{\omega}_{b}(\alpha)$ and $\hat{\mathrm{T}}^{\omega}_{b}(\beta)$ would stand in the same ordering as α and β. This is a consequence of the fact that the criteria for comparing ordinals in Cantor normal form are the same as for natural numbers in complete base b-representation. There is a Cantor normal form for positive integers m to any base b with $b \geq 2$, namely we can express m uniquely in the form

$$m \quad = \quad b^{n_1} \cdot k_1 + \ldots + b^{n_r} \cdot k_r \tag{15}$$

where $m > n_1 > \ldots > n_r \geq 0$ and $0 < k_1, \ldots, k_r < b$. As each $n_i > 0$ is itself of this form we can repeat this procedure, arriving at what is called the **complete** b**-representation** of m. In this way we get a unique representation of m over the alphabet $0, 1, \ldots, b, +, \cdot$.

For example $7\,625\,597\,485\,157 = 3^{27} \cdot 1 + 3^4 \cdot 2 + 3^1 \cdot 2 + 3^0 \cdot 2 = 3^{3^3} + 3^{3+1} \cdot 2 + 3^1 \cdot 2 + 2$.

In [57] Goodstein defined what came to be called *Goodstein sequences.*

Definition 12.4 For naturals $m > 0$ and $c \geq b \geq 2$ let $\mathrm{S}^b_c(m)$ be the integer resulting from m by replacing the base b in the complete b-representation of m everywhere by c. For example $\mathrm{S}^3_4(34) = 265$, since $34 = 3^3 + 3 \cdot 2 + 1$ and $4^4 + 4 \cdot 2 + 1 = 265$.

Given any natural number m and non-decreasing function

$$f : \mathbb{N} \rightarrow \mathbb{N}$$

with $f(0) \geq 2$ define

$$m^f_0 = m, \quad \ldots \quad , m^f_{i+1} = \mathrm{S}^{f(i)}_{f(i+1)}(m^f_i) \dot{-} 1$$

where $k \dot{-} 1$ is the predecessor of k if $k > 0$, and $k \dot{-} 1 = 0$ if $k = 0$.

We shall call $(m^f_i)_{i \in \mathbb{N}}$ a **Goodstein sequence**. Note that a sequence $(m^f_i)_{i \in \mathbb{N}}$ is uniquely determined by f once we fix its starting point $m = m^f_0$.

The case when f is just a shift function has received special attention. Given any m we define $m_0 = m$ and $m_{i+1} := \mathrm{S}^{i+2}_{i+3}(m_i) \dot{-} 1$ and call $(m_i)_{i \in \mathbb{N}}$ a **special Goodstein sequence**. Thus $(m_i)_{i \in \mathbb{N}} = (m^{\mathrm{id}_2}_i)_{i \in \mathbb{N}}$, where $\mathrm{id}_2(x) = x + 2$. Special Goodstein sequences can differ only with respect to their starting points. They give rise to a recursive function f_{good} defined as follows: $f_{good}(m)$ is the least i such that $m_i = 0$ where $(m_i)_{i \in \mathbb{N}}$ is the special Goodstein sequence starting with $m_0 = m$.

Goodstein proved that all Goodstein sequences are finite. From his work combined with that of [45] he could have concluded the following result (see [113, Theorem 2.9]).

Theorem 12.5 *Termination of primitive recursive Goodstein sequences is not provable in* **PA**.

