Explicit Mathematics With The Monotone Fixed Point Principle. II: Models

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Abstract

This paper continues investigations of the monotone fixed point principle in the context of Feferman's explicit mathematics begun in [14]. Explicit mathematics is a versatile formal framework for representing Bishop-style constructive mathematics and generalized recursion theory. The object of investigation here is the theory of explicit mathematics augmented by the monotone fixed point principle, which asserts that any monotone operation on classifications (Feferman's notion of set) possesses a least fixed point. To be more precise, the new axiom not merely postulates the existence of a least solution, but, by adjoining a new constant to the language, it is ensured that a fixed point is uniformly presentable as a function of the monotone operation. Let $\mathbf{T_0}+\mathbf{UMID}$ denote this extension of explicit mathematics. [14] gave lower bounds for the strength of two subtheories of $\mathbf{T_0}+\mathbf{UMID}$ in relating them to fragments of second order arithmetic based on $\mathbf{\Pi}^1_2$ comprehension. [14] showed that $\mathbf{T_0}\restriction+\mathbf{UMID}$ and $\mathbf{T_0}\restriction+\mathbf{IND}_{\mathbb{N}}+\mathbf{UMID}$ have at least the strength of $(\mathbf{\Pi}^1_2-\mathbf{CA})\restriction$ and $(\mathbf{\Pi}^1_2-\mathbf{CA})$, respectively.

Here we are concerned with the exact reversals. Let $\mathbf{UMID}_{\mathbb{N}}$ be the monotone fixed-point principle for subclassifications of the natural numbers. Among other results, it is shown that $\mathbf{T_0}\restriction+\mathbf{UMID}_{\mathbb{N}}$ and $\mathbf{T_0}\restriction+\mathbf{IND}_{\mathbb{N}}+\mathbf{UMID}_{\mathbb{N}}$ have the same strength as $(\mathbf{\Pi}^1_2-\mathbf{CA})\restriction$ and $(\mathbf{\Pi}^1_2-\mathbf{CA})$, respectively.

The results are achieved by constructing set-theoretic models for the aforementioned systems of explicit mathematics in certain extensions of Kripke-Platek set theory and subsequently relating these set theories to subsystems of second order arithmetic.

1 Introduction

The present paper is a continuation of [14] and therefore familiarity with [14] will be presumed and the same notations will be used.

The main results of [14] were stated as follows (cf. [14], Theorem 5.3):

Theorem *Let ϕ be a Π^1_3 sentence of second order arithmetic and ϕ^* be its canonical translation into the language of explicit mathematics.*

(i) If $(\mathbf{\Pi}^1_2-\mathbf{CA})\restriction\vdash\phi$*, then* $\mathbf{T_0}\restriction+\mathbf{UMID}\vdash\phi^*$.

(ii) If $(\mathbf{\Pi}^1_2-\mathbf{CA})\vdash\phi$*, then* $\mathbf{T_0}\restriction+\mathbf{IND}_{\mathbb{N}}+\mathbf{UMID}\vdash\phi^*$.

Inspection of the proofs in [14, 13] readily reveals that the interpretations do not require the principle **UMID** for the entire universe. A restricted version of **UMID**, termed $\mathbf{UMID}_{\mathbb{N}}$, suffices. $\mathbf{UMID}_{\mathbb{N}}$ asserts that any monotone operator f which sends subclassifications of $\mathbb{N}$ to subclassifications of $\mathbb{N}$ has a least fixed point $\mathbf{lfp}(f)$. Thus [14] Theorem 5.3 can be sharpened as follows:

Theorem 1.1 *Let ϕ be a Π^1_3 sentence of second order arithmetic.*

(i) If $(\mathbf{\Pi}^1_2-\mathbf{CA})\restriction\vdash\phi$, then $\mathbf{T_0}\restriction+\mathbf{UMID}_{\mathbb{N}}\vdash\phi^*$.

(ii) If $(\mathbf{\Pi}^1_2-\mathbf{CA})\vdash\phi$, then $\mathbf{T_0}\restriction+\mathbf{IND}_{\mathbb{N}}+\mathbf{UMID}_{\mathbb{N}}\vdash\phi^*$.

The intent of this paper is to show that the reversals of the above theorem hold too. They are obtained by carefully constructing set-theoretic models for the aforementioned systems of explicit mathematics in certain extensions of Kripke-Platek set theory and subsequently relating these set theories to subsystems of second order arithmetic. To be more precise, we construct a class model $\widetilde{\mathfrak{M}}$ of $\mathbf{T_0}\restriction+\mathbf{IND}_{\mathbb{N}}+\mathbf{UMID}_{\mathbb{N}}$ in $\mathbf{KP}^w+\Sigma_1$-$\mathbf{Sep}$. The latter system is Kripke-Platek set theory (which is taken to include the axiom of infinity) augmented by Σ_1 Separation with Foundation restricted to sets but retaining induction on ω for all formulae. The model $\widetilde{\mathfrak{M}}$ arises as a union of (set) models $\mathfrak{M}^{\mathbf{V}_n,\mathbf{h}_n}$. For meta n, the existence of $\mathfrak{M}^{\mathbf{V}_n,\mathbf{h}_n}$ can already be proved in $\mathbf{KP}^r+\Sigma_1$-$\mathbf{Sep}$ which is the theory $\mathbf{KP}^w+\Sigma_1$-$\mathbf{Sep}$ shorn of the schema of induction on ω. With regard to the theory $\mathbf{T_0}\restriction+\mathbf{UMID}_{\mathbb{N}}$ we then show that those theorems of $\mathbf{T_0}\restriction+\mathbf{UMID}_{\mathbb{N}}$, wherein all the classification quantifiers are existential, hold in any of the models $\mathfrak{M}^{\mathbf{V}_n,\mathbf{h}_n}$. The latter reduces $\mathbf{T_0}\restriction+\mathbf{UMID}_{\mathbb{N}}$ to $\mathbf{KP}^r+\Sigma_1$-$\mathbf{Sep}$. Since $\mathbf{KP}^r+\Sigma_1$-$\mathbf{Sep}$ and $\mathbf{KP}^w+\Sigma_1$-$\mathbf{Sep}$ prove the same theorems of second order arithmetic as $(\mathbf{\Pi}^1_2-\mathbf{CA})\restriction$ and $(\mathbf{\Pi}^1_2-\mathbf{CA})$, respectively, the direction "$\Leftarrow$" in Theorem 1.1 ensues.

There are some indications that $\widetilde{\mathfrak{M}}$ might actually be a model of **UMID**. But a proof seems to require more "fine structure theory" of $\widetilde{\mathfrak{M}}$ than is provided in the present paper. However, we pursue the question of how much of **UMID** is actually modelled by $\widetilde{\mathfrak{M}}$ at some length. It is proved that $\widetilde{\mathfrak{M}}$ satisfies the monotone fixed point principle on a subuniverse **I**. This stronger principle, dubbed $\mathbf{UMID_I}$, asserts the monotone fixed point principle for subclassifications of a classification **I** which comprises $\mathbb{N}$ and contains all the standard constants and is also closed under application.

The paper is organized as follows: Section 2 introduces restrictions of **UMID** and the axioms pertaining to the classification constant **I**, and further describes the subsystems of set theory, $\mathbf{KP}^r+\Sigma_1$-$\mathbf{Sep}$ and $\mathbf{KP}^w+\Sigma_1$-$\mathbf{Sep}$, needed for the interpretations.

Section 3 is devoted to the construction of the model $\widetilde{\mathfrak{M}}$, yielding the reduction of $\mathbf{T_0}\restriction+\mathbf{IND}_{\mathbb{N}}+\mathbf{UMID}_{\mathbb{N}}$ to $\mathbf{KP}^w+\Sigma_1$-$\mathbf{Sep}$.

Section 4 is concerned with the reduction of $\mathbf{T_0}\restriction+\mathbf{UMID}_{\mathbb{N}}$ to $\mathbf{KP}^r+\Sigma_1$-$\mathbf{Sep}$. Since the full model $\widetilde{\mathfrak{M}}$ cannot be formalized in $\mathbf{T_0}\restriction+\mathbf{UMID}_{\mathbb{N}}$, the approach is to first show partial cut elimination for $\mathbf{T_0}\restriction+\mathbf{UMID}_{\mathbb{N}}$ in a Tait-style calculus and subsequently interpret partially normalized derivations of Σ^{EM} formulae in the approximations $\mathfrak{M}^{\mathbf{V}_n,\mathbf{h}_n}$. The Σ^{EM} formulae are those wherein all the classification quantifiers are existential.

The circle of reductions is completed in Section 5 by showing that $\mathbf{KP}^r+\Sigma_1$-$\mathbf{Sep}$ and $\mathbf{KP}^w+\Sigma_1$-$\mathbf{Sep}$ prove the same theorems of second order arithmetic as $(\mathbf{\Pi}^1_2-\mathbf{CA})\restriction$ and $(\mathbf{\Pi}^1_2-\mathbf{CA})$, respectively.

2 Preliminaries

2.1 Variants of the monotone fixed point principle

Definition 2.1 As in [14] we use the abbreviations $X \mathrel{\mathring{=}} Y \;:=\; \forall v(v \mathrel{\mathring{\in}} X \leftrightarrow v \mathrel{\mathring{\in}} Y)$ and $X \mathrel{\mathring{\subseteq}} Y \;:=\; \forall v(v \mathrel{\mathring{\in}} X \to v \mathrel{\mathring{\in}} Y)$. To state the monotone fixed point principle for subclassifications of a given classification A we introduce the following shorthands:

$$\begin{array}{lll} \mathbf{Clop(f,A)} & \text{if} & \forall X \mathrel{\mathring{\subseteq}} A\, \exists Y \mathrel{\mathring{\subseteq}} A\; fX \simeq Y \\ \mathbf{Ext(f,A)} & \text{if} & \forall X \mathrel{\mathring{\subseteq}} A\, \forall Y \mathrel{\mathring{\subseteq}} A\; [X \mathrel{\mathring{=}} Y \to fX \mathrel{\mathring{=}} fY] \\ \mathbf{Mon}(f,A) & \text{if} & \forall X \mathrel{\mathring{\subseteq}} A\, \forall Y \mathrel{\mathring{\subseteq}} A\, [X \mathrel{\mathring{\subseteq}} Y \to fX \mathrel{\mathring{\subseteq}} fY]. \\ \mathbf{Lfp}(Y,f,A) & \text{if} & fY \mathrel{\mathring{\subseteq}} Y \;\wedge\; Y \mathrel{\mathring{\subseteq}} A \;\wedge\; \forall X \mathrel{\mathring{\subseteq}} A\; \big[fX \mathrel{\mathring{\subseteq}} X \;\to\; Y \mathrel{\mathring{\subseteq}} X\big] \end{array}$$

When f satisfies $\mathbf{Clop(f,A)}$, we call f a *classification operation on A*. When f satisfies $\mathbf{Clop(f,A)}$ and $\mathbf{Ext(f,A)}$, we call f *extensional* or an *extensional operation on A*. When f satisfies $\mathbf{Clop(f,A)}$ and $\mathbf{Mon}(f,A)$, we say that f is a *monotone operation on A*. Since monotonicity entails extensionality, a monotone operation is always extensional.

Now we state $\mathbf{UMID}_A$.

$\mathbf{UMID}_A$ (Uniform Monotone Inductive Definition on A)

$$\forall f\,[\mathbf{Clop(f,A)} \wedge \mathbf{Mon}(f,A) \to \mathbf{Lfp}(\mathbf{lfp}(f),f,A)].$$

$\mathbf{UMID}_A$ states that if f is monotone on subclassifications of A, $\mathbf{lfp}(f)$ is a least fixed point of f.

We also introduce an extension of $\mathbf{T_0}$, dubbed $\mathbf{T_0(I)}$, which has an additional classification constant $\mathbf{I}$ and axioms pertaining to $\mathbf{I}$, asserting that $\mathbf{I}$ is a classification which is closed under application and contains all the basic constants, i.e. $\mathbf{e} \mathrel{\mathring{\in}} \mathbf{I}$ where $\mathbf{e}$ is any of the constants $\mathbf{0}, \mathbf{s}_{\mathbb{N}}, \mathbf{p}_{\mathbb{N}}, \mathbf{k}, \mathbf{s}, \mathbf{d}, \mathbf{p}, \mathbf{p_0}, \mathbf{p_1}, \mathbf{i}, \mathbf{j}, \mathbf{c}_{\phi}, \mathbb{N}, \mathbf{lfp}, \mathbf{I}$, and

$$\forall x \mathrel{\mathring{\in}} \mathbf{I}\, \forall y \mathrel{\mathring{\in}} \mathbf{I}\, \forall z\,(\,xy \simeq z \to z \mathrel{\mathring{\in}} \mathbf{I}\,).$$

Since $\mathbf{T_0(I)}\!\restriction\, \vdash \mathbb{N} \mathrel{\mathring{\subseteq}} \mathbf{I}$, we have $\mathbf{T_0}\!\restriction +\mathbf{UMID}_{\mathbb{N}} \subseteq \mathbf{T_0(I)}\!\restriction +\mathbf{UMID_I}$.

2.2 Subsystems of set theory

The axiom systems for set theory considered in this paper are formulated in the usual language of set theory (called $\mathcal{L}_\in$ hereafter) containing $\in$ as the only non-logical symbol besides $=$. Formulae are built from prime formulae $a \in b$ and $a = b$ by use of propositional connectives and quantifiers $\forall x, \exists x$. Bounded quantifiers $\forall x \in a$, $\exists x \in a$ are defined as usual. Δ_0-formulae are the formulae wherein all quantifiers are bounded; Σ_1-formulae are those of the form $\exists x \varphi(x)$ where $\varphi(a)$ is a Δ_0-formula. For $n > 0$, Π_n-formulae (Σ_n-formulae) are the formulae with a prefix of n alternating unbounded quantifiers starting with a universal (existential) one followed by a Δ_0-formula. The class of Σ-formulae is the smallest class of formulae containing the Δ_0-formulae which is closed under $\wedge$, $\vee$, bounded quantification and unbounded existential quantification.

The exact details of the formulation do not really matter for the purpose of this paper, any standard formulation will work. Also, we use the standard Δ_0-definitions of predicates like $x = \emptyset$, $\mathrm{Tran}(x)$, $\mathrm{On}(x)$ and the like. In what follows we shall assume familiarity with the basics of admissible set theory as presented in [3], Chap. I,II.

Definition 2.2 We use Kripke-Platek set theory **KP** (cf. [3]) as our basic theory. It consists of the axioms of Extensionality, Pairing, Union, Infinity[1] and of the axiom schemata of Separation and Collection for Δ_0-formulae as well as the Foundation schema for arbitrary formulae.

$\mathbf{KP}^r$ arises from **KP** by replacing the axiom schema of Foundation by the Foundation axiom

$$\forall x(\exists y(y \in x) \to \exists y(y \in x \land \forall z \in x(z \notin y))).$$

$\mathbf{KP}^w$ is obtained from $\mathbf{KP}^r$ by adding the schema

$$\mathbf{IND}_\omega \qquad \forall x \in \omega(\forall y \in x\phi(y) \to \phi(x)) \to \forall x \in \omega\phi(x)$$

of induction on ω to $\mathbf{KP}^r$ (for all formulae ϕ).

Definition 2.3 Σ_n-Separation (abbreviated Σ_n-**Sep**) is the schema of axioms

$$\exists z\,\forall u\,\big(u{\in}z \;\leftrightarrow\; [u{\in}a \land \phi(u)]\big)$$

for all set-theoretic Σ_n-formulae ϕ with z not free in ϕ.

2.3 Some results derivable in $\mathbf{KP}^r + \Sigma_1$-Sep

For later use we collect some derivable consequences of $\mathbf{KP}^r + \Sigma_1$-**Sep**.

Definition 2.4 The schema of Σ *Foundation* consists of all formulae

$$\forall x\,[(\forall y \in x)\phi(y) \;\to\; \phi(x)] \;\to\; \forall x\phi(x),$$

where ϕ is a Σ formula.

Proposition 2.5 *All instances of* Σ *Foundation are provable in* $\mathbf{KP}^r + \Sigma_1$-**Sep**.

Proof: First, we show that $\mathbf{KP}^r + \Sigma_1$-**Sep** proves that every set a has a transitive closure $\mathbf{TC}(a)$. For $n \in \omega$ let $\psi(n, f)$ be the formula expressing that f is a function with domain $n + 1$ such that $f(0) = a \land (\forall k < n)\, f(k + 1) = f(k) \cup \bigcup f(k)$. The class $A := \{n \in \omega : \exists f\psi(n, f)\}$ is a set by Σ_1 Separation. Using the axiom of Foundation, one easily proves that $A = \omega$. Moreover, $(\forall n \in \omega)\exists! y\exists f[\psi(n, f) \land \mathbf{ran}(f) = y]$. Thus, by Σ Replacement (cf. [3], I.4.6) there exists a function F with domain ω such that $(\forall n \in \omega)\exists f[\psi(n, f) \land \mathbf{ran}(f) = F(n)]$. Obviously, we have $\mathbf{TC}(a) = \mathbf{ran}(F)$.

Now, for a contradiction, suppose we have a failure of Σ Foundation. Then

$$\forall x\,[(\forall y \in x)\phi(y) \;\to\; \phi(x)]$$

but $\neg\phi(a)$ for some a. Set $B := \{y \in \mathbf{TC}(\{a\}) : \neg\phi(y)\}$. Since every Σ formula is equivalent to a Σ_1 formula (using Δ_0 Collection), $C := \{y \in \mathbf{TC}(\{a\}) : \phi(y)\}$ is a set by Σ_1 Separation. As $B = \mathbf{TC}(\{a\}) \setminus C$, B is a set by Δ_0 Separation. Observe that $B \neq \emptyset$ since $a \in B$. Using the Foundation axiom there exists $c \in B$ such that $(\forall z \in c)(z \notin B)$. Since $\mathbf{TC}(\{a\})$ is transitive the latter implies $(\forall z \in c)\phi(z)$, yielding $\phi(c)$. But that collides with $c \in B$. □

[1] For the results of this paper it is crucial that *Infinity* is assumed to be among the axioms of **KP**.

Lemma 2.6 *In* $\mathbf{KP}^r + \Sigma_1$-**Sep** *we can define functions by* Σ *Recursion. To put it more formally, if* $\Phi(\vec{x}, y, z)$ *is a* Σ *formula, then*

$$\mathbf{KP}^r + \Sigma_1\text{-}\mathbf{Sep} \vdash \forall \vec{x}\, \forall y \exists! z\, \Phi(\vec{x}, y, z) \rightarrow$$
$$\forall \vec{x}\, \forall u \exists! f \left[\mathbf{Fun}(f) \wedge \mathbf{dom}(f) = \mathbf{TC}(u) \wedge \forall v{\in}\mathbf{TC}(u)\, \Phi(\vec{x}, \{\langle y, f(y)\rangle : y{\in}v\}, f(v))\right],$$

where $\mathbf{Fun}(f)$ *asserts that* f *is a function and* $\mathbf{dom}(f)$ *denotes its domain.*

Proof: As far as Foundation is concerned, the proof of Σ Recursion in **KP** only requires Σ Foundation as can immediately be gleaned from the proof of [3], I.6.4 or [5], I.11.8. $\Box$

As a consequence, all the familiar functions defined via Σ Recursion are at our disposal in $\mathbf{KP}^r + \Sigma_1$-**Sep**. Prominent examples are the rank function, the Mostowski collapsing function as well as the constructible hierarchy.

Definition 2.7 We will use Gödel's constructible hierarchy $\mathbf{L} = (\mathbf{L}_\alpha)_{\alpha \in \mathrm{On}}$ in one of its usual formulations. For definiteness let

$$\mathbf{L}_0 = \emptyset, \;\; \mathbf{L}_{\alpha+1} = \mathbf{Def}(\mathbf{L}_\alpha), \;\; \mathbf{L}_\lambda = \bigcup_{\alpha<\lambda} \mathbf{L}_\alpha \text{ for } \lambda \in \mathrm{Lim}.$$

Here $\mathbf{Def}(x)$ is the set of all definable subsets of x.

For subsets $U \subseteq \omega$ we will also consider the relativized constructible hierarchy $\mathbf{L}(U) = \bigcup_{\alpha \in \mathrm{On}} \mathbf{L}_\alpha(U)$ which is defined as follows:

$$\mathbf{L}_0(U) = \emptyset, \;\; \mathbf{L}_{\alpha+1}(U) = \mathbf{Def}_U(\mathbf{L}_\alpha(U)), \;\; \mathbf{L}_\lambda(U) = \bigcup_{\alpha<\lambda} \mathbf{L}_\alpha(U) \text{ for } \lambda \in \mathrm{Lim}.$$

Here $\mathbf{Def}_U(x)$ is the set of all subsets definable over the structure $(x, \in\restriction x^2, U \cap x)$ in the language $\mathcal{L}_\in(\mathsf{R})$ which contains an additional relation symbol R.

Definition 2.8 Let $r \subseteq V \times V$. For a set x let $r_x := \{y : \langle y, x\rangle \in r\}$. Set

$$\mathbf{wfp}(a, r) := \bigcap \{u \subseteq a : \forall x{\in}a\, [\{r_x \subseteq u \rightarrow x{\in}u],$$

i.e. the *well-founded part of* r *on* a. r is said to be *well-founded on* a if if $\mathbf{wfp}(a, r) = a$ holds or, in equivalent terms, if

$$\forall b \left[b \subseteq a \wedge b \neq \emptyset \rightarrow \exists x{\in}b\, \forall y{\in}b\, (\langle y, x\rangle {\notin} r)\right].$$

r is *well-founded* if r is well-founded on a for (any) some a satisfying $r \subseteq a \times a$.

Lemma 2.9 *The function* $(a, r) \mapsto \mathbf{wfp}(a, r)$ *is* Σ_1 *in* $\mathbf{KP}^r + \Sigma_1$-**Sep**.

The predicate of being well-founded is provably Δ_1 *in* $\mathbf{KP}^r + \Sigma_1$-**Sep**.

