Fragments of Kripke–Platek Set Theory with Infinity

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Abstract

In this paper we shall investigate fragments of Kripke–Platek set theory with Infinity which arise from the full theory by restricting Foundation to Π_n Foundation, where $n \geq 2$. The strength of such fragments will be characterized in terms of the smallest ordinal α such that L_α is a model of every Π_2 sentence which is provable in the theory.

1 Introduction

Kripke-Platek set theory plus Infinity (hereinafter called KPω) is a truly remarkable subsystem of ZF. Though considerably weaker than ZF, a great deal of set theory requires only the axioms of this subsystem (cf.[Ba]). KPω consists of the axioms Extensionality, Pair, Union, (Set)Foundation, Infinity, along with the schemas of Δ_0–Collection, Δ_0–Separation, and Foundation for Definable Classes. So KPω arises from ZF by completely omitting Power Set and restricting Separation and Collection to Δ_0–formulas. These alterations are suggested by the informal notion of "predicative". KPω is an impredicative theory, notwithstanding. It is known from [Ho1], [Ho2] and [J] that KPω proves the same arithmetical sentences as Feferman's system ID_1 of positive inductive definitions (cf.[Fe]). Its proof–theoretic ordinal is the Howard ordinal $\theta\varepsilon_{\Omega+1}0$.

This article deals with fragments resulting from KPω by restricting the amount of foundation. The *Foundation Schema* is considered in the form

$$\forall x[(\forall y \in x)A(y) \rightarrow A(x)] \rightarrow \forall x A(x).$$

For a class of set–theoretic formulas $\mathcal{H}$, we denote by $\mathcal{H}$*–Foundation* this schema with $A(x)$ belonging to $\mathcal{H}$.

KPω^- is KPω without the Foundation Schema.

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As usual, L_α denotes the α–th level of the constructible hierarchy.

A set–theoretic formula is said to be Π_k (respectively Σ_k) if it consists of a string of k alternating quantifiers beginning with an universal (respectively existential) one, followed by a Δ_0–formula. A Δ_0–*formula* is a set–theoretic formula in which all quantifiers appear restricted.
The division of Foundation into Σ_k– and Π_k–Foundation is reminiscent of the commonly used hierarchy of subsystems of PA (= Peano Arithmetic). However, while Σ^0_n–Induction and Π^0_n–Induction are equivalent over PA without Induction, neither Π_n–Foundation nor Σ_n–Foundation needs to imply the other over KPω^- (according to R. Lubarsky; personal communication).

1.1 Definition. Let T be a subtheory of ZFC. For a collection of set theoretic sentences $\mathcal{H}$, we call L_α an $\mathcal{H}$–*model of* T if $\mathsf{L}_\alpha \models A$ holds for all theorems A of T with A from $\mathcal{H}$. By $|T|_{\mathcal{H}}$ we denote the least ordinal $\alpha > 0$ such that L_α is an $\mathcal{H}$ model of T.

$\alpha = |T|_{\Pi_2}$ will serve as a measure of strength for T. This is because for theories T (entailing KPω^- + Σ_1–Foundation), L_α is the least (non empty) transitive set closed under functions Σ_1–*definable in* T. Here a function $f : V \longrightarrow V$ (V:= universe of sets) is called Σ_1–definable in T if there is a Σ_1–formula $A(x,y)$ such that $V \models \forall x A(x, f(x))$ and $T \vdash \forall x\ \exists! y A(x,y)$.
Another justification for viewing $|T|_{\Pi_2}$ as a good measure of strength is that this ordinal equals the proof-theoretic ordinal $|T|$ of T (defined in [P]) provided that T is an impredicative theory.

In terms of Feferman-Aczel functions θ_α (cf. [Schü IX]), the main result of this paper reads as follows (with $\Omega := \Omega_1$):

1.2 Theorem. *Let* $\delta_1 = \Omega^\omega$, $\delta_{k+1} = \Omega^{\delta_k}$. *Then*

$$|\,\mathrm{KP}\omega^- + \Pi_{n+1}\text{–Foundation}\,| = |\,\mathrm{KP}\omega^- + \Pi_{n+1}\text{–Foundation}\,|_{\Pi_2} = \theta\delta_n 0$$

holds for $n \geq 1$.

Let $\mathcal{H}$–*Induction* denote the schema

$$F(0)\ \wedge\ (\forall x \in \omega)[F(x) \rightarrow F(x+1)] \rightarrow (\forall x \in \omega) F(x)$$

where $F(x)$ is an $\mathcal{H}$–formula and ω stands for the first limit ordinal.
By *IND* we denote $\mathcal{H}$–Induction with $\mathcal{H}$ the collection of all set theoretic formulas.

