Relativized ordinal analysis: The case of Power Kripke-Platek set theory

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Abstract

The paper relativizes the method of ordinal analysis developed for Kripke-Platek set theory to theories which have the power set axiom. We show that it is possible to use this technique to extract information about Power Kripke-Platek set theory, $\mathbf{KP}(\mathcal{P})$.

As an application it is shown that whenever $\mathbf{KP}(\mathcal{P}) + \mathbf{AC}$ proves a $\Pi_2^{\mathcal{P}}$ statement then it holds true in the segment V_τ of the von Neumann hierarchy, where τ stands for the Bachmann-Howard ordinal.

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1. Introduction

Ordinal analyses of ever stronger theories have been obtained over the last 20 years (cf. [1, 2, 3, 20, 21, 24, 25, 27, 28, 29]). The strongest theories for which proof-theoretic ordinals have been determined are subsystems of second order arithmetic with comprehension restricted to Π^1_2-comprehension (or even Δ^1_3-comprehension). Thus it appears that it is currently impossible to furnish an ordinal analysis of any set theory which has the power set axiom among its axioms as such a theory would dwarf the strength of second order arithmetic. Notwithstanding the foregoing, the current paper relativizes the techniques of ordinal analysis developed for Kripke-Platek set theory, **KP**, to obtain useful information about Power Kripke-Platek set theory, $\mathbf{KP}(\mathcal{P})$, culminating in a bound for the transfinite iterations of the power set operation that are provable in the latter theory. It is perhaps worthwhile comparing the results in this paper with other approaches to relativizing the ordinal analysis of **KP**. T. Arai [4] has used an ordinal representation system of Bachmann-Howard type enriched by Skolem functions to provide an analysis of Zermelo-Fraenkel set theory. In the approach of the present paper the ordinal representation is not changed at all. Rather than obtaining a characterization of the proof-theoretic ordinal of $\mathbf{KP}(\mathcal{P})$, we characterize the smallest segment of the von Neumann hierarchy which is closed under the provable power-recursive functions of $\mathbf{KP}(\mathcal{P})$ whereby one also obtains a proof-theoretic reduction of $\mathbf{KP}(\mathcal{P})$ to Zermelo set theory plus iterations of the powerset operation to any ordinal below the Bachmann-Howard ordinal.[1] The same bound also holds for the theory $\mathbf{KP}(\mathcal{P}) + \mathbf{AC}$, where $\mathbf{AC}$ stands for the *axiom of choice*. These theorems considerably sharpen results of H. Friedman to the extent that $\mathbf{KP}(\mathcal{P}) + \mathbf{AC}$ does not prove the existence of the first non-recursive ordinal ω_1^{CK} (cf. [12, Theorem 2.5] and [17, Theorem 10]).

Technically we draw on tools that have been developed more than 30 years ago. With the pioneering work of Jäger [14] on Kripke-Platek set theory and its extensions to stronger theories by Jäger and Pohlers [15] the forum of ordinal analysis switched from subsystems of second-order arithmetic to set theory, shaping what is called *admissible proof theory*, after the standard models of **KP**. We also draw on the framework of operator controlled derivations developed by Buchholz [23] that allows one to express the uniformity of infinite derivations and to carry out their bookkeeping in an elegant way.

The results and techniques of this paper have important applications. The characterization of the strength of $\mathbf{KP}(\mathcal{P})$ in terms of the von Neumann hierarchy is used in [32, Theorem 1.1] to calibrate the strength of the calculus of construction with one type universe (which is an intuitionistic type theory). Another application is made in connection with the so-called *existence property*, **EP**, that intuitionistic set theories may or may not have. Full intuitionistic Zermelo-Fraenkel set theory, **IZF**, does not have the existence property, where **IZF** is formulated with Collection (cf. [13]). By contrast, an ordinal analysis of intuitionistic $\mathbf{KP}(\mathcal{P})$ similar to the one given in this paper together with results from [31] can be utilized to show that **IZF** with only bounded separation has the **EP**.

[1] The theories share the same $\Sigma_1^{\mathcal{P}}$ theorems, but are still distinct since Zermelo set theory does not prove $\Delta_0^{\mathcal{P}}$-Collection whereas $\mathbf{KP}(\mathcal{P})$ does not prove full Separation.

2. Power Kripke-Platek set theory

A particularly interesting (classical) subtheory of **ZF** is Kripke-Platek set theory, **KP**. Its standard models are called *admissible sets*. One of the reasons that this is an important theory is that a great deal of set theory requires only the axioms of **KP**. An even more important reason is that admissible sets have been a major source of interaction between model theory, recursion theory and set theory (cf. [6]). Roughly **KP** arises from **ZF** by completely omitting the power set axiom and restricting separation and collection to set bounded formulae but adding set induction (or class foundation). These alterations are suggested by the informal notion of 'predicative'.

To be more precise, quantifiers of the forms $\forall x \in a$, $\exists x \in a$ are called *set bounded*. *Set bounded* or Δ_0*-formulae* are formulae wherein all quantifiers are set bounded. The axioms of **KP** consist of *Extensionality, Pair, Union, Infinity,* Δ_0*-Separation*

$$\exists x\, \forall u\, [u \in x \leftrightarrow (u \in a \wedge A(u))]$$

for all Δ_0-formulae $A(u)$, Δ_0-*Collection*

$$\forall x \in a\, \exists y\, G(x,y) \rightarrow \exists z\, \forall x \in a\, \exists y \in z\, G(x,y)$$

for all Δ_0-formulae $G(x,y)$, and *Set Induction*

$$\forall x\ [(\forall y \in x\, C(y)) \rightarrow C(x)] \rightarrow \forall x\, C(x)$$

for all formulae $C(x)$.

A transitive set A such that $(A, \in)$ is a model of **KP** is called an *admissible set*. Of particular interest are the models of **KP** formed by segments of Gödel's *constructible hierarchy* **L**. The constructible hierarchy is obtained by iterating the definable powerset operation through the ordinals

$$\begin{aligned}
\mathbf{L}_0 &= \emptyset, \\
\mathbf{L}_\lambda &= \bigcup \{\mathbf{L}_\beta : \beta < \lambda\}\ \lambda \text{ limit} \\
\mathbf{L}_{\beta+1} &= \big\{X :\ X \subseteq \mathbf{L}_\beta;\ X \text{ definable over } \langle \mathbf{L}_\beta, \in \rangle\big\}.
\end{aligned}$$

So any element of **L** of level α is definable from elements of **L** with levels $< \alpha$ and the parameter $\mathbf{L}_\alpha$. An ordinal α is *admissible* if the structure $(\mathbf{L}_\alpha, \in)$ is a model of **KP**.

If the power set operation is considered as a definite operation, but the universe of all sets is regarded as an indefinite totality, we are led to systems of set theory having Power Set as an axiom but only Bounded Separation axioms and intuitionistic logic for reasoning about the universe at large. The study of subsystems of **ZF** formulated in intuitionistic logic with Bounded Separation but containing the Power Set axiom was apparently initiated by Pozsgay [18, 19] and then pursued more systematically by Tharp [34], Friedman [11] and Wolf [36]. These systems are actually semi-intuitionistic as they contain the law of excluded middle for bounded formulae.

In the classical context, weak subsystems of **ZF** with Bounded Separation and Power Set have been studied by Thiele [35], Friedman [12] and more recently at great length by Mathias [17]. Mac Lane has singled out and championed a particular fragment of **ZF**, especially in his book *Form and Function* [16]. *Mac Lane Set Theory*, christened **MAC** in [17], comprises the axioms of Extensionality, Null Set, Pairing, Union, Infinity, Power Set, Bounded Separation, Foundation, and Choice. **MAC** is naturally related to systems derived from topos-theoretic notions and, moreover, to type theories.

Definition 2.1. We use subset bounded quantifiers $\exists x \subseteq y\ \ldots$ and $\forall x \subseteq y\ \ldots$ as abbreviations for $\exists x(x \subseteq y \wedge \ldots)$ and $\forall x(x \subseteq y \rightarrow \ldots)$, respectively.

The $\Delta_0^{\mathcal{P}}$-formulae are the smallest class of formulae containing the atomic formulae closed under $\wedge, \vee, \rightarrow, \neg$ and the quantifiers

$$\forall x \in a,\ \exists x \in a,\ \forall x \subseteq a,\ \exists x \subseteq a.$$

Definition 2.2. $\mathbf{KP}(\mathcal{P})$ has the same language as **ZF**. Its axioms are the following: Extensionality, Pairing, Union, Infinity, Powerset, $\Delta_0^{\mathcal{P}}$-Separation, $\Delta_0^{\mathcal{P}}$-Collection and Set Induction (or Class Foundation).

The transitive models of $\mathbf{KP}(\mathcal{P})$ have been termed **power admissible** sets in [12].

Remark 2.3. Alternatively, $\mathbf{KP}(\mathcal{P})$ can be obtained from **KP** by adding a function symbol $\mathcal{P}$ for the powerset function as a primitive symbol to the language and the axiom

$$\forall y\, [y \in \mathcal{P}(x) \leftrightarrow y \subseteq x]$$

and extending the schemes of Δ_0 Separation and Collection to the Δ_0 formulae of this new language.

Lemma 2.4. $\mathbf{KP}(\mathcal{P})$ *is* **not** *the same theory as* $\mathbf{KP} + \mathbf{Pow}$*. Indeed,* $\mathbf{KP} + \mathbf{Pow}$ *is a much weaker theory than* $\mathbf{KP}(\mathcal{P})$ *in which one cannot prove the existence of* $V_{\omega+\omega}$*.*

Proof: Note that in the presence of full Separation and Infinity there is no difference between our system **KP** and Mathias's [17] **KP**. It follows from [17, Theorem 14] that $\mathbf{Z}+\mathbf{KP}+\mathbf{AC}$ is conservative over $\mathbf{Z}+\mathbf{AC}$ for stratifiable sentences. **Z** and $\mathbf{Z}+\mathbf{AC}$ are of the same proof-theoretic strength as the constructible hierarchy can be simulated in **Z**; a stronger statement is given in [17, Theorem 16]. As a result, **Z** and $\mathbf{Z}+\mathbf{KP}$ are of the same strength. As $\mathbf{KP}+\mathbf{Pow}$ is a subtheory of $\mathbf{Z}+\mathbf{KP}$, we have that $\mathbf{KP}+\mathbf{Pow}$ is not stronger than **Z**. If $\mathbf{KP}+\mathbf{Pow}$ could prove the existence of $V_{\omega+\omega}$ it would prove the consistency of **Z**. On the other hand $\mathbf{KP}(\mathcal{P})$ proves the existence of V_α for every ordinal α and hence proves the existence of arbitrarily large transitive models of **Z**. □

Remark 2.5. Our system $\mathbf{KP}(\mathcal{P})$ is not quite the same as the theory $\mathbf{KP}^{\mathcal{P}}$ in Mathias' paper [17, 6.10]. The difference between $\mathbf{KP}(\mathcal{P})$ and $\mathbf{KP}^{\mathcal{P}}$ is that in the latter system set induction only holds for $\Sigma_1^{\mathcal{P}}$ formulae, or what amounts to the same, $\Pi_1^{\mathcal{P}}$ foundation ($A \neq \emptyset \rightarrow \exists x \in A\; x \cap A = \emptyset$ for $\Pi_1^{\mathcal{P}}$ classes A).

Friedman [12] includes only Set Foundation in his formulation of a formal system $\mathbf{PAdm}^s$ appropriate to the concept of recursion in the power set operation $\mathcal{P}$.

3. A Tait-style formalization of KP($\mathcal{P}$)

For technical reasons we shall use a Tait–style sequent calculus version of $\mathbf{KP}(\mathcal{P})$ in which finite sets of formulae can be derived. In addition, formulae have to be in negation normal form (cf. [33]). The language consists of: free variables $a_0, a_1, \cdots$, bound variables $x_0, x_1, \cdots$; the predicate symbol $\in$; the logical symbols $\neg, \vee, \wedge, \forall, \exists$. One peculiarity will be that we treat bounded quantifiers and subset bounded quantifiers as quantifiers in their own right.

We will use $a, b, c, \cdots,\ x, y, z, \cdots,\ A, B, C, \cdots$ as metavariables whose domains are the domain of the free variables, bound variables, formulae, respectively.

The *atomic formulae* are those of the form $(a \in b), \neg(a \in b)$.

The *formulae* are defined inductively as follows:

(i) Atomic formulae are formulae.

(ii) If A and B are formulae, then so are $(A \wedge B)$ and $(A \vee B)$.

(iii) If $A(b)$ is a formula in which x does not occur, then $\forall x A(x)$, $\exists x A(x)$, $(\forall x \in a)A(x)$, $(\exists x \in a)A(x)$, $(\forall x \subseteq a)A(x)$, and $(\exists x \subseteq a)A(x)$ are formulae.

The quantifiers $\exists x, \forall x$ will be called *unbounded*, whereas the other quantifiers will be referred to as *bounded quantifiers*. A $\Delta_0^{\mathcal{P}}$*–formula* is a formula which contains no unbounded quantifiers. The Δ_0–formulae are those $\Delta_0^{\mathcal{P}}$-formulae that do not contain subset bounded quantifiers.