Proof: Let us work in $\mathbf{KP} + \Sigma_1$-**Sep**. We write $x{\prec}y$ for $\langle x, y\rangle \in r$. Define an operation F on the ordinals by Σ Recursion (Lemma 2.6):

$$\begin{aligned} F(\alpha) &= \{x \in a : \forall y{\in}a\, (y{\prec}x \rightarrow \exists \beta < \alpha\, y{\in}F(\beta))\} \\ &= \text{the set of all } x{\in}a \text{ such that } \{y{\in}a : y{\prec}x\} \subseteq \bigcup_{\beta<\alpha} F(\beta). \end{aligned}$$

Note that $\beta \leq \alpha$ implies $F(\beta) \subseteq F(\alpha)$. There is a set

$$b_0 = \{x{\in}a : \exists \alpha\, (x{\in}F(\alpha))\},$$

this being the place where we need Σ_1 Separation. Note that if $x{\in}a$ and $r_x \subseteq b_0$, then, by Σ Reflection, there exists α such that $r_x \subseteq F(\alpha)$ and hence $x{\in}F(\alpha+1)$, showing that $x \in b_0$. On the other hand, if a set $u \subseteq a$ satisfies $\forall x{\in}a\,[r_x \subseteq u \rightarrow x{\in}u]$, then one easily verifies by induction on α that $F(\alpha) \subseteq u$; whence $b_0 \subseteq u$. The upshot of the above is that $b_0 = \mathbf{wfp}(a, r)$.

To show that b_0 has a Σ_1 definition, observe first that by Σ Reflection there exists an ordinal ρ such that $b_0 = \{x{\in}a : \exists\alpha < \rho\,(x{\in}F(\alpha))\}$. Thus the desired Σ_1 definition of $\mathbf{wfp}(a, r)$ is given by

$$\begin{aligned} b = \mathbf{wfp}(a, r) \quad \leftrightarrow \quad & \exists f\,\exists\rho\,\big[\mathbf{dom}(f) = \rho + 1 \;\wedge\; \bigcup \mathbf{ran}(f) = b \;\wedge\; \forall\alpha \leq \rho \\ & f(\alpha) = \text{ the set of all } x{\in}a \text{ such that } \{y{\in}a :\, y{\prec}x\} \subseteq \bigcup_{\beta<\alpha} f(\beta) \\ & \wedge\; f(\rho) = \bigcup_{\beta<\rho} f(\beta). \end{aligned}$$

Further, since the predicate $b = \mathbf{wfp}(a, r)$ is Π_1 by definition, the above also yields the second assertion of the present lemma. □

Corollary 2.10 (**KP** + Σ_1-**Sep**) *If r is well-founded on a and*

$$\forall x{\in}a\,[\forall y{\in}a\,(\langle y, x\rangle \in r \rightarrow \phi(y)) \;\rightarrow\; \phi(x)],$$

then $\forall x{\in}a\,\phi(x)$.

Proof: By the previous lemma, we can pick a function f whose domain is an ordinal $\rho + 1$ such that $\bigcup_{\beta<\rho} f(\beta) = \mathbf{wfp}(a, r)$ and

$$\forall\alpha \leq \rho\; f(\alpha) = \text{ the set of all } x{\in}a \text{ such that } \{y{\in}a :\, y{\prec}x\}.$$

¿From $\forall x{\in}a\,[\forall y{\in}a\,(\langle y, x\rangle \in r \rightarrow \phi(y)) \;\rightarrow\; \phi(x)]$ one then deduces

$$\forall\alpha \leq \rho\,[\forall\beta < \alpha\,\forall y{\in}f(\beta)\phi(y) \;\rightarrow\; \forall x{\in}f(\alpha)\phi(x)].$$

Whence, using Foundation, it follows $\forall\alpha \leq \rho\,\forall x{\in}f(\alpha)\,\phi(x)$, and thus $\forall x{\in}\mathbf{wfp(a, r)}\,\phi(x)$. □

2.4 Projection functions

In this section we assume some results from recursion theory on admissible sets, in particular results about projectibility and characterization of stable ordinals. Details can be found in [3].

To begin with we recall some definitions from ordinal recursion theory.

Definition 2.11 An ordinal κ is said to be stable if $\mathbf{L}_\kappa \prec_1 \mathbf{L}$, i.e. $\mathbf{L}_\kappa$ is a Σ_1-elementary substructure of $\mathbf{L}$.

Let $\rho > \kappa$. κ is ρ-stable if $\mathbf{L}_\kappa \prec_1 \mathbf{L}_\rho$.

Another rendering of stability comes in terms of ordinal recursion theory (cf. [11], VIII.5.1):

κ is stable iff κ is closed under all ∞-partial recursive ordinal functions.

Likewise,

κ is ρ-stable iff κ is closed under all (∞,ρ)-partial recursive functions.

Definition 2.12 For $n \in \omega$ define

$$\begin{aligned} \boldsymbol{\sigma}_0 &:= \{\alpha : \alpha \text{ is } \Sigma_1 \text{ definable in } \mathbf{L} \text{ without parameters}\} \\ \boldsymbol{\sigma}_{n+1} &:= \{\alpha : \alpha \text{ is } \Sigma_1 \text{ definable in } \mathbf{L} \text{ in the parameters } \boldsymbol{\sigma}_0,\ldots,\boldsymbol{\sigma}_n\}. \end{aligned}$$

The existence of the ordinals $\boldsymbol{\sigma}_n$ follows via Σ Separation. $(n \mapsto \boldsymbol{\sigma}_n)_{n\in\omega}$ is a definable class function in $\mathbf{KP}^w+\Sigma_1$-**Sep** whereas in $\mathbf{KP}^r+\Sigma_1$-**Sep** we can only prove the existence of $\boldsymbol{\sigma}_n$ for external n.

Lemma 2.13 *$\mathbf{L}_{\boldsymbol{\sigma}_n} \prec_1 \mathbf{L}_{\boldsymbol{\sigma}_{n+1}}$, i.e. $\mathbf{L}_{\boldsymbol{\sigma}_n}$ is a Σ_1 elementary substructure of $\mathbf{L}_{\boldsymbol{\sigma}_{n+1}}$.*

Proof: [3], V.7.9. □

Lemma 2.14 *For each n there exists a function*

$$\mathbf{f}_n : \mathbf{L}_{\boldsymbol{\sigma}_n} \stackrel{1-1}{\to} \omega$$

which is Σ_1 definable on $\mathbf{L}_{\boldsymbol{\sigma}_n}$ with the aid of the parameters $\sigma_0,\ldots,\sigma_{n-1}$.

Proof: Combine [3], V.7.10 and [3], V.6.2. □

Definition 2.15 Let $\partial(k,m) := 5^{2^k\cdot 3^m}$. The functions $\mathbf{g}_n : \mathbf{L}_{\boldsymbol{\sigma}_n} \stackrel{1-1}{\to} \omega$ are defined by recursion on n as follows:

$$\begin{aligned} \mathbf{g}_0(x) &= \partial(0,\mathbf{f}_0(x)) \\ \mathbf{g}_{n+1}(x) &= \begin{cases} \mathbf{g}_n(x) & \text{if } x \in \mathbf{L}_{\boldsymbol{\sigma}_n} \\ \partial(n+1,\mathbf{f}_{n+1}(x)) & \text{if } x \notin \mathbf{L}_{\boldsymbol{\sigma}_n} \end{cases} \\ \mathbf{g}_\infty &:= \bigcup_{n\in\omega} \mathbf{g}_n. \end{aligned}$$

Lemma 2.16 *(i)* $\mathbf{g}_n \subseteq \mathbf{g}_{n+1}$.

(ii) $\mathbf{g}_\infty : \bigcup_{n\in\omega} \mathbf{L}_{\boldsymbol{\sigma}_n} \stackrel{1-1}{\to} \omega$.

(iii) $\mathbf{ran}(\mathbf{g}_\infty) \subseteq \{5^{k+1} : k \in \omega\}$.

Proof: Obvious. □

Remark 2.17 The existence of the functions $\mathbf{g}_n$ can be proved in $\mathbf{KP}^w + \Sigma_1$-**Sep**. $(n \mapsto \mathbf{g}_n)_{n<\omega}$ is a definable class function of $\mathbf{KP}^w + \Sigma_1$-**Sep**. Consequently, $\mathbf{g}_\infty$ is a definable class function of $\mathbf{KP}^w + \Sigma_1$-**Sep**, too.

In the case of $\mathbf{KP}^r + \Sigma_1$-**Sep** the functions $\mathbf{g}_n$ can be proved to exist for meta n only.

2.5 Relativizations

We will also have use for relativizations of the ordinals and functions of the previous subsection with regard to a set $U \subseteq \omega$.

Definition 2.18 For $n \in \omega$ define

$$\begin{aligned} \boldsymbol{\sigma}_0^U &:= \{\alpha : \ \alpha \text{ is } \Sigma_1 \text{ definable in } \mathbf{L}(U) \text{ without parameters}\} \\ \boldsymbol{\sigma}_{n+1}^U &:= \{\alpha : \ \alpha \text{ is } \Sigma_1 \text{ definable in } \mathbf{L}(U) \text{ in the parameters } \boldsymbol{\sigma}_0^U, \dots, \boldsymbol{\sigma}_n^U\}. \end{aligned}$$

$\mathbf{f}_n^U, \mathbf{g}_n^U$, and $\mathbf{g}_\infty^U$ are defined in a similar vein.

The relativizations of the results of the previous subsection can be proved in the same way and in the same theories as their unrelativized cousins.

3 Models

3.1 Applicative structures

Models for the applicative part of $\mathbf{T_0}$ (dubbed *applicative structures*) can be developed in very weak systems of set theory, since only recursively enumerable sets are required. The basic method for building models of $\mathbf{T_0}$ upon applicative structures goes back to [6, 104–107]. Here we shall extend a particular model construction over a "free" applicative structure introduced in [7, 3.3].

Since the models we use are already well described in the literature, cf. [7, 8, 16], we do not present the full details.

We start off with the pairing structure $\mathfrak{S}^{\text{pair}} = (\mathbf{S}, \pi, \pi_0, \pi_1, 0)$, where $\mathbf{S} = \omega$ and $\pi : \mathbf{S}^2 \to \mathbf{S} \setminus \{0\}$ is an injective (recursive) pairing function with (recursive) inverses π_0, π_1 such that $\pi_0(0) = \pi_1(0) = 0$. For technical reasons we moreover fix a special such function π, namely $\pi(x, y) = 2^x \cdot 3^y$. As its inverses, we fix π_0, π_1 where $\pi_0(z) = x$ and $\pi_1(z) = y$ if $z = 2^x \cdot 3^y$ and $\pi_0(z) = \pi_1(z) = z$ if z cannot be written in this form.

We call the base set $\mathbf{S}$ (and not ω) since we will have other "natural numbers" in this model and we want to avoid confusion between those two sets. Moreover, the intuition about $\mathbf{S}$ is that $\mathbf{S}$ consists of general objects and not only of the natural numbers.

For each $n \in \omega$, the representation $n^\circ \in \mathbf{S}$ of n in the structure $\mathfrak{S}^{\text{pair}}$ is defined inductively by $0^\circ = 0$, $(n+1)^\circ = \pi(0, n^\circ)$. The classification constant $\mathbb{N}$ will be interpreted as the set of all n°, where $n \in \omega$. More generally, for $X \subseteq \omega$ let $X^\circ = \{n^\circ : n \in X\}$. In the following, we use the codes

$$\begin{gathered} \mathbf{0} = 0, \mathbf{k} = 1^\circ, \mathbf{s} = 2^\circ, \mathbf{p} = 3^\circ, \mathbf{p}_0 = 4^\circ, \mathbf{p}_1 = 5^\circ, \mathbf{d} = 6^\circ, \mathbf{s}_\mathbb{N} = 7^\circ, \mathbf{p}_\mathbb{N} = 8^\circ, \\ \mathbf{i} = 9^\circ, \mathbf{j} = 10^\circ, \mathbf{lfp} = 11^\circ, \mathbb{N} = 12^\circ, \mathbf{I} = 13^\circ, \mathbf{U} = 14^\circ, \text{ and } \mathbf{c}_m = (15 + m)^\circ. \end{gathered}$$

The relation $\mathbf{App_S} \subseteq \mathbf{S}^3$ is inductively defined by the following clauses, where we use the abbreviations $xy \simeq z :\equiv \mathbf{App_S}(x, y, z)$, (x, y) for $\pi(x, y)$ and inductively $(x_1, x_2, \dots, x_{3+n}) := (x_1, (x_2, \dots, x_{3+n}))$.

- $\mathbf{k}x \simeq (\mathbf{k}, x), (\mathbf{k}, x)y \simeq x$
- $\mathbf{s}x \simeq (\mathbf{s}, x), (\mathbf{s}, x)y \simeq (\mathbf{s}, x, y)$.
- If $xz \simeq u, yz \simeq v$ and $uv \simeq w$, then $(\mathbf{s}, x, y)z \simeq w$.

- $\mathbf{p}x \simeq (\mathbf{p}, x), (\mathbf{p}, x)y \simeq \pi(x, y), \mathbf{p_0}x \simeq \pi_0(x), \mathbf{p_1}x = \pi_1(x)$
- $\mathbf{d}x \simeq (\mathbf{d}, x), (\mathbf{d}, x)y \simeq (\mathbf{d}, x, y), (\mathbf{d}, x, y)z_1 \simeq (\mathbf{d}, x, y, z_1)$
  $(\mathbf{d}, x, y, z_1)z_2 = \begin{cases} x & \text{if } z_1 = z_2 \\ y & \text{if } z_1 \neq z_2 \end{cases}$
- $\mathbf{s_{\mathbb{N}}}x \simeq (0, x),\ \mathbf{p_{\mathbb{N}}}(0, x) \simeq x$
- $\mathbf{c}_m x \simeq (\mathbf{c}_m, x),\ \mathbf{i}(x, y) \simeq (\mathbf{i}, x, y),\ \mathbf{j}(x, y) \simeq (\mathbf{j}, x, y),\ \mathbf{lfp}\, x \simeq (\mathbf{lfp}, x)$

This defines an applicative structure

$$\mathfrak{S}^{\text{app}} := \langle \mathbf{S}, \mathbf{App_S}, \mathbf{k}, \mathbf{s}, \mathbf{p}, \mathbf{p}_0, \mathbf{p}_1, \mathbf{d}, \mathbf{s_{\mathbb{N}}}, \mathbf{p_{\mathbb{N}}}, \mathbf{0}, \mathbb{N}, \mathbf{I}, (\mathbf{c}_m)_{m\in\omega}, \mathbf{i}, \mathbf{j}, \mathbf{lfp} \rangle$$

such that $\mathfrak{S}^{\text{app}}$ models the applicative part of $\mathbf{T_0}$. $\mathfrak{S}^{\text{app}}$ can be shown to be an element of $\mathbf{L}_{\omega_1^{\text{CK}}}$, actually, it can be shown to be coded by recursively enumerable sets.

Definition 3.1 a) Let $\mathbf{B} \subseteq \mathbf{S}$ be defined as $\mathbf{B} := \mathbf{S} \setminus (\pi[\mathbf{S}^2] \cup \{0\})$. Denoting the closure of $\mathbf{B}$ under π by $\text{Gen}(\mathbf{B})$, we see that $\pi : \mathbf{S}^2 \to \mathbf{S} \setminus \mathbf{B}$, $\text{Gen}(\mathbf{B} \cup \{0\}) = \mathbf{S}$ and $\pi_0(x) = \pi_1(x) = x$ for all $x \in \mathbf{B}$. We say that $\mathbf{B}$ is an *atomic base* for $\mathbf{S}$.

b) For $x \in \mathbf{S}$ let $\mathbf{supp_B}(x) \subseteq \mathbf{B}$ be defined by recursion on the definition of $\text{Gen}(\mathbf{B} \cup \{0\})$ by $\mathbf{supp_B}(x) = \{x\}$ for $x \in \mathbf{B}$, $\mathbf{supp_B}(0) = \emptyset$ and $\mathbf{supp_B}(\pi(x, y)) = \mathbf{supp_B}(x) \cup \mathbf{supp_B}(y)$.

c) Let

$$\mathbf{Aut}(\mathbf{B}) := \{\sigma : \sigma : \mathbf{B} \to \mathbf{B} \text{ and } \sigma \text{ is bijective}\}.$$

For $F \subseteq \mathbf{B}$ let

$$\mathbf{Aut}(\mathbf{B}/F) := \{\sigma \in \mathbf{Aut}(\mathbf{B}) :\ (\forall x \in F)(\sigma(x) = x)\}$$

Each $\sigma \in \mathbf{Aut}(\mathbf{B})$ induces a mapping $\hat{\sigma} : \mathbf{S} \to \mathbf{S}$ via $\hat{\sigma} \restriction \mathbf{B} = \sigma$, $\hat{\sigma}(0) = 0$, and $\hat{\sigma}(\pi(x, y)) = \pi(\hat{\sigma}(x), \hat{\sigma}(y))$. Since $\hat{\sigma}$ is uniquely determined by σ, we shall identify $\hat{\sigma}$ with σ.

Lemma 3.2 *a) If $xy \simeq z$, then $\mathbf{supp_B}(z) \subseteq \mathbf{supp_B}(x) \cup \mathbf{supp_B}(y)$.*
b) If $\sigma \in \mathbf{Aut}(\mathbf{B})$, then $xy \simeq z \Leftrightarrow \sigma(x)\sigma(y) \simeq \sigma(z)$.
c) If $\sigma \in \mathbf{Aut}(\mathbf{B}/F)$ and $\mathbf{supp_B}(x) \subseteq F$, then $\sigma(x) = x$.

Proof: a), b), c) can be proved by induction over the definition of $\mathbf{App_S}$. For details cf. [7, 3.3] or [16]. □

3.2 Iterating operators along well-founded relations

The inductive model construction to be introduced in the next subsection involves a clause which asks whether an operator can be iterated along certain well-orderings to produce a fixed point. Here we provide the pertinent terminology.

For each elementary $\phi(a, b_1, \ldots, b_m, A_1, \ldots, A_n)$ we shall write

$$\hat{x}.\phi(x, b_1, \ldots, b_m, A_1, \ldots, A_n) \text{ for } \mathbf{c}_\phi(b_1, \ldots, b_m, A_1, \ldots, A_n).$$

Given a family (A, g) of classifications over A, i.e. gx is a classification for all $x \mathrel{\mathring{\in}} A$, we set

$$\bigcup\{gx : x \mathrel{\mathring{\in}} A\} \;:=\; \hat{z}.\exists x\,(x, z) \mathrel{\mathring{\in}} B,$$

where $B \simeq \mathbf{j}(A, g)$.

Let $\mathbf{Pd}_{\mathbb{N}}(R, a) := \hat{y}.(y, a) \mathrel{\mathring{\in}} R \wedge y \mathrel{\mathring{\in}} \mathbb{N}$. Let $\mathbf{F}$ be an application term such for every operator Φ and every operation g which satisfies $\forall y \mathrel{\mathring{\in}} \mathbf{Pd}_{\mathbb{N}}(R, a)\, \exists Y g(R, \Phi, y) \simeq Y$,

$$\begin{aligned} \mathbf{F}(R, \Phi, a, g) \;&\simeq\; \bigcup\{g(R, \Phi, y) : y \mathrel{\mathring{\in}} \mathbf{Pd}_{\mathbb{N}}(R, a)\} \\ &\quad \cup \Phi(\bigcup\{g(R, \Phi, y) : y \mathrel{\mathring{\in}} \mathbf{Pd}_{\mathbb{N}}(R, a)\}). \end{aligned}$$

Definition 3.3 Using the recursion theorem (cf. [14], 2.2 or [6], 3.3), we find an application term $\mathbf{it}_{\mathbb{N}}$ such that

$$\forall xyz\, \mathbf{it}_{\mathbb{N}}(x, y, z) \simeq \mathbf{F}(x, y, z, \mathbf{it}_{\mathbb{N}}).$$

As a result, if R is well-founded and Φ is an operator, then $\mathbf{it}_{\mathbb{N}}(R, \Phi, a) \downarrow$ and

$$\begin{aligned} \mathbf{it}_{\mathbb{N}}(R, \Phi, a) \;&\simeq\; \bigcup\{\mathbf{it}_{\mathbb{N}}(R, \Phi, a) : y \mathrel{\mathring{\in}} \mathbf{Pd}_{\mathbb{N}}(R, a)\} \\ &\quad \cup \Phi(\bigcup\{\mathbf{it}_{\mathbb{N}}(R, \Phi, a) : y \mathrel{\mathring{\in}} \mathbf{Pd}_{\mathbb{N}}(R, a)\}). \end{aligned} \tag{1}$$

Finally, let

$$\mathbf{It}_{\mathbb{N}}(R, \Phi) := \bigcup\{\mathbf{it}_{\mathbb{N}}(R, \Phi, x) : x \mathrel{\mathring{\in}} \mathbb{N}\}.$$

Note that $\mathbf{it}_{\mathbb{N}}$ and $\mathbf{It}_{\mathbb{N}}$ are closed application terms of $\mathbf{T_0}$, and that (1) can be proved in $\mathbf{T_0}$.

3.3 Models for $\mathbf{T_0}$ and more

Definition 3.4 For λ a limit ordinal, let

$$\begin{aligned} \mathbf{well}(\alpha) \;&:=\; \{R \in \mathbf{L}_\alpha : \; R \textit{ is a well-ordering of a subset of } \omega\}; \\ \mathbf{WELL} \;&:=\; \{\mathbf{well}(\lambda) : \lambda \text{ limit}\}. \end{aligned}$$

In what follows variables $\mathcal{X}, \mathcal{Y}$ will always range over $\mathbf{WELL}$.

In this subsection we work in $\mathbf{KP}^r + \Sigma_1\textbf{-Sep}$. Starting from a set $\mathcal{X} \in \mathbf{WELL}$ and a $1-1$ mapping $\ell : \mathcal{X} \to \mathbf{B}$, a model $\mathfrak{M}^{\mathcal{X},\ell}$ of $\mathbf{T_0}$ is built inductively above this given family of well-orderings. In the model $\mathfrak{M}^{\mathcal{X},\ell}$, each set $A \in \mathcal{X}$ induces a classification named by $\ell(A)$. This method of constructing models of $\mathbf{T_0}$ originates with [6] and [8], however, it will be crucial for our purposes to extend the inductive definition of those models by a new clause (iv) to produce least fixed points for operators.