Already the termination of special Goodstein sequences, i.e. those where the base change is governed by the shift function, is not provable in **PA**. This result was obtained only much later by Kirby and Paris in 1982 [73] using model-theoretic tools. [73] prompted Cichon [21] to find a different (short) proof that harked back to older proof-theoretic work of Kreisel's [79] from 1952 which identified the so-called $< \varepsilon_0$-recursive functions as the provably recursive functions of **PA**. Other results pivotal to [21] were ordinal-recursion-theoretic classifications of Schwichtenberg [123] and Wainer [147] from around 1970 which showed that the latter class of recursive functions consists exactly of those elementary in one of the fast growing functions F_α with $\alpha < \varepsilon_0$. As F_{ε_0} eventually dominates any of these functions it is not provably total in **PA**. Cichon verified that F_{ε_0} is elementary in the function f_{good} of Definition 12.4. Thus termination of special Goodstein sequences is not provable in **PA**. In [113] the question is pondered whether the latter result could have been proved much earlier (from a technical as well as sociological point of view).

Mathematical independence results enjoyed great popularity in the 1970s and 1980s. Perhaps the most elegant of these is a strengthening of the Finite Ramsey Theorem due to Paris and Harrington (cf. [89]). The original proofs of the independence of combinatorial statements from **PA** all used techniques from non-standard models of arithmetic. Only later on alternative proofs using proof-theoretic techniques were found. However, results from ordinal-theoretic proof theory turned out to be pivotal in providing independence results for theories stronger than **PA**, and even led to a new combinatorial statement. The stronger theories referred to are Friedman's system $\mathbf{ATR}_0$ of *arithmetical transfinite recursion* and the system $(\Pi^1_1\text{-}\mathbf{CA})_0$ based on Π^1_1-comprehension. The independent combinatorial statements have their origin in certain embeddability questions in the theory of finite graphs. The first is a famous theorem of Kruskal asserting that every set of finite trees has only finitely many minimal elements.

Definition 12.6 A *finite tree* is a finite partially ordered set $\mathbb{B} = (B, \leq)$ such that:

(i) B has a smallest element (called the *root* of $\mathbb{B}$);

(ii) for each $s \in B$ the set $\{t \in B : t \leq s\}$ is a totally ordered subset of B.

Definition 12.7 For finite trees $\mathbb{B}_1$ and $\mathbb{B}_2$, an *embedding* of $\mathbb{B}_1$ into $\mathbb{B}_2$ is a one-to-one mapping $f : \mathbb{B}_1 \to \mathbb{B}_2$ such that $f(a \wedge b) = f(a) \wedge f(b)$ for all $a, b \in \mathbb{B}_1$, where $a \wedge b$ denotes the infimum of a and b.

We write $\mathbb{B}_1 \leq \mathbb{B}_2$ to mean that there exists an embedding $f : \mathbb{B}_1 \to \mathbb{B}_2$.

Theorem 12.8 (Kruskal's theorem) *For every infinite sequence of trees $\big(\mathbb{B}_k : k < \omega\big)$, there exist indices i and j such that $i < j < \omega$ and $\mathbb{B}_i \leq \mathbb{B}_j$. (In particular, there is no infinite set of pairwise nonembeddable trees.)*

Theorem 12.9 *Kruskal's Theorem is not provable in* $\mathbf{ATR}_0$ *(*cf. [129]*).*

The proof of the above independence result exploits a connection between finite trees and ordinal representations for ordinals $< \Gamma_0$ and the fact that Γ_0 is the proof-theoretic ordinal of $\mathbf{ATR}_0$. Each ordinal representation $\mathfrak{a}$ is assigned a finite tree $\mathbb{B}_{\mathfrak{a}}$ to the effect that for two representations $\mathfrak{a}$ and $\mathfrak{b}$, $\mathbb{B}_{\mathfrak{a}} \leq \mathbb{B}_{\mathfrak{b}}$ implies $\mathfrak{a} \leq \mathfrak{b}$. Hence Kruskal's theorem implies the well-foundedness of Γ_0 and is therefore not provable in $\mathbf{ATR}_0$. The connection between finite trees and ordinal representations for ordinals $< \Gamma_0$ was noticed by Friedman (cf. [129]) and independently by Diana Schmidt (cf. [116]).