The *negation* $\neg A$ of a formula A is defined to be the formula obtained from A by (i) putting $\neg$ in front of any atomic formula, (ii) replacing $\wedge, \vee, \forall x, \exists x, (\forall x \in a), (\exists x \in a), (\forall x \subseteq a), (\exists x \subseteq a)$ by $\vee, \wedge, \exists x, \forall x, (\exists x \in a), (\forall x \in a), (\exists x \subseteq a), (\forall x \subseteq a)$, respectively, and (iii) dropping double negations. $A \to B$ stands for $\neg A \vee B$.

$\vec{a}, \vec{b}, \vec{c}, \cdots$ and $\vec{x}, \vec{y}, \vec{z}, \cdots$ will be used to denote finite sequences of free and bound variables, respectively.

We use $F[a_1, \cdots, a_n]$ (by contrast with $F(a_1, \cdots, a_n)$) to denote a formula the free variables of which are among $a_1, \cdots, a_n$. We will write
$a = \{x \in b : G(x)\}$ for $(\forall x \in a)[x \in b \wedge G(x)] \wedge (\forall x \in b)[G(x) \to x \in a]$.

$a = b$ stands for $(\forall x \in a)(x \in b) \wedge (\forall x \in b)(x \in a)$. $a \subseteq b$ stands for $(\forall x \in a)(x \in b)$. However, as part of a subset bounded quantifier $(\forall x \subseteq a)$ or $(\exists x \subseteq b)$, $\subseteq$ is considered to be a primitive symbol.

Definition 3.1. The sequent-style version of $\mathbf{KP}(\mathcal{P})$ derives finite sets of formulae denoted by $\Gamma, \Delta, \Theta, \Xi, \cdots$. The intended meaning of Γ is the disjunction of all formulae of Γ. We use the notation Γ, A for $\Gamma \cup \{A\}$, and Γ, Ξ for $\Gamma \cup \Xi$.

The *axioms of* $\mathbf{KP}(\mathcal{P})$ are the following:

Logical axioms:	$\Gamma, A, \neg A$ for every $\Delta_0^{\mathcal{P}}$–formula A.
Extensionality:	$\Gamma, a = b \wedge B(a) \to B(b)$ for every $\Delta_0^{\mathcal{P}}$-formula $B(a)$.
Pair:	$\Gamma, \exists x[a \in x \wedge b \in x]$
Union:	$\Gamma, \exists x(\forall y \in a)(\forall z \in y)(z \in x)$
$\Delta_0^{\mathcal{P}}$ *–Separation:*	$\Gamma, \exists y(y = \{x \in a : G(x)\})$ for every $\Delta_0^{\mathcal{P}}$–formula $G(b)$.
Set Induction:	$\Gamma, \forall u\, [(\forall x \in u)\, G(x) \to G(u)] \to \forall u\, G(u)$ for every formula $G(b)$.
Infinity:	$\Gamma, \exists x\, [(\exists y \in x)\, y \in x \wedge (\forall y \in x)(\exists z \in x)\, y \in z]$.
Power Set:	$\Gamma, \exists z\, (\forall x \subseteq a) x \in z$.

The *logical rules of inference* are:

$(\wedge)$	$\vdash \Gamma, A$ and $\vdash \Gamma, B$	$\Rightarrow$	$\vdash \Gamma, A \wedge B$
$(\vee)$	$\vdash \Gamma, A_i$ for $i \in \{0,1\}$	$\Rightarrow$	$\vdash \Gamma, A_0 \vee A_1$
$(b\forall)$	$\vdash \Gamma, a \in b \rightarrow F(a)$	$\Rightarrow$	$\vdash \Gamma, (\forall x \in b) F(x)$
$(pb\forall)$	$\vdash \Gamma, a \subseteq b \rightarrow F(a)$	$\Rightarrow$	$\vdash \Gamma, (\forall x \subseteq b) F(x)$
$(\forall)$	$\vdash \Gamma, F(a)$	$\Rightarrow$	$\vdash \Gamma, \forall x F(x)$
$(b\exists)$	$\vdash \Gamma, a \in b \wedge F(a)$	$\Rightarrow$	$\vdash \Gamma, (\exists x \in b) F(x)$
$(pb\exists)$	$\vdash \Gamma, a \subseteq b \wedge F(a)$	$\Rightarrow$	$\vdash \Gamma, (\exists x \subseteq b) F(x)$
$(\exists)$	$\vdash \Gamma, F(a)$	$\Rightarrow$	$\vdash \Gamma, \exists x F(x)$
(Cut)	$\vdash \Gamma, A$ and $\vdash \Gamma, \neg A$	$\Rightarrow$	$\vdash \Gamma.$

In the foregoing rules $F(a)$ is an arbitrary formula. Of course, it is demanded that in $(b\forall)$, $(pb\forall)$ and $(\forall)$ the free variable a is not to occur in the conclusion; a is called the *eigenvariable* of that inference.

The *non–logical rule of inference* is:

$$(\Delta_0^{\mathcal{P}}\text{–COLLR}) \quad \vdash \Gamma, (\forall x \in a) \exists y H(x,y) \quad \Rightarrow \quad \vdash \Gamma, \exists z (\forall x \in a)(\exists y \in z) H(x,y)$$

for every $\Delta_0^{\mathcal{P}}$–formula $H(b,c)$.

This rule is not weaker than the schema of $\Delta_0^{\mathcal{P}}$-Collection since side formulae (those in Γ) are allowed: Using logical rules we have

$$\vdash \neg(\forall x \in a)\, \exists y\, H(x,y),\ (\forall x \in a)\, \exists y\, H(x,y).$$

Thus if $H(b,c)$ is $\Delta_0^{\mathcal{P}}$ we can employ ($\Delta_0^{\mathcal{P}}$–COLLR) to conclude

$$\vdash \neg(\forall x \in a)\, \exists y\, H(x,y),\ \exists z\, (\forall x \in a)\, (\exists y \in z)\, H(x,y)$$

so that, by applying $(\vee)$ twice, we arrive at

$$\vdash (\forall x \in a)\, \exists y\, H(x,y) \rightarrow \exists z\, (\forall x \in a)(\exists y \in z)\, H(x,y).$$

We shall conceive of axioms as inferences with an empty set of premisses. The *minor formulae* (m.f.) of an inference are those formulae which are rendered prominently in its premises. The *principal formulae* (p.f.) of an inference are the formulae rendered prominently in its conclusion. (Cut) has no p.f. So any inference has the form

$$(*) \quad \text{For all } i < k \ \vdash \Gamma, \Xi_i \quad \Rightarrow \quad \vdash \Gamma, \Xi$$

$(0 \leq k \leq 2)$, where Ξ consists of the p.f. and Ξ_i is the set of m.f. in the i–th premise. The formulae in Γ are called *side formulae* (s.f.) of $(*)$.

Derivations are defined inductively, as usual. $\mathcal{D}$,$\mathcal{D}$',$\mathcal{D}_0, \cdots$ range as syntactic variables over derivations. All this is completely standard, and we refer to [33] for notions like "*length of a derivation* $\mathcal{D}$" (abbreviated by $|\mathcal{D}|$), "*last inference of* $\mathcal{D}$", "*direct subderivation of* $\mathcal{D}$". We write $\mathcal{D} \vdash \Gamma$ if $\mathcal{D}$ is a derivation of Γ.

4. A representation system for the Bachmann-Howard ordinal

Definition 4.1. Let Ω be a "big" ordinal, e.g. $\Omega = \aleph_1$ or ω_1^{ck}. By recursion on α we define sets $C^\Omega(\alpha, \beta)$ and the ordinal $\psi_\Omega(\alpha)$ as follows:

$$C^\Omega(\alpha,\beta) \quad = \quad \begin{cases} \ \textit{closure of } \beta \cup \{0,\Omega\} \\ \ \textit{under:} \\ +, (\xi \mapsto \omega^\xi) \\ (\xi \longmapsto \psi_\Omega(\xi))_{\xi<\alpha} \end{cases} \tag{1}$$

$$\psi_\Omega(\alpha) \quad \simeq \quad \min\{\rho < \Omega :\ C^\Omega(\alpha,\rho) \cap \Omega = \rho\}. \tag{2}$$

It can be shown that $\psi_\Omega(\alpha)$ is always defined and thus

$$\psi_\Omega(\alpha) < \Omega.$$

In the case of Ω being ω_1^{ck}, this follows from [23]. Moreover,

$$[\psi_\Omega(\alpha),\Omega) \ \cap\ C^\Omega(\alpha,\psi_\Omega(\alpha)) \ =\ \emptyset\,.$$

Thus the order-type of the ordinals below Ω which belong to the set $C^\Omega(\alpha,\psi_\Omega(\alpha))$ is $\psi_\Omega(\alpha)$. $\psi_\Omega(\alpha)$ is also a countable ordinal. In more pictorial terms, $\psi_\Omega(\alpha)$ is the α^{th} collapse of Ω.

Let $\varepsilon_{\Omega+1}$ be the least ordinal $\alpha > \Omega$ such that $\omega^\alpha = \alpha$. The set of ordinals $C^\Omega(\varepsilon_{\Omega+1}, 0)$ gives rise to an elementary computable ordinal representation system (cf. [14, 8, 23, 26]). In what follows, $C^\Omega(\varepsilon_{\Omega+1}, 0)$ will sometimes be denoted by $\mathcal{T}(\Omega)$.

In point of fact,

$$C^\Omega(\varepsilon_{\Omega+1},0) \,\cap\, \Omega = \psi_\Omega(\varepsilon_{\Omega+1}).$$

The ordinal $\psi_\Omega(\varepsilon_{\Omega+1})$ is known as the **Bachmann-Howard ordinal**. Its relation to **KP** is that it is the proof-theoretic ordinal of this theory as was shown by Jäger [14]. Moreover it is the smallest ordinal such that $L_{\psi_\Omega(\varepsilon_{\Omega+1})}$ is a Π_2-model of **KP** (see [22, Theorem 2.1] or [30, theorem 4.3]), i.e., whenever **KP** proves a Π_2 sentence C of set theory, then $L_{\psi_\Omega(\varepsilon_{\Omega+1})} \models C$.

For later it is also worthwhile recording the following fact.

Lemma 4.2. *For all* α, $C^\Omega(\alpha,0) = C^\Omega(\alpha,\psi_\Omega(\alpha))$.

5. The infinitary proof system $\boldsymbol{RS_\Omega^{\mathcal{P}}}$

The purpose of this section is to introduce an infinitary proof system $RS_\Omega^{\mathcal{P}}$. The letter combination "RS" is used for traditional reasons. They stand for "ramified set theory", following [14].

Henceforth all ordinals will be assumed to belong to $C^\Omega(\varepsilon_{\Omega+1}, 0)$.

The problem of "naming" sets will be solved by building a formal von Neumann hierarchy using the ordinals $< \Omega$ belonging to this set (i.e., ordinals $< \psi_\Omega(\varepsilon_{\Omega+1})$).

Definition 5.1. We define the $RS_\Omega^{\mathcal{P}}$–terms. To each $RS_\Omega^{\mathcal{P}}$–term t we also assign its *level*, $|t|$.

1. For each $\alpha < \Omega$, $\mathbb{V}_\alpha$ is an $RS_\Omega^{\mathcal{P}}$–term with $|\,\mathbb{V}_\alpha\,| = \alpha$.
2. For each $\alpha < \Omega$, we have infinitely many free variables $a_1^\alpha, a_2^\alpha, a_3^\alpha, \ldots$ which are $RS_\Omega^{\mathcal{P}}$–terms with $|\,a_i^\alpha\,| = \alpha$.
3. If $F(x, \vec{y}\,)$ is a $\Delta_0^{\mathcal{P}}$-formula of $\mathbf{KP}(\mathcal{P})$ (whose free variables are exactly those indicated) and $\vec{s} \equiv s_1, \cdots, s_n$ are $RS_\Omega^{\mathcal{P}}$–terms, then the formal expression
$$\{x \in \mathbb{V}_\alpha \mid F(x, \vec{s}\,)\}$$
is an $RS_\Omega^{\mathcal{P}}$–term with $|\,\{x \in \mathbb{V}_\alpha \mid F(x, \vec{s}\,)\}\,| = \alpha$.

The $RS_\Omega^{\mathcal{P}}$*–formulae* are the expressions of the form $F(s_1, \ldots, s_n)$, where $F(a_1, \ldots, a_n)$ is a formula of $\mathbf{KP}(\mathcal{P})$ with all free variables exhibited and $s_1, \ldots, s_n$ are $RS_\Omega^{\mathcal{P}}$-terms. We set

$$|\,F(s_1, \ldots, s_n)\,| \;\;=\;\; \{|\,s_1\,|, \ldots, |\,s_n\,|\}.$$

A formula is a $\Delta_0^{\mathcal{P}}$**-formula** of $RS_\Omega^{\mathcal{P}}$ if it is of the form $F(s_1, \ldots, s_n)$ with $F(a_1, \ldots, a_n)$ being $\Delta_0^{\mathcal{P}}$-formula of $\mathbf{KP}(\mathcal{P})$ and $s_1, \ldots, s_n$ $RS_\Omega^{\mathcal{P}}$-terms.

As in the case of the Tait-style version of $\mathbf{KP}(\mathcal{P})$, we let $\neg A$ be the formula which arises from A by (i) putting $\neg$ in front of each atomic formula, (ii) replacing $\wedge, \vee, (\forall x\,{\in}\, s), (\exists x\,{\in}\, s), (\forall x \subseteq s), (\exists x \subseteq s), \forall x, \exists x$ by $\vee, \wedge, (\exists x\,{\in}\, s), (\forall x\,{\in}\, s), (\exists x \subseteq s), (\forall x \subseteq s), \exists x, \forall x$, respectively, and (iii) dropping double negations. $A \to B$ stands for $\neg A \vee B$.