Definition 3.5 By induction on α we define structures

$$\mathfrak{M}^{\mathcal{X},\ell}_\alpha = \langle \mathbf{S}, \mathbf{CL}^{\mathcal{X},\ell}_\alpha, \mathring{\in}^{\mathcal{X},\ell}_\alpha, \mathbf{App_S}, \mathbf{k}, \mathbf{s}, \mathbf{p}, \mathbf{p}_0, \mathbf{p}_1, \mathbf{d}, \mathbf{s}_{\mathbb{N}}, \mathbf{p}_{\mathbb{N}}, \mathbf{0}, \mathbb{N}, \mathbf{I}, (\mathbf{c}_m)_{m\in\omega}, \mathbf{i}, \mathbf{j}, \mathbf{lfp}\rangle$$

extending $\mathfrak{S}^{\mathrm{app}}$. So we only have to define $\mathbf{CL}^{\mathcal{X},\ell}_\alpha \subseteq \mathbf{S}$ and $\mathring{\in}^{\mathcal{X},\ell}_\alpha \subseteq \mathbf{S} \times \mathbf{CL}^{\mathcal{X},\ell}_\alpha$.

a) $\mathbf{CL}_0^{\mathcal{X},\ell} = \{\mathbb{N}, \mathbf{I}\} \cup \mathbf{ran}(\ell)$.
$z \mathring{\in}_0^{\mathcal{X},\ell} \mathbb{N} \;:\Leftrightarrow\; z \in \mathbf{S}^\circ$.
$z \mathring{\in}_0^{\mathcal{X},\ell} \mathbf{I}$ iff z is in the closure of $\mathbf{S}^\circ$ under π (i.e. $z \in \mathrm{Gen}(\mathbf{S}^\circ)$).
For $R \in \mathcal{X}$ and $z \in \mathbf{S}$ set

$$z \mathring{\in}_0^{\mathcal{X},\ell} \ell(R) \;:\Leftrightarrow\; \exists n, m \in \omega\, [nRm \,\wedge\, z = (n^\circ, m^\circ)].$$

b) If $\alpha = \beta + 1$ is a successor, then let $\mathbf{CL}_\beta^{\mathcal{X},\ell} \subseteq \mathbf{CL}_\alpha^{\mathcal{X},\ell}$ and $\mathring{\in}_\beta^{\mathcal{X},\ell} \subseteq \mathring{\in}_\alpha^{\mathcal{X},\ell}$. In addition, new classifications at level α are generated by the following rules:

(i) If ϕ is an elementary formula with Gödelnumber m, then let $\mathbf{c}_m(\vec{x}, \vec{a}) \in \mathbf{CL}_\alpha^{\mathcal{X},\ell}$ for all $\vec{x} \in S$ and $\vec{a} \in \mathbf{CL}_\beta^{\mathcal{X},\ell}$. Further define

$$z \mathring{\in}_\alpha^{\mathcal{X},\ell} \mathbf{c}_m(\vec{x}, \vec{a}) \;:\Leftrightarrow\; \mathfrak{M}_\beta^{\mathcal{X},\ell} \models \phi[z, \vec{x}, \vec{a}].$$

(ii) If $a \in \mathbf{CL}_\beta^{\mathcal{X},\ell}$, $f \in \mathbf{S}$ and $\mathfrak{M}_\beta^{\mathcal{X},\ell} \models \forall x \mathring{\in} a \exists Y (fx \simeq Y)$, then let $\mathbf{j}(f, a) \in \mathbf{CL}_\alpha^{\mathcal{X},\ell}$ and, for $z \in \mathbf{S}$,

$$z \mathring{\in}_\alpha^{\mathcal{X},\ell} \mathbf{j}(f, a) \;:\Leftrightarrow\; \mathfrak{M}_\beta^{\mathcal{X},\ell} \models \exists x \mathring{\in} a \exists y \mathring{\in} fx (z = (x, y)).$$

(iii) For $a, b \in \mathbf{CL}_{M_0,\beta}$ let $\mathbf{i}(a, b) \in \mathbf{CL}_{M_0,\alpha}$ and for $z \in S$ let

$$z \mathring{\in}_\alpha^{\mathcal{X},\ell} \mathbf{i}(a, b) \;:\Leftrightarrow\; \forall X \subseteq \mathbf{S} (\mathrm{Prog}(a, b, X) \to z \in X)$$

where $\mathrm{Prog}(a, b, X) :\equiv$

$$\forall x \mathring{\in}_\beta^{\mathcal{X},\ell} a \left(\forall y \mathring{\in} \mathbf{S}\, [(y, x) \mathring{\in}_\beta^{\mathcal{X},\ell} b \to y \in X] \to x \in X\right).$$

(iv) Let $f \in \mathbf{S}$. Suppose there is a well-ordering R of ω in $\mathcal{X}$ such that the following hold:

- $\mathfrak{M}_\beta^{\mathcal{X},\ell} \models \forall x\, \exists Y\, \mathbf{it}_\mathbb{N}(\ell(R), f, x) \simeq Y$.
- $\mathfrak{M}_\beta^{\mathcal{X},\ell} \models \exists Z\, \exists X\, [\mathbf{It}_\mathbb{N}(\ell(R), f) \simeq Z \;\wedge\; f(Z) \simeq X \;\wedge\; X \mathring{\subseteq} Z]$.
- For all $S \in \mathcal{X}$ if $S <_\mathbf{L} R$, then
  (a) $\mathfrak{M}_\beta^{\mathcal{X},\ell} \models \forall x\, \exists Y\, \mathbf{it}_\mathbb{N}(\ell(S), f, x) \simeq Y$
  (b) $\mathfrak{M}_\beta^{\mathcal{X},\ell} \models \exists Z\, \exists X\, [\mathbf{It}_\mathbb{N}(\ell(S), f) \simeq Z \;\wedge\; f(Z) \simeq X \;\wedge\; X \mathring{\not\subseteq} Z]$.

Then $\mathbf{lfp}(f) \in \mathbf{CL}_\alpha^{\mathcal{X},\ell}$ and

$$z \mathring{\in}_\alpha^{\mathcal{X},\ell} \mathbf{lfp}(f) \;:\Leftrightarrow\; \mathfrak{M}_\beta^{\mathcal{X},\ell} \models z \mathring{\in} \mathbf{It}_\mathbb{N}(\ell(R), f).$$

c) If α is a limit ordinal, let $\mathbf{CL}_\alpha^{\mathcal{X},\ell} := \bigcup_{\beta<\alpha} \mathbf{CL}_\beta^{\mathcal{X},\ell}$ and $\mathring{\in}_\alpha^{\mathcal{X},\ell} := \bigcup_{\beta<\alpha} \mathring{\in}_\beta^{\mathcal{X},\ell}$.

Finally, set $\mathbf{CL}^{\mathcal{X},\ell} := \bigcup_\alpha \mathbf{CL}_\alpha^{\mathcal{X},\ell}$ and $\mathring{\in}^{\mathcal{X},\ell} := \bigcup_\alpha \mathring{\in}_\alpha^{\mathcal{X},\ell}$. Let

$$\mathfrak{M}^{\mathcal{X},\ell} := \langle \mathbf{S}, \mathbf{CL}^{\mathcal{X},\ell}, \mathring{\in}^{\mathcal{X},\ell}, \mathbf{App_S}, \mathbf{k}, \mathbf{s}, \mathbf{p}, \mathbf{p}_0, \mathbf{p}_1, \mathbf{d}, \mathbf{s}_\mathbb{N}, \mathbf{p}_\mathbb{N}, \mathbf{0}, \mathbb{N}, \mathbf{I}, (\mathbf{c}_m)_{m\in\omega}, \mathbf{i}, \mathbf{j}, \mathbf{lfp} \rangle$$

Remark 3.6 (i) Regarding part (iv) of Definition 3.5,b), note that the well-ordering R is uniquely determined by being the $<_\mathbf{L}$-least satisfying the pertaining conditions.

(ii) Note that $e\overset{\circ}{\in}^{\mathcal{X},\ell}\mathbf{I}$ holds if e is any of the constants $\mathbf{k}, \mathbf{s}, \mathbf{p}, \mathbf{p}_0, \mathbf{p}_1, \mathbf{d}, \mathbf{s}_{\mathbb{N}}, \mathbf{p}_{\mathbb{N}}, \mathbf{0}, \mathbb{N}, \mathbf{I}$, $(\mathbf{c}_m)_{m\in\omega}$,$\mathbf{i}$,$\mathbf{j}$,$\mathbf{lfp}$, or $e \in \mathbf{S}^\circ$. Moreover, if $x, y\overset{\circ}{\in}^{\mathcal{X},\ell}\mathbf{I}$ and $xy \simeq z$, then $z\overset{\circ}{\in}^{\mathcal{X},\ell}\mathbf{I}$.

Lemma 3.7 *The following are provable in* $\mathbf{KP}^r + \Sigma_1$-$\mathbf{Sep}$*:*

(i) For all α*,* $\mathbf{well}(\alpha)$ *is a set. The map* $\alpha \mapsto \mathbf{well}(\alpha)$ *is* Σ_1*.*

(ii) $\mathfrak{M}^{\mathcal{X},\ell}$ *is a set.*

(iii) If $\mathcal{X} = \mathbf{well}(\rho)$*, then the the map*

$$\big(\lambda \mapsto \mathfrak{M}^{\mathbf{well}(\lambda),\ell\restriction\mathbf{well}(\lambda)}\big)_{\lambda<\rho}$$

is Σ_1 *(in the parameter* ℓ*).*

Proof: (i): The class $\mathbf{well}(\alpha)$ is Δ_1 definable in $\mathbf{KP}^r + \Sigma_1$-$\mathbf{Sep}$ by 2.9; thus, using Δ_1 Separation, $\mathbf{KP}^r + \Sigma_1$-$\mathbf{Sep}$ proves that $\mathbf{well}(\alpha)$ is a set. As a result, the function $\alpha \mapsto \mathbf{well}(\alpha)$ is Σ_1.

(ii): First, notice that the function $\alpha \mapsto \langle \overset{\circ}{\in}_\alpha^{\mathcal{X},\ell}, \mathbf{CL}_\alpha^{\mathcal{X},\ell}\rangle$ is Σ_1. This is immediate for the clauses (i) and (ii) of Definition 3.5. For the clauses (iii) and (iv) which involve the well-founded part of a relation on a set and the notion of being well-founded, respectively, we draw on Lemma 2.9.

The applicative part of $\mathfrak{M}^{\mathcal{X},\ell}$ is clearly a set. $\mathbf{CL}^{\mathcal{X},\ell}$ is a set by Σ_1-$\mathbf{Sep}$ since $\mathbf{CL}^{\mathcal{X},\ell} = \{a \in \mathbf{S} :\ \exists\alpha\,(a \in \mathbf{CL}_\alpha^{\mathcal{X},\ell})\,\}$. Likewise, $\overset{\circ}{\in}^{\mathcal{X},\ell}$ is a set by Σ_1-$\mathbf{Sep}$ as

$$\overset{\circ}{\in}^{\mathcal{X},\ell} = \{\langle x, a\rangle \in \mathbf{S}\times\mathbf{S} :\ \exists\alpha\,(\langle x, a\rangle \in \overset{\circ}{\in}_\alpha^{\mathcal{X},\ell})\,\}.$$

(iii): For $\lambda < \rho$, put $\mathcal{X}_\lambda := \mathbf{well}(\lambda)$ and $\ell_\lambda := \ell\restriction\mathbf{well}(\lambda)$. $\mathbf{CL}^{\mathcal{X}_\lambda,\ell_\lambda}$ is a set by (i) and (ii), and we can use Σ Replacement to obtain a function $f : \mathbf{CL}^{\mathcal{X}_\lambda,\ell_\lambda} \to \mathbf{ON}$ defined by

$$f(a) \ = \ \textit{the least}\ \alpha.\, a \in \mathbf{CL}_\alpha^{\mathcal{X}_\lambda,\ell_\lambda}.$$

As a result, there exists β such that $\mathbf{CL}^{\mathcal{X}_\lambda,\ell_\lambda} = \mathbf{CL}_{<\beta}^{\mathcal{X}_\lambda,\ell_\lambda} = \mathbf{CL}_\beta^{\mathcal{X}_\lambda,\ell_\lambda}$. Whence,

$$\lambda \ \mapsto\ \textit{the least}\ \beta.\, \mathbf{CL}_{<\beta}^{\mathcal{X}_\lambda,\ell_\lambda} = \mathbf{CL}_\beta^{\mathcal{X}_\lambda,\ell_\lambda}$$

is Σ_1. Therefore $\lambda \mapsto \mathfrak{M}^{\mathcal{X}_\lambda,\ell_\lambda}$ is Σ_1 too. □

Lemma 3.8 *Let* $\mathcal{X}, \mathcal{Y} \in \mathbf{WELL}$*,* $\mathcal{X} \subseteq \mathcal{Y}$*,* $\ell : \mathcal{X} \overset{1-1}{\to} \mathbf{B}$*,* $\wp : \mathcal{Y} \overset{1-1}{\to} \mathbf{B}$*, and* $\wp\restriction\mathcal{X} = \ell$*. Then:*

a) For all α*,* $\mathbf{CL}_\alpha^{\mathcal{X},\ell} \subseteq \mathbf{CL}_\alpha^{\mathcal{Y},\wp}$*.*
b) If $a \in \mathbf{CL}_\alpha^{\mathcal{X},\ell}$*, then for all* $x \in \mathbf{S}$*,* $x\overset{\circ}{\in}_\alpha^{\mathcal{X},\ell} a \Leftrightarrow x\overset{\circ}{\in}_\alpha^{\mathcal{Y},\wp} a$*.*
c) If $\alpha \le \beta$ *and* $a \in \mathbf{CL}_\alpha^{\mathcal{X},\ell}$*, then for all* $x \in \mathbf{S}$*,* $x\overset{\circ}{\in}_\alpha^{\mathcal{X},\ell} a \Leftrightarrow x\overset{\circ}{\in}_\beta^{\mathcal{X},\ell} a$*.*
d) If $\alpha \le \beta$ *and* $a \in \mathbf{CL}_\alpha^{\mathcal{X},\ell}$*, then for all* $x \in \mathbf{S}$*,* $x\overset{\circ}{\in}_\alpha^{\mathcal{X},\ell} a \Leftrightarrow x\overset{\circ}{\in}_\beta^{\mathcal{Y},\wp} a$*.*

Proof: *a)* and *b)* are proved simultaneously by induction on α. *c)* is proved by induction on β. *d)* follows from *a),b),c)*. As the proofs are routine, we restrict ourselves to the most interesting case in *a),b)*. Let $\alpha = \beta + 1$. Suppose $a \in \mathbf{CL}_\alpha^{\mathcal{X},\ell}$ via clause (iv). Then $a = \mathbf{lfp}(f)$ for some $f \in \mathbf{S}$ and there exists a well-ordering R of ω in $\mathcal{X}$ such that the following hold:

- $\mathfrak{M}^{\mathcal{X},\ell}_{\beta} \models \forall x\, \exists Y\, \mathbf{it}_{\mathbb{N}}(\ell(R), f, x) \simeq Y$.
- $\mathfrak{M}^{\mathcal{X},\ell}_{\beta} \models \exists Z\, \exists X\, [\mathbf{It}_{\mathbb{N}}(\ell(R), f) \simeq Z \;\wedge\; f(Z) \simeq X \;\wedge\; X \mathrel{\mathring{\subsetneq}} Z]$.
- For all $S \in \mathcal{X}$ if $S <_{\mathbf{L}} R$, then
  - (a) $\mathfrak{M}^{\mathcal{X},\ell}_{\beta} \models \forall x\, \exists Y\, \mathbf{it}_{\mathbb{N}}(\ell(S), f, x) \simeq Y$
  - (b) $\mathfrak{M}^{\mathcal{X},\ell}_{\beta} \models \exists Z\, \exists X\, [\mathbf{It}_{\mathbb{N}}(\ell(S), f) \simeq Z \;\wedge\; f(Z) \simeq X \;\wedge\; X \mathring{\not\subseteq} Z]$.

Let $\mathcal{X} = \mathbf{well}(\lambda)$ and $\mathcal{Y} = \mathbf{well}(\lambda')$. Since $\mathcal{X} \subseteq \mathcal{Y}$, we have $\lambda \le \lambda'$. Hence, if $S' \in \mathcal{Y}$ and $S' <_{\mathbf{L}} R$, then $S' \in \mathcal{X}$ since λ is a limit. Thus, using the induction hypothesis and noting that $\ell(R) = \wp(R)$, we obtain

a) $\mathfrak{M}^{\mathcal{Y},\wp}_{\beta} \models \forall x\, \exists Y\, \mathbf{it}_{\mathbb{N}}(\wp(R), f, x) \simeq Y$.
b) $\mathfrak{M}^{\mathcal{Y},\wp}_{\beta} \models \exists Z\, \exists X\, [\mathbf{It}_{\mathbb{N}}(\wp(R), f) \simeq Z \;\wedge\; f(Z) \simeq X \;\wedge\; X \mathrel{\mathring{\subsetneq}} Z]$.
c) For all $S \in \mathcal{X}$ if $S <_{\mathbf{L}} R$, then
  (i) $\mathfrak{M}^{\mathcal{Y},\wp}_{\beta} \models \forall x\, \exists Y\, \mathbf{it}_{\mathbb{N}}(\wp(S), f, x) \simeq Y$
  (ii) $\mathfrak{M}^{\mathcal{Y},\wp}_{\beta} \models \exists Z\, \exists X\, [\mathbf{It}_{\mathbb{N}}(\wp(S), f) \simeq Z \;\wedge\; f(Z) \simeq X \;\wedge\; X \mathring{\not\subseteq} Z]$.

As a result of the above, $a = \mathbf{lfp}(f) \in \mathbf{CL}^{\mathcal{Y},\wp}_{\alpha}$ and thus, by induction hypothesis,

$$z \mathring{\in}^{\mathcal{X},\ell}_{\alpha} a \Leftrightarrow \mathfrak{M}^{\mathcal{X},\ell}_{\beta} \models z \mathring{\in} \mathbf{It}_{\mathbb{N}}(\ell(R), f) \Leftrightarrow \mathfrak{M}^{\mathcal{Y},\wp}_{\beta} \models z \mathring{\in} \mathbf{It}_{\mathbb{N}}(\wp(R), f) \Leftrightarrow z \mathring{\in}^{\mathcal{Y},\wp}_{\alpha} a.$$

□

Remark 3.9 In the following we may omit the indices α in the predicate $x \mathring{\in}^{\mathcal{X},\ell}_{\alpha} a$ as the preceding lemma shows that the relation is independent of these parameters as long as $a \in \mathbf{CL}^{\mathcal{X},\ell}_{\alpha}$.

Corollary 3.10 *Let $\mathcal{X}, \mathcal{Y} \in \mathbf{WELL}$, $\mathcal{X} \subseteq \mathcal{Y}$, $\ell : \mathcal{X} \overset{1-1}{\to} \mathbf{B}$, $\wp : \mathcal{Y} \overset{1-1}{\to} \mathbf{B}$, and $\wp \restriction \mathcal{X} = \ell$. Then:*

a) $\mathbf{CL}^{\mathcal{X},\ell} \subseteq \mathbf{CL}^{\mathcal{Y},\wp}$.
b) If $a \in \mathbf{CL}^{\mathcal{X},\ell}$, then for all $x \in \mathbf{S}$, $x \mathring{\in}^{\mathcal{X},\ell} a \;\Leftrightarrow\; x \mathring{\in}^{\mathcal{Y},\wp} a$.

Convention. From now on, when considering a structure $\mathfrak{M}^{\mathcal{X},\ell}$, upper case letters $X, Y, Z, \ldots$ will be understood to vary over $\mathbf{CL}^{\mathcal{X},\ell}$; this will save space and is in keeping with the syntax of $\mathbf{T_0}$.

Definition 3.11 The collection of Σ^{EM}-formulae is the smallest set of $\mathbf{T_0}(\mathbf{I})$-formulae which contains the quantifier-free formulae and is closed under $\vee, \wedge, \forall x, \exists x$, and existential quantification over classifications.

Corollary 3.12 (Σ^{EM}-persistence) *Let $\mathcal{X}, \mathcal{Y} \in \mathbf{WELL}$, $\mathcal{X} \subseteq \mathcal{Y}$, $\ell : \mathcal{X} \overset{1-1}{\to} \mathbf{B}$, $\wp : \mathcal{Y} \overset{1-1}{\to} \mathbf{B}$, and $\wp \restriction \mathcal{X} = \ell$. If ψ is a Σ^{EM}-sentence (with parameters) whose classification parameters are in $\mathbf{CL}^{\mathcal{X},\ell}$, then*

$$\mathfrak{M}^{\mathcal{X},\ell} \models \psi \;\Rightarrow\; \mathfrak{M}^{\mathcal{Y},\wp} \models \psi.$$

Proof: This is proved by induction on the generation of ψ. The atomic case follows from 3.10 and the fact that $\mathfrak{M}^{\mathcal{X},\ell}$ and $\mathfrak{M}^{\mathcal{Y},\wp}$ share the same applicative part. If ψ is of the form $\psi_0 \wedge \psi_1$ or $\psi_0 \wedge \psi_1$, then the assertion is immediate by induction hypothesis. If ψ is of the form $\exists x \psi_0(x)$ or $\forall x \psi_0(x)$, then the assertion follows from the induction hypothesis and the fact that these quantifiers range over the same realm of objects in both structures. Finally, let ψ be of the form $\exists U \psi_0(U)$. Then there exists $Z \in \mathbf{CL}^{\mathcal{X},\ell}$ such that $\mathfrak{M}^{\mathcal{X},\ell} \models \psi_0[Z]$. Then $Z \in \mathbf{CL}^{\mathcal{Y},\wp}$ and $\mathfrak{M}^{\mathcal{Y},\wp} \models \psi$ follows by induction hypothesis. □

Definition 3.13 Given $\mathcal{Y} \in \mathbf{WELL}$, $\imath : \mathcal{Y} \overset{1-1}{\to} \mathbf{B}$, $\hbar : \mathcal{Y} \overset{1-1}{\to} \mathbf{B}$, and $\sigma \in \mathbf{Aut}(\mathbf{B})$, we say that σ is an *isomorphism* between $\mathfrak{M}^{\mathcal{Y},\imath}$ and $\mathfrak{M}^{\mathcal{Y},\hbar}$, written

$$\sigma : \mathfrak{M}^{\mathcal{Y},\imath} \cong \mathfrak{M}^{\mathcal{Y},\hbar},$$

if $\hbar = \sigma \circ \imath$, i.e., for all $u \in \mathcal{Y}$, $\hbar(u) = \sigma(\imath(u))$.