A hope in connection with ordinal analyses is that they may lead to discoveries of new combinatorial principles which encapsulate considerable proof-theoretic strength. Examples are still scarce. One case where ordinal notations led to a new combinatorial result was Friedman's extension of Kruskal's Theorem, EKT, which asserts that finite trees are well-quasi-ordered under gap embeddability (see [129]). The gap condition imposed on the embeddings is directly related to an ordinal notation system that was used for the analysis of Π^1_1 comprehension. The principle EKT played a role in the proof of the graph minor theorem of Robertson and Seymour (see [39]).

Definition 12.10 For $n < \omega$, let $\mathcal{B}_n$ be the set of all finite trees with labels from n, i.e. $(\mathbb{B}, \ell) \in \mathcal{B}_n$ if $\mathbb{B}$ is a finite tree and $\ell : B \to \{0, \ldots, n-1\}$. The set $\mathcal{B}_n$ is quasiordered by putting $(\mathbb{B}_1, \ell_1) \leq (\mathbb{B}_2, \ell_2)$ if there exists an embedding $f : \mathbb{B}_1 \to \mathbb{B}_2$ with the following properties:

1. for each $b \in B_1$ we have $\ell_1(b) = \ell_2(f(b))$;
2. if b is an immediate successor of $a \in \mathbb{B}_1$, then for each $c \in \mathbb{B}_2$ in the interval $f(a) < c < f(b)$ we have $\ell_2(c) \geq \ell_2(f(b))$.

The condition (ii) above is called a *gap condition.*

Theorem 12.11 *For each $n < \omega$, $\mathcal{B}_n$ is a well quasi ordering (abbreviated* $\mathrm{WQO}(\mathcal{B}_n)$*), i.e. there is no infinite set of pairwise nonembeddable trees.*

Theorem 12.12 $\forall n < \omega\ \mathrm{WQO}(\mathcal{B}_n)$ *is not provable in* $\Pi^1_1 - \mathbf{CA}_0$*.*

The proof of Theorem 12.12 employs the ordinal representation system of section 9. for the proof-theoretic ordinal of $\Pi^1_1 - \mathbf{CA}_0$ which is $\psi_{\Omega_1}(\Omega_\omega)$. The connection between $< \omega$ labelled trees and this ordinal is that $\forall n < \omega\ \mathrm{WQO}(\mathcal{B}_n)$ implies the wellfoundedness of $\psi_{\Omega_1}(\Omega_\omega)$ (on the basis of $\mathbf{ACA}_0$ say). The connection is even closer in that the gap condition imposed on the embeddings between trees is actually gleaned from the ordering of the ordinal representations. If one views these terms as labelled trees, then the gap condition is exactly what one needs to ensure that an embedding of two such trees implies that the ordinal corresponding to the first tree is less than the ordinal corresponding to the second tree.

It is also for that reason that criticism had been levelled against the principle EKT for being too contrived or too metamathematical. But this was superseded by the role that EKT played in the proof of the graph minor theorem of Robertson and Seymour (see [39]).

As to the importance attributed to the graph minor theorem, let us quote from a book on Graph Theory [24], p. 249.

> *Our goal* [...] *is a single theorem, one which dwarfs any other result in graph theory and may doubtless be counted among the deepest theorems that mathematics has to offer:* in every infinite set of graphs there are two such that one is a minor of the other. *This* minor theorem, *inconspicuous though it may look at first glance, has made a fundamental impact both outside graph theory and within. Its proof, due to Neil Robertson and Paul Seymour, takes well over 500 pages.*

Definition 12.13 Let $e = xy$ be an edge of a graph $G = (V, E)$, where V and E denote its vertex and edge set, respectively. By G/e we denote the graph obtained from G by *contracting* the edge e into a new vertex v_e, which becomes adjacent to all the former neighbours of x and of y. Formally, G/e is a graph (V', E') with vertex set $V' := (V \setminus \{x, y\}) \cup \{v_e\}$ (where v_e is the "new" vertex, i.e. $v \notin V \cup E$) and edge set

$$\begin{aligned} E' \ &:= \ \big\{vw \in E | \{v, w\} \cap \{x, y\} = \emptyset\big\} \\ &\quad \cup \big\{v_e w | xw \in E \setminus \{e\} \ \ \vee \ \ yw \in E \setminus \{e\}\big\}. \end{aligned}$$