Remark 5.2. Note that in contrast to the infinitary system used for the ordinal of $\mathbf{KP}$ (see [14, 8]) the terms of $RS_\Omega^{\mathcal{P}}$ may contain free variables. This will be crucial in proving the Soundness Theorem 8.1.

Observe that the impredicativity of Powerset is reflected in the formation rules for $RS_\Omega^{\mathcal{P}}$-terms in that, owing to clause 3 of Definition 5.1, terms of level α can be generated by referring to terms of higher levels.

Convention: In the sequel, $RS_\Omega^{\mathcal{P}}$–formulae will simply be referred to as formulae. The same usage applies to $RS_\Omega^{\mathcal{P}}$–terms.

We denote by upper case Greek letters $\Gamma, \Delta, \Lambda, \dots$ finite sets of $RS_\Omega^{\mathcal{P}}$–formulae. The intended meaning of $\Gamma = \{A_1, \cdots, A_n\}$ is the disjunction $A_1 \vee \cdots \vee A_n$. Γ, Ξ stands for $\Gamma \cup \Xi$ and Γ, A stands for $\Gamma \cup \{A\}$.

Definition 5.3. The *axioms* of $RS_\Omega^{\mathcal{P}}$ are:

(A1) Γ, A, $\neg A$ for A in $\Delta_0^{\mathcal{P}}$.

(A2) Γ, $t = t$.

(A3) Γ, $s_1 \neq t_1, \dots, s_n \neq t_n, \neg A(s_1, \dots, s_n)$, $A(t_1, \dots, t_n)$
for $A(s_1, \dots, s_n)$ in $\Delta_0^{\mathcal{P}}$.

(A4) Γ, $s \in \mathbb{V}_\alpha$ if $|\, s \,| < \alpha$.

(A5) Γ, $s \subseteq \mathbb{V}_\alpha$ if $|\, s \,| \leq \alpha$.

(A6) $\Gamma, t \notin \{x \in \mathbb{V}_\alpha \mid F(x, \vec{s})\}, F(t, \vec{s})$
whenever $F(t, \vec{s})$ is $\Delta_0^{\mathcal{P}}$ and $|\, t \,| < \alpha$.

(A7) $\Gamma, \neg F(t, \vec{s}), t \in \{x \in \mathbb{V}_\alpha \mid F(x, \vec{s})\}$
whenever $F(t, \vec{s})$ is $\Delta_0^{\mathcal{P}}$ and $|\, t \,| < \alpha$.

The *inference rules* of $RS_{\Omega}^{\mathcal{P}}$ are:

$$(\wedge) \qquad \frac{\Gamma, A \quad \Gamma, A'}{\Gamma, A \wedge A'}$$

$$(\vee) \qquad \frac{\Gamma, A_i}{\Gamma, A_0 \vee A_1} \quad \text{if } i = 0 \text{ or } i = 1$$

$$(b\forall)_\infty \qquad \frac{\Gamma,\ s \in t \rightarrow F(s) \ \text{ for all } |\, s\,| < |\, t\,|}{\Gamma,\ (\forall x \in t) F(x)}$$

$$(b\exists) \qquad \frac{\Gamma,\ s \in t \wedge F(s)}{\Gamma, (\exists x \in t) F(x)} \quad \text{if } |\, s\,| < |\, t\,|$$

$$(pb\forall)_\infty \qquad \frac{\Gamma, s \subseteq t \rightarrow F(s) \ \text{ for all } |\, s\,| \leq |\, t\,|}{\Gamma, (\forall x \subseteq t) F(x)}$$

$$(pb\exists) \qquad \frac{\Gamma, s \subseteq t \wedge F(s)}{\Gamma, (\exists x \subseteq t) F(x)} \quad \text{if } |\, s\,| \leq |\, t\,|$$

$$(\forall)_\infty \qquad \frac{\Gamma,\ F(s) \ \text{ for all } s}{\Gamma, \forall x\, F(x)}$$

$$(\exists) \qquad \frac{\Gamma,\ F(s)}{\Gamma, \exists x\, F(x)}$$

$$(\notin)_\infty \qquad \frac{\Gamma, r \in t \rightarrow r \neq s \ \text{ for all } |\, r\,| < |\, t\,|}{\Gamma, s \notin t}$$

$$(\in) \qquad \frac{\Gamma,\ r \in t \ \wedge \ r = s}{\Gamma, s \in t} \quad \text{if } |\, r\,| < |\, t\,|$$

$$(\not\subseteq)_\infty \qquad \frac{\Gamma, r \subseteq t \rightarrow r \neq s \ \text{ for all } |\, r\,| \leq |\, t\,|}{\Gamma, s \not\subseteq t}$$

$$(\subseteq) \qquad \frac{\Gamma,\ r \subseteq t \ \wedge \ r = s}{\Gamma, s \subseteq t} \quad \text{if } |\, r\,| \leq |\, s\,|$$

$$(\text{Cut}) \qquad \frac{\Gamma, A \qquad \Gamma, \neg A}{\Gamma}$$

$$(\Sigma^{\mathcal{P}}\text{-}Ref) \quad \frac{\Gamma, A}{\Gamma,\ \exists z\, A^z} \quad \text{if } A \text{ is a } \Sigma^{\mathcal{P}}\text{-formula,}$$

where a formula is said to be in $\Sigma^{\mathcal{P}}$ if all its unbounded quantifiers are existential.

A^z results from A by restricting all unbounded quantifiers to z.

5.1. $\mathcal{H}$–controlled derivations

In general in $RS_\Omega^{\mathcal{P}}$ we cannot remove cuts that have $\Delta_0^{\mathcal{P}}$ cut formulae. What's more, the rule $(\Sigma^{\mathcal{P}}\text{-}Ref)$ poses an obstacle to removing cuts involving $\Sigma_1^{\mathcal{P}}$ formulae. Notwithstanding that, it will turn out that cuts of a complexity higher than $\Delta_0^{\mathcal{P}}$ can be removed from derivations of $\Sigma^{\mathcal{P}}$ formulae if they are of a very uniform kind.

For the presentation of infinitary proofs we draw on [8]. Buchholz developed a very elegant and flexible setting for describing uniformity in infinitary proofs, called operator controlled derivations.

Definition 5.4. Let

$$P(ON) = \{X : X \text{ is a set of ordinals}\}.$$

A class function

$$\mathcal{H} : P(ON) \to P(ON)$$

will be called an **operator** if $\mathcal{H}$ is a closure operator, i.e monotone, inclusive and idempotent, and satisfies the following conditions for all $X \in P(ON)$:

1. $0 \in \mathcal{H}(X)$ and $\Omega \in \mathcal{H}(X)$.
2. If α has Cantor normal form $\omega^{\alpha_1} + \cdots + \omega^{\alpha_n}$, then

$$\alpha \in \mathcal{H}(X) \iff \alpha_1, ..., \alpha_n \in \mathcal{H}(X).$$

The latter ensures that $\mathcal{H}(X)$ will be closed under $+$ and $\sigma \mapsto \omega^\sigma$, and decomposition of its members into additive and multiplicative components.

For a sequent $\Gamma = \{A_1, \ldots, A_n\}$ we define

$$|\,\Gamma\,| \;\; := \;\; |\,A_1\,| \cup \ldots \cup |\,A_n\,|.$$

If s is an $RS_\Omega^{\mathcal{P}}$-term, the operator $\mathcal{H}[s]$ is defined by

$$\mathcal{H}[s](X) \;\; = \;\; \mathcal{H}(X \cup \{|\,s\,|\}).$$

Likewise, if $\mathfrak{X}$ is a formula or a sequent we define

$$\mathcal{H}[\mathfrak{X}](X) \;\; = \;\; \mathcal{H}(X \cup |\,\mathfrak{X}\,|).$$

If $\mathfrak{Y}_i$ is a term, or a formula, or a sequent for all $1 \le i \le n$, we let $\mathcal{H}[\mathfrak{Y}_1, \mathfrak{Y}_2] = (\mathcal{H}[\mathfrak{Y}_1])[\mathfrak{Y}_2]$, $\mathcal{H}[\mathfrak{Y}_1, \mathfrak{Y}_2, \mathfrak{Y}_3] = (\mathcal{H}[\mathfrak{Y}_1, \mathfrak{Y}_2])[\mathfrak{Y}_3]$, etc.

Lemma 5.5. *Let $\mathcal{H}$ be an operator. Let s be a term and $\mathfrak{X}$ be a formula or a sequent.*

(i) $\forall X, X' \in P(ON)[X' \subseteq X \Longrightarrow \mathcal{H}(X') \subseteq \mathcal{H}(X)]$.

(ii) $\mathcal{H}[s]$ *and* $\mathcal{H}[\mathfrak{X}]$ *are operators.*

(iii) $|\,\mathfrak{X}\,| \subseteq \mathcal{H}(\emptyset) \implies \mathcal{H}[\mathfrak{X}] = \mathcal{H}$.

(iv) $|\,s\,| \in \mathcal{H}(\emptyset) \implies \mathcal{H}[s] = \mathcal{H}$.

Since we also want to keep track of the complexity of cuts appearing in derivations, we endow each formula with an ordinal rank.

Definition 5.6. The *rank* of a formula is determined as follows.

1. $rk(s \in t) := rk(s \notin t) := max\{|\,s\,| + 1, |\,t\,| + 1\}$.

2. $rk((\exists x \in t)F(x)) := rk((\forall x \in t)F(x)) := max\{|\,t\,|, rk(F(\mathbb{V}_0)) + 2\}$.

3. $rk((\exists x \subseteq t)F(x)) := rk((\forall x \subseteq t)F(x)) := max\{|\,t\,|+1, rk(F(\mathbb{V}_0))+2\}$.

4. $rk(\exists x\, F(x)) := rk(\forall x\, F(x)) := max\{\Omega, rk(F(\mathbb{V}_0)) + 2\}$.

5. $rk(A \wedge B) := rk(A \vee B) := max\{rk(A), rk(B)\} + 1$.

Note that for a $\Delta_0^{\mathcal{P}}$ formula A we have $rk(A) < \Omega$.
There is plenty of leeway in designing the actual rank of a formula.

Definition 5.7. Let $\mathcal{H}$ be an operator and let Λ be a finite set of $RS_\Omega^{\mathcal{P}}$–formulae. $\mathcal{H} \,\substack{\alpha \\ \rho}\!\!\mid\!\!\!-\, \Lambda$ is defined by recursion on α.

If Λ is an **axiom** and $|\,\Lambda\,| \cup \{\alpha\} \subseteq \mathcal{H}(\emptyset)$, then $\mathcal{H} \,\substack{\alpha \\ \rho}\!\!\mid\!\!\!-\, \Lambda$.

Moreover, we have inductive clauses pertaining to the inference rules of $RS_\Omega^{\mathcal{P}}$, which all come with the additional requirement that

$$|\,\Lambda\,| \cup \{\alpha\} \subseteq \mathcal{H}(\emptyset)$$

where Λ is the sequent of the conclusion. We shall not repeat this requirement below.

Below the third column gives the requirements that the ordinals have to satisfy for each of the inferences. For instance in the case of $(\forall)_\infty$, to be able to conclude that $\mathcal{H} \,\substack{\alpha \\ \rho}\!\!\mid\!\!\!-\, \Gamma, \forall x F(x)$, it is required that for all terms s

there exists α_s such that $\mathcal{H}[s] \,\Big|\!\frac{\alpha_s}{\rho}\, \Gamma, F(s)$ and $|\,s\,| < \alpha_s + 1 < \alpha$. The side conditions for the rules $(b\forall)_\infty, (pb\forall)_\infty, (\notin)_\infty, (\not\subseteq)_\infty$ below have to read in the same vein.