Proposition 3.14 *If* $\sigma : \mathfrak{M}^{\mathcal{Y},\imath} \cong \mathfrak{M}^{\mathcal{Y},\hbar}$, *then*

$$\mathfrak{M}^{\mathcal{Y},\imath} \models \phi[\vec{x}, \vec{X}] \;\Leftrightarrow\; \mathfrak{M}^{\mathcal{Y},\hbar} \models \phi[\sigma(\vec{x}), \sigma(\vec{X})]$$

holds for all formulae $\phi(\vec{u}, \vec{U})$ *of* $\mathbf{T_0}(\mathbf{I})$, $\vec{x} \in \mathbf{S}$, *and* $\vec{X} \in \mathbf{CL}^{\mathcal{Y},\imath}$. *The latter comprises that*

$$\mathbf{CL}^{\mathcal{X},\hbar} = \{\sigma(Y) : Y \in \mathbf{CL}^{\mathcal{X},\imath}\}.$$

Proof: One first shows by induction on α:

a) $X \in \mathbf{CL}_{\alpha}^{\mathcal{Y},\imath} \Leftrightarrow \sigma(X) \in \mathbf{CL}_{\alpha}^{\mathcal{Y},\hbar}$;
b) If $X \in \mathbf{CL}_{\alpha}^{\mathcal{Y},\imath}$, then for all x,

$$x \mathring{\in}_{\alpha}^{\mathcal{Y},\imath} X \;\Leftrightarrow\; \sigma(x) \mathring{\in}_{\alpha}^{\mathcal{Y},\hbar} \sigma(X).$$

If $\alpha > 0$, then the assertions follow immediately from the induction hypotheses. The only interesting case is $\alpha = 0$. Let $X \in \mathbf{CL}_0^{\mathcal{Y},\imath}$. If $X = \mathbb{N}$, then $\sigma(X) = X \in \mathbf{CL}_0^{\mathcal{Y},\hbar}$ and $x \mathring{\in}^{\mathcal{Y},\imath} \mathbb{N} \Leftrightarrow \sigma(x) \mathring{\in}^{\mathcal{Y},\hbar} \sigma(\mathbb{N})$ follows from the fact that for all $x \in \mathbf{S}^{\circ}$, $x = \sigma(x)$.

Now suppose $X = \imath(R)$, where $R \in \mathcal{Y}$. As $\sigma(X) = \sigma(\imath(R)) = \hbar(R)$, it follows $\sigma(X) \in \mathbf{CL}_{\alpha}^{\mathcal{Y},\hbar}$. Further, since for all $n, m \in \omega$, $\sigma((n^{\circ}, m^{\circ})) = (n^{\circ}, m^{\circ})$, we get

$$\begin{aligned} z \mathring{\in}_0^{\mathcal{Y},\imath} \imath(R) &\Leftrightarrow \exists n, m \in \omega\, [nRm \wedge z = (n^{\circ}, m^{\circ})] \\ &\Leftrightarrow \exists n, m \in \omega\, [nRm \wedge \sigma(z) = (n^{\circ}, m^{\circ})] \\ &\Leftrightarrow \sigma(z) \mathring{\in}_0^{\mathcal{Y},\hbar} \hbar(R) \\ &\Leftrightarrow \sigma(z) \mathring{\in}_0^{\mathcal{Y},\hbar} \sigma(\imath(R)). \end{aligned}$$

Conversely, if $\sigma(X) \in \mathbf{CL}_0^{\mathcal{Y},\hbar}$ and $\sigma(X) \neq \mathbb{N}$, then $\sigma(X) = \hbar(R)$ for some $R \in \mathcal{Y}$. As $\hbar = \sigma \circ \imath$ and all maps are injective, this implies $X = \imath(R)$; thus $X \in \mathbf{CL}^{\mathcal{Y},\imath}$.

As a result of the above, we have

a) $X \in \mathbf{CL}^{\mathcal{Y},\imath} \Leftrightarrow \sigma(X) \in \mathbf{CL}^{\mathcal{Y},\hbar}$;
b) If $X \in \mathbf{CL}^{\mathcal{Y},\imath}$, then for all x, $x \mathring{\in}^{\mathcal{Y},\imath} X \Leftrightarrow \sigma(x) \mathring{\in}^{\mathcal{Y},\hbar} \sigma(X)$.

The desired assertion is now (straightforwardly) proved by formula induction on ϕ. Note that if ϕ is an atomic formula other than $t \mathrel{\mathring{\in}} X$, then this follows from Lemma 3.2, taking into account that $\mathfrak{M}^{\mathcal{Y},\imath}$ and $\mathfrak{M}^{\mathcal{Y},\hbar}$ both have the same applicative structure. $\Box$

Notation Let $\mathcal{Y} \in \mathbf{WELL}$ and $\imath : \mathcal{Y} \stackrel{1-1}{\to} \mathbf{B}$. If $\mathcal{Z} \subseteq \mathcal{Y}$ we use the notation $\imath[\mathcal{Z}]$ for the set $\{\imath(a) : a \in \mathcal{Z}\}$.

If $\vec{x}$ is a tuple $x_1, \ldots, x_n$, $\sigma(\vec{x})$ stands for $\sigma(x_1), \ldots, \sigma(x_n)$.

Lemma 3.15 *Let $\mathcal{Z} \subseteq \mathcal{Y} \in \mathbf{WELL}$, $\imath : \mathcal{Y} \stackrel{1-1}{\to} \mathbf{B}$ and $\hbar : \mathcal{Y} \stackrel{1-1}{\to} \mathbf{B}$. Further, suppose that $\imath \restriction \mathcal{Z} = \hbar \restriction \mathcal{Z}$ and $\imath[\mathcal{Y} \setminus \mathcal{Z}] \cap \hbar[\mathcal{Y} \setminus \mathcal{Z}] = \emptyset$. Then, letting $F := \imath[\mathcal{Z}]$, there exists $\sigma \in \mathbf{Aut}(\mathbf{B}/F)$, such that $\sigma = \sigma^{-1}$ and*

$$\sigma : \mathfrak{M}^{\mathcal{Y},\imath} \cong \mathfrak{M}^{\mathcal{Y},\hbar}.$$

Proof: Define

$$\sigma(x) := \begin{cases} x & \text{if } x \notin \imath[\mathcal{Y} \setminus \mathcal{Z}] \cup \hbar[\mathcal{Y} \setminus \mathcal{Z}] \\ \hbar(\imath^{-1}(x)) & \text{if } x \in \imath[\mathcal{Y} \setminus \mathcal{Z}] \\ \imath(\hbar^{-1}(x)) & \text{if } x \in \hbar[\mathcal{Y} \setminus \mathcal{Z}] \end{cases}$$

Obviously, we have $\forall x \in F\, \sigma(x) = x$ and $\sigma = \sigma^{-1}$. Further, we have $\sigma \circ \imath = \hbar$ as well as $\sigma \circ \hbar = \imath$. $\Box$

Lemma 3.16 *Let $\mathcal{Z} \subseteq \mathcal{Y} \in \mathbf{WELL}$, $\imath : \mathcal{Y} \stackrel{1-1}{\to} \mathbf{B}$ and $\jmath : \mathcal{Y} \stackrel{1-1}{\to} \mathbf{B}$. Furthermore, suppose that $\imath \restriction \mathcal{Z} = \jmath \restriction \mathcal{Z}$ and $\mathbf{B} \setminus (\imath[\mathcal{Y}] \cup \jmath[\mathcal{Y}])$ is infinite. Then, for all formulae $\phi(\vec{u}, \vec{U}, W)$, $\vec{x} \in \mathbf{S}$ and $\vec{X} \in \mathbf{CL}^{\mathcal{Y},\imath}$ satisfying $(\mathbf{supp_B}(\vec{x}) \cup \mathbf{supp_B}(\vec{X})) \cap (\imath[\mathcal{Y} \setminus \mathcal{Z}] \cup \jmath[\mathcal{Y} \setminus \mathcal{Z}]) = \emptyset$,*

$$\forall R \in \mathcal{Y}\ \big(\ \mathfrak{M}^{\mathcal{Y},\imath} \models \phi[\vec{x}, \vec{X}, \imath(R)] \ \Leftrightarrow\ \mathfrak{M}^{\mathcal{Y},\jmath} \models \phi[\vec{x}, \vec{X}, \jmath(R)]\ \big).$$

In particular, the latter holds if $\mathbf{supp_B}(\vec{x}) \cup \mathbf{supp_B}(\vec{X}) \subseteq \imath[\mathcal{Z}]$.

Proof: Since $\mathbf{B} \setminus (\imath[\mathcal{Y}] \cup \jmath[\mathcal{Y}])$ is infinite, we can find an $A \subseteq \mathbf{B} \setminus (\imath[\mathcal{Y}] \cup \jmath[\mathcal{Y}])$ and a bijection $\partial : (\mathcal{Y} \setminus \mathcal{Z}) \to A$. Now define $\hbar$ by $\hbar \restriction \mathcal{Z} := \imath \restriction \mathcal{Z}$ and $\hbar \restriction (\mathcal{Y} \setminus \mathcal{Z}) := \partial$.

By Lemma 3.15, we may select $\rho, \tau \in \mathbf{Aut}(\mathbf{B}/F)$ such that $\rho : \mathfrak{M}^{\mathcal{Y},\imath} \cong \mathfrak{M}^{\mathcal{Y},\hbar}$ and $\tau : \mathfrak{M}^{\mathcal{Y},\jmath} \cong \mathfrak{M}^{\mathcal{Y},\hbar}$. As a result, $\tau^{-1} \circ \rho \in \mathbf{Aut}(\mathbf{B}/F)$, $\tau^{-1} \circ \rho : \mathfrak{M}^{\mathcal{Y},\imath} \cong \mathfrak{M}^{\mathcal{Y},\jmath}$, and $\tau^{-1} \circ \rho(\imath(R)) = \jmath(R)$. Any $b \in F$ satisfies $\rho(b) = b = \tau(b)$. Hence, letting $\sigma := \tau^{-1} \circ \rho$, we obtain $\sigma(b) = b$ for any $b \in \mathbf{B}$ with $\mathbf{supp_B}(b) \subseteq F$. Thus the desired assertion follows by Proposition 3.14. $\Box$

3.4 The model $\widetilde{\mathfrak{M}}$

In this subsection we introduce the structures $\mathfrak{M}^{\mathbf{V}_n,\mathbf{h}_n}$ and the class model $\widetilde{\mathfrak{M}}$ and show that $\widetilde{\mathfrak{M}}$ satisfies Join and $\mathbf{UMID_I}$. Here we are going to reason in $\mathbf{KP}^w + \Sigma_1$-$\mathbf{Sep}$ unless indicated otherwise.

Definition 3.17 Let $\mathbf{V}_n := \mathbf{WELL} \cap \mathbf{L}_{\boldsymbol{\sigma}_n}$, $\mathbf{h}_n := \mathbf{g}_n \restriction \mathbf{V}_n$ and $\mathbf{h}_\infty := \bigcup_{n \in \omega} \mathbf{h}_n$. Put

$$\widetilde{\mathfrak{M}} := \bigcup_n \mathfrak{M}^{\mathbf{V}_n,\mathbf{h}_n}$$

i.e. $\widetilde{\mathfrak{M}} := \langle \mathbf{S}, \mathbf{CL}_{\widetilde{\mathfrak{M}}}, \mathring{\in}_{\widetilde{\mathfrak{M}}}, \mathbf{App_S}, \mathbf{k}, \mathbf{s}, \mathbf{p}, \mathbf{p}_0, \mathbf{p}_1, \mathbf{d}, \mathbf{s}_{\mathbb{N}}, \mathbf{p}_{\mathbb{N}}, \mathbf{0}, \mathbb{N}, \mathbf{I}, (\mathbf{c}_m)_{m \in \omega}, \mathbf{i}, \mathbf{j}, \mathbf{lfp} \rangle$,

where $\mathbf{CL}_{\widetilde{\mathfrak{M}}} := \bigcup_n \mathbf{CL}^{\mathbf{V}_n,\mathbf{h}_n}$ and $\mathring{\in}_{\widetilde{\mathfrak{M}}} := \bigcup_n \mathring{\in}^{\mathbf{V}_n,\mathbf{h}_n}$.

Lemma 3.18 *Let $\psi(\vec{u},\vec{U})$ be an arbitrary formula and $\vec{X} \in \mathbf{CL}^{\mathbf{V}_{n_0},\mathbf{h}_{n_0}}$. If $\mathfrak{M}^{\mathbf{V}_{n_0},\mathbf{h}_{n_0}} \models \psi[\vec{x},\vec{X}]$, then there exists $\rho \in \mathbf{Aut}(\mathbf{B}/\mathbf{ran}(\mathbf{h}_{n_0}))$ such that $\rho = \rho^{-1}$, $\rho(\vec{X}) \in \mathbf{CL}^{\mathbf{V}_{n_0},\mathbf{h}_{n_0}}$, $(\mathbf{supp_B}(\rho(\vec{x})) \cup \mathbf{supp_B}(\rho(\vec{X}))) \cap \mathbf{ran}(\mathbf{h}_\infty) \subseteq \mathbf{ran}(\mathbf{h}_{n_0})$, and $\mathfrak{M}^{\mathbf{V}_{n_0},\mathbf{h}_{n_0}} \models \psi[\rho(\vec{x}),\rho(\vec{X})]$.*

Proof: Let

$$G := ((\mathbf{supp_B}(\vec{x}) \cup \mathbf{supp_B}(\vec{X})) \cap \mathbf{ran}(\mathbf{h}_\infty)) \setminus \mathbf{ran}(\mathbf{h}_{n_0}).$$

As $\mathbf{supp_B}(\vec{x}) \cup \mathbf{supp_B}(\vec{X})$ is a finite set there exists a prime number $p > 5$ such that

$$(\mathbf{supp_B}(\vec{x}) \cup \mathbf{supp_B}(\vec{X})) \cap \{p^{i+1} : i \in \omega\} = \emptyset.$$

Pick $U \subseteq \{p^{i+1} : i \in \omega\}$ such that U and G have the same number of elements, and let $\Im : G \to U$ be a bijection. Note that $\mathbf{ran}(\mathbf{h}_{n_0}) \subseteq \{5^{i+1} : i \in \omega\}$ according to 2.15. We may then define $\rho \in \mathbf{Aut}(\mathbf{B}/\mathbf{ran}(\mathbf{h}_{n_0}))$ by:

$$\rho(x) := \begin{cases} x & \text{if } x \notin (G \cup U) \\ \Im(x) & \text{if } x \in G \\ \Im^{-1}(x) & \text{if } x \in U. \end{cases}$$

As a result,

$$\rho[G] \subseteq \{p^{i+1} : i \in \omega\} \text{ and } \rho = \rho^{-1}.$$

On account of $\mathfrak{M}^{\mathbf{V}_{n_0},\mathbf{h}_{n_0}} \models \psi[\vec{x},\vec{X}]$ and Proposition 3.14, we obtain $\mathfrak{M}^{\mathbf{V}_{n_0},\rho\circ\mathbf{h}_{n_0}} \models \psi[\rho(\vec{x}),\rho(\vec{X})]$. But $\rho \circ \mathbf{h}_{n_0} = \mathbf{h}_{n_0}$, and hence $\mathfrak{M}^{\mathbf{V}_{n_0},\mathbf{h}_{n_0}} \models \psi[\rho(\vec{x}),\rho(\vec{X})]$. □

Corollary 3.19 *Let $\psi(\vec{u},\vec{U})$ be an arbitrary formula. For meta $n_0 < n_1$ the following is provable in $\mathbf{KP}^r + \Sigma_1$-$\mathbf{Sep}$: If $\vec{X} \in \mathbf{CL}^{\mathbf{V}_{n_0},\mathbf{h}_{n_0}}$ and $\mathfrak{M}^{\mathbf{V}_{n_0},\mathbf{h}_{n_0}} \models \psi[\vec{x},\vec{X}]$, then there exists $\rho \in \mathbf{Aut}(\mathbf{B}/\mathbf{ran}(\mathbf{h}_{n_0}))$ such that $\rho = \rho^{-1}$, $\rho(\vec{X}) \in \mathbf{CL}^{\mathbf{V}_{n_0},\mathbf{h}_{n_0}}$, $(\mathbf{supp_B}(\rho(\vec{x})) \cup \mathbf{supp_B}(\rho(\vec{X}))) \cap \mathbf{ran}(\mathbf{h}_{n_1}) \subseteq \mathbf{ran}(\mathbf{h}_{n_0})$, and $\mathfrak{M}^{\mathbf{V}_{n_0},\mathbf{h}_{n_0}} \models \psi[\rho(\vec{x}),\rho(\vec{X})]$.*

Proof: Replace $\mathbf{h}_\infty$ with $\mathbf{h}_{n_1}$ in the proof of 3.18. □

Theorem 3.20 *Let $\phi(\vec{u},\vec{U},W)$ be a Σ^{EM}-formula and $n_0 \leq n_1$. Suppose*

$$(\mathbf{supp_B}(\vec{x}) \cup \mathbf{supp_B}(\vec{X})) \cap \mathbf{ran}(\mathbf{h}_{n_1}) \subseteq \mathbf{ran}(\mathbf{h}_{n_0})$$

and $\vec{X} \in \mathbf{CL}^{\mathbf{V}_{n_0},\mathbf{h}_{n_0}}$. If $\exists R \in \mathbf{V}_{n_1}\ \mathfrak{M}^{\mathbf{V}_{n_1},\mathbf{h}_{n_1}} \models \phi[\vec{x},\vec{X},\mathbf{h}_{n_1}(R)]$, then $\exists R \in \mathbf{V}_{n_0}\ \mathfrak{M}^{\mathbf{V}_{n_0},\mathbf{h}_{n_0}} \models \phi[\vec{x},\vec{X},\mathbf{h}_{n_0}(R)]$.

Proof: Let

$$\begin{aligned} F &:= (\mathbf{supp_B}(\vec{x}) \cup \mathbf{supp_B}(\vec{X})) \cap \mathbf{ran}(\mathbf{h}_{n_1}); \\ \mathcal{Z} &:= (\mathbf{h}_{n_1})^{-1}[F] = (\mathbf{h}_{n_0})^{-1}[F]. \end{aligned}$$

Then $\mathcal{Z} \in \mathbf{L}_{\boldsymbol{\sigma}_{n_0}}$ and $\mathbf{h}_{n_1} \restriction \mathcal{Z} = \mathbf{h}_{n_0} \restriction \mathcal{Z} \in \mathbf{L}_{\boldsymbol{\sigma}_{n_0}}$. We may pick $\ell : \mathbf{V}_{n_1} \stackrel{1-1}{\to} \mathbf{B}$ such that

$$\ell \restriction \mathcal{Z} = \mathbf{h}_{n_1} \restriction \mathcal{Z} \ \wedge\ \ell\,[\mathbf{V}_{n_1} \setminus \mathcal{Z}] \subseteq \{p^{i+1} : i \in \omega\},$$

where p is a prime number $p > 5$ such that $(\mathbf{supp_B}(\vec{x}) \cup \mathbf{supp_B}(\vec{X})) \cap \{p^{i+1} : i \in \omega\} = \emptyset$. This is possible since $\mathbf{supp_B}(\vec{x}) \cup \mathbf{supp_B}(\vec{X})$ is a finite set. We then have $(\mathbf{supp_B}(\vec{x}) \cup$

$\mathbf{supp_B}(\vec{X})) \cap (\ell\,[\mathbf{V}_{n_1}\backslash\mathcal{Z}] \cup \mathbf{h}_{n_1}\,[\mathbf{V}_{n_1}\backslash\mathcal{Z}]) = \emptyset$. Therefore, as $\mathfrak{M}^{\mathbf{V}_{n_1},\mathbf{h}_{n_1}} \models \phi[\vec{x},\vec{X},\mathbf{h}_{n_1}(R)]$, Lemma 3.16 yields $\mathfrak{M}^{\mathbf{V}_{n_1},\ell} \models \phi[\vec{x},\vec{X},\ell(R)]$, and hence

$$\exists \mathcal{X} \in \mathbf{WELL}\, \exists R \in \mathcal{X}\, \exists \jmath : \mathcal{X} \stackrel{1-1}{\to} \mathbf{B}\, \big[\mathcal{Z} \subseteq \mathcal{X} \ \wedge\ \jmath \restriction \mathcal{Z} = \mathbf{h}_{n_0} \restriction \mathcal{Z} \ \wedge \\ \forall u \in (\mathcal{X} \setminus \mathcal{Z})\, (\jmath(u) \in \{p^{i+1} : i \in \omega\}) \ \wedge\ \mathfrak{M}^{\mathcal{X},\jmath} \models \phi[\vec{x},\vec{X},\jmath(R)]\,\big]. \tag{2}$$

The latter statement is a Σ formula whose parameters are $\vec{X},\vec{x},\mathcal{Z}$, and $\mathbf{h}_{n_0} \restriction \mathcal{Z}$. The latter being elements of $\mathbf{L}_{\boldsymbol{\sigma}_{n_0}}$, (2) must already hold in $\mathbf{L}_{\boldsymbol{\sigma}_{n_0}}$. As a result, we may choose $\mathcal{X}$ and $\jmath$ from $\mathbf{L}_{\boldsymbol{\sigma}_{n_0}}$. We have $(\mathbf{supp_B}(\vec{x}) \cup \mathbf{supp_B}(\vec{X})) \cap (\jmath\,[\mathcal{X}\backslash\mathcal{Z}] \cup \mathbf{h}_{n_0}\,[\mathcal{X}\backslash\mathcal{Z}]) = \emptyset$, thus, putting to use Lemma 3.16 again, $\exists R \in \mathcal{X}\; \mathfrak{M}^{\mathcal{X},\jmath} \models \phi[\vec{x},\vec{X},\jmath(R)]$ implies

$$\exists R \in \mathcal{X}\; \mathfrak{M}^{\mathcal{X},\mathbf{h}_{n_0}\restriction\mathcal{X}} \models \phi[\vec{x},\vec{X},\mathbf{h}_{n_0}(R)].$$

Recall that ϕ is a Σ^{EM}-formula. Hence, by Corollary 3.12, we then get $\exists R \in \mathcal{X}\; \mathfrak{M}^{\mathbf{V}_{n_0},\mathbf{h}_{n_0}} \models \phi[\vec{x},\vec{X},\mathbf{h}_{n_0}(R)]$. □

Corollary 3.21 *Let $\phi(\vec{u},\vec{U},W)$ be a Σ^{EM}-formula. For meta $n_0 < n_1$ the following is provable in $\mathbf{KP}^r + \Sigma_1$-$\mathbf{Sep}$: If $(\mathbf{supp_B}(\vec{x}) \cup \mathbf{supp_B}(\vec{X})) \cap \mathbf{ran}(\mathbf{h}_{n_1}) \subseteq \mathbf{ran}(\mathbf{h}_{n_0})$, $\vec{X} \in \mathbf{CL}^{\mathbf{V}_{n_0},\mathbf{h}_{n_0}}$ and $\exists R \in \mathbf{V}_{n_1}\; \mathfrak{M}^{\mathbf{V}_{n_1},\mathbf{h}_{n_1}} \models \phi[\vec{x},\vec{X},\mathbf{h}_{n_1}(R)]$, then $\exists R \in \mathbf{V}_{n_0}\; \mathfrak{M}^{\mathbf{V}_{n_0},\mathbf{h}_{n_0}} \models \phi[\vec{x},\vec{X},\mathbf{h}_{n_0}(R)]$.*

Proposition 3.22 *$\widetilde{\mathfrak{M}}$ is a model of Join.*

Proof: Suppose $A \in \mathbf{CL}_{\widetilde{\mathfrak{M}}}$ and $\widetilde{\mathfrak{M}} \models \forall x \mathrel{\mathring{\in}} A\, \exists Y\, fx \simeq Y$. Set

$$F := (\mathbf{supp_B}(A) \cup \mathbf{supp_B}(f)) \cap \mathbf{ran}(\mathbf{h}_\infty).$$

Note that F is finite. Thus, by definition of $\widetilde{\mathfrak{M}}$, there exists n_0 such that $A \in \mathbf{CL}^{\mathbf{V}_{n_0},\mathbf{h}_{n_0}}$ and $F \subseteq \mathbf{ran}(\mathbf{h}_{n_0})$.