By employing an infinitary calculus with ω–rule, the methods used for establishing 1.2 can also be utilized to show the following result:

1.2^* **Theorem.** *For $n \geq 1$, we have*

$$|\mathrm{KP}\omega^- + \Pi_{n+1}\text{–Foundation} + IND|_{\Pi_2} = \theta\eta_n 0$$

where $\eta_1 = \Omega^{\varepsilon_0}$ and $\eta_{k+1} = \Omega^{\eta_k}$.

There are some results known from the literature which we want to go into. They require some notations.

1.3 Definition. For ordinals α, we define a function φ_α from ordinals to ordinals by the following recursion: $\varphi_0(\xi)$ is ω^ξ; for $\alpha > 0$, $\varphi_\alpha(\xi)$ is the ξth simultaneous fixed point of all functions φ_β with $\beta < \alpha$.
We write ε_α for $\varphi_1(\alpha)$ and $\varphi\alpha\beta$ for $\varphi_\alpha(\beta)$. The least α such that $\varphi\alpha 0 = \alpha$ is usually denoted by Γ_0. For further background information on these functions cf.[Schü] and [P].

Cantini ([Ca2]) proves $|\ \mathrm{KP}\omega^- + \Sigma_1\text{–Foundation}\ |_\Sigma = \varphi\omega 0$ and $|\ \mathrm{KP}\omega^- + \Sigma_1 - \text{Foundation} + \mathit{IND}\ |_\Sigma = \varphi\varepsilon_o 0$. Furthermore, it is known from Cantini [Ca1] that $|\,\mathrm{KP}\omega^- + \Pi_1 - \text{Foundation} + \mathit{IND}\,|_\Sigma = \varepsilon_0$. Here Σ means the smallest collection of formulas containing the Δ_0–formulas closed under $\wedge, \vee, (\exists x \in a), (\forall x \in a)$, and $\exists x$ (cf.[Ba]). The methods of [Ca1] can be easily adapted to yield $|\,\mathrm{KP}\omega^- + \Pi_1\text{–Foundation} + \Sigma_1\text{–Induction}\,|_{\Pi_2} = \omega^\omega$. However, Π_1–Foundation is not Foundation enough to yield an interesting fragment of KPω. $\mathrm{KP}\omega^- + \Pi_1$–Foundation + *IND* is even too weak to prove totality of the ordinal function $\alpha \mapsto \alpha + \alpha$.

We commence with a brief description of the content of this paper. In Section 2 we set up sequent calculus versions of $\mathrm{KP}\omega^-$ and $\mathrm{KP}\omega^- + \omega$–rule, the benefit of which is to admit partial cut–elimination. This technique will be exploited in Section 3. We also show that $|\ T\ |_{\Sigma_1}$ and $|\ T\ |_{\Pi_2}$ coincide for reasonable theories T. Section 3 is devoted to establishing upper bounds for $|\ \mathrm{KP}\omega^- + \Pi_n\text{–Foundation}\ |_{\Pi_2}$ where $n \geq 2$. This requires elaborated techniques from impredicative proof theory. In part we shall build on Pohlers [P] (this volume). Finally, we show in Section 4 that the upper bounds obtained in Section 3 are best possible. Unfortunately, it is by no means clear how to adapt the techniques used for Π_n–Foundation to Σ_n–Foundation.

2 Partial models, partial cut–elimination

The usual proof of the Σ Reflection Principle goes through in $\mathrm{KP}\omega^-$ (cf.[Ba I.4.3]). In particular, every Σ–formula is equivalent to a Σ_1–formula in $\mathrm{KP}\omega^-$. Therefore, $\mathrm{KP}\omega^- + \Sigma_1$–Foundation implies Σ-Foundation. Moreover, if T comprises $\mathrm{KP}\omega^-$, every Σ_1–model of T needs to be a Σ–model of T. By the next theorem we can even go further. Hereinafter, we use the following notations:

If B is a formula then B^x results from B by replacing each unrestricted quantifier $\forall y(\cdots)$ and $\exists y(\cdots)$ by $(\forall y\in a)(\cdots)$ and $(\exists y\in a)(\cdots)$, respectively (cf.[Ba I.4]).