The clauses are the following:

$$(\wedge) \qquad \frac{\mathcal{H} \,\Big|\!\frac{\alpha_0}{\rho}\, \Gamma, A_0 \qquad \mathcal{H} \,\Big|\!\frac{\alpha_0}{\rho}\, \Gamma, A_1}{\mathcal{H} \,\Big|\!\frac{\alpha}{\rho}\, \Gamma, A_0 \wedge A_1} \qquad \alpha_0 < \alpha$$

$$(\vee) \qquad \frac{\mathcal{H} \,\Big|\!\frac{\alpha_0}{\rho}\, \Lambda, A_i}{\mathcal{H} \,\Big|\!\frac{\alpha}{\rho}\, \Gamma, A_0 \vee A_1} \qquad \begin{array}{r} \alpha_0 < \alpha \\ i \in \{0,1\} \end{array}$$

$$(\mathrm{Cut}) \qquad \frac{\mathcal{H} \,\Big|\!\frac{\alpha_0}{\rho}\, \Lambda, B \qquad \mathcal{H} \,\Big|\!\frac{\alpha_0}{\rho}\, \Lambda, \neg B}{\mathcal{H} \,\Big|\!\frac{\alpha}{\rho}\, \Lambda} \qquad \begin{array}{r} \alpha_0 < \alpha \\ rk(B) < \rho \end{array}$$

$$(b\forall)_\infty \qquad \frac{\mathcal{H}[s] \,\Big|\!\frac{\alpha_s}{\rho}\, \Gamma, s \in t \to F(s) \text{ for all } |\,s\,| < |\,t\,|}{\mathcal{H} \,\Big|\!\frac{\alpha}{\rho}\, \Gamma, (\forall x \in t)F(x)} \qquad |\,s\,| \leq \alpha_s < \alpha$$

$$(b\exists) \qquad \frac{\mathcal{H} \,\Big|\!\frac{\alpha_0}{\rho}\, \Gamma, s \in t \wedge F(s)}{\mathcal{H} \,\Big|\!\frac{\alpha}{\rho}\, \Gamma, (\exists x \in t)F(x)} \qquad \begin{array}{r} \alpha_0 < \alpha \\ |\,s\,| < |\,t\,| \\ |\,s\,| < \alpha \end{array}$$

$$(pb\forall)_\infty \qquad \frac{\mathcal{H}[s] \,\Big|\!\frac{\alpha_s}{\rho}\, \Gamma, s \subseteq t \to F(s) \text{ for all } |\,s\,| \leq |\,t\,|}{\mathcal{H} \,\Big|\!\frac{\alpha}{\rho}\, \Gamma, (\forall x \subseteq t)F(x)} \qquad |\,s\,| \leq \alpha_s < \alpha$$

$$(pb\exists) \qquad \frac{\mathcal{H} \,\Big|\!\frac{\alpha_0}{\rho}\, \Gamma, s \subseteq t \wedge F(s)}{\mathcal{H} \,\Big|\!\frac{\alpha}{\rho}\, \Gamma, (\exists x \subseteq t)F(x)} \qquad \begin{array}{r} \alpha_0 < \alpha \\ |\,s\,| \leq |\,t\,| \\ |\,s\,| < \alpha \end{array}$$

$$(\forall)_\infty \qquad \frac{\mathcal{H}[s] \,\Big|\!\frac{\alpha_s}{\rho}\, \Gamma, F(s) \text{ for all } s}{\mathcal{H} \,\Big|\!\frac{\alpha}{\rho}\, \Gamma, \forall x F(x)} \qquad |\,s\,| < \alpha_s + 1 < \alpha$$

$$(\exists) \qquad \frac{\mathcal{H} \,\Big|\!\frac{\alpha_0}{\rho}\, \Gamma, F(s)}{\mathcal{H} \,\Big|\!\frac{\alpha}{\rho}\, \Gamma, \exists x F(x)} \qquad \begin{array}{r} \alpha_0 + 1 < \alpha \\ |\,s\,| < \alpha \end{array}$$

$$(\notin)_\infty \qquad \frac{\mathcal{H}[r] \,\big|\!\frac{\alpha_r}{\rho}\, \Gamma, r \in t \rightarrow r \neq s \text{ for all } |\,r\,| < |\,t\,|}{\mathcal{H} \,\big|\!\frac{\alpha}{\rho}\, \Gamma, s \notin t} \qquad |\,r\,| \leq \alpha_r < \alpha$$

$$(\in) \qquad \frac{\mathcal{H} \,\big|\!\frac{\alpha_0}{\rho}\, \Gamma, r \in t \wedge r = s}{\mathcal{H} \,\big|\!\frac{\alpha}{\rho}\, \Gamma, s \in t} \qquad \begin{array}{r} \alpha_0 < \alpha \\ |\,r\,| < |\,t\,| \\ |\,r\,| < \alpha \end{array}$$

$$(\not\subseteq)_\infty \qquad \frac{\mathcal{H}[r] \,\big|\!\frac{\alpha_r}{\rho}\, \Gamma, r \subseteq t \rightarrow r \neq s \text{ for all } |\,r\,| \leq |\,t\,|}{\mathcal{H} \,\big|\!\frac{\alpha}{\rho}\, \Gamma, s \not\subseteq t} \qquad |\,r\,| \leq \alpha_r < \alpha$$

$$(\subseteq) \qquad \frac{\mathcal{H} \,\big|\!\frac{\alpha_0}{\rho}\, \Gamma, r \subseteq t \wedge r = s}{\mathcal{H} \,\big|\!\frac{\alpha}{\rho}\, \Gamma, s \subseteq t} \qquad \begin{array}{r} \alpha_0 < \alpha \\ |\,r\,| \leq |\,t\,| \\ |\,r\,| < \alpha \end{array}$$

$$(\Sigma^{\mathcal{P}}\text{-}Ref) \qquad \frac{\mathcal{H} \,\big|\!\frac{\alpha_0}{\rho}\, \Gamma, A}{\mathcal{H} \,\big|\!\frac{\alpha}{\rho}\, \Gamma, \exists z\, A^z} \qquad \begin{array}{r} \alpha_0 + 1, \Omega < \alpha \\ A \in \Sigma^{\mathcal{P}} \end{array}$$

Remark 5.8. Suppose $\mathcal{H} \,\big|\!\frac{\alpha}{\rho}\, \Gamma(s_1, \ldots, s_n)$ where $\Gamma(a_1, \ldots, a_n)$ is a sequent of $\mathbf{KP}(\mathcal{P})$ such that all variables $a_1, \ldots, a_n$ do occur in $\Gamma(a_1, \ldots, a_n)$ and $s_1, \ldots, s_n$ are $RS^{\mathcal{P}}_{\Omega}$-terms. Then we have that $|\,s_1\,|, \ldots, |\,s_n\,| \in \mathcal{H}(\emptyset)$. Standing in sharp contrast to the ordinal analysis of $\mathbf{KP}$ (cf. [14, 8]), however, the terms s_i may and often will contain subterms that the operator $\mathcal{H}$ does **not** control, that is, subterms t with $|\,t\,| \notin \mathcal{H}(\emptyset)$.

The following observation is easily established by induction on α.

Lemma 5.9 (Weakening)**.**

$$\mathcal{H} \,\big|\!\frac{\alpha}{\rho}\, \Gamma \;\wedge\; \alpha \leq \alpha' \in \mathcal{H}(\emptyset) \;\wedge\; \rho \leq \rho' \;\wedge\; |\,\Lambda\,| \subseteq \mathcal{H}(\emptyset) \implies \mathcal{H} \,\big|\!\frac{\alpha'}{\rho'}\, \Gamma, \Lambda\,.$$

Lemma 5.10 (Inversion)**.** *(i) If* $\mathcal{H} \,\big|\!\frac{\alpha}{\rho}\, \Gamma, A \vee B$ *and* $rk(A \vee B) \geq \Omega$*, then* $\mathcal{H} \,\big|\!\frac{\alpha}{\rho}\, \Gamma, A, B\,.$

(ii) If $\mathcal{H} \,\big|\!\frac{\alpha}{\rho}\, \Gamma, A_0 \wedge A_1\,,\; i \in \{0,1\}$ *and* $rk(A_0 \wedge A_1) \geq \Omega$*, then* $\mathcal{H} \,\big|\!\frac{\alpha}{\rho}\, \Gamma, A_i\,.$

(iii) $\mathcal{H} \,\big|\!\frac{\alpha}{\rho}\, \Gamma, \forall x\, F(x) \;\wedge\; \gamma \in \mathcal{H}(\emptyset) \;\wedge\; \gamma < \Omega \implies \mathcal{H} \,\big|\!\frac{\alpha}{\rho}\, \Gamma, (\forall x \in \mathbb{V}_\gamma) F(x)\,.$

(iv) *If* $\mathcal{H} \overset{\alpha}{\underset{\rho}{\mid}} \Gamma, (\forall x \in t)\, F(x)$ *and* $rk(F(\mathbb{V}_0)) \geq \Omega$*, then for all* $|\, s\,| < |\, t\,|$ *we have* $\mathcal{H}[s] \overset{\alpha}{\underset{\rho}{\mid}} \Gamma, s \in t \rightarrow F(s)$.

(v) *If* $\mathcal{H} \overset{\alpha}{\underset{\rho}{\mid}} \Gamma, (\forall x \subseteq t)\, F(x)$ *and* $rk(F(\mathbb{V}_0)) \geq \Omega$*, then for all* $|\, s\,| \leq |\, t\,|$ *we have* $\mathcal{H}[s] \overset{\alpha}{\underset{\rho}{\mid}} \Gamma, s \subseteq t \rightarrow F(s)$.

Proof: All proofs are by induction on α. Note that a formula C of $rk(C) \geq \Omega$ cannot be an active part of an axiom, i.e., if C occurred in an axiom sequent the sequent obtained by deleting C or replacing C with another formula would still be an axiom.

We show (iii). Firstly, suppose that $\forall x\, F(x)$ was the principal formula of the last inference. Then we have $\mathcal{H}[s] \overset{\alpha_s}{\underset{\rho}{\mid}} \Gamma, \forall x\, F(x), F(s)$ for all terms s, using weakening (Lemma 5.9) if $\forall x\, F(x)$ was not a side formula of the inference. Moreover, $|\, s\,| \leq \alpha_s + 1 < \alpha$ holds for all s. Inductively we have $\mathcal{H}[s] \overset{\alpha_s}{\underset{\rho}{\mid}} \Gamma, (\forall x \in \mathbb{V}_\gamma)F(x), F(s)$ for all $|\, s\,| < \gamma$. Hence, using $(\vee)$, $\mathcal{H}[s] \overset{\alpha_s+1}{\underset{\rho}{\mid}} \Gamma, (\forall x \in \mathbb{V}_\gamma)F(x), s \in \mathbb{V}_\gamma \rightarrow F(s)$ holds for all $|\, s\,| < \gamma$, so that via an inference $(b\forall)$ we arrive at $\mathcal{H} \overset{\alpha}{\underset{\rho}{\mid}} \Gamma, (\forall x \in \mathbb{V}_\gamma)F(x)$.

Now assume that $\forall x\, F(x)$ was not the principal formula of the last inference. Then the assertion follows by applying the induction hypothesis to its premisses and performing the same inference. □

6. Embedding

To relate $\mathbf{KP}(\mathcal{P})$ to the infinitary system $RS_\Omega^{\mathcal{P}}$ we show that $\mathbf{KP}(\mathcal{P})$ can be embedded into $RS_\Omega^{\mathcal{P}}$. Indeed, the finite $\mathbf{KP}(\mathcal{P})$-derivations give rise to very uniform infinitary derivations.

Definition 6.1. For $\Gamma = \{A_1, \ldots, A_n\}$ let

$$no(\Gamma) := \omega^{rk(A_1)} \# \cdots \# \omega^{rk(A_n)}.$$

Here "*no*" stands for "norm". We define

$$\Vdash \Gamma \quad :\Longleftrightarrow \quad \text{for all operators } \mathcal{H},\ \mathcal{H}[\Gamma] \overset{no(\Gamma)}{\underset{0}{\mid}} \Gamma$$

and

$$\Vdash^{\xi}_{\rho} \Gamma \quad :\Longleftrightarrow \quad \text{for all operators } \mathcal{H},\ \mathcal{H}[\Gamma] \overset{no(\Gamma)\#\xi}{\underset{\rho}{\mid}} \Gamma\, .$$

Lemma 6.2. *(i) For all formulae A,*

$$\Vdash A, \neg A.$$

(ii) ($\Delta_0^{\mathcal{P}}$-Collection)

$$\Vdash_0^4 (\forall x \in s)\, \exists y\, F(x,y) \to \exists z\, (\forall x \in s)(\exists y \in z)\, F(x,y)$$

if $F(\mathbb{V}_0, \mathbb{V}_0)$ is in $\Delta_0^{\mathcal{P}}$.

Proof: (i): We proceed by induction on the syntactic complexity of A. For A in $\Delta_0^{\mathcal{P}}$ this is an axiom of $RS_\Omega^{\mathcal{P}}$. Suppose A is of the form $\forall x F(x)$. Let $\mathcal{H}$ be an arbitrary operator. Let $\alpha_s := |\, s\,| + no(\{F(s), \neg F(s)\})$ and $\alpha := no(\{\forall x F(x), \exists x \neg F(x)\})$. Note that $|\, s\,| < \alpha_s + 1 < \alpha$ since $rk(\forall x F(x)) = max\{\Omega, rk(F(\mathbb{V}_0)) + 2\}$. Inductively we have

$$\mathcal{H}[F(s), s] \;\overset{\alpha_s}{\underset{0}{|}}\; F(s), \neg F(s)$$

for all terms s. Using an inference ($\exists$) we get

$$\mathcal{H}[F(s), s] \;\overset{no(\{F(s), \exists x \neg F(x)\})}{\underset{0}{|}}\; F(s), \exists x \neg F(x)\,.$$

Hence, via an inference ($\forall$), we arrive at $\mathcal{H}[\forall x F(x)] \;\overset{\alpha}{\underset{0}{|}}\; \forall x F(x), \exists x \neg F(x)\,,$ noting that $\mathcal{H}[F(s), s] \subseteq (\mathcal{H}[\forall x \neg F(x)])[s]$.

The other cases are similar.