Now suppose $\widetilde{\mathfrak{M}} \models x_0 \mathrel{\mathring{\in}} A$. Then, using 3.10,

$$\mathfrak{M}^{\mathbf{V}_{n_0},\mathbf{h}_{n_0}} \models x_0 \mathrel{\mathring{\in}} A. \tag{3}$$

By 3.18, there exists $\rho \in \mathbf{Aut}(\mathbf{B}/\mathbf{ran}(\mathbf{h}_{n_0}))$ such that $\rho = \rho^{-1}$, $\rho(A) = A$, $\rho(f) = f$, $\mathbf{supp_B}(\rho(x_0)) \cap \mathbf{ran}(\mathbf{h}_\infty) \subseteq \mathbf{ran}(\mathbf{h}_{n_0})$, and $\mathfrak{M}^{\mathbf{V}_{n_0},\mathbf{h}_{n_0}} \models \rho(x_0) \mathrel{\mathring{\in}} A$. Consequently,

$$\widetilde{\mathfrak{M}} \models \exists Y\, [f\rho(x_0) \simeq Y\,].$$

Whence, for some $n_1 \geq n_0$,

$$\mathfrak{M}^{\mathbf{V}_{n_1},\mathbf{h}_{n_1}} \models \exists Y\, [f\rho(x_0) \simeq Y\,]. \tag{4}$$

Note that $\big(\mathbf{supp_B}(\rho(x_0)) \cup \mathbf{supp_B}(f) \cup \mathbf{supp_B}(A)\big) \cap \mathbf{ran}(\mathbf{h}_{n_1}) \subseteq \mathbf{ran}(\mathbf{h}_{n_0})$. Therefore, by 3.20, (4) implies

$$\mathfrak{M}^{\mathbf{V}_{n_0},\mathbf{h}_{n_0}} \models \exists Y\, [f\rho(x_0) \simeq Y\,].$$

Since $\mathbf{h}_{n_0} = \rho \circ \mathbf{h}_{n_0}$, we may deduce

$$\mathfrak{M}^{\mathbf{V}_{n_0},\mathbf{h}_{n_0}} \models \exists Y\, [\rho(f)\rho(\rho(x_0)) \simeq Y\,]$$

by Proposition 3.14. As $\rho(f) = f$ and $\rho = \rho^{-1}$, the latter provides

$$\mathfrak{M}^{\mathbf{V}_{n_0},\mathbf{h}_{n_0}} \models \exists Y\,[fx_0 \simeq Y\,].$$

As $x_0 \mathrel{\mathring{\in}} A$ was arbitrary, we have shown

$$\mathfrak{M}^{\mathbf{V}_{n_0},\mathbf{h}_{n_0}} \models \forall x \mathrel{\mathring{\in}} A\,\exists Y\,[fx \simeq Y\,]. \tag{5}$$

On account of the definition of $\mathfrak{M}^{\mathbf{V}_{n_0},\mathbf{h}_{n_0}}$, (5) yields

$$\mathfrak{M}^{\mathbf{V}_{n_0},\mathbf{h}_{n_0}} \models \exists X\,\mathbf{j}(A,f) \simeq X,$$

and thus

$$\widetilde{\mathfrak{M}} \models \exists X\,\mathbf{j}(A,f) \simeq X.$$

□

Lemma 3.23 *If* $A \in \mathbf{CL}^{\mathbf{V}_{n_0},\mathbf{h}_{n_0}}$, $\widetilde{\mathfrak{M}} \models \forall X \mathrel{\mathring{\subseteq}} A\,\exists Y \mathrel{\mathring{\subseteq}} A\,fX \simeq Y$ *and* $\big(\mathbf{supp_B}(f) \cup \mathbf{supp_B}(A)\big) \cap \mathbf{ran}(\mathbf{h}_\infty) \subseteq \mathbf{ran}(\mathbf{h}_{n_0})$, *then*

$$\mathfrak{M}^{\mathbf{V}_{n_0},\mathbf{h}_{n_0}} \models \forall X \mathrel{\mathring{\subseteq}} A\,\exists Y \mathrel{\mathring{\subseteq}} A\,fX \simeq Y.$$

Proof: Suppose $\widetilde{\mathfrak{M}} \models \forall X \mathrel{\mathring{\subseteq}} A\,\exists Y \mathrel{\mathring{\subseteq}} A\,fX \simeq Y$ and $\big(\mathbf{supp_B}(f) \cup \mathbf{supp_B}(A)\big) \cap \mathbf{ran}(\mathbf{h}_\infty) \subseteq \mathbf{ran}(\mathbf{h}_{n_0})$. Fix $X_0 \in \mathbf{CL}^{\mathbf{V}_{n_0},\mathbf{h}_{n_0}}$ such that $\mathfrak{M}^{\mathbf{V}_{n_0},\mathbf{h}_{n_0}} \models X_0 \mathrel{\mathring{\subseteq}} A$. By 3.18 there exists $\rho \in \mathbf{Aut}(\mathbf{B}/\mathbf{ran}(\mathbf{h}_{n_0}))$ such that $\rho = \rho^{-1}$, $\rho(f) = f$, $\rho(A) = A$, $\mathbf{supp_B}(\rho(X_0)) \cap \mathbf{ran}(\mathbf{h}_\infty) \subseteq \mathbf{ran}(\mathbf{h}_{n_0})$, $\rho(X_0) \in \mathbf{CL}^{\mathbf{V}_{n_0},\mathbf{h}_{n_0}}$ and $\mathfrak{M}^{\mathbf{V}_{n_0},\mathbf{h}_{n_0}} \models \rho(X_0) \mathrel{\mathring{\subseteq}} A$. Consequently,

$$\widetilde{\mathfrak{M}} \models \rho(X_0) \mathrel{\mathring{\subseteq}} A \;\wedge\; \exists Y\,[f\rho(X_0) \simeq Y\,].$$

Whence, for some $n_1 \geq n_0$,

$$\mathfrak{M}^{\mathbf{V}_{n_1},\mathbf{h}_{n_1}} \models \exists Y\,[f\rho(X_0) \simeq Y\,]. \tag{6}$$

As $(\mathbf{supp_B}(f) \cup \mathbf{supp_B}(\rho(X_0))) \cap \mathbf{ran}(\mathbf{h}_{n_1}) \subseteq \mathbf{ran}(\mathbf{h}_{n_0})$, we obtain

$$\mathfrak{M}^{\mathbf{V}_{n_0},\mathbf{h}_{n_0}} \models \exists Y\,[f\rho(X_0) \simeq Y\,], \tag{7}$$

utilizing (6) and 3.20. The latter implies $\mathfrak{M}^{\mathbf{V}_{n_0},\mathbf{h}_{n_0}} \models \exists Y\,[\rho(f)\rho(\rho(X_0)) \simeq Y\,]$ by 3.14 since $\mathbf{h}_{n_0} = \rho \circ \mathbf{h}_{n_0}$. But $\rho(f) = f$ and $\rho(\rho(X_0)) = X_0$. Hence

$$\mathfrak{M}^{\mathbf{V}_{n_0},\mathbf{h}_{n_0}} \models \exists Y\,[fX_0 \simeq Y\,].$$

As $X_0 \in \mathbf{CL}^{\mathbf{V}_{n_0},\mathbf{h}_{n_0}}$ was arbitrary, we get $\mathfrak{M}^{\mathbf{V}_{n_0},\mathbf{h}_{n_0}} \models \forall X \mathrel{\mathring{\subseteq}} A\,\exists Y \mathrel{\mathring{\subseteq}} A\,fX \simeq Y$. □

Corollary 3.24 *For meta* $n_0 < n$ *the following is provable in* $\mathbf{KP}^r + \Sigma_1$-**Sep**: *If* $A \in \mathbf{CL}^{\mathbf{V}_{n_0},\mathbf{h}_{n_0}}$, $\mathfrak{M}^{\mathbf{V}_n,\mathbf{h}_n} \models \forall X \mathrel{\mathring{\subseteq}} A\,\exists Y \mathrel{\mathring{\subseteq}} A\,fX \simeq Y$ *and* $\big(\mathbf{supp_B}(f) \cup \mathbf{supp_B}(A)\big) \cap \mathbf{ran}(\mathbf{h}_n) \subseteq \mathbf{ran}(\mathbf{h}_{n_0})$, *then*

$$\mathfrak{M}^{\mathbf{V}_{n_0},\mathbf{h}_{n_0}} \models \forall X \mathrel{\mathring{\subseteq}} A\,\exists Y \mathrel{\mathring{\subseteq}} A\,fX \simeq Y.$$

Proof: Replace $\widetilde{\mathfrak{M}}$ with $\mathfrak{M}^{\mathbf{V}_n,\mathbf{h}_n}$ and $\mathbf{h}_\infty$ with $\mathbf{h}_n$ in the proof of 3.23. $\square$

Proposition 3.25 $\widetilde{\mathfrak{M}} \models \mathbf{UMID_I}$.

Proof: Suppose $\widetilde{\mathfrak{M}} \models \mathbf{Clop}(f,\mathbf{I}) \wedge \mathbf{Mon}(f,\mathbf{I})$. Let

$$F := \mathbf{supp_B}(f) \cap \mathbf{ran}(\mathbf{h}_\infty).$$

Then $F \subseteq \mathbf{ran}(\mathbf{h}_{n_0})$ for some n_0. Let $n_1 > n_0$. Owing to Lemma 3.23, we have

$$\mathfrak{M}^{\mathbf{V}_{n_1},\mathbf{h}_{n_1}} \models \forall X \mathrel{\mathring{\subseteq}} \mathbf{I}\, \exists Y \mathrel{\mathring{\subseteq}} \mathbf{I}\, fX \simeq Y. \tag{8}$$

Further, (8) and $\widetilde{\mathfrak{M}} \models \mathbf{Mon}(f,\mathbf{I})$ imply

$$\mathfrak{M}^{\mathbf{V}_{n_1},\mathbf{h}_{n_1}} \models \mathbf{Clop}(f,\mathbf{I}) \wedge \mathbf{Mon}(f,\mathbf{I}). \tag{9}$$

Letting

$$R := \{\langle \mathbf{f}_{n_0}(\eta), \mathbf{f}_{n_0}(\xi)\rangle : \eta < \xi < \boldsymbol{\sigma}_{n_0}\}$$

($\mathbf{f}_{n_0}$ has been defined in Lemma 2.14), we obtain a well-ordering $R \in \mathbf{V}_{n_1}$ of ω of length $\boldsymbol{\sigma}_{n_0}$. Using (9) and the fact that $\mathfrak{M}^{\mathbf{V}_{n_1},\mathbf{h}_{n_1}} \models \mathbf{T_0}$, there exist $A, B \in \mathbf{CL}^{\mathbf{V}_{n_1},\mathbf{h}_{n_1}}$ such that

$$\mathfrak{M}^{\mathbf{V}_{n_1},\mathbf{h}_{n_1}} \models \mathbf{It}_{\mathbb{N}}(\hat{R}, f) \simeq A \;\wedge\; fA \simeq B, \tag{10}$$

where $\hat{R} := \mathbf{h}_{n_1}(R)$. Next, we aim at showing

$$\mathfrak{M}^{\mathbf{V}_{n_1},\mathbf{h}_{n_1}} \models B \mathrel{\mathring{\subseteq}} A. \tag{11}$$

Suppose $\mathfrak{M}^{\mathbf{V}_{n_1},\mathbf{h}_{n_1}} \models x_0 \mathrel{\mathring{\in}} B$. Then

$$\exists R \in \mathbf{V}_{n_1}\, \mathfrak{M}^{\mathbf{V}_{n_1},\mathbf{h}_{n_1}} \models \exists Y\, \exists Z\, [\mathbf{It}_{\mathbb{N}}(\tau(\mathbf{h}_{n_1}(R)), f) \simeq Y \;\wedge\; fY \simeq Z \wedge x_0 \mathrel{\mathring{\in}} Z].$$

Note that $\mathbf{supp_B}(x_0) = \emptyset$ as $\mathfrak{M}^{\mathbf{V}_{n_1},\mathbf{h}_{n_1}} \models x_0 \mathrel{\mathring{\in}} \mathbf{I}$. Therefore, by Theorem 3.20, the above implies

$$\exists R^* \in \mathbf{V}_{n_0}\, \mathfrak{M}^{\mathbf{V}_{n_0},\mathbf{h}_{n_0}} \models \exists Y\, \exists Z\, [\mathbf{It}_{\mathbb{N}}(\tau(\mathbf{h}_{n_0}(R^*)), f) \simeq Y \;\wedge\; fY \simeq Z \wedge x_0 \mathrel{\mathring{\in}} Z].$$

As the order-type of R^* is strictly smaller than the order-type of R, the latter implies $\mathfrak{M}^{\mathbf{V}_{n_0},\mathbf{h}_{n_0}} \models x_0 \mathrel{\mathring{\in}} A$, confirming (11). Using (11), the definition of $\mathfrak{M}^{\mathbf{V}_{n_1},\mathbf{h}_{n_1}}$ then ensures $\mathbf{lfp}(f) \in \mathbf{CL}^{\mathbf{V}_{n_1},\mathbf{h}_{n_1}}$. As $\widetilde{\mathfrak{M}} \models \mathbf{Clop}(f,\mathbf{I}) \wedge \mathbf{Mon}(f,\mathbf{I})$, $\mathbf{lfp}(f)$ is clearly the least fixed-point of f in $\widetilde{\mathfrak{M}}$. Therefore $\widetilde{\mathfrak{M}} \models \mathbf{UMID_I}$. $\square$

Corollary 3.26 *For meta* $n_0 < n_1$ *the following is provable in* $\mathbf{KP}^r + \Sigma_1$-$\mathbf{Sep}$*: If* $\mathfrak{M}^{\mathbf{V}_{n_1},\mathbf{h}_{n_1}} \models \mathbf{Clop}(f,\mathbf{I}) \wedge \mathbf{Mon}(f,\mathbf{I})$ *and* $\mathbf{supp_B}(f) \cap \mathbf{ran}(\mathbf{h}_{n_1}) \subseteq \mathbf{ran}(\mathbf{h}_{n_0})$*, then* $\mathbf{lfp}(f) \in \mathbf{CL}^{\mathbf{V}_{n_1},\mathbf{h}_{n_1}}$ *and* $\mathfrak{M}^{\mathbf{V}_{n_1},\mathbf{h}_{n_1}} \models \mathbf{Lfp}(\mathbf{lfp}(f), f, \mathbf{I})$.

Proof: Replace $\widetilde{\mathfrak{M}}$ with $\mathfrak{M}^{\mathbf{V}_{n_1},\mathbf{h}_{n_1}}$ and $\mathbf{h}_\infty$ with $\mathbf{h}_{n_1}$ in the proof of 3.25 and use 3.24 instead of 3.23. $\square$

Open Problem 3.27 The proof of Proposition 3.25 lends itself to the question whether $\widetilde{\mathfrak{M}}$ is actually a model of **UMID**. In the proof of $\widetilde{\mathfrak{M}} \models \mathbf{UMID_I}$ we exploited the central fact that the elements of $\mathbf{I}$ are invariant under automorphisms. However, we conjecture that $\widetilde{\mathfrak{M}} \models \mathbf{UMID}$.

3.5 Some upper bounds

Theorem 3.28 $(\mathbf{KP}^w + \Sigma_1\text{-}\mathbf{Sep})$ $\widetilde{\mathfrak{M}} \models \mathbf{T_0}(\mathbf{I})\!\restriction + \mathbf{IND}_{\mathbb{N}} + \mathbf{UMID}_{\mathbf{I}}$.

(To be more precise, for every theorem ψ *of* $\mathbf{T_0}(\mathbf{I})\!\restriction + \mathbf{IND}_{\mathbb{N}} + \mathbf{UMID}_{\mathbf{I}}$ *we have* $\mathbf{KP}^w + \Sigma_1\text{-}\mathbf{Sep} \vdash \widetilde{\mathfrak{M}} \models \psi$*.)*

Proof: Clearly, $\widetilde{\mathfrak{M}}$ is a model of the applicative axioms and the axioms for natural numbers. Furthermore, in view of Corollary 3.10 and the definition of the $\mathfrak{M}^{\mathbf{V}_n,\mathbf{h}_n}$'s, it follows that $\widetilde{\mathfrak{M}}$ satisfies **(ECA)**, $\mathbf{IND}_{\mathbb{N}}$, and $\mathbf{IG}\!\restriction$. Further, by Proposition 3.22, $\widetilde{\mathfrak{M}}$ is a model of Join, and, by Proposition 3.25, $\widetilde{\mathfrak{M}}$ is a model of $\mathbf{UMID}_{\mathbf{I}}$. □

Theorem 3.29 $(\mathbf{KP} + \Sigma_1\text{-}\mathbf{Sep})$ $\widetilde{\mathfrak{M}} \models \mathbf{T_0}(\mathbf{I}) + \mathbf{UMID}_{\mathbf{I}}$.

(To be more precise, for every theorem ψ *of* $\mathbf{T_0}(\mathbf{I}) + \mathbf{UMID}_{\mathbf{I}}$ *we have* $\mathbf{KP}+\Sigma_1\text{-}\mathbf{Sep} \vdash \widetilde{\mathfrak{M}} \models \psi$*.)*

Proof: One uses Foundation to verify $\widetilde{\mathfrak{M}} \models \mathbf{IG}$. The rest follows from 3.28.

To show $\widetilde{\mathfrak{M}} \models \mathbf{IG}$ in more detail, assume

$$\widetilde{\mathfrak{M}} \models \forall x \mathrel{\mathring{\in}} A\,[\forall y\,((y,x) \mathrel{\mathring{\in}} B \to \phi(y)) \to \phi(x)]. \tag{12}$$

We need to verify that $\widetilde{\mathfrak{M}} \models \forall x \mathrel{\mathring{\in}} \mathbf{i}(A,B)\,\phi(x)$. Let $a := \{z \in \mathbf{S} : \widetilde{\mathfrak{M}} \models z \mathrel{\mathring{\in}} A\}$ and $r := \{\langle y,x\rangle \in \mathbf{S}\times\mathbf{S} : \widetilde{\mathfrak{M}} \models (y,x) \mathrel{\mathring{\in}} B\}$. The definition of $\mathbf{i}(A,B)$ yields that $\{x \in \mathbf{S} : \widetilde{\mathfrak{M}} \models x \mathrel{\mathring{\in}} \mathbf{i}(A,B)\} = \mathbf{wfp}(a,r)$. Thus, using induction on the well-founded part of r on a (Corollary 2.10), the assumption (12) implies $\forall x \in \mathbf{wfp}(a,r)\,\widetilde{\mathfrak{M}} \models \phi(x)$, and hence $\widetilde{\mathfrak{M}} \models \forall x \mathrel{\mathring{\in}} \mathbf{i}(A,B)\,\phi(x)$. □

For theories T_1, T_2, we use the notation $T_1 \leq T_2$ to signify that T_1 is proof-theoretically reducible to T_2.

Theorem 3.30 *(i)* $\mathbf{T_0}(\mathbf{I})\!\restriction + \mathbf{IND}_{\mathbb{N}} + \mathbf{UMID}_{\mathbf{I}} \;\leq\; \mathbf{KP}^w + \Sigma_1\text{-}\mathbf{Sep}$.

(ii) $\mathbf{T_0}(\mathbf{I}) + \mathbf{UMID}_{\mathbf{I}} \;\leq\; \mathbf{KP} + \Sigma_1\text{-}\mathbf{Sep}$.

3.6 Relativizations

To obtain more detailed information than Theorem 3.30 (which will be presented in section 5) one should relativize the model constructions of this section to a given set $U \subseteq \omega$.