If X is obtained from Y by first deleting some vertices and edges, and then contracting some further edges, X is said to be a *minor* of Y. In point of fact, the order in which deletions and contractions are applied is immaterial as any graph obtained from another by repeated deletions and contractions in any order is its minor.

Theorem 12.14 (Robertson and Seymour 1986-1997) *If $G_0, G_1, G_2, \ldots$ is an infinite sequence of finite graphs, then there exist $i < j$ so that G_i is isomorphic to a minor of G_j.*

Corollary 12.15 *(i)* (Vázsonyi's conjecture) *If all the G_k are trivalent, then there exist $i < j$ so that G_i is embeddable into G_j.*

(ii) (Wagner's conjecture) *For any 2-manifold M there are only finitely many graphs which are not embeddable in M and are minimal with this property.*

Theorem 12.16 (Friedman, Robertson, Seymour [39]) *(i)* GMT *implies* EKT *within, say,* $\mathbf{RCA}_0$.

(ii) GMT *is not provable in* $\Pi^1_1 - \mathbf{CA}_0$.

A further independence result that ensues from ordinal analysis is due to Buchholz [11]. It concerns an extension of the hydra game of Kirby and Paris. It is shown in [11] that the assertion that Hercules has a winning strategy in this game is not provable in the theory $\Pi^1_1 - \mathbf{CA} + \mathbf{BI}$.

It would be very desirable to also find mathematically fruitful combinatorial principles hidden in stronger representation systems such as the ones based on Mahlo cardinals and weakly compact cardinals used for analyzing Kripke-Platek set theory with a recursively Mahlo universe and with Π_3-reflection, respectively.

12.2.1 Provable functions

One aim of proof theory is to find uniform scales against which one can measure the computational complexity of functions verifiably computable in "known" theories. Given a theory T, one is often interested in its provably recursive (or computable) functions. One of the oldest results of this sort is due to many people (at least Mints, Parsons, Takeuti).

Theorem 12.17 *The provably computable functions of* $\mathbf{I}\Sigma^0_1$ *are the primitive recursive functions, where* $\mathbf{I}\Sigma^0_1$ *is the fragment of* $\mathbf{PA}$ *with induction restricted to* Σ^0_1 *formulae.*

A not too difficult proof is obtained via partial cut elimination followed by "reading-off" primitive recursive bounds for existential quantifiers in such proofs.

For full $\mathbf{PA}$ there is Kreisel's classification of the provably computable functions as the $< \varepsilon_0$ recursive functions [79]. Here an ordinal representation system provides the uniform scale. Such a characterization can actually be extracted from the ordinal analysis of any theory. Indeed, it is a general fact that an ordinal analysis of a theory T yields, as a by-product, a characterization of the provably recursive functions of T. An ordinal analysis of T via an ordinal representation system $\langle A, \lhd, \ldots \rangle$ provides a reduction (also ensuring at least Π^0_2-conservativity) of T to

$$\mathbf{PA} + \bigcup_{\alpha \in A} \mathrm{TI}(\lhd_{\bar{\alpha}}) \tag{16}$$

where $\bigcup_{\alpha \in A} \mathrm{TI}(\lhd_{\bar{\alpha}})$ denotes the schema of transfinite induction for all initial segments $\lhd_{\bar{\alpha}}$ of the wellordering $\lhd$ (indexed externally). On the strength of the latter, it suffices to characterize the provably recursive functions of theories of type (16).