(ii): By (i) we have $\Vdash \neg(\forall x \in s)\, \exists y\, F(x,y),\ (\forall x \in s)\, \exists y\, F(x,y)$. Since the formula $(\forall x \in s)\, \exists y\, F(x,y)$ is $\Sigma^{\mathcal{P}}$ an inference ($\Sigma^{\mathcal{P}}$-Ref) yields

$$\Vdash_0^2 \neg(\forall x \in s)\, \exists y\, F(x,y),\ \exists z\, (\forall x \in s)\, (\exists y \in z)\, F(x,y).$$

Thus, by applying ($\vee$) twice, we arrive at

$$\Vdash_0^4 (\forall x \in s)\, \exists y\, F(x,y) \to \exists z\, (\forall x \in s)(\exists y \in z)\, F(x,y).$$

$\square$

Lemma 6.3. (Equality and Extensionality)

$$\Vdash_\rho s_1 \neq t_1, \ldots, s_n \neq t_n, \neg A(s_1, \ldots, s_n), A(t_1, \ldots, t_n)$$

where $\rho = \max(rk(s_1 \neq t_1), \ldots, rk(s_n \neq t_n)) + 1$.

Proof: We proceed by induction on the buildup of $A(\vec{s})$. Let $\mathcal{H}$ be an arbitrary operator.

If $A(\vec{s})$ is $\Delta_0^{\mathcal{P}}$ then this is an axiom.

Suppose $A(\vec{s})$ is a formula $\forall x\, F(x,\vec{s})$. Let $\vec{s} \neq \vec{t}$ stand for $s_1 \neq t_1, \ldots, s_n \neq t_n$. Let $\Gamma_r := \{\vec{s} \neq \vec{t}, \neg F(r,\vec{s}), F(r,\vec{t})\}$ and $\alpha_r := no(\Gamma_r)$. Inductively we have

$$\mathcal{H}[\Gamma_r] \,\big|\!\frac{\alpha_r}{\rho}\, \Gamma_r$$

for all terms r. Using an inference $(\exists)$ we obtain $\mathcal{H}[\tilde{\Gamma}_r] \,\big|\!\frac{\tilde{\alpha}_r}{\rho}\, \tilde{\Gamma}_r$ where

$$\tilde{\Gamma}_r := \{\vec{s} \neq \vec{t}, \exists x \neg F(x,\vec{s}), F(r,\vec{t})\}$$

and $\tilde{\alpha}_r := no(\tilde{\Gamma}_r)$, noting that $|\,r\,| < \Omega \leq no(\exists x \neg F(x,\vec{s}))$. Thus, using an inference $(\forall)_\infty$, we have

$$\mathcal{H}[\Gamma] \,\big|\!\frac{no(\Gamma)}{\rho}\, \Gamma$$

where $\Gamma := \{\vec{s} \neq \vec{t}, \exists x \neg F(x,\vec{s}), \forall x F(x,\vec{t})\}$. In the latter we used the fact that $\mathcal{H}[\tilde{\Gamma}_r] \subseteq (\mathcal{H}[\Gamma])[r]$.

Suppose $A(\vec{s})$ is a formula $(\forall x \subseteq s_1)\, F(x,\vec{s})$. Inductively we have

$$\Vdash_\rho q \neq r, \vec{s} \neq \vec{t}, \neg F(q,\vec{s}), F(r,\vec{t})$$

where $|\,q\,| \leq |\,s_1\,|$ and $|\,r\,| \leq |\,t_1\,|$. As $q \not\subseteq s_1, q \subseteq s_1$ is an axiom we can use $(\wedge)$ to infer

$$\Vdash_\rho q \neq r, \vec{s} \neq \vec{t}, \neg F(r,\vec{t}), q \not\subseteq s_1, q \subseteq s_1 \,\wedge\, F(q,\vec{s}). \tag{3}$$

Via $(bp\exists)$ followed by two $(\vee)$ inferences, (3) yields

$$\Vdash_\rho q \subseteq s_1 \to q \neq r, \vec{s} \neq \vec{t}, \neg F(r,\vec{t}), (\exists x \subseteq s_1)F(x,\vec{s}) \tag{4}$$

for all q satisfying $|\,q\,| \leq |\,s_1\,|$. Thus, applying $(\not\subseteq)_\infty$ to (4) we have

$$\Vdash_\rho r \not\subseteq s_1, \vec{s} \neq \vec{t}, \neg F(r,\vec{t}), (\exists x \subseteq s_1)F(x,\vec{s}). \tag{5}$$

Since $s_1 \neq t_1, r \not\subseteq t_1, r \subseteq s_1$ is an axiom we can apply a cut with (5), obtaining

$$\Vdash_\rho^{\delta_r} \vec{s} \neq \vec{t}, r \not\subseteq t_1, \neg F(r,\vec{t}), (\exists x \subseteq s_1)F(x,\vec{s}) \tag{6}$$

where $\delta_r = no(r \subseteq s_1)$. To (6) we can apply $(\vee)$ twice so that

$$\Vdash_\rho^{\delta_r+2} \vec{s} \neq \vec{t}, r \subseteq t_1 \to \neg F(r,\vec{t}), (\exists x \subseteq s_1)F(x,\vec{s}) \tag{7}$$

holds for all r with $|\, r\,| \leq |\, t_1\,|$. Hence, by applying $(bp\forall)_\infty$ to (7), we arrive at

$$\Vdash_\rho \vec{s} \neq \vec{t}, (\forall x \subseteq t_1)\neg F(r, \vec{t}\,), (\exists x \subseteq s_1)F(x, \vec{s}\,)$$

The other cases are similar. □

Lemma 6.4. (Set Induction)

$$\Vdash \forall x\, [(\forall y \in x)F(y) \to F(x)] \;\longrightarrow\; \forall x F(x).$$

Proof. Fix an operator $\mathcal{H}$. Let $A \equiv \forall x\, [(\forall y \in x)F(y) \to F(x)]$. First, we show, by induction on $|\, s\,|$, that

$$(+) \qquad \mathcal{H}[A, s] \mid\!\frac{\omega^{rk(A)}\#\omega^{|s|+1}}{0}\, \neg A, F(s)\,.$$

So assume that

$$\mathcal{H}[A, t] \mid\!\frac{\omega^{rk(A)}\#\omega^{|t|+1}}{0}\, \neg A, F(t)$$

holds for all $|\, t\,| < |\, s\,|$. Using $(\vee)$, this yields

$$\mathcal{H}[A, s, t] \mid\!\frac{\omega^{rk(A)}\#\omega^{|t|+1}+1}{0}\, \neg A, t \in s \to F(t)$$

for all $|\, t\,| < |\, s\,|$, and hence

$$(1) \qquad \mathcal{H}[A, s] \mid\!\frac{\omega^{rk(A)}\#\omega^{|s|}+2}{0}\, \neg A, (\forall x \in s)F(x)$$

via $(b\forall)_\infty$. Set $\eta_s := \omega^{rk(A)}\#\omega^{|s|} + 2$. By Lemma 6.2 we have $\mathcal{H}[A, s] \mid\!\frac{\eta_s}{0}\, \neg F(s), F(s)\,$. Therefore, using (1) and $(\wedge)$,

$$\mathcal{H}[A, s] \mid\!\frac{\eta_s+1}{0}\, \neg A, (\forall y \in s)F(y) \wedge \neg F(s), F(s)\,.$$

From the latter we obtain

$$\mathcal{H}[A, s] \mid\!\frac{\eta_s+2}{0}\, \neg A, \exists x\, [(\forall y \in x)F(y) \wedge \neg F(x)], F(s)$$

via $(\exists)$. This shows (+).

Finally, (+) enables us to deduce, via $(\forall)_\infty$, that

$$\mathcal{H}[A, s] \mid\!\frac{\omega^{rk(A)}+\Omega}{0}\, \neg A, \forall x F(x)\,.$$

From this the assertion follows by applying $(\vee)$ twice. □

Lemma 6.5. (Infinity Axiom) *For any operator $\mathcal{H}$ we have*

$$\mathcal{H} \mid\!\frac{\omega+2}{0}\; \exists x\,[(\exists y \in x)\, y \in x \;\wedge\; (\forall y \in x)(\exists z \in x)\, y \in z]\,.$$

Proof: Let s be a term with $|\,s\,| = n < \omega$. Then $\mathcal{H} \mid\!\frac{0}{0}\, s \in \mathbb{V}_{n+1}$ and $\mathcal{H} \mid\!\frac{0}{0}\, \mathbb{V}_{n+1} \in \mathbb{V}_\omega$ since these formulae are axioms. Via $(\wedge)$ we deduce

$$\mathcal{H} \mid\!\frac{1}{0}\, \mathbb{V}_{n+1} \in \mathbb{V}_\omega \;\wedge\; s \in \mathbb{V}_{n+1}$$

and hence $\mathcal{H} \mid\!\frac{n+2}{0}\, (\exists z \in \mathbb{V}_\omega) s \in z$, using $(b\exists)$. An inference $(\vee)$ yields

$$\mathcal{H} \mid\!\frac{n+3}{0}\, s \in \mathbb{V}_\omega \rightarrow (\exists z \in \mathbb{V}_\omega) s \in z\,.$$

Since this holds for all terms s with $|\,s\,| < \omega$, we conclude that

$$\mathcal{H} \mid\!\frac{\omega}{0}\, (\forall y \in \mathbb{V}_\omega)(\exists z \in \mathbb{V}_\omega) y \in z\,. \tag{8}$$

Since $\mathbb{V}_0 \in \mathbb{V}_\omega$ is an axiom we have $\mathcal{H} \mid\!\frac{1}{0}\, \mathbb{V}_0 \in \mathbb{V}_\omega \;\wedge\; \mathbb{V}_0 \in \mathbb{V}_\omega$ via $(\wedge)$ and thus

$$\mathcal{H} \mid\!\frac{2}{0}\, (\exists z \in \mathbb{V}_\omega) z \in \mathbb{V}_\omega\,, \tag{9}$$

using $(b\exists)$. Combining (8) and (9) we arrive at

$$\mathcal{H} \mid\!\frac{\omega+1}{0}\, (\exists z \in \mathbb{V}_\omega) z \in \mathbb{V}_\omega \;\wedge\; (\forall y \in \mathbb{V}_\omega)(\exists z \in \mathbb{V}_\omega) y \in z\,.$$

Thus an inference $(b\exists)$ furnishes us with

$$\mathcal{H} \mid\!\frac{\omega+2}{0}\, \exists x\,[(\exists z \in x) z \in x \;\wedge\; (\forall y \in x)(\exists z \in x) y \in z]\,.$$

$\square$

Lemma 6.6. ($\Delta_0^{\mathcal{P}}$–Separation) *Let $A(a, b, c_1, \ldots, c_n)$ be a $\Delta_0^{\mathcal{P}}$–formula of $\mathcal{L}$ with all free variables among the exhibited. Let $r, s_1, \ldots, s_n$ be $RS_\Omega^{\mathcal{P}}$-terms. Let $\mathcal{H}$ be an arbitrary operator. Then:*

$$\mathcal{H}[r, \vec{s}\,] \mid\!\frac{\alpha+8}{\rho}\, \exists y\,[(\forall x \in y)(x \in r \wedge A(x, r, \vec{s}\,) \;\wedge\; (\forall x \in r)(A(x, r, \vec{s}\,) \rightarrow x \in y)]\,,$$

where $\alpha = |\,r\,|$ and $\rho = \max\{|\,r\,|, |\,s_1\,|, \ldots, |\,s_n\,|\} + \omega$.

Proof: Define the $RS^{\mathcal{P}}_{\Omega}$-term p by

$$p := \{x \in \mathbb{V}_\alpha \mid x \in r \wedge A(x, r, \vec{s}\,)\}.$$

Then $|\,p\,| = \alpha$. Let $\tilde{\mathcal{H}} := \mathcal{H}[r, \vec{s}\,]$. We have $\tilde{\mathcal{H}}[t] \,\big|\!\frac{0}{0}\, t \notin p, t \in r \wedge A(t, r, \vec{s}\,)$ for all $|\,t\,| < \alpha$ since this is an axiom (A6). Hence, using ($\vee$) twice, $\tilde{\mathcal{H}}[t] \,\big|\!\frac{\alpha+2}{0}\, t \in p \rightarrow t \in r \wedge A(t, r, \vec{s}\,)$, and therefore

$$\tilde{\mathcal{H}} \,\big|\!\frac{\alpha+3}{0}\, (\forall x \in p)(x \in r \wedge A(x, r, \vec{s}\,)) \tag{10}$$

by applying $(b\forall)_\infty$. We also have, on account of being axioms, $\tilde{\mathcal{H}}[t] \,\big|\!\frac{0}{0}\, t \notin r, t \in r$ and $\tilde{\mathcal{H}}[t] \,\big|\!\frac{0}{0}\, \neg A(t, r, \vec{s}\,), A(t, r, \vec{s}\,)$. Using ($\wedge$) and weakening (Lemma 5.9) we conclude that

$$\tilde{\mathcal{H}}[t] \,\big|\!\frac{1}{0}\, t \notin r, \neg A(t, r, \vec{s}\,), t \in r \wedge A(t, r, \vec{s}\,)\,. \tag{11}$$

Since $\tilde{\mathcal{H}}[t] \,\big|\!\frac{0}{0}\, \neg(t \in r \wedge A(t, r, \vec{s}\,)), t \in p$ holds on account of being an axiom (A7), a cut applied to (11) and the latter yields

$$\tilde{\mathcal{H}}[t] \,\big|\!\frac{1}{\rho}\, t \notin r, \neg A(t, r, \vec{s}\,), t \in p \tag{12}$$

since $rk(t \in r \wedge A(t, r, \vec{s}\,)) < \rho$ holds for terms t with $|\,t\,| < \alpha$. Now use ($\vee$) four times to arrive at

$$\tilde{\mathcal{H}}[t] \,\big|\!\frac{\alpha+5}{\rho}\, t \in r \rightarrow (A(t, r, \vec{s}\,) \rightarrow t \in p)\,. \tag{13}$$

Applying $(b\forall)_\infty$ to (13) yields

$$\tilde{\mathcal{H}} \,\big|\!\frac{\alpha+6}{\rho}\, (\forall x \in r)(A(x, r, \vec{s}\,) \rightarrow x \in p)\,. \tag{14}$$

Combining (10) and (14) via ($\wedge$) we have

$$\tilde{\mathcal{H}} \,\big|\!\frac{\alpha+7}{\rho}\, (\forall x \in p)(x \in r \wedge A(x, r, \vec{s}\,)) \;\wedge\; (\forall x \in r)(A(x, r, \vec{s}\,) \rightarrow x \in p)\,.$$

Consequently, by means of ($b\exists$),

$$\tilde{\mathcal{H}} \,\big|\!\frac{\alpha+8}{\rho}\, \exists y[(\forall x \in y)(x \in r \wedge A(x, r, \vec{s}\,)) \;\wedge\; (\forall x \in r)(A(x, r, \vec{s}\,) \rightarrow x \in y)]\,.$$

$\square$

Lemma 6.7. (Pair and Union) *For any operator $\mathcal{H}$ the following hold:*

(i) $\mathcal{H}[s,t] \mathrel{|\!\frac{\alpha+2}{0}} \exists z\,(s \in z \wedge t \in z)$ *where* $\alpha = \max(|\,s\,|, |\,t\,|) + 1$.