Definition 3.31 The structure

$$\mathfrak{M}^{\mathcal{X},\ell}_{U,\alpha} = \langle \mathbf{S}, \mathbf{CL}^{\mathcal{X},\ell}_{U,\alpha}, \mathring{\in}^{\mathcal{X},\ell}_{U,\alpha}, \mathbf{App_S}, \mathbf{k}, \mathbf{s}, \mathbf{p}, \mathbf{p}_0, \mathbf{p}_1, \mathbf{d}, \mathbf{s}_{\mathbb{N}}, \mathbf{p}_{\mathbb{N}}, \mathbf{0}, \mathbb{N}, \mathbf{I}, (\mathbf{c}_m)_{m\in\omega}, \mathbf{i}, \mathbf{j}, \mathbf{lfp}, \mathbf{U}\rangle$$

is defined as $\mathfrak{M}^{\mathcal{X},\ell}_{\alpha}$ in Definition 3.5 except that $\mathbf{CL}^{\mathcal{X},\ell}_{U,0}$ has an additional classification $\mathbf{U} := 14^\circ$ satisfying

$$z \mathring{\in}^{\mathcal{X},\ell}_{U,0} \mathbf{U} \;:\Leftrightarrow\; z \in U^\circ.$$

The structures $\mathfrak{M}^{\mathcal{X},\ell}_{U,\alpha}$ give rise to the model

$$\mathfrak{M}^{\mathcal{X},\ell}_{U} \;:=\; \langle \mathbf{S}, \mathbf{CL}^{\mathcal{X},\ell}_{U}, \mathring{\in}^{\mathcal{X},\ell}_{U}, \mathbf{App_S}, \mathbf{k}, \mathbf{s}, \mathbf{p}, \mathbf{p}_0, \mathbf{p}_1, \mathbf{d}, \mathbf{s}_{\mathbb{N}}, \mathbf{p}_{\mathbb{N}}, \mathbf{0}, \mathbb{N}, \mathbf{I}, (\mathbf{c}_m)_{m\in\omega}, \mathbf{i}, \mathbf{j}, \mathbf{lfp}, \mathbf{U}\rangle,$$

where $\mathbf{CL}^{\mathcal{X},\ell}_U := \bigcup_\alpha \mathbf{CL}^{\mathcal{X},\ell}_{U,\alpha}$ and $\mathring{\in}^{\mathcal{X},\ell}_U := \bigcup_\alpha \mathring{\in}^{\mathcal{X},\ell}_{U,\alpha}$. Finally let $\mathbf{V}^U_n := \mathbf{WELL} \cap \mathbf{L}_{\sigma^U_n}(U)$ and $\mathbf{h}^U_n := \mathbf{g}^U_n \upharpoonright \mathbf{V}^U_n$ and put

$$\widetilde{\mathfrak{M}}_U \;:=\; \bigcup_n \mathfrak{M}_U^{\mathbf{V}^U_n,\mathbf{h}^U_n}.$$

The results of this section all carry over to $\widetilde{\mathfrak{M}}_U$. As a strengthening of Theorem 3.29 we obtain:

Theorem 3.32 *If* $\mathbf{T_0}(\mathbf{I}) + \mathbf{UMID_I} \vdash \psi$, *then* $\mathbf{KP} + \Sigma_1\text{-}\mathbf{Sep} \vdash \forall U \subseteq \omega\ \widetilde{\mathfrak{M}}_U \models \psi$.

4 Reducing $\mathbf{T_0}(\mathbf{I})\upharpoonright + \mathbf{UMID_I}$ to $\mathbf{KP}^r + \Sigma_1$-Sep

Since the model $\widetilde{\mathfrak{M}}$ cannot be formalized in $\mathbf{KP}^r + \Sigma_1$-**Sep** we cannot use it to reduce $\mathbf{T_0}(\mathbf{I})\upharpoonright + \mathbf{UMID_I}$ to $\mathbf{KP}^r + \Sigma_1$-**Sep**. However, for meta n, the existence of $\mathfrak{M}^{\mathbf{V}_n,\mathbf{h}_n}$ can already be proved in $\mathbf{KP}^r + \Sigma_1$-**Sep**. Settling for a partial interpretation, we shall show that those theorems of $\mathbf{T_0}\upharpoonright + \mathbf{UMID}_{\mathbb{N}}$, wherein all the classification quantifiers are existential, hold in any of the models $\mathfrak{M}^{\mathbf{V}_n,\mathbf{h}_n}$. The latter reduces $\mathbf{T_0}(\mathbf{I})\upharpoonright + \mathbf{UMID_I}$ to $\mathbf{KP}^r + \Sigma_1$-**Sep**. Technically, the first step consists in proving a partial cut-elimination result for $\mathbf{T_0}(\mathbf{I})\upharpoonright + \mathbf{UMID_I}$.

4.1 A Tait-style calculus for explicit mathematics

The Tait-style calculus to be developed in this subsection relies on a slightly different account of the language of explicit mathematics. Namely, the Tait language $\mathcal{L}_T$ only contains the logical symbols $\wedge$, $\vee$, $\forall$, $\exists$, but has the additional relation symbols $\neq$, $\sim$App, $\mathring{\notin}$ to express the complements of $=$, App, $\mathring{\in}$, respectively. Negation then becomes a defined operation on the formulae in the obvious way, using the de Morgan laws to push it through to the prime formulae.

Definition 4.1 The Σ^{EM}-formulae form the least class of formulae containing the quantifier-free formulae which is closed under $\wedge$, $\vee$, object quantification, and $\exists$-quantification over classifications.

The Π^{EM}-formulae form the least class of formulae containing the quantifier-free formulae which is closed under $\wedge$, $\vee$, object quantification, and $\forall$-quantification over classifications.

Δ^{EM}-formulae of $\mathcal{L}_T$ are formulae which are both Σ^{EM}- and Π^{EM}-formulae, i.e. formulae which do not contain any unbounded classification quantifiers.

Σ^{EM}_1-formulae are formulae of the form $\exists X_1 \ldots \exists X_k \phi(X_1, \ldots, X_k)$ where ϕ does not contain classification quantifiers. Π^{EM}_1-formulae are the negations of Σ^{EM}_1-formulae.

The idea now is to embed theories from explicit mathematics into the Tait-calculus and then to perform a partial cut-elimination which only leaves us with cuts on Σ^{EM}_1- and Π^{EM}_1-formulae. For this to work we have to make some minor adjustments. First, we need an adequate definition of the rank of a formula.

Definition 4.2 The *rank* of an $\mathcal{L}_T$-formula is the rank over its Σ^{EM}_1- and Π^{EM}_1-subformulae. Formally, if ϕ is a Σ^{EM}_1- or Π^{EM}_1-formula, then $\mathrm{rk}(\phi) = 0$. If ϕ is not of the latter form, the rank is defined as follows:

a) If ϕ is of the form $\phi_0 \wedge \phi_1$ or $\phi_0 \vee \phi_1$, then $\mathrm{rk}(\phi) = \max\{\mathrm{rk}(\phi_0), \mathrm{rk}(\phi_1)\} + 1$.
b) If ϕ is of the form $\exists x\psi(x), \forall x\psi(x), \exists X\psi(X)$, or $\forall X\psi(X)$, then $\mathrm{rk}(\phi) = \mathrm{rk}(\psi) + 1$.

The second adjustment is to make sure that all formulae introduced by non-logical axioms and rules are Σ_1^{EM}. For this it is convenient to switch to a slightly different formulation of the join axiom which has a syntactically simpler form.

Lemma 4.3 *The applicative fragment of* $\mathbf{EM}_0\restriction$ *proves that under the hypothesis* $\forall x \mathrel{\mathring{\in}} A\,\exists X(fx \simeq X)$ *the following assertions are equivalent:*

(i) $\exists Z\,\mathbf{Join}(f, A, Z)$, *i.e.* $\exists Z(Z \simeq \mathbf{j}(f, A) \wedge \forall z(z \mathrel{\mathring{\in}} Z \leftrightarrow \exists x \mathrel{\mathring{\in}} A \exists y(z \simeq (x, y) \wedge y \mathrel{\mathring{\in}} fx))$.

(ii) $\forall z \exists Z\,\mathbf{Join}^v(f, z, A, Z)$ *where*

$$\begin{aligned}\mathbf{Join}^v(f, z, A, Z) \equiv \exists Y \exists X(&Z \simeq \mathbf{j}(f, a) \wedge \\ &(z \mathrel{\mathring{\in}} Z \rightarrow \mathbf{p}_0 z \mathrel{\mathring{\in}} A \wedge Y \simeq f(\mathbf{p}_0 z) \wedge \mathbf{p}_1 z \mathrel{\mathring{\in}} Y) \wedge \\ &(\mathbf{p}_0 z \mathrel{\mathring{\in}} A \wedge (X \simeq f(\mathbf{p}_0 z) \rightarrow \mathbf{p}_1 z \mathrel{\mathring{\in}} X) \rightarrow z \mathrel{\mathring{\in}} Z)).\end{aligned}$$

Proof: Argue in the applicative fragment of $\mathbf{EM}_0\restriction$. If $\forall x \mathrel{\mathring{\in}} A\,(\exists X(fx \simeq X))$, then these X are unique. Therefore

$$\begin{aligned}\exists Z\,\mathbf{Join}(f, A, Z) \;&\Leftrightarrow\; \forall z \exists Z(Z \simeq \mathbf{j}(f, A) \wedge (z \mathrel{\mathring{\in}} Z \leftrightarrow \exists x \mathrel{\mathring{\in}} A \exists y(z \simeq (x, y) \wedge y \mathrel{\mathring{\in}} fx)) \\ &\Leftrightarrow\; \forall z \exists Z\,\big[\,Z \simeq \mathbf{j}(f, A) \wedge \\ &\qquad (z \mathrel{\mathring{\in}} Z \rightarrow \mathbf{p}_0 z \mathrel{\mathring{\in}} A \wedge \exists Y(Y \simeq f(\mathbf{p}_0 z) \wedge \mathbf{p}_1 z \mathrel{\mathring{\in}} Y)) \wedge \\ &\qquad (\mathbf{p}_0 z \mathrel{\mathring{\in}} A \wedge \forall X(X \simeq f(\mathbf{p}_0 z) \rightarrow \mathbf{p}_1 z \mathrel{\mathring{\in}} X) \rightarrow z \mathrel{\mathring{\in}} Z)\,\big] \\ &\Leftrightarrow\; \forall z \exists Z\,\mathbf{Join}^v(f, z, A, Z)\end{aligned}$$

□

Definition 4.4 The calculus $\mathcal{T}$ is defined as follows:

a) Logical axioms

$$(\mathrm{Ax}) \quad \Gamma, \neg\phi, \phi \text{ where } \mathrm{rk}(\phi) = 0.$$

b) Equality axioms

$$\begin{aligned}&(\mathrm{Eq1}) \quad \Gamma, t = t \text{ for object terms } t. \\ &(\mathrm{Eq2}) \quad \Gamma, s \neq t, \neg\phi(s), \phi(t) \text{ where } \mathrm{rk}(\phi) = 0.\end{aligned}$$

c) Logical rules

$$(\wedge)\ \frac{\Gamma, \phi_0 \quad \Gamma, \phi_1}{\Gamma, \phi_0 \wedge \phi_1} \qquad (\vee)\ \frac{\Gamma, \phi_i}{\Gamma, \phi_0 \vee \phi_1} \quad i = 0, 1$$

$$(\forall^0)\ \frac{\Gamma, \phi(a)}{\Gamma, \forall x\phi(x)} \qquad (\exists^0)\ \frac{\Gamma, \phi(t)}{\Gamma, \exists x\phi(x)}$$

$$(\forall^1)\ \frac{\Gamma, \phi(A)}{\Gamma, \forall X\phi(X)} \qquad (\exists^1)\ \frac{\Gamma, \phi(A)}{\Gamma, \exists X\phi(X)}$$

As usual we have the proviso that the free variables a and A in $(\forall^0)$ and $(\forall^1)$, respectively, are not to occur in the respective conclusion.

d) Non-logical axioms

$$\Gamma, \phi$$

where ϕ is one of the following:

- an instance of an applicative axiom.
- an instance of (**ECA**), i.e. $\exists X(X \simeq \mathbf{c}_m(\vec{t}, \vec{A}) \wedge \forall x(x \mathring{\in} X \leftrightarrow F(x, \vec{t}, \vec{A})))$ for certain terms $\vec{t}$ and classification variables and constants $\vec{A}$.
- the *induction axiom*

$$0 \mathring{\in} A \wedge \forall x \mathring{\in} \mathbf{N}(x \mathring{\in} A \rightarrow \mathbf{s_N} x \mathring{\in} A) \rightarrow \forall x \mathring{\in} \mathbf{N}(x \mathring{\in} A).$$

- the open form of $(\mathbf{IG})\restriction$, which is separated into two axioms,

$$\begin{array}{ll} (\mathbf{IG}1) & \Gamma, \exists X(X \simeq \mathbf{i}(A, B) \wedge \mathbf{Prog}_A(B, X)), \\ (\mathbf{IG}2)\restriction & \Gamma, \mathbf{i}(A, B) \simeq D \wedge \mathbf{Prog}_A(B, C) \rightarrow \forall x \mathring{\in} D(x \mathring{\in} C). \end{array}$$

- the rule for join

$$(\mathbf{JR}) \quad \frac{\Gamma, \forall x \mathring{\in} A \exists X(fx \simeq X)}{\Gamma, \exists Z\, \mathbf{Join}^v(f, t, A, Z)}$$

  for terms f and t.

In the following we write $\mathcal{T} \mid\!\frac{n}{k}\, \Gamma$ to convey that there exists a derivation in $\mathcal{T}$ in which all cut-formulae have rank less than k and which is of length $\leq n$.

The definition of the calculus $\mathcal{T}$ is tailored so that the following proposition holds:

Proposition 4.5 *If $\mathbf{EM}_0\restriction + (\mathbf{IG})\restriction + (\mathbf{J}) \vdash \phi$, then there are $n, k < \omega$ such that $\mathcal{T} \mid\!\frac{n}{k}\, \phi$.*

Proof: This is standard. □

Since all non-logical axioms and rules only introduce formulae of rank 0, we can eliminate all cuts of higher complexity from our derivations. In other words:

Proposition 4.6 *If $\mathcal{T} \mid\!\frac{m}{k}\, \Gamma$, then there is some n such that $\mathcal{T} \mid\!\frac{n}{1}\, \Gamma$. In point of fact, one has $n \leq 2_{k-1}(m)$ where $2_0(m) = m$ and $2_{r+1}(m) = 2^{2_r(m)}$.*

Proof: Standard cut-elimination. □

When we combine the previous propositions, we obtain:

Proposition 4.7 *If $\mathbf{EM}_0\restriction + (\mathbf{IG})\restriction + (\mathbf{J}) \vdash \phi$, then there is some $n < \omega$ such that $\mathcal{T} \mid\!\frac{n}{1}\, \phi$.*

To treat $\mathbf{UMID_I}$ in this context we again (as in the case of (**J**)) prefer to use a slight variant of the axiom which is in a syntactic form that can be dealt with in an easier way in the following.

Lemma 4.8 *The applicative fragment of $\mathbf{EM}_0\restriction$ proves: If $\mathbf{Clop}(\mathbf{f}, \mathbf{I})$, then the following formulations of the least fixed-point axiom are equivalent.*

(i) $\mathbf{Lfp}(A, f, \mathbf{I})$

(ii)

$$\mathbf{Lfp}^v(A, f, \mathbf{I}) :=$$
$$\forall U \forall Y \forall Z\,[\,Y \simeq fA \wedge Z \simeq fU \wedge U \mathrel{\overset{\circ}{\subseteq}} \mathbf{I} \;\rightarrow\; Y \mathrel{\overset{\circ}{\subseteq}} A \mathrel{\overset{\circ}{\subseteq}} \mathbf{I} \wedge (Z \mathrel{\overset{\circ}{\subseteq}} U \rightarrow A \mathrel{\overset{\circ}{\subseteq}} U)\,]$$

Therefore, the axiom $\mathbf{UMID_I}$ *is equivalent to*

$$(\mathbf{Umid_I}) \qquad \forall f(\mathbf{Clop(f, I)} \wedge \mathbf{Mon}(f, \mathbf{I}) \rightarrow \exists X\,[\mathbf{lfp}(f) \simeq X \;\wedge\; \mathbf{Lfp}^v(X, f, \mathbf{I})]),$$

where the official renderings of $\mathbf{Clop(f, A)}$ *and* $\mathbf{Mon}(f, A)$ *are as follows:*

$$\mathbf{Clop(f, A)} \;:=\; \forall X\,\exists Y\,[X \mathrel{\overset{\circ}{\subseteq}} A \;\rightarrow\; Y \mathrel{\overset{\circ}{\subseteq}} A \wedge fX \simeq Y\,]$$
$$\mathbf{Mon}(f, A) \;:=\; \forall X\,\forall Y\,[X \mathrel{\overset{\circ}{\subseteq}} A \wedge Y \mathrel{\overset{\circ}{\subseteq}} A \wedge X \mathrel{\overset{\circ}{\subseteq}} Y \;\rightarrow\; fX \mathrel{\overset{\circ}{\subseteq}} fY]$$

with $fX \mathrel{\overset{\circ}{\subseteq}} fY$ *being a shorthand for* $\exists Z \exists W\,(fX \simeq Z \wedge fY \simeq W \wedge Z \mathrel{\overset{\circ}{\subseteq}} W)$.

Proof: Similar to Lemma 4.3. □

4.2 The interpretation

In the following we shall use variables $\mathcal{A}, \mathcal{B}, \mathcal{C}, \ldots$ to range over classifications of structures of the form $\mathfrak{M}^{\mathcal{X},\ell}$, i.e. $\mathbf{CL}^{\mathcal{X},\ell}$. $\mathfrak{a}, \mathfrak{b}, \mathfrak{c}, \ldots$ will be ranging over elements of $\mathbf{S}$. These conventions are necessitated by our desire to differentiate between the variables of the formal theory $\mathbf{T_0(I)}\!\restriction + \mathbf{Umid_I}$ and variables ranging over the realms over their interpretations.

Proposition 4.9 *Let* $\Gamma[\vec{a}, \vec{A}]$ *be a set of* Σ^{EM}*-formulae. If* $\mathcal{T} \;\overset{n}{\underset{1}{\vdash}}\; \neg(\mathbf{Umid_I}), \Gamma[\vec{a}, \vec{A}]$*, then, for all (meta)* m*, the theory* $\mathbf{KP}^r + \Sigma_1$*-***Sep** *proves:*

$$\forall \vec{\mathcal{A}} \in \mathbf{CL}^{\mathbf{V}_m, \mathbf{h}_m}\, \forall \vec{\mathfrak{a}} \in \mathbf{S}\;(\mathfrak{M}^{\mathbf{V}_m, \mathbf{h}_m} \models \Gamma[\vec{\mathfrak{a}}, \vec{\mathcal{A}}]).$$

Here $\models \Gamma[\vec{\mathfrak{a}}, \vec{\mathcal{A}}]$ *is short for* $\models \bigvee \Gamma[\vec{\mathfrak{a}}, \vec{\mathcal{A}}]$*, where* $\bigvee \Gamma[\vec{\mathfrak{a}}, \vec{\mathcal{A}}]$ *is the disjunction over all formulae of* $\Gamma[\vec{\mathfrak{a}}, \vec{\mathcal{A}}]$.

Proof: We will use the abbreviations

$$\mathbf{CL}^m \;:=\; \mathbf{CL}^{\mathbf{V}_m, \mathbf{h}_m}, \qquad \mathfrak{M}^m \;:=\; \mathfrak{M}^{\mathbf{V}_m, \mathbf{h}_m}.$$

The proof proceeds by (meta) induction on n. We shall restrict our attention to the most important cases, as the remaining ones easily follow using the induction hypotheses.

If Γ is an axiom, then we discern two subcases.

In the first one, Γ is a Δ^{EM}-formula (i.e. in the cases of (Ax), (Eq), applicative axioms, induction axiom and (**IG2**)$\restriction$). Then the assertion holds by construction of $\mathfrak{S}_{M,\gamma}$. In the case of the induction axiom we have to note that for each $\mathcal{A} \in \mathbf{CL}^m$ the set $\{\mathfrak{a} \in S : \mathfrak{M}^m \models \mathfrak{a} \mathrel{\overset{\circ}{\in}} \mathcal{A}\}$ is in $\mathbf{L}_{\boldsymbol{\sigma}_m}$ and therefore we can use induction in $\mathbf{L}_{\boldsymbol{\sigma}_m}$ (on the set $\{n^\circ : n \in \omega\}$) to prove the instance of the induction axiom.

In the second axiom case we have an instance of (**ECA**) or one of (**IG1**) in its open formulation. For example, let us treat (**IG1**). For arbitrary m and $\mathcal{A}_0, \mathcal{A}_1 \in \mathbf{CL}^m$ we have $\mathbf{i}(\mathcal{A}_0, \mathcal{A}_1) \in \mathbf{CL}^m$ and so the assertion is established.

We leave out the propositional, quantifier and equality rules, since they can be treated using the induction hypotheses. But note that it is important that there are no $(\forall^1)$-rules to be considered as fact that Γ consists of Σ^{EM}-formulae solely.

Case 2: The last inference was a cut. As the cut formulae possess rank 0, we then have premises of the shape

$$\mathcal{T} \mid\!\frac{n_0}{1}\, \Gamma[\vec{a}, \vec{A}], \exists\vec{Y}\phi[\vec{a}, \vec{b}, \vec{Y}, \vec{A}, \vec{B}]$$

and

$$\mathcal{T} \mid\!\frac{n_1}{1}\, \Gamma[\vec{a}, \vec{A}], \forall\vec{Y}\neg\phi[\vec{a}, \vec{b}, \vec{Y}, \vec{A}, \vec{B}],$$

where ϕ is a Δ^{EM}-formula and $n_0, n_1 < n$. Application of the induction hypothesis to the first premise yields

$$\forall\vec{\mathcal{A}} \in \mathbf{CL}^m\ \mathfrak{M}^m \models \Gamma[\vec{\mathfrak{a}}, \vec{\mathcal{A}}], \exists\vec{\mathcal{B}}\phi[\vec{\mathfrak{a}}, \vec{\mathbf{0}}, \vec{\mathcal{B}}, \vec{\mathcal{A}}, \vec{\emptyset}] \tag{13}$$

for all $m \in \mathbb{N}$ and $\vec{\mathfrak{a}} \in \mathbf{S}$, where $\vec{\emptyset}$ is a tuple of $\emptyset$'s and $\vec{\mathbf{0}}$ is a tuple of $\mathbf{0}$'s of appropriate lengths with $\emptyset$ being the empty classification. Applying $(\forall^1)$ inversions to the second premise we get

$$\mathcal{T} \mid\!\frac{n_1}{1}\, \Gamma[\vec{a}, \vec{A}], \neg\phi[\vec{a}, \vec{b}, \vec{C}, \vec{A}, \vec{B}]$$

for new classification variables $\vec{C}$. Applying the induction hypothesis to this derivation we get

$$\forall\vec{\mathcal{A}}\vec{\mathcal{B}} \in \mathbf{CL}^m \big(\mathfrak{M}^m \models \Gamma[\vec{\mathfrak{a}}, \vec{\mathcal{A}}], \neg\phi[\vec{\mathfrak{a}}, \vec{\mathbf{0}}, \vec{\mathcal{B}}, \vec{\mathcal{A}}, \vec{\emptyset}]\big) \tag{14}$$

for all $\vec{\mathfrak{a}} \in \mathbf{S}$.