Definition 12.18 Let $\alpha \in A$ such that $0 \lhd \alpha$. A number-theoretic function f is called α-recursive if it can be generated by the usual schemes for generating primitive recursive functions plus the following scheme:

$$f(m, \vec{n}) = \begin{cases} h(m, \vec{n}, f(\theta(m, \vec{n}), \vec{n})) & \text{if } 0 \lhd m \lhd \alpha \\ g(m, \vec{n}) & \text{otherwise,} \end{cases}$$

where g, h, θ are α-recursive and θ satisfies $\theta(\beta, \vec{x}) \lhd \beta$ whenever $0 \lhd \beta \lhd \alpha$.

Theorem 12.19 *The provably recursive functions of* $\mathbf{PA} + \bigcup_{\alpha \in A} \mathrm{TI}(A_{\bar{\alpha}}, \lhd_{\bar{\alpha}})$ *are excactly the recursive functions which are* α*-recursive for some* $\alpha \in A$.

The proof of Theorem 12.19 poses, however, fascinating technical problems since the cut elimination usually takes place in infinitary calculi. A cut-free proof of a Σ^0_1 statement can still be infinite and one needs a further "collapse" into the finite to be able to impose a numerical bound on the existential quantifier. One technical tool for achieving this characterization is to embed $\mathbf{PA} + \bigcup_{\alpha \in A} \mathrm{TI}(\lhd_{\bar{\alpha}})$ into a system of Peano arithmetic with the ω-rule and a *repetition rule*, `Rep`, which simply repeats the premise as the conclusion. The addition of the `Rep` rule enables one to carry out a *continuous cut elimination*, due to Mints [86], which is a continuous operation in the usual tree topology on proof trees. A further pivotal step consists in making the ω-rule more constructive by assigning codes to proofs, where codes for applications of finitary rules contain codes for the proofs of the premises, and codes for applications of the ω-rule contain Gödel numbers for partial functions enumerating codes of the premises. The aforementioned enumerating functions can be required to be partial recursive, making the proof trees recursive, or even primitive recursive in the presence of the rule `Rep` which enables one to stretch recursive trees into primitive recursive trees. Theorem 12.19 can be extracted from Kreisel-Mints-Simpson [83], Lopez-Escobar [84], or Schwichtenberg [124] and was certainly known to these authors. A variant of the characterization of Theorem 12.19 is given in Friedman-Sheard [40], where the provable functions of $\mathbf{PA} + \bigcup_{\alpha \in A} \mathrm{TI}(\lhd_{\bar{\alpha}})$ are classified as the *descent recursive functions over* A. But before discussing this and related results, we would like to draw attention to a more recent approach which has the great advantage over the previous one that one need not bother with codes for infinite derivations. In this approach one adds an extra feature to infinite derivations by which one can exert a greater control on derivations so as to be able to directly read off numerical bounds from cut free proofs of Σ^0_1 statements. This has been carried out by Buchholz-Wainer [16] for the special case of **PA**. In much greater generality and flexibility this approach has been developed by Weiermann [148].

Ordinal analysis can also be used to extract information about other types of provable functions and higher type functionals, for example, hyperarithmetic functions, set recursive functions and ordinal recursive functions (cf. [110]).

13 Epilogue

Proof theory has become a large subject with many specialized areas. As a result we have only been able to sketch developments close to the main artery of its body, starting from its inception at the beginning of the 20th century. One of the omissions is a study of different proof systems and their relationships. In their "Basic Proof theory", [144] gives a good selection, but some important calculi such as the Schütte proof system are not covered. Another major omission is Bounded Arithmetic where one is concerned with feasibility issues and thus studies theories whose provable functions are subclasses of the elementary recursive functions.

Among other topics we had to omit that by and large belong to the remit of proof theory, one can currently finds the following entries in the Stanford Encyclopedia of Philosophy:

1. Linear Logic
2. Type Theory
3. Set Theory: Constructive and Intuitionistic ZF
4. Automated Reasoning (discusses resolution, unification etc).

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