(ii) $\mathcal{H}[s] \mathrel{|\!\frac{\beta+4}{0}} \exists z\,(\forall y \in s)(\forall x \in y)(x \in z)$ *where* $\beta = |\,s\,|$.

Proof: (i): $s \in \mathbb{V}_\alpha$ and $t \in \mathbb{V}_\alpha$ are axioms. Thus $\mathcal{H}[s,t] \mathrel{|\!\frac{1}{0}} s \in \mathbb{V}_\alpha \wedge t \in \mathbb{V}_\alpha$, and hence $\mathcal{H}[s,t] \mathrel{|\!\frac{\alpha+2}{0}} \exists z\,(s \in z \wedge t \in z)$ by means of $(b\exists)$.
(ii): Let r and t be terms of levels $< \beta$. Since $r \in \mathbb{V}_\beta$ is an axiom, we have

$$\mathcal{H}[s,r] \mathrel{|\!\frac{0}{0}} r \in \mathbb{V}_\beta\ .$$

Thus we get

$$\begin{aligned}
&\mathcal{H}[s,t,r] \mathrel{|\!\frac{\beta}{0}} r \in t \to r \in \mathbb{V}_\beta \\
&\mathcal{H}[s,t] \mathrel{|\!\frac{\beta+1}{0}} (\forall x \in t)x \in \mathbb{V}_\beta \\
&\mathcal{H}[s,t] \mathrel{|\!\frac{\beta+2}{0}} t \in s \to (\forall x \in t)x \in \mathbb{V}_\beta \\
&\mathcal{H}[s] \mathrel{|\!\frac{\beta+3}{0}} (\forall y \in s)(\forall x \in t)x \in \mathbb{V}_\beta \\
&\mathcal{H}[s] \mathrel{|\!\frac{\beta+4}{0}} \exists z\,(\forall y \in s)(\forall x \in t)x \in z\ .
\end{aligned}$$

$\square$

Lemma 6.8. (Power Set) *For any operator* $\mathcal{H}$ *the following holds:*

$$\mathcal{H}[s] \mathrel{|\!\frac{\alpha+3}{0}} \exists z\,(\forall x \subseteq s)\,x \in z\,,$$

where $\alpha = |\,s\,|$.

Proof: Let t be a term with $|\,t\,| \leq \alpha$. Then $t \in \mathbb{V}_{\alpha+1}$ is an axiom. Whence, using $(\vee)$, $(pb\forall)_\infty$, and $(\exists)$, we have

$$\begin{aligned}
&\mathcal{H}[s,t] \mathrel{|\!\frac{0}{0}} t \in \mathbb{V}_{\alpha+1} \\
&\mathcal{H}[s,t] \mathrel{|\!\frac{\alpha+1}{0}} t \subseteq s \to t \in \mathbb{V}_{\alpha+1} \\
&\mathcal{H}[s] \mathrel{|\!\frac{\alpha+2}{0}} (\forall x \subseteq s)x \in \mathbb{V}_{\alpha+1} \\
&\mathcal{H}[s] \mathrel{|\!\frac{\alpha+3}{0}} \exists z\,(\forall x \subseteq s)x \in z\,.
\end{aligned}$$

$\square$

Theorem 6.9. *If*

$$\mathbf{KP}(\mathcal{P}) \vdash \Gamma(a_1, \ldots, a_l)$$

then there exist $m, n < \omega$ such that

$$\mathcal{H}[s_1, \ldots, s_l] \;\Big|\!\frac{\omega^{\Omega+m}}{\Omega+n}\, \Gamma(s_1, \ldots, s_l)$$

holds for all $RS_\Omega^{\mathcal{P}}$-terms $s_1, \ldots, s_l$ and operators $\mathcal{H}$. m and n depend solely on the $\mathbf{KP}(\mathcal{P})$-derivation of $\Gamma(\vec{a})$.

Proof: One proceeds by induction on the length of the $\mathbf{KP}(\mathcal{P})$-derivation of $\Gamma(\vec{a}\,)$. Note that the rank of an $RS_\Omega^{\mathcal{P}}$-formula A is always $< \Omega + \omega$ and thus the norms of $RS_\Omega^{\mathcal{P}}$-sequents will always be $< \omega^{\Omega+\omega}$.

If $\Gamma(\vec{a}\,)$ is an axiom of $\mathbf{KP}(\mathcal{P})$ then the assertion follows from the earlier results of this section.

If the last inference was ($\Delta_0^{\mathcal{P}}$-COLLR) then $\Gamma(\vec{a}\,)$ contains a formula $(\forall x \in a_i)\exists y\, F(x, y, \vec{a}\,)$ with $F(b, c, \vec{a}\,)$ being $\Sigma^{\mathcal{P}}$ and inductively we have $n_0, m_0 < \omega$ such that

$$\mathcal{H}[\vec{s}] \;\Big|\!\frac{\omega^{\Omega+m_0}}{\Omega+n_0}\, \Gamma(\vec{s}), (\forall x \in s_i)\exists y\, F(x, y, \vec{s}\,)$$

holds for all terms $\vec{s}$. Since $(\forall x \in s_i)\exists y\, F(x, y, \vec{s}\,)$ is $\Sigma^{\mathcal{P}}$ an application of ($\Sigma^{\mathcal{P}}$-*Ref*) yields

$$\mathcal{H}[\vec{s}] \;\Big|\!\frac{\omega^{\Omega+m_0+1}}{\Omega+n_0}\, \Gamma(\vec{s}), \exists z\, (\forall x \in s_i)(\exists y \in z)\, F(x, y, \vec{s}\,)\,,$$

i.e., $\mathcal{H}[\vec{s}] \;\Big|\!\frac{\omega^{\Omega+m_0+1}}{\Omega+n_0}\, \Gamma(\vec{s}\,)$.

As an example of a logical rule we shall treat ($pb\exists$). So suppose the last inference of our $\mathbf{KP}(\mathcal{P})$-derivation $\mathcal{D}$ was an instance of ($pb\exists$). Then $\Gamma(\vec{a}\,)$ contains a formula of the form $(\exists x \subseteq a_i) \wedge F(x, \vec{a}\,)$ and there exists a shorter $\mathbf{KP}(\mathcal{P})$-derivation $\mathcal{D}_0$ whose end sequent is either of the form $\Gamma(\vec{a}), c \subseteq a_i \wedge F(c, \vec{a}\,)$ with c not occurring in $\Gamma(\vec{a}\,)$ or c is a_j for some $1 \le j \le l$. In the former case the induction hypothesis supplies us with $n_0, m_0 < \omega$ such that

$$\mathcal{H}[\vec{s}] \;\Big|\!\frac{\omega^{\Omega+m_0}}{\Omega+n_0}\, \Gamma(\vec{s}), \mathbb{V}_0 \subseteq s_i \wedge F(\mathbb{V}_0, \vec{s}\,) \tag{15}$$

holds for all terms $\vec{s}$. As $|\,\mathbb{V}_0\,| = 0 \le |\,s_i\,|$ we can apply an inference ($pb\exists$) in yielding

$$\mathcal{H}[\vec{s}] \;\Big|\!\frac{\omega^{\Omega+m_0+2}}{\Omega+n_0}\, \Gamma(\vec{s}), (\exists x \subseteq s_i) F(x, \vec{s}\,) \tag{16}$$

and thus $\mathcal{H}[\vec{s}]\ \Big|\!\frac{\omega^{\Omega+m_0+2}}{\Omega+n_0}\ \Gamma(\vec{s}\,)$ as $(\exists x \subseteq s_i)F(x,\vec{s}\,)$ belongs to $\Gamma(\vec{s}\,)$.

Now let's turn to the case where c is a_j. Then, by the induction hypothesis, there are $n_0, m_0 < \omega$ such that

$$\mathcal{H}[\vec{s}\,]\ \Big|\!\frac{\omega^{\Omega+m_0}}{\Omega+n_0}\ \Gamma(\vec{s}\,), s_j \subseteq s_i \wedge F(s_j,\vec{s}\,) \tag{17}$$

holds for all terms $\vec{s}$. Owing to Lemma 6.3 we can find m_1 such that with $\rho := \omega^{\Omega+m_1}$ we have

$$\mathcal{H}[\vec{s}, r]\ \Big|\!\frac{\rho}{0}\ r \neq s_j, s_j \not\subseteq s_i, r \subseteq s_i$$

and $\mathcal{H}[\vec{s}, r]\ \Big|\!\frac{\rho}{0}\ s_j \neq r, \neg F(s_j,\vec{s}\,), F(r,\vec{s}\,)$ hold for all $r, \vec{s}$. By applying weakening and $(\wedge)$ we thus get

$$\mathcal{H}[\vec{s}, r]\ \Big|\!\frac{\rho+1}{0}\ r \not\subseteq s_i, r \neq s_j, \neg F(s_j,\vec{s}\,), r \subseteq s_i \wedge F(r,\vec{s}\,)$$

for all r with $|\,r\,| \leq |\,s_i\,|$. Now apply $(pb\exists)$, $(\vee)$ (twice), $(\not\subseteq)_\infty$, and $(\vee)$ (twice):

$$\begin{aligned}
&\mathcal{H}[\vec{s}, r]\ \Big|\!\frac{\rho+2}{0}\ r \not\subseteq s_i, r \neq s_j, \neg F(s_j,\vec{s}\,), (\exists x \subseteq s_i)F(x,\vec{s}\,)\\
&\mathcal{H}[\vec{s}, r]\ \Big|\!\frac{\rho+4}{0}\ r \subseteq s_i \to r \neq s_j, \neg F(s_j,\vec{s}\,), (\exists x \subseteq s_i)F(x,\vec{s}\,)\\
&\mathcal{H}[\vec{s}\,]\ \Big|\!\frac{\rho+5}{0}\ s_j \not\subseteq s_i, \neg F(s_j,\vec{s}\,), (\exists x \subseteq s_i)F(x,\vec{s}\,)\\
&\mathcal{H}[\vec{s}\,]\ \Big|\!\frac{\rho+7}{0}\ \neg(s_j \subseteq s_i\ \wedge\ F(s_j,\vec{s}\,)), (\exists x \subseteq s_i)F(x,\vec{s}\,)\,.
\end{aligned} \tag{18}$$

Finally, by applying a cut to (17) and (18) we have

$$\mathcal{H}[\vec{s}\,]\ \Big|\!\frac{\omega^{\Omega+m}}{\Omega+n}\ \Gamma(\vec{s}\,), (\exists x \subseteq s_i)F(x,\vec{s}\,)$$

i.e., $\mathcal{H}[\vec{s}\,]\ \Big|\!\frac{\omega^{\Omega+m}}{\Omega+n}\ \Gamma(\vec{s}\,)$, where $m = \max(m_0, m_1) + 1$ and n is chosen such that $n > n_0$ and $rk(s_j \subseteq s_i\ \wedge\ F(s_j,\vec{s}\,)) < \Omega + n$ for all $\vec{s}$.

The case of the last inference being $(b\exists)$ is treated in the same vein as $(pb\exists)$. All the other inferences are straightforward as the desired assertion can be obtained immediately from the induction hypothesis applied to the premisses followed by the corresponding inference in $RS^{\mathcal{P}}_{\Omega}$. For example, in the case of the $(\Delta^{\mathcal{P}}_0$-COLLR) one inductively finds $m_0, n_0 < \omega$ such that

$$\mathcal{H}[\vec{s}\,]\ \Big|\!\frac{\omega^{\Omega+m}}{\Omega+n}\ \Gamma_0(\vec{s}\,), (\forall x \in s_i)\exists y\, H(x,y,\vec{s}\,)$$

holds for all $\vec{s}$, where $H(x,y,\vec{a}\,)$ is $\Sigma^{\mathcal{P}}$. Using $(\Sigma^{\mathcal{P}}\text{-}Ref)$ one obtains

$$\mathcal{H}[\vec{s}\,]\ \Big|\!\frac{\omega^{\Omega+m}}{\Omega+n}\ \Gamma_0(\vec{s}\,), \exists z(\forall x \in s_i)(\exists y \in z)\, H(x,y,\vec{s}\,)\,.$$

$\square$

7. Cut elimination

The usual cut elimination procedure works as long as the cut formulae are not in $\Delta_0^{\mathcal{P}}$ and have not been introduced by an inference $(\Sigma^{\mathcal{P}}\text{-}Ref)$. As the principal formula of an inference $(\Sigma^{\mathcal{P}}\text{-}Ref)$ has rank Ω one gets the following result.