Now assume that there are $\vec{\mathfrak{a}} \in S$, $\vec{\mathcal{A}} \in \mathbf{CL}^m$ such that $\mathfrak{M}^m \not\models \Gamma[\vec{\mathfrak{a}}, \vec{\mathcal{A}}]$. Then (13) supplies us with $\vec{\mathcal{B}} \in \mathbf{CL}^m$ such that

$$\mathfrak{M}^m \models \phi[\vec{\mathfrak{a}}, \vec{\mathbf{0}}, \vec{\mathcal{B}}, \vec{\mathcal{A}}, \vec{\emptyset}]. \tag{15}$$

Using (14), we get

$$\mathfrak{M}^m \models \Gamma[\vec{\mathfrak{a}}, \vec{\mathcal{A}}], \neg\phi[\vec{\mathfrak{a}}, \vec{0}, \vec{\mathcal{B}}, \vec{\mathcal{A}}, \vec{\emptyset}]. \tag{16}$$

From (15) and (16) we deduce

$$\mathfrak{M}^m \models \Gamma[\vec{\mathfrak{a}}, \vec{\mathcal{A}}]$$

contradicting our assumption. Thus $\mathfrak{M}^m \models \Gamma[\vec{\mathfrak{a}}, \vec{\mathcal{A}}]$ must be true.

Case 3: The last inference is (**JR**). Then a formula $\exists Z\, \mathbf{Join}^v(t, s, A_i, Z)$ is in Γ and we have

$$\mathcal{T} \mid\!\frac{n_0}{1}\, \Gamma[\vec{a}, \vec{A}], \forall y \mathrel{\mathring{\in}} A_i \exists Y(ty \simeq Y)$$

for some $n_0 < n$. Fix $\vec{\mathcal{A}} \in \mathbf{CL}^m$ and $\vec{\mathfrak{a}} \in \mathbf{S}$ and let $f := t[\vec{\mathfrak{a}}, \vec{\mathcal{A}}]$. Assume $\mathfrak{M}^m \not\models \Gamma[\vec{\mathfrak{a}}, \vec{\mathcal{A}}]$. The induction hypothesis then yields $\mathfrak{M}^m \models \forall\mathfrak{b} \mathrel{\mathring{\in}} \mathcal{A}_i \exists\mathcal{B}(f\mathfrak{b} \simeq \mathcal{B})$ and therefore $\mathbf{j}(f, \mathcal{A}_i) \in \mathbf{CL}^m$ by the definition of $\mathbf{CL}^{\mathbf{V}_m, \mathbf{h}_m}$. Consequently, $\mathfrak{M}^m \models \exists\mathcal{C}\, \mathbf{Join}(f, \mathcal{A}_i, \mathcal{C})$ and therefore $\mathfrak{M}^m \models \forall\mathfrak{c}\exists\mathcal{C}\, \mathbf{Join}^v(f, \mathfrak{c}, \mathcal{A}, \mathcal{C})$. But the latter implies $\mathfrak{M}^m \models \Gamma[\vec{\mathfrak{a}}, \vec{\mathcal{A}}]$, colliding with our assumption. Therefore $\mathfrak{M}^m \models \Gamma[\vec{\mathfrak{a}}, \vec{\mathcal{A}}]$ holds.

Case 4: Assume now (this is the pivotal case) that the last inference was an $(\exists^0)$-inference with main formula $\neg(\mathbf{Umid_I})$. Then we have a scenario of the form

$$\mathcal{T} \mid\!\frac{n_0}{1}\, \neg(\mathbf{Umid_I}), \Gamma[\vec{a}, \vec{A}], \mathbf{Clop}(\mathbf{t}, \mathbf{I}) \wedge \mathbf{Mon}(t, \mathbf{I}) \wedge \forall X\, (\mathbf{lfp}(f) \simeq X \rightarrow \neg\mathbf{Lfp}^v(X, t, \mathbf{I}))$$

for $n_0 < n$ and a term t whose free variables are among $\vec{a}, \vec{b}, \vec{A}, \vec{B}$. Using ($\wedge$) inversions followed by a ($\forall^1$) inversion, we get the following derivations

$$\mathcal{T} \mid\frac{n_0}{1} \neg(\mathbf{Umid_I}), \Gamma[\vec{a}, \vec{A}], \exists Y\, (B \mathrel{\mathring{\subseteq}} \mathbf{I} \to Y \mathrel{\mathring{\subseteq}} \mathbf{I} \wedge tB \simeq Y) \tag{17}$$

$$\mathcal{T} \mid\frac{n_0}{1} \neg(\mathbf{Umid_I}), \Gamma[\vec{a}, \vec{A}], B \mathrel{\mathring{\subseteq}} \mathbf{I} \wedge C \mathrel{\mathring{\subseteq}} \mathbf{I} \wedge B \mathrel{\mathring{\subseteq}} C \to tB \mathrel{\mathring{\subseteq}} tC \tag{18}$$

$$\mathcal{T} \mid\frac{n_0}{1} \neg(\mathbf{Umid_I}), \Gamma[\vec{a}, \vec{A}], \mathbf{lfp}(f) \simeq B \to \neg\mathbf{Lfp}^v(B, t, \mathbf{I}) \tag{19}$$

where B, C are new free classification variables.

Now assume $m \in \mathbb{N}$, $\vec{\mathcal{A}} \in \mathbf{CL}^m$ and $\vec{\mathfrak{a}} \in \mathbf{S}$. Set $f := t[\vec{\mathfrak{a}}, \vec{\mathbf{0}}, \vec{\mathcal{A}}, \vec{\emptyset}]$.

Define $k := m + 1$. By 3.19 there exists $\rho \in \mathbf{Aut}(\mathbf{B}/\mathbf{ran}(\mathbf{h}_m))$ such that $\rho = \rho^{-1}$,

$$\Big(\bigcup_i \mathbf{supp_B}(\rho(\mathfrak{a}_i)) \;\cup\; \bigcup_j \mathbf{supp_B}(\rho(\mathcal{A}_j)) \;\cup\; \mathbf{supp_B}(\rho(f))\Big) \;\cap\; \mathbf{ran}(\mathbf{h}_k) \subseteq \mathbf{ran}(\mathbf{h}_m)$$

and $\rho(\vec{\mathcal{A}}) \in \mathbf{CL}^m$.

We now distinguish two subcases.

Subcase 4.1: Assume $\mathfrak{M}^k \models \Gamma[\rho(\vec{\mathfrak{a}}), \rho(\vec{\mathcal{A}})]$. Employing Corollary 3.21, we then obtain $\mathfrak{M}^m \models \Gamma[\rho(\vec{\mathfrak{a}}), \rho(\vec{\mathcal{A}})]$. The latter yields $\mathfrak{M}^m \models \Gamma[\rho(\rho(\vec{\mathfrak{a}})), \rho(\rho(\vec{\mathcal{A}}))]$ due to 3.14; thence $\mathfrak{M}^m \models \Gamma[\vec{\mathfrak{a}}, \vec{\mathcal{A}}]$ as $\rho = \rho^{-1}$.

Subcase 4.2: Assume $\mathfrak{M}^k \not\models \Gamma[\rho(\vec{\mathfrak{a}}), \rho(\vec{\mathcal{A}})]$. Set $\bar{f} := \rho(f)$. Substituting $\rho(\vec{\mathfrak{a}}), \rho(\vec{\mathcal{A}})$ for $\vec{a}, \vec{A}$, the induction hypotheses pertaining to (18) and (17) supply

$$\forall \mathcal{B} \in \mathbf{CL}^k\; \mathfrak{M}^k \models \exists Y\, (\mathcal{B} \mathrel{\mathring{\subseteq}} \mathbf{I} \to Y \mathrel{\mathring{\subseteq}} \mathbf{I} \wedge \bar{f}\mathcal{B} \simeq Y), \tag{20}$$

$$\forall \mathcal{B}, \mathcal{C} \in \mathbf{CL}^k\; \mathfrak{M}^k \models \mathcal{B} \mathrel{\mathring{\subseteq}} \mathbf{I} \wedge \mathcal{C} \mathrel{\mathring{\subseteq}} \mathbf{I} \wedge \mathcal{B} \mathrel{\mathring{\subseteq}} \mathcal{C} \to \bar{f}\mathcal{B} \mathrel{\mathring{\subseteq}} \bar{f}\mathcal{C}, \tag{21}$$

$$\forall \mathcal{B} \in \mathbf{CL}^k\; \mathfrak{M}^k \models \mathbf{lfp}(\bar{f}) \simeq \mathcal{B} \to \neg\mathbf{Lfp}^v(\mathcal{B}, \bar{f}, \mathbf{I}). \tag{22}$$

Recall that $\mathbf{supp_B}(\bar{f}) \cap \mathbf{ran}(\mathbf{h}_k) \subseteq \mathbf{ran}(\mathbf{h}_m)$. (20) and (21) yield $\mathfrak{M}^k \models \mathbf{Clop}(\bar{f}, \mathbf{I}) \wedge \mathbf{Mon}(\bar{f}, \mathbf{I})$. Thus, putting to use Corollary 3.26, we obtain $\mathbf{lfp}(\bar{f}) \in \mathbf{CL}^{\mathbf{V}_k, \mathbf{h}_k}$ and $\mathfrak{M}^k \models \mathbf{Lfp}(\mathbf{lfp}(\bar{f}), \bar{f}, \mathbf{I})$. The latter yields $\mathfrak{M}^k \models \mathbf{Lfp}^v(\mathbf{lfp}(\bar{f}), \bar{f}, \mathbf{I})$ which collides with (22). Consequently, this subcase cannot occur and we are back to the first subcase. As a result we have $\mathfrak{M}^m \models \Gamma[\vec{\mathfrak{a}}, \vec{\mathcal{A}}]$ as desired. □

Corollary 4.10 *If $\mathbf{T_0}(\mathbf{I}){\restriction} + \mathbf{UMID_I} \vdash \phi(\vec{x}, \vec{X})$ for a Σ^{EM}-formula ϕ, then, for all m, $\mathbf{KP}^r + \Sigma_1$-$\mathbf{Sep}$ proves that for all $\vec{\mathfrak{a}} \in \mathbf{S}$ and $\vec{\mathcal{A}} \in \mathbf{CL}^{Vm, \mathbf{h}_m}$, $\mathfrak{M}^{\mathbf{V}_m, \mathbf{h}_m} \models \phi[\vec{\mathfrak{a}}, \vec{\mathcal{A}}]$.*

Theorem 4.11 $\mathbf{T_0}(\mathbf{I}){\restriction} + \mathbf{UMID_I} \;\leq\; \mathbf{KP}^r + \Sigma_1$-$\mathbf{Sep}$.

4.3 Relativizations

The results of this section can be relativized to the models $\mathfrak{M}_U^{\mathbf{V}_n^U, \mathbf{h}_n^U}$ of Definition 3.31. In particular we get:

Corollary 4.12 *If $\mathbf{T_0}(\mathbf{I}){\restriction} + \mathbf{UMID_I} \vdash \phi(\vec{x}, \vec{X})$ for a Σ^{EM}-formula ϕ, then, for all m, $\mathbf{KP}^r + \Sigma_1$-$\mathbf{Sep}$ proves that for all $U \subseteq \omega$, $\vec{\mathfrak{a}} \in \mathbf{S}$ and $\vec{\mathcal{A}} \in \mathbf{CL}^{\mathbf{V}_m^U, \mathbf{h}_m^U}$, $\mathfrak{M}_U^{\mathbf{V}_m^U, \mathbf{h}_m^U} \models \phi[\vec{\mathfrak{a}}, \vec{\mathcal{A}}]$.*

5 Reducing set theory to analysis and vice versa

In this section the circle of reductions is completed by showing that $\mathbf{KP}^r + \Sigma_1$-**Sep** and $\mathbf{KP}^w + \Sigma_1$-**Sep** prove the same theorems of second order arithmetic as $(\mathbf{\Pi}^1_2 - \mathbf{CA}) \restriction$ and $(\mathbf{\Pi}^1_2 - \mathbf{CA})$, respectively. The language of second-order arithmetic is regarded as a sublanguage of set theory via the translation mapping numerical quantifiers $\exists x$ to $\exists x(x \in \omega \wedge \dots)$ and set quantifiers $\exists X$ to $\exists X(X \subseteq \omega \wedge \dots)$. We also show that $\mathbf{KP} + \Sigma_1$-Separation is a conservative extension of $(\mathbf{\Pi}^1_2 - \mathbf{CA}) + \mathbf{BI}$, where **BI** is the so-called principle of *Bar Induction*, i.e. the axiom schema

$$\forall X \left(\mathrm{WO}(<_X) \wedge \forall n \left[\forall m <_X n\, \Phi(m) \rightarrow \Phi(n)\right] \rightarrow \forall n \Phi(n)\right)$$

for all formulae Φ of the language of second order arithmetic, where $m <_X n := 2^m \cdot 3^n \in X$.

Definition 5.1 a) A non-empty transitive set A is called an *admissible set* if

$$\langle A, \in \rangle \models \mathbf{KP}.$$

b) An ordinal α is called *admissible*, if $\mathbf{L}_\alpha$ is an admissible set.

Recall that *Infinity* is assumed to be among the axioms of **KP**. The next lemma is needed for proving $\mathbf{\Pi}^1_2$ Comprehension in $\mathbf{KP}^r + \Sigma_1$-**Sep**.

Lemma 5.2 $\mathbf{KP}^r + \Sigma_1$-$\mathbf{Sep} \vdash \forall x \subseteq \omega\, \exists y\, [x \in y \wedge y$ *is an admissible set*$]$.

Proof: Suppose $X \subseteq \omega$. Let $\mathbf{L}(X)$ be the class of all sets constructible from X. Note that $\mathbf{L}(X)$ is naturally equipped with a Δ_1 definable well-ordering $<_{\mathbf{L}(X)}$ since X inherits a well-ordering from ω.

Let A_X be the set of those $a \in \mathbf{L}(X)$ for which there is a Σ_1 definition of a in $\mathbf{L}(X)$ using the parameter X. To be more precise, let

$$B_X = \{\ulcorner \phi(u,v,w) \urcorner : \ulcorner \phi(u,v,w) \urcorner \text{ is the Gödel number of a } \Delta_0 \text{ formula } \phi \text{ s.t. } \mathbf{L}(X) \vDash \exists y \exists z \phi(y,z,X)\} \tag{23}$$

and define $F : B_X \longrightarrow \mathbf{L}(X)$ by

$$F(\ulcorner \phi(u,v,w) \urcorner) = <_{\mathbf{L}(X)}\text{–least pair } \langle c,d \rangle \text{ s.t. } \mathbf{L}(X) \vDash \exists y \exists z \phi(y,z,X) \tag{24}$$

and finally put

$$A_X = \{c : \exists d\, (\langle c,d \rangle \in \mathbf{ran}(F))\}. \tag{25}$$

Using a Σ_1 satisfaction predicate, one sees that B_X is Σ_1 definable and thus B_X is a set by Σ_1 Separation. Then $\mathbf{ran}(F)$ is a set by Σ collection and consequently A_X is a set. Obviously

$$\langle A_X, \in \cap\, A \times A \rangle \prec_1 \mathbf{L}(X). \tag{26}$$

Now let c_{A_X} be the Mostowski collapsing function on A_X. Then $\mathbf{ran}(c_{A_X})$ is an admissible set due to (26), and, in addition, this set contains X since

$$c_{A_X}(X) = \{c_{A_X}(n) : n \in X\} = \{n : n \in X\} = X.$$

This proves our lemma. □

Theorem 5.3 $\mathbf{KP}^r + \Sigma_1$-**Sep** *and* $(\mathbf{\Pi}^1_2 - \mathbf{CA})\restriction$ *prove the same sentences of second order arithmetic.*

Proof: "$\supseteq$": We shall be arguing informally in $\mathbf{KP}^r + \Sigma_1$-**Sep**.

First, we address Comprehension. Instead of Π^1_2 Comprehension we may as well show Σ^1_2 Comprehension.

In the set-theoretic language a Σ^1_2 formula becomes a formula

$$\exists x \subseteq \omega\ \forall y \subseteq \omega\ \psi(n, x, y)$$

where ψ is arithmetic, i.e. all quantifiers in ψ are bounded by ω. Now with any Π^1_1 formula $\theta(u, U)$ with free variables u and U ranging over ω and subsets of ω, respectively, one can associate an arithmetic formula $y \prec_{u,U} z$ such that for all $X \subseteq \omega$, $\prec_{n,X}$ is a binary relation on ω and given $n \in \omega$,

$$\theta(n, X) \quad \text{iff} \quad \prec_{n,X} \text{ is well-founded.} \tag{27}$$

For any binary relation $\prec$ on ω we define an operation $C_\prec$ via Σ Recursion on the ordinals:

$$C_\prec(\alpha) \quad = \quad \{n \in \omega : \forall m[m \prec n \to m \in \bigcup_{\beta<\alpha} C_\prec(\beta)]\}. \tag{28}$$

Hence

$$\begin{aligned} \{n \in \omega : \exists x \subseteq \omega\ \forall y \subseteq \omega\ \psi(n, x, y)\} \ = \\ \{n \in \omega : \exists X \subseteq \omega\ [\prec_{n,X} \text{ is well-founded}]\}. \end{aligned} \tag{29}$$

Suppose now that A is an admissible set, that $\prec$ is well-founded and $\prec$ is an element of A. Let $F_\prec$ be the restriction of $C_\prec$ to the ordinals of A. Then $F_\prec$ is a set which is Σ_1 definable on A. We claim that

$$\forall n \in \omega \exists \alpha \in A\ [n \in F_\prec(\alpha)]. \tag{30}$$

If this were not the case, we would let n_0 be a $\prec$-least integer n such that $n \notin \bigcup \mathbf{ran}(F_\prec)$. Consequently,

$$\forall m \prec n_0 \exists \alpha \in A[m \in F_\prec(\alpha)].$$

But then, by Σ Reflection in A, there would exist $\alpha_0 \in A$ such that $\forall m \prec n_0\ [m \in F_\prec(\alpha_0)]$, yielding the contradiction $n_0 \in F_\prec(\alpha_0 + 1)$.

¿From (30), using Σ Reflection in A, we obtain an $\alpha \in A$ such that $\omega \subseteq F_\prec(\alpha)$. Thus we obtain a function $H \in A$ with $H : \omega \longrightarrow \mathbf{ON}$ by letting

$$H(n) \quad = \quad \text{least } \alpha.\ n \in F_\prec(\alpha). \tag{31}$$

The important property that H satisfies is

$$\forall n \forall m[n \prec m \to H(n) < H(m)]. \tag{32}$$

On the other hand, (32) always implies that $\prec$ is well-founded. So the upshot is that, in view of Lemma 5.2, the well-foundedness of a relation $\prec$ on ω is equivalent to the existence of an admissible set A which contains $\prec$ and a function $H\in A$ satisfying (32). Let $\phi(H,\prec)$ be a shorthand for (32). By the preceding, the right hand side of (29) gives the same class as

$$\{n\in\omega : \exists A[A \text{ is admissible} \wedge x\in A \wedge \exists H\in A\, \phi(H,\prec_{n,x})]\}, \tag{33}$$

rendering $\{n\in\omega : \exists x\subseteq\omega\ \forall y\subseteq\omega\ \psi(n,x,y)\}$ a Σ_1 class and therefore a set via Σ Separation.

"$\subseteq$": In the course of the proof we employ the method of trees which has been used by several people (see [2], Sec. 5). Within $(\mathbf{\Pi}^1_2-\mathbf{CA})\restriction$ we make the following definitions:

A *tree* is a non–empty set T of (codes for) finite sequences of natural numbers such that $s\subseteq t\wedge t\in T\ \rightarrow\ s\in T$. A tree T is said to be *well founded* if there is no function f such that $\forall n f[n]\in T$, where $f[n]=\langle f(0),\cdots,f(n-1)\rangle$. Trees T and T' are said to be *isomorphic*, written $T\cong T'$, if there exists an isomorphism between them, i.e. an order preserving bijection of T onto T'. If s and t are finite sequences of natural numbers, $s\star t$ denotes the concatenation of s followed by t. If T is a tree and $s\in T$, we write $T_s=\{t : s\star t\in T\}$.

A tree T is said to be *suitable*, written $\mathbf{ST}(T)$, if it is well founded and, for all $s\in T$, if $s\star\langle m\rangle\in T$ and $s\star\langle n\rangle\in T$ and $T_{s\star\langle m\rangle}\cong T_{s\star\langle n\rangle}$, then $m=n$.

Clearly the predicate $\mathbf{ST}$ is Π^1_1. The point of the definition is that if T and T' are suitable then there is at most one order preserving bijection of T onto T'. For suitable trees T and T' we write $T\breve{\in}T'$ to mean $\exists n\left[\langle n\rangle\in T'\ \wedge\ T\cong T'_{\langle n\rangle}\right]$. The relations $\cong$ and $\breve{\in}$ are Σ^1_1 on $\mathbf{ST}$.

The idea is now to identify a suitable tree T with the inductively defined set

$$|T| = \left\{|T_{\langle n\rangle}| : \langle n\rangle\in T\right\}$$

and in this way to model hereditarily countable sets within second order arithmetic (cf. [2], Sect.5, [12], [15]. The nice thing about suitable trees is that we have

$$|T|=|T'| \quad\text{iff}\quad T\cong T', \quad\text{and}\quad |T|\in|T'| \quad\text{iff}\quad T\breve{\in}T'.$$

Specifically, if T,T' are suitable trees and, for all S, $S\breve{\in}T$ iff $S\breve{\in}T'$, then $T\cong T'$.

Now let $\mathfrak{A}=\langle\mathcal{M},\mathcal{X},...,\in\rangle$ be a model of $(\mathbf{\Pi}^1_2-\mathbf{CA})\restriction$, where $\mathcal{M}=(M,\cdots)$ is a model of the first order theorems of $(\mathbf{\Pi}^1_2-\mathbf{CA})\restriction$ and $\mathcal{X}$ is a subset of the power set of M. To be precise, this notation means that in $\mathfrak{A}$ the set quantifiers range over the elements of $\mathcal{X}$.

Let $B=\left\{T\in\mathcal{X} : \mathfrak{A}\models\mathbf{ST}(T)\right\}$. For $T,T'\in B$ set $[T]=\{S\in B : \mathfrak{A}\models S\cong T\}$ and

$$[T]\in_{\mathfrak{B}}[T'] \quad\text{iff}\quad \mathfrak{A}\models T\breve{\in}T'.$$

Let $\mathfrak{B}$ be the structure $\left\langle\{[T] : T\in B\},\in_{\mathfrak{B}}\right\rangle$ for the language $\mathcal{L}_{ST}$.

By the above considerations, we know that $\mathfrak{B}\models$ *Extensionality.* We intend to show that $\mathfrak{B}$ is a model of $\mathbf{KP}^r+\Sigma_1$-$\mathbf{Sep}$. The set-theoretic language can be interpreted into the language of second order arithmetic as follows. Set theoretic variables are interpreted as ranging over suitable trees. The equality relation $=$ between set theoretic variables is interpreted as $\cong$, and $\in$ is interpreted as $\breve{\in}$.