Theorem 7.1 (Cut elimination I)**.**

$$\mathcal{H} \overset{\alpha}{\underset{\Omega+n+1}{\vdash}} \Gamma \quad \Rightarrow \quad \mathcal{H} \overset{\omega_n(\alpha)}{\underset{\Omega+1}{\vdash}} \Gamma$$

where $\omega_0(\beta) := \beta$ *and* $\omega_{k+1}(\beta) := \omega^{\omega_k(\beta)}$.

Proof: The proof is standard. For details see [8, Lemma 3.14]. □

Lemma 7.2 (Boundedness)**.** *Let* A *be a* $\Sigma^{\mathcal{P}}$*-formula,* $\alpha \le \beta < \Omega$*, and* $\beta \in \mathcal{H}(\emptyset)$*. If*

$$\mathcal{H} \overset{\alpha}{\underset{\rho}{\vdash}} \Gamma, A$$

then

$$\mathcal{H} \overset{\alpha}{\underset{\rho}{\vdash}} \Gamma, A^{\mathbb{V}_\beta} .$$

Proof: Note that the derivation contains no instances of $(\Sigma^{\mathcal{P}}\text{-}Ref)$. The proof is by induction on α. For details see [8, Lemma 3.17]. □

The obstacle to pushing cut elimination further is exemplified by the following scenario:

$$\dfrac{\dfrac{\mathcal{H} \overset{\delta}{\underset{\Omega}{\vdash}} \Gamma, A}{\mathcal{H} \overset{\xi}{\underset{\Omega}{\vdash}} \Gamma, \exists z\, A^z}(\Sigma^{\mathcal{P}}\text{-}Ref) \qquad \dfrac{\cdots \mathcal{H}[s] \overset{\xi_s}{\underset{\Omega}{\vdash}} \Gamma, \neg A^s \cdots (s \in \mathcal{T})}{\mathcal{H} \overset{\xi}{\underset{\Omega}{\vdash}} \Gamma, \forall z\, \neg A^z}(\forall)}{\mathcal{H} \overset{\alpha}{\underset{\Omega+1}{\vdash}} \Gamma}(Cut)$$

Fortunately, it is possible to eliminate cuts in the above situation provided that the side formulae Γ are of complexity $\Sigma^{\mathcal{P}}$. The technique is known as "collapsing" of derivations.

If the length of a derivation of $\Sigma^{\mathcal{P}}$-formulae is $\ge \Omega$, then "collapsing" results in a shorter derivation, however, at the cost of a much more complicated controlling operator.

Definition 7.3.

$$\mathcal{H}_\delta(X) = \bigcap \{C^\Omega(\alpha, \beta) : \; X \subseteq C^\Omega(\alpha, \beta) \;\wedge\; \delta < \alpha\}$$

Lemma 7.4. *(i) $\mathcal{H}_\eta$ is an operator.*

(ii) $\eta < \eta' \implies \mathcal{H}_\eta(X) \subseteq \mathcal{H}_{\eta'}(X)$.

(iii) If $\xi \in \mathcal{H}_\eta(X)$ and $\xi < \eta + 1$ then $\psi_\Omega(\xi) \in \mathcal{H}_\eta(X)$.

Proof: See [8, Lemma 4.6]. □

Lemma 7.5. *Suppose $\eta \in \mathcal{H}_\eta(\emptyset)$. Define $\hat{\beta} := \eta + \omega^{\Omega+\beta}$.*

(i) If $\alpha \in \mathcal{H}_\eta$ then $\hat{\alpha}, \psi_\Omega(\hat{\alpha}) \in \mathcal{H}_{\hat{\alpha}}$.

(ii) If $\alpha_0 \in \mathcal{H}_\eta$ and $\alpha_0 < \alpha$ then $\psi_\Omega(\hat{\alpha_0}) < \psi_\Omega(\hat{\alpha})$.

Proof: See [8, Lemma 4.7]. □

Theorem 7.6 (Collapsing Theorem)**.** *Let Γ be a set of $\Sigma^{\mathcal{P}}$-formulae and $\eta \in \mathcal{H}_\eta(\emptyset)$. Then we have*

$$\mathcal{H}_\eta \;\frac{\;\alpha\;}{\;\Omega+1\;}\; \Gamma \quad \Rightarrow \quad \mathcal{H}_{\hat{\alpha}} \;\frac{\;\psi_\Omega(\hat{\alpha})\;}{\;\psi_\Omega(\hat{\alpha})\;}\; \Gamma$$

where $\hat{\alpha} = \eta + \omega^{\Omega+\alpha}$.

Proof by induction on α. Suppose $\mathcal{H}_\eta \vdash^{\alpha}_{\Omega+1} \Gamma$. We shall distinguish cases according to the last inference of $\mathcal{H}_\eta \vdash^{\alpha}_{\Omega+1} \Gamma$. Note that this cannot be $(\forall)_\infty$ since Γ consists of $\Sigma^{\mathcal{P}}$-formulae. Note also that $\eta \in \mathcal{H}_\eta(\emptyset)$ implies $\eta \in \mathcal{H}_{\hat{\alpha}}(\emptyset)$, and therefore

$$\alpha \in \mathcal{H}_\eta(\emptyset) \quad \Rightarrow \quad \psi_\Omega(\hat{\alpha}) \in \mathcal{H}_{\hat{\alpha}}(\emptyset). \tag{19}$$

Case 0: Suppose Γ is an axiom. Then $\mathcal{H}_{\hat{\alpha}} \vdash^{\psi_\Omega(\hat{\alpha})}_{\psi_\Omega(\hat{\alpha})} \Gamma$ follows immediately by (19).

Case 1: Suppose the last inference was $(pb\forall)_\infty$. Then there is an $A \in \Gamma$ of the form $(\forall x \subseteq t)F(x)$ and $\mathcal{H}_\eta[s] \vdash^{\alpha_s}_{\Omega+1} \Gamma, s \subseteq t \to F(s)$ and $\alpha_s < \alpha$ hold for all s with $|s| \leq |t|$. By Lemma 4.2 we have

$$\mathcal{H}_\eta(\emptyset) = C^\Omega(\eta + 1, 0) = C^\Omega(\eta + 1, \psi_\Omega(\eta + 1)).$$

Since $|t| \in \mathcal{H}_\eta(\emptyset)$ it follows that $|t| \in C^\Omega(\eta + 1, \psi_\Omega(\eta + 1) \cap \Omega$, whence $|t| < \psi_\Omega(\eta + 1)$ and hence $|s| < \psi_\Omega(\eta + 1)$ whenever $|s| \leq |t|$. As a result, $|\,s\,| \in C^{\Omega}(\eta+1, \psi_{\Omega}(\eta+1)) = \mathcal{H}_{\eta}(\emptyset)$ holds for all $|\,s\,| \leq |\,t\,|$. Whence $\mathcal{H}_{\eta}[s] = \mathcal{H}_{\eta}$ for all $|\,s\,| \leq |\,t\,|$. Therefore, by the induction hypothesis,

$$\mathcal{H}_{\hat{\alpha}_s} \,\big|\!\frac{\psi_{\Omega}(\hat{\alpha}_s)}{\psi_{\Omega}(\hat{\alpha}_s)}\, \Gamma, s \subseteq t \to F(s) \tag{20}$$

for all $|\,s\,| \leq |\,t\,|$. Let $|\,s\,| \leq |\,t\,|$. Since $|\,s\,| < \psi_{\Omega}(\eta+1)$ one computes that $\psi_{\Omega}(\hat{\alpha}_s) < \psi_{\Omega}(\hat{\alpha})$. Therefore, an inference $(pb\forall)_{\infty}$ applied to (20) yields $\mathcal{H}_{\hat{\alpha}} \,\big|\!\frac{\psi_{\Omega}(\hat{\alpha})}{\psi_{\Omega}(\hat{\alpha})}\, \Gamma$.

The cases were the last inference is an instance of $(b\forall)_{\infty}$, $(\notin)_{\infty}$, $(\not\subseteq)_{\infty}$, or $(\wedge)$ are dealt with in a similar manner.

Case 2: Suppose the last inference was $(\exists)$. Then there is a formula $A \in \Gamma$ of the form $\exists x\, F(x)$ such that $\mathcal{H}_{\eta} \,\big|\!\frac{\alpha_0}{\Omega+1}\, \Gamma, F(s)$ holds for some term s and $\alpha_0 + 1 < \alpha$. The induction hypothesis yields

$$\mathcal{H}_{\hat{\alpha}_0} \,\big|\!\frac{\psi_{\Omega}(\hat{\alpha}_0)}{\psi_{\Omega}(\hat{\alpha}_0)}\, \Gamma, F(s)\,.$$

Since $\alpha_0, |\,s\,| \in \mathcal{H}_{\eta}(\emptyset)$ we see that

$$\alpha_0, |\,s\,| \in C^{\Omega}(\eta+1, \psi_{\Omega}(\eta+1))\,.$$

Consequently we have $|\,s\,|, \psi_{\Omega}(\hat{\alpha}_0) < \psi_{\Omega}(\hat{\alpha})$. Thus, via $(\exists)$ we conclude that $\mathcal{H}_{\hat{\alpha}} \,\big|\!\frac{\psi_{\Omega}(\hat{\alpha})}{\psi_{\Omega}(\hat{\alpha})}\, \Gamma$.

The cases were the last inference is an instance of $(b\exists)$, $(pb\exists)$, $(\in)$, $(\subseteq)$, or $(\vee)$ are dealt with in a similar manner.

Case 3: Suppose $\exists z\, A^z \in \Gamma$ and $\mathcal{H}_{\eta} \,\big|\!\frac{\alpha_0}{\Omega+1}\, \Gamma, A$ with $\alpha_0 < \alpha$. This means that the last inference was $(\Sigma^{\mathcal{P}}\text{-}Ref)$. The induction hypothesis yields $\mathcal{H}_{\hat{\alpha}_0} \,\big|\!\frac{\psi_{\Omega}(\hat{\alpha}_0)}{\psi_{\Omega}(\hat{\alpha}_0)}\, \Gamma, A$ and therefore, as A is a $\Sigma^{\mathcal{P}}$-formula, we get

$$\mathcal{H}_{\hat{\alpha}_0} \,\big|\!\frac{\psi_{\Omega}(\hat{\alpha}_0)}{\psi_{\Omega}(\hat{\alpha}_0)}\, \Gamma, A^{\mathbb{V}_{\psi_{\Omega}(\hat{\alpha}_0)}}$$

by Lemma 7.2. Since $\psi_{\Omega}(\hat{\alpha}_0) \in \mathcal{H}_{\hat{\alpha}}$ and $\psi_{\Omega}(\hat{\alpha}_0) < \psi_{\Omega}(\hat{\alpha})$, an inference $(\exists)$ yields $\mathcal{H}_{\hat{\alpha}} \,\big|\!\frac{\psi_{\Omega}(\hat{\alpha})}{\psi_{\Omega}(\hat{\alpha})}\, \Gamma, \exists z\, A^z$, i.e. $\mathcal{H}_{\hat{\alpha}} \,\big|\!\frac{\psi_{\Omega}(\hat{\alpha})}{\psi_{\Omega}(\hat{\alpha})}\, \Gamma$.

Case 4: Suppose the last inference was (Cut). Then

$$\mathcal{H}_{\eta} \,\big|\!\frac{\alpha_0}{\Omega+1}\, \Gamma, A \quad \text{and} \quad \mathcal{H}_{\eta} \,\big|\!\frac{\alpha_0}{\Omega+1}\, \Gamma, \neg A\;,$$

where $\alpha_0 < \alpha$ and A is a formula with $rk(A) \leq \Omega$.

Case 4.1: Suppose that $rk(A) < \Omega$. This implies

$$rk(A) \in C^{\Omega}(\eta+1, \psi_{\Omega}(\eta+1))$$

and hence $rk(A) < \psi_{\Omega}(\eta+1) < \psi_{\Omega}(\hat{\alpha})$. Inductively we have

$$\mathcal{H}_{\hat{\alpha}_0} \,\frac{|\psi_{\Omega}(\hat{\alpha}_0)}{\psi_{\Omega}(\hat{\alpha}_0)}\, \Gamma, A \;\text{ and }\; \mathcal{H}_{\hat{\alpha}_0} \,\frac{|\psi_{\Omega}(\hat{\alpha}_0)}{\psi_{\Omega}(\hat{\alpha}_0)}\, \Gamma, \neg A\ .$$

Thus $\mathcal{H}_{\hat{\alpha}} \,\frac{|\psi_{\Omega}(\hat{\alpha})}{\psi_{\Omega}(\hat{\alpha})}\, \Gamma$ by means of (Cut).