For a set theoretic formula φ let φ^A be the corresponding second order arithmetic formula. We then get for $T_1, \dots, T_k \in B$ that

$$\mathfrak{B} \models \varphi([T_1], \dots, [T_k]) \quad \text{iff} \quad \mathfrak{A} \models \varphi^A(T_1, \dots, T_k). \tag{34}$$

Note that if φ happens to be a Δ_0 formula, then φ^A will be equivalent to a Δ^1_2 formula within the structure $\mathfrak{A}$, because any universal bounded quantifier in φ gets translated into a quantifier of the form

$$\forall S\,[\mathbf{ST}(\mathbf{S}) \wedge \mathbf{S}\breve{\in}\mathbf{T} \ \rightarrow \ \dots \mathbf{S} \dots],$$

where T is a suitable tree. The latter is equivalent to

$$\forall S\,[\exists n\, S \cong T_{\langle n\rangle} \ \rightarrow \ \dots S \dots],$$

and thus equivalent to

$$\forall n[\dots T_{\langle n\rangle} \dots],$$

employing Extensionality.

First, we want to very that $\mathfrak{B}$ is a model of Δ_0 Collection. Suppose

$$\mathfrak{B} \models \forall x \in [T] \exists y \varphi(x, y)$$

where ψ is Δ_0. For convenience, let us assume that ψ has no free variables other than x, y. By (34) it follows

$$\mathfrak{A} \models \forall S \exists R[\mathbf{ST}(S) \wedge S \breve{\in} T \ \rightarrow \ \mathbf{ST}(R) \wedge \psi^A(S, R));$$

hence $\mathfrak{A} \models \forall n \exists R \Theta(n, R, T)$, where

$$\Theta(S, R, T) \ \text{ stands for } \ \langle n\rangle \in T \ \rightarrow \ \mathbf{ST}(R) \wedge \psi^A(T_{\langle n\rangle}, R).$$

Now $\mathbf{\Delta^1_2} - \mathbf{CA}$ proves the Σ^1_2 Axiom of Choice, ($\Sigma^1_2 - \mathbf{AC}$), since the proof of the Kondo–Addison uniformization theorem can be done within $\mathbf{\Delta^1_2} - \mathbf{CA}$. $\Theta(S, R, T)$ is equivalent to a Σ^1_2 formula. So, by $\Sigma^1_2 - \mathbf{AC}$ in $\mathfrak{A}$, there is an $X \in \mathcal{X}$ such that

$$\mathfrak{A} \models \forall n \Theta(n, (X)_n, T). \tag{35}$$

Setting

$$V = \{t \in M : \mathfrak{A} \models \exists n \exists s (t = \langle n\rangle \star s \wedge \langle n\rangle \in T \wedge s \in (X)_n \wedge \forall m < n\,[(X)_m \ncong (X)_n])\},$$

we have $V \in \mathcal{X}$ by $\Delta^1_2 - \mathbf{CA}$ in $\mathfrak{A}$. By the very definition of V it follows

$$\mathfrak{A} \models \mathbf{ST}(V) \wedge \forall n \exists R \breve{\in} V \Theta(n, R, T), \tag{36}$$

thus $\mathfrak{B} \models \forall x \in [T] \exists y \in [V] \psi(x, y)$. This verifies $\mathfrak{B} \models \Delta_0$ Collection.

Next we verify Σ_1 Separation. Consider a Σ_1 class in $\mathfrak{B}$:

$$K \ = \ \{[S] : \mathfrak{B} \models [S] \in [T] \ \wedge \ \exists y \psi([S], y, [T], [P])\} \tag{37}$$

with ψ being Δ_0. Then

$$K \;=\; \{[T_{\langle m\rangle}] : \langle m\rangle \in T \;\wedge\; \mathfrak{B} \models \exists y \psi([T_{\langle m\rangle}], y, [T], [P])\}. \tag{38}$$

Put

$$Y \;=\; \{m : \langle m\rangle \in T\,; \mathfrak{A} \models \exists R\big[\mathbf{ST}(R) \wedge \psi^A(T_{\langle m\rangle}, R, T, P)\big]. \tag{39}$$

Since the defining formula can be rendered Σ^1_2, we have $Y \in \mathcal{M}$. Now define

$$T^* \;=\; \{\langle m\rangle \star s : \langle m\rangle \star s \in T \;\wedge\; m \in Y\}. \tag{40}$$

Then T^* is a suitable tree in $\mathfrak{A}$ and

$$\mathfrak{B} \models \forall \mathfrak{x}[\mathfrak{x} \in [\mathfrak{T}^*] \;\leftrightarrow\; \mathfrak{x} \in [\mathfrak{T}] \;\wedge\; \exists \mathfrak{y} \psi(\mathfrak{x}, \mathfrak{y}, [\mathfrak{T}], [\mathfrak{P}]). \tag{41}$$

This shows that $\mathfrak{B}$ is a model of Σ_1 Separation.

The verification of the remaining axioms of $\mathbf{KP}^r + \Sigma_1$-**Sep** is routine.

In the rest of the proof we are going to show that the second order arithmetic part of $\mathfrak{B}$, that is to say

$$\big\langle \omega^{\mathfrak{B}}, \mathrm{Pow}(\omega)^{\mathfrak{B}}, \in^{\mathfrak{B}} \cap\, (\omega^{\mathfrak{B}} \times \mathrm{Pow}(\omega)^{\mathfrak{B}})\big\rangle$$

is isomorphic to $\mathfrak{A}$, so that the same sentences of second order arithmetic hold true in $\mathfrak{A}$ and $\mathfrak{B}$.

Within $(\mathbf{\Pi}^1_2 - \mathbf{CA})\!\restriction$ we define, for $n \in \mathbb{N}$ and $X \subseteq \mathbb{N}$,

$$T^n \;=\; \big\{\langle k_1, \ldots, k_r\rangle :\; k_1, \ldots k_r \in \mathbb{N};\; n > k_1 > \ldots > k_r\,\big\} \tag{42}$$

$$T^X \;=\; \big\{\langle n\rangle \star s : s \in T^n \;\; ; n \in X\big\}. \tag{43}$$

Then $S \,\breve{\in}\, T^n$ iff $S \cong (T^n)_{\langle m\rangle} = T^m$ for some $m < n$, and $T^n \,\breve{\in}\, T^X$ iff $n \in X$.

The mapping $i : \mathfrak{A} \to \big(\omega^{\mathcal{B}}, \mathrm{Pow}(\omega)^{\mathfrak{B}}\big)$ determined by $n \mapsto [T^n]$ and $X \mapsto [T^X]$ then provides the desired isomorphism. Therefore the same sentences of second order arithmetic hold in $\mathfrak{A}$ and $\mathfrak{B}$.

It should be clear how the model $\mathfrak{B}$ lends itself to a syntactic translation of $\mathbf{KP}^r$ + Σ_1-**Sep** into $(\mathbf{\Pi}^1_2 - \mathbf{CA})\!\restriction$. When in doubt, consult [12], Sec. 7. □

Theorem 5.4 *The following theories prove the same sentences of second order arithmetic:*

(i) $\mathbf{KP}^w + \Sigma_1$-**Sep** *and* $(\mathbf{\Pi}^1_2 - \mathbf{CA})$.

(ii) $\mathbf{KP} + \Sigma_1$-**Sep** *and* $(\mathbf{\Pi}^1_2 - \mathbf{CA}) + \mathbf{BI}$.

Proof: In the proof we shall refer to definitions made in the proof of 5.3.

(i): It is obvious that $\mathbf{KP}^w + \Sigma_1$-**Sep** proves the (translation) of **IND**. Thus, in view of Theorem 5.3 and its proof, it remains to verify that $\mathfrak{B}$ is a model of the schema of induction over ω. Well, the role of ω in $\mathfrak{B}$ is played by

$$T^\omega := \big\{\langle n\rangle \star s : s \in T^n \;\; ; n \text{ is a natural number}\big\},$$

where T^n is defined as in (42). Consequently, the schema of induction over ω in $\mathfrak{B}$ follows from the induction schema **IND** in $\mathfrak{A}$.

(ii): "$\supseteq$": By Theorem 5.3, it remains to verify Bar Induction. Let $\prec$ be a well-founded relation on ω. We have to verify transfinite induction along $\prec$ for arbitrary classes in our background theory.

To this end, we define an operation $C_\prec$ via Σ Recursion on the ordinals:

$$C_\prec(\alpha) \quad = \quad \{n \in \omega : \forall m[m \prec n \rightarrow m \in \bigcup_{\beta<\alpha} C_\prec(\beta)]\}. \tag{44}$$

Employing Σ Separation,

$$X_\prec \quad = \quad \{n \in \omega : \exists \alpha \, [n \in C_\prec(\alpha)]\} \tag{45}$$

is a set. We claim that

$$X_\prec \;=\; \omega. \tag{46}$$

If this were not the case, let n_0 be a $\prec$-least integer such that $n_0 \notin X_\prec$. This implies $\forall m \prec n_0 \, [m \in X_\prec]$, and thus $\forall m \prec n_0 \exists \alpha \in A \, [m \in C_\prec(\alpha)]$. But then, by Σ Reflection, there would exist α_0 such that $\forall m \prec n_0 \; [m \in C_\prec(\alpha_0)]$, yielding the contradiction $n_0 \in C_\prec(\alpha) \subseteq X_\prec$.

By virtue of (46), we obtain a function $G : \omega \longrightarrow \mathbf{ON}$ by letting

$$G(n) \quad = \quad \text{least } \alpha.\, n \in C_\prec(\alpha). \tag{47}$$

Since G satisfies $\forall n \forall m[n \prec m \rightarrow G(n) < G(m)]$, transfinite induction along $\prec$ for arbitrary classes follows from induction over ordinals, i.e. Foundation.

"$\subseteq$": In view of Theorem 5.3 it remains to verify that $\mathfrak{B}$ is a model of Foundation. Let φ be Δ_0. Assume

$$\mathfrak{B} \models \forall x \big[\forall y \in x \varphi(y) \rightarrow \; \varphi(x)\big].$$

Then, for $[T] \in \mathfrak{B}$, we must prove $\mathfrak{B} \models \varphi([T])$. By (34) we have

$$\mathfrak{A} \models \forall S \big[\mathbf{ST}(S) \wedge \forall n\big(\langle n \rangle \in S \;\rightarrow\; \varphi^A(S_{\langle n \rangle})\big) \;\rightarrow\; \varphi^A(S)\big]. \tag{48}$$

Let $s \prec t$ iff $s, t \in T$ and $\mathfrak{A} \models s = t \star \langle m \rangle$ for some $m \in M$. Then $\mathfrak{A} \models \mathbf{WF}(\prec)$.
Now (48) implies

$$\mathfrak{A} \models \forall t \in T \big[(\forall s \prec t)\varphi^A(T_s) \;\rightarrow\; \varphi^A(T_t)\big].$$

By Bar Induction in $\mathfrak{A}$, this gives $\mathfrak{A} \models \varphi_A(T_{\langle\rangle})$, hence $\mathfrak{B} \models \varphi([T])$. $\square$

6 Conclusion

Recall that for theories T_1, T_2, we use the notation $T_1 \leq T_2$ to signify that T_1 is proof-theoretically reducible to T_2. $T_1 < T_2$ signifies that T_2 is proof-theoretically stronger than T_1. $T_1 \equiv T_2$ stands for proof-theoretic equivalence.

Due to Theorem 3.30, Theorem 4.11, Theorem 5.3, Theorem 5.4 and Theorem 1.1, we obtain:

Theorem 6.1 *(i)* $\mathbf{T_0}\!\restriction + \mathbf{UMID}_{\mathbb{N}} \;\equiv\; \mathbf{T_0(I)}\!\restriction + \mathbf{UMID_I} \;\equiv\; (\mathbf{\Pi^1_2} - \mathbf{CA})\!\restriction \;\equiv\; \mathbf{KP^r} + \Sigma_1$**-Sep**.

(ii) $\mathbf{T_0}\!\restriction + \mathbf{IND}_{\mathbb{N}} + \mathbf{UMID}_{\mathbb{N}} \;\equiv\; \mathbf{T_0(I)}\!\restriction + \mathbf{IND}_{\mathbb{N}} + \mathbf{UMID_I} \;\equiv\; (\mathbf{\Pi}^1_2 - \mathbf{CA}) \;\equiv\; \mathbf{KP^w} + \Sigma_1\text{-}\mathbf{Sep}$.

(iii) $(\mathbf{\Pi}^1_2 - \mathbf{CA}) \;<\; \mathbf{T_0} + \mathbf{UMID}_{\mathbb{N}} \;\leq\; \mathbf{T_0(I)} + \mathbf{UMID_I} \;\leq\; (\mathbf{\Pi^1_2} - \mathbf{CA}) + \mathbf{BI} \;\equiv\; \mathbf{KP} + \Sigma_1\text{-}\mathbf{Sep}$.

Regarding the full system $\mathbf{T_0} + \mathbf{UMID}$, we conjecture that

$$\mathbf{T_0} + \mathbf{UMID}_{\mathbb{N}} \;\equiv\; \mathbf{T_0(I)} + \mathbf{UMID_I} \;\equiv\; \mathbf{T_0} + \mathbf{UMID} \;\equiv\; (\mathbf{\Pi}^1_2 - \mathbf{CA}) + \mathbf{BI}.$$

In point of fact, we obtain more specific results than the previous theorem. Recall that any sentence ϕ of second order arithmetic has a canonical translation ϕ^* in the language of $\mathbf{T_0}$ (see [14], Definition 5.1).

Theorem 6.2 *Let ϕ be a Π^1_3 sentence of second order arithmetic.*

(i) $(\mathbf{\Pi}^1_2 - \mathbf{CA})\!\restriction \vdash \phi$ *iff* $\mathbf{T_0}\!\restriction + \mathbf{UMID}_{\mathbb{N}} \vdash \phi^*$ *iff* $\mathbf{T_0(I)}\!\restriction + \mathbf{UMID_I} \vdash \phi^*$.

(ii) $(\mathbf{\Pi}^1_2 - \mathbf{CA}) \vdash \phi$ *iff* $\mathbf{T_0}\!\restriction + \mathbf{IND}_{\mathbb{N}} + \mathbf{UMID}_{\mathbb{N}} \vdash \phi^*$ *iff* $\mathbf{T_0(I)}\!\restriction + \mathbf{IND}_{\mathbb{N}} + \mathbf{UMID_I} \vdash \phi^*$.

(iii) If $\mathbf{T_0(I)} + \mathbf{UMID_I} \vdash \phi^*$*, then* $(\mathbf{\Pi}^1_2 - \mathbf{CA}) + \mathbf{BI} \vdash \phi$.

Proof: The directions "$\Rightarrow$" in (i) and (ii) are due to Theorem 1.1. For "$\Leftarrow$" let ϕ be $\forall X \exists Y \forall Z \theta(X, Y, Z)$ with θ arithmetic.

(i): Suppose $\mathbf{T_0(I)} \restriction + \mathbf{UMID_I} \vdash \phi^*$. Rendering the part "$\forall Z\, \theta(X, Y, Z)$" in $\mathbf{\Pi}^1_1$ normal form, one obtains an arithmetical relation $<^\theta_{X,Y}$ such that

$$(\mathbf{\Pi}^0_\infty - \mathbf{CA})\!\restriction \vdash \forall \mathbf{X} \forall \mathbf{Y} [\forall \mathbf{Z} \theta(\mathbf{X}, \mathbf{Y}, \mathbf{Z}) \leftrightarrow \mathbf{WF}(<^\theta_{\mathbf{X},\mathbf{Y}})].$$

Since $(\mathbf{\Pi}^0_\infty - \mathbf{CA})\!\restriction$ is a subtheory of $\mathbf{T_0}\!\restriction$ via *, the latter equivalence also holds in $\mathbf{T_0}\!\restriction$. As a result, $\mathbf{T_0(I)}\!\restriction + \mathbf{UMID_I} \vdash \phi^*$ yields

$$\mathbf{T_0(I)}\!\restriction + \mathbf{UMID_I} \vdash \forall X \mathrel{\mathring{\subseteq}} \mathbb{N}\, \exists Y, Z \mathrel{\mathring{\subseteq}} \mathbb{N}\, [\mathbf{i}(\mathbb{N}, <^{\theta^*}_{X,Y}) \simeq Z \wedge Z \mathrel{\mathring{=}} \mathbb{N}].$$

As "$\exists Y, Z \mathrel{\mathring{\subseteq}} \mathbb{N}\, [\mathbf{i}(\mathbb{N}, <^{\theta^*}_{X,Y}) \simeq Z \wedge Z \mathrel{\mathring{=}} \mathbb{N}]$" is Σ^{EM}, we can employ Corollary 4.12 to obtain

$$\mathbf{KP}^r + \Sigma_1\text{-}\mathbf{Sep} \vdash \forall X \subseteq \omega\, \mathfrak{M}_X^{\mathbf{V}^X_n, \mathbf{h}^X_n} \models \exists Y, Z \mathrel{\mathring{\subseteq}} \mathbb{N}\, [\mathbf{i}(\mathbb{N}, <^{\theta^*}_{X^\circ,Y}) \simeq Z \wedge Z \mathrel{\mathring{=}} \mathbb{N}].$$

The latter yields $\mathbf{KP}^r + \Sigma_1\text{-}\mathbf{Sep} \vdash \forall X \subseteq \omega\, \mathbf{L}_{\sigma^X_n}(X) \models \exists Y \subseteq \omega\, \mathbf{WF}(<^\theta_{X,Y})$, and hence $\mathbf{KP}^r + \Sigma_1\text{-}\mathbf{Sep} \vdash \forall X \subseteq \omega\, \exists Y \subseteq \omega\, \forall Z \subseteq \omega\, \theta(X, Y, Z)$ as well-foundedness is absolute for the structures $\mathbf{L}_{\sigma^X_n}(X)$.

Thus, by Theorem 5.4, $(\mathbf{\Pi}^1_2 - \mathbf{CA})\!\restriction \vdash \phi$.

(ii) "$\Leftarrow$": In this proof we don't resort to $\mathbf{\Pi}^1_1$ normal form.
Suppose $\mathbf{T_0(I)}\!\restriction + \mathbf{IND}_{\mathbb{N}} + \mathbf{UMID_I} \vdash \phi^*$. Then, using Theorem 3.32, $\mathbf{KP}^w + \Sigma_1\text{-}\mathbf{Sep}$ proves

$$\forall X \subseteq \omega\, \widetilde{\mathfrak{M}}_X \models \exists Y \mathrel{\mathring{\subseteq}} \mathbb{N}\, \forall Z \mathrel{\mathring{\subseteq}} \mathbb{N}\, \theta(X^\circ, Y, Z)^*.$$

Arguing in $\mathbf{KP}^w + \Sigma_1\text{-}\mathbf{Sep}$, fix $X \subseteq \omega$ and let $Y \mathrel{\mathring{\subseteq}} \mathbb{N}$ be a classification in $\widetilde{\mathfrak{M}}_X$ such that

$$\widetilde{\mathfrak{M}}_X \models \forall Z \mathrel{\mathring{\subseteq}} \mathbb{N}\, \theta(X^\circ, Y, Z)^*. \tag{49}$$

Set $\mathbf{L}_\infty^X := \bigcup_{n\in\omega} \mathbf{L}_{\sigma_n^X}(X)$. We claim that for each $W \in \mathbf{L}_\infty^X$ with $W \subseteq \omega$ there exists $Z \in \mathbf{CL}_{\widetilde{\mathfrak{M}}_X}$ such that $W^\circ = Z$. To see this note that W gives rise to a well-ordering $<_W$ on ω which is defined by $k <_W m$ if $k < m \wedge k, m \in W$. $\mathbf{h}_\infty(<_W)$ is a classification in $\widetilde{\mathfrak{M}}_X$ from which the desired classification Z can be obtained via elementary comprehension.

The latter combined with (49) yields

$$\widetilde{\mathfrak{M}}_X \models \theta(X^\circ, Y, W^\circ)^*,$$

which implies

$$\mathbf{L}_\infty^X \models \theta(X, \bar{Y}, W),$$

where $\bar{Y} := \{k \in \omega : \ k^\circ \mathrel{\mathring{\in}} Y\}$. Since $W \in \mathbf{L}_\infty^X$ was arbitrary, it follows

$$\mathbf{L}_\infty^X \models \exists Y \subseteq \omega\, \forall W \subseteq \omega\, \theta(X, Y, W).$$

Since the preceding formula is $\mathbf{\Sigma}^1_2$ in X and $\mathbf{L}_\infty^X$ is absolute for such formulae, it follows $\forall X \subseteq \omega\, \exists Y \subseteq \omega\, \forall Z \subseteq \omega\, \theta(X, Y, Z)$. Thus, by 5.4, $(\mathbf{\Pi}^1_2 - \mathbf{CA}) \vdash \psi$.

(iii) is proved in the same way as (ii) "$\Leftarrow$". □

Open problems The main problem left open in this paper is whether the equivalences of Theorem 6.2 also hold with **UMID** in place of $\mathbf{UMID_I}$. To my knowledge, the only construction of a model of $\mathbf{T_0} + \mathbf{UMID}$ was given by Takahashi in Appendix 2 of [16]. Unfortunately, there is just a sketch of a model of $\mathbf{T_0} + \mathbf{UMID}$ in [16]. I must confess that I never understood that construction. I conjecture though, that the structure $\widetilde{\mathfrak{M}}$ of subsection 3.4 (or a slight variant of it) is already a model of **UMID**. A proof of $\widetilde{\mathfrak{M}} \models \mathbf{UMID}$ seems to require more "fine structure theory" of $\widetilde{\mathfrak{M}}$ than is provided in the present paper. To show that $\widetilde{\mathfrak{M}} \models \mathbf{UMID_I}$ we utilized the fact that the elements of **I** are invariant under automorphisms. This part of the proof wouldn't work for **UMID**. But similar problems emerged in the construction of models for **MID** in [10, 16]. A fruitful avenue to pursue might be to combine the techniques of [10] with the ones in the present paper.

Other interesting problems for further investigations in this area are raised in the last section ("Outlook") of [14]. To determine the strength of **MID** and **UMID** on the basis of *intuitionistic* explicit mathematics are the most challenging among them. Here it might be useful to address first the related problem of fixed points for monotone inductive definitions in constructive Zermelo-Fraenkel set theory (cf. [1]).

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