Case 4.2: Suppose that $rk(A) = \Omega$. Then A or $\neg A$ will be of the form $\exists z\, F(z)$ with $F(\mathbb{V}_0)$ being $\Delta_0^{\mathcal{P}}$. We may assume that the former is the case. Then the induction hypothesis applied to $\mathcal{H}_{\eta} \,\frac{|\alpha_0}{\Omega+1}\, \Gamma, A$ yields $\mathcal{H}_{\hat{\alpha}_0} \,\frac{|\psi_{\Omega}(\hat{\alpha}_0)}{\psi_{\Omega}(\hat{\alpha}_0)}\, \Gamma, A$. Since $\psi_{\Omega}(\hat{\alpha}_0) \in \mathcal{H}_{\hat{\alpha}_0}(\emptyset)$, we can apply the Boundedness Lemma 7.2, obtaining

$$\mathcal{H}_{\hat{\alpha}_0} \,\frac{|\psi_{\Omega}(\hat{\alpha}_0)}{\psi_{\Omega}(\hat{\alpha}_0)}\, \Gamma, A^{\mathbb{V}_{\psi_{\Omega}(\hat{\alpha}_0)}}\ . \tag{21}$$

By applying inversion (Lemma 5.10(iii)) to $\mathcal{H}_{\hat{\alpha}_0} \,\frac{|\alpha_0}{\Omega+1}\, \Gamma, \neg A$ we also get

$$\mathcal{H}_{\hat{\alpha}_0} \,\frac{|\alpha_0}{\Omega+1}\, \Gamma, \neg A^{\mathbb{V}_{\psi_{\Omega}(\hat{\alpha}_0)}}\ . \tag{22}$$

Observing that $\Gamma, \neg A^{\mathbb{V}_{\psi_{\Omega}(\hat{\alpha}_0)}}$ is a set of $\Sigma^{\mathcal{P}}$-formulae, we can apply the induction hypothesis to (22), yielding

$$\mathcal{H}_{\alpha_1} \,\frac{|\psi_{\Omega}(\alpha_1)}{\psi_{\Omega}(\alpha_1)}\, \Gamma, \neg A^{\mathbb{V}_{\psi_{\Omega}(\hat{\alpha}_0)}}\ , \tag{23}$$

where $\alpha_1 = \hat{\alpha}_0 + \omega^{\Omega+\alpha_0} = \eta + \omega^{\Omega+\alpha_0} + \omega^{\Omega+\alpha_0} < \eta + \omega^{\Omega+\alpha} = \hat{\alpha}$. Moreover, we have $\psi_{\Omega}(\alpha_1) < \psi_{\Omega}(\hat{\alpha})$. Therefore (Cut) applied to (21) and (23) furnishes $\mathcal{H}_{\hat{\alpha}} \,\frac{|\psi_{\Omega}(\hat{\alpha})}{\psi_{\Omega}(\hat{\alpha})}\, \Gamma$. □

Note that the Collapsing Theorem removes all instances of $(\Sigma^{\mathcal{P}}\text{-}Ref)$.

Also note that we cannot eliminate cuts with $\Delta_0^{\mathcal{P}}$-formulae since we don't have predicative cut elimination [8, Theorem 3.16] as in the case **KP**.

Corollary 7.7. *Let A be a $\Sigma^{\mathcal{P}}$-sentence of* $\mathbf{KP}(\mathcal{P})$*. Suppose that* $\mathbf{KP}(\mathcal{P}) \vdash A$*. Then there exists an operator $\mathcal{H}$ and an ordinal $\rho < \psi_{\Omega}(\varepsilon_{\Omega+1})$ such that*

$$\mathcal{H} \,\frac{|\rho}{\rho}\, A\ .$$

Proof: Let $\mathcal{H}_0$ be defined as in Definition 7.3. By Theorem 6.9 we have

$$\mathcal{H}_0 \mathrel{\Big|\!\frac{\omega^{\Omega+m}}{\Omega+m+1}} A$$

for some $0 < m < \omega$. Applying ordinary cut elimination, Theorem 7.1, we get

$$\mathcal{H}_0 \mathrel{\Big|\!\frac{\omega_m(\omega^{\Omega+m})}{\Omega+1}} A\ .$$

Finally, using the Collapsing Theorem 7.6 we arrive at

$$\mathcal{H}_{\omega_{m+1}(\omega^{\Omega+m})} \mathrel{\Big|\!\frac{\rho}{\rho}} A$$

with $\rho := \psi_\Omega(\omega_{m+1}(\omega^{\Omega+m}))$. $\square$

8. Soundness

For the main Theorem of this paper, we want to show that derivability in $RS_\Omega^{\mathcal{P}}$ entails truth. Since $RS_\Omega^{\mathcal{P}}$-formulae contain variables we need the notion of assignment. Let VAR be the set of free variables of $RS_\Omega^{\mathcal{P}}$. A variable assignment ℓ is a function

$$\ell : VAR \longrightarrow V_{\psi_\Omega(\varepsilon_{\Omega+1})}$$

satisfying $\ell(a^\alpha) \in V_{\alpha+1}$, where as per usual V_α denotes the α^{th} level of the von Neumann hierarchy.

ℓ can be canonically lifted to all $RS_\Omega^{\mathcal{P}}$-terms as follows:

$$\begin{aligned} \ell(\mathbb{V}_\alpha) &= V_\alpha \\ \ell(\{x \in \mathbb{V}_\alpha \mid F(x, s_1, \ldots, s_n)\}) &= \{x \in V_\alpha \mid F(x, \ell(s_1), \ldots, \ell(s_n))\}\,. \end{aligned}$$

Note that $\ell(s) \in V_{\psi_\Omega(\varepsilon_{\Omega+1})}$ holds for all $RS_\Omega^{\mathcal{P}}$-terms s. Moreover, we have $\ell(s) \in V_{|\,s\,|+1}$.

Theorem 8.1 (Soundness). *Let $\mathcal{H}$ be an operator with $\mathcal{H}(\emptyset) \subseteq C^\Omega(\varepsilon_{\Omega+1}, 0)$ and $\alpha, \rho < \psi_\Omega(\varepsilon_{\Omega+1})$. Let $\Gamma(s_1, \ldots, s_n)$ be a sequent consisting only of $\Sigma^{\mathcal{P}}$-formulae. Suppose*

$$\mathcal{H} \mathrel{\Big|\!\frac{\alpha}{\rho}} \Gamma(s_1, \ldots, s_n)\ .$$

Then, for all variable assignments ℓ,

$$V_{\psi_\Omega(\varepsilon_{\Omega+1})} \models \Gamma(\ell(s_1), \ldots, \ell(s_n))\,,$$

where the latter, of course, means that $V_{\psi_\Omega(\varepsilon_{\Omega+1})}$ is a model of the disjunction of the formulae in $\Gamma(s_1, \ldots, s_n)$.

Proof: The proof proceeds by induction on α. Note that, owing to $\alpha, \rho < \Omega$, the proof tree pertaining to $\mathcal{H} \mid\!\frac{\alpha}{\rho}\, \Gamma(s_1, \ldots, s_n)$ neither contains any instances of $(\Sigma^{\mathcal{P}}\text{-}Ref)$ nor of $(\forall)_\infty$, and that all cuts are performed with $\Delta_0^{\mathcal{P}}$-formulae. The proof is straightforward as all the axioms of $RS_\Omega^{\mathcal{P}}$ are true under the interpretation and all other rules are truth preserving with respect to this interpretation. Observe that we make essential use of the free variables when showing the soundness of $(b\forall)_\infty$, $(pb\forall)_\infty$, $(\notin)_\infty$ and $(\not\subseteq)_\infty$. We treat $(pb\forall)_\infty$ as an example. So assume $(\forall x \subseteq s_i)F(x, \vec{s}) \in \Gamma(\vec{s})$ and

$$\mathcal{H}[r] \mid\!\frac{\alpha_r}{\rho}\, \Gamma(s_1, \ldots, s_n), r \subseteq s_i \to F(r, \vec{s}\,)$$

holds for all terms r with $|\, r\,| \leq |\, s_i\,|$ for some $\alpha_r < \alpha$. In particular we have $\mathcal{H}[a^\beta] \mid\!\frac{\alpha'}{\rho}\, \Gamma(s_1, \ldots, s_n), a^\beta \subseteq s_i \to F(a^\beta, \vec{s}\,)$, where $\beta = |\, s_i\,|$ and a^β is a free variable not occurring in $\Gamma(s_1, \ldots, s_n)$ and $\alpha' = \alpha_{a^\beta}$. By the induction hypothesis we have

$$V_{\psi_\Omega(\varepsilon_{\Omega+1})} \models \Gamma(\ell(s_1), \ldots, \ell(s_n)), \ell'(a^\beta) \subseteq \ell(s_i) \to F(\ell'(a^\beta), \ell(s_1), \ldots, \ell(s_n)\,)$$

where ℓ' is an arbitrary variable assignment. This entails that either

$$V_{\psi_\Omega(\varepsilon_{\Omega+1})} \models \Gamma(\ell(s_1), \ldots, \ell(s_n))$$

or

$$V_{\psi_\Omega(\varepsilon_{\Omega+1})} \models \ell'(a^\beta) \subseteq \ell(s_i) \to F(\ell'(a^\beta), \ell(s_1), \ldots, \ell(s_n)\,)$$

for all assignments ℓ'. In the former case we have found what we want and in the latter case we arrive at $V_{\psi_\Omega(\varepsilon_{\Omega+1})} \models (\forall x \subseteq \ell(s_i))F(x, \ell(s_1), \ldots, \ell(s_n)\,)$ and therefore also have $V_{\psi_\Omega(\varepsilon_{\Omega+1})} \models \Gamma(\ell(s_1), \ldots, \ell(s_n))$. $\square$

Combining Theorem 8.1 and Corollary 7.7 we have the following:

Theorem 8.2. *If A is a $\Sigma^{\mathcal{P}}$-sentence and*

$$\mathbf{KP}(\mathcal{P}) \vdash A$$

then

$$V_{\psi_\Omega(\varepsilon_{\Omega+1})} \models A.$$

The bound of this Corollary is actually sharp, that is, $\psi_\Omega(\varepsilon_{\Omega+1})$ is the first ordinal with that property. This follows immediately from [22, Theorem 4.9].

Corollary 8.3. *Assume* **AC** *in the background theory. If A is a $\Sigma^{\mathcal{P}}$-sentence and*

$$\mathbf{KP}(\mathcal{P}) + \mathbf{AC} \vdash A$$

then $V_{\psi_\Omega(\varepsilon_{\Omega+1})} \models A$.

Proof: This follows from Theorem 8.2 since the axiom of choice, **AC**, can be formulated as a $\Pi_1^{\mathcal{P}}$-sentence. □

The previous results can be extended to $\Pi_2^{\mathcal{P}}$ sentences, basically by the same proof as for [22, Theorem 2.1].

Theorem 8.4. *Let A be a $\Pi_2^{\mathcal{P}}$-sentence.*

(i) If $\mathbf{KP}(\mathcal{P}) \vdash A$ then $V_{\psi_\Omega(\varepsilon_{\Omega+1})} \models A$.

(ii) If $\mathbf{KP}(\mathcal{P}) + \mathbf{AC} \vdash A$ then $V_{\psi_\Omega(\varepsilon_{\Omega+1})} \models A$.

Proof: (i): Assume $\mathbf{KP}(\mathcal{P}) \vdash \forall u \exists w H(u,w)$ with $H(u,w)$ being $\Delta_0^{\mathcal{P}}$. Let $\sigma := \psi_\Omega(\varepsilon_{\Omega+1})$. Let $b \in V_\sigma$. We have to verify that $V_\sigma \models \exists w H(b,w)$. σ is a limit, so there is $\xi < \sigma$ such that $b \in V_\xi$. Since V_ξ does not satisfy all $\Sigma^{\mathcal{P}}$-sentences provable in $\mathbf{KP}(\mathcal{P})$, we have $\mathbf{KP}(\mathcal{P}) \vdash B$ and $V_\xi \models \neg B$ for some $\Sigma^{\mathcal{P}}$-sentence B. Since $\Sigma^{\mathcal{P}}$-reflection is provable in $\mathbf{KP}(\mathcal{P})$, we also get $\mathbf{KP}(\mathcal{P}) \vdash \exists \alpha \exists x (x = V_\alpha \wedge B^x)$. Then, using $\Delta_0^{\mathcal{P}}$–Collection, we obtain

$$\mathbf{KP}(\mathcal{P}) \vdash \exists z \exists \alpha \exists x [x = V_\alpha \wedge B^x \wedge (\forall u \in x)(\exists w \in z) H(u,w)] \,.$$

Since this formula is equivalent to a $\Sigma^{\mathcal{P}}$–formula in $\mathbf{KP}(\mathcal{P})$, we get

$$V_\sigma \models \exists \alpha \exists x [x = V_\alpha \wedge B^x \wedge (\forall u \in x) \exists w H(u,w)]$$

As the formula "$x = V_\alpha$" has the same meaning in V_σ as it has in V, there exists $\alpha < \sigma$ such that $V_\alpha \models B$ and $(\forall u \in V_\alpha)(\exists w \in V_\sigma) H(u,w)$. By the choice of B, this implies $\xi < \alpha$, hence $b \in V_\alpha$, thus $V_\sigma \models \exists w H(b,w)$.

(ii) follows from (i) since **AC** can be expressed as a $\Pi_1^{\mathcal{P}}$ statement. □

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