2.1 Theorem. *Let* $\mathrm{KP}\omega^- \subseteq T$. *Furthermore, suppose that* $T\vdash B$ *implies* $T\vdash \exists\alpha\exists x(x=L_\alpha \wedge B^x)$ *for all* Σ_1*–sentences* B. *If* T *has a* Σ_1*–model then* T *has a* Π_2*–model and*

$$|\,T\,|_{\Sigma_1} = |\,T\,|_{\Pi_2}\ .$$

Proof. Let L_σ be the minimal Σ_1–model of T. Assume $T\vdash \forall u\exists w H(u,w)$ with $H(u,w)$ being Δ_0. Let $a\in\mathsf{L}_\sigma$. We have to verify that $\mathsf{L}_\sigma\models\exists w H(a,w)$. σ is a limit, so there is $\xi<\sigma$ such that $a\in\mathsf{L}_\xi$. Since L_ξ is not a Σ_1 model of T, we have $T\vdash B$ and $\mathsf{L}_\xi\models\neg B$ for some Σ_1 sentence B. By assumption, we also get $T\vdash\exists\alpha\exists x(x=\mathsf{L}_\alpha\wedge B^x)$. Then, using Δ_0–Collection, we obtain

$$T\vdash \exists z\exists\alpha\exists x[x=\mathsf{L}_\alpha\wedge B^x\wedge(\forall u\in x)(\exists w\in z)H(u,w)]$$

Since this formula is equivalent to a Σ_1–formula in $\mathrm{KP}\omega^-$, we get $\mathsf{L}_\sigma\models\exists\alpha\exists x[x=\mathsf{L}_\alpha\wedge B^x\wedge(\forall u\in x)\exists w H(u,w)]$. As σ is a limit $>\omega$, the formula "$x=\mathsf{L}_\alpha$" doesn't shift its meaning when we move from L_σ to the universe (see [D II.2.12]). Hence there exists $\alpha<\sigma$ such that $\mathsf{L}_\alpha\models B$ and $(\forall u\in\mathsf{L}_\alpha)(\exists w\in\mathsf{L}_\sigma)H(u,w)$. By the choice of B, this implies $\xi<\alpha$, hence $a\in\mathsf{L}_\alpha$, thus $\mathsf{L}_\sigma\models\exists w H(a,w)$. □

2.2 Remark. The construction of the constructible hierarchy can be carried out in $\mathrm{KP}\omega^- + \Sigma_1$–Foundation, and it can be shown that for every theorem A of $\mathrm{KP}\omega^-$, we have $\mathrm{KP}\omega^- + \Sigma_1$–Foundation $\vdash A^{\mathsf{L}}$ (cf.[Ba]). Hence the theorem above applies to such theories as $\mathrm{KP}\omega^- + \Pi_k$–Foundation $+\,\Pi_r$–*IND* and $\mathrm{KP}\omega^- + \Sigma_n$–Foundation $+\,\Pi_r$–*IND*, where $k\geq 2$ and $n,r\geq 1$.

When using $|T|_{\Pi_2}$ as a measure of strength, one is naturally led to ask for the relation of this ordinal to the proof–theoretic ordinal $|\,T\,|$ of T (cf.[P]). As a rule of thumb we have for $\alpha=\omega^\alpha$ and $\mathrm{KP}\omega^-\subseteq T\subseteq\mathrm{KP}\omega$

$$|\,T\,|_{\Pi_2}=\alpha\ \Rightarrow |\,T\,|=\varphi\alpha 0.$$

Why? Usually, the proof of $|\,T\,|_{\Pi_2}\geq\alpha$ lends itself to an interpretation of the system $RA_{<\alpha}$ of ramified analysis in T (cf.[FS] and [Schü]). Schütte established that $|\,RA_{<\alpha}\,|=\varphi\alpha 0$. Since for the theories T we have in mind here, the determination of $|\,T\,|_{\Pi_2}$ also yields an embedding of T into a system $RS_{<\alpha}$ of ramified set theory, we also get $|\,T\,|\leq\varphi\alpha 0$ by the methods of [P Theorem 25].

For technical reasons we shall diverge from the usual presentation of $\mathrm{KP}\omega$.

As our basic system underlying the various theories we choose a Tait–style sequent calculus version of $\mathrm{KP}\omega^-$ in which finite sets of formulas can be derived. In addition, formulas have to be in negation normal form (cf.[Schw]).
The language consists of: free variables $a_0, a_1, \cdots$, bound variables $x_0, x_1, \cdots$; the predicate symbol $\in$; the logical symbols $\neg, \vee, \wedge, \forall, \exists$.
We will use $a, b, c, \cdots,\ x, y, z, \cdots,\ A, B, C, \cdots$ as metavariables whose domains are the domain of the free variables, bound variables, formulas, respectively.
The *atomic formulas* are those of the form $(s \in t), \neg(s \in t)$.
The *formulas* are defined inductively as follows:
(i) Atomic formulas are formulas.
(ii) If A and B are formulas, then so are $(A \wedge B)$ and $(A \vee B)$.
(iii) If $A(b)$ is a formula in which x does not occur, then $\forall x A(x), \exists x A(x)$, $(\forall x \in a)A(x), (\exists x \in a)A(x)$ are formulas.
The quantifiers $\exists x, \forall x$ will be called *unrestricted*. A Δ_0–*formula* is a formula which contains no unrestricted quantifiers.
The *negation* $\neg A$ of a formula A is defined to be the formula obtained from A by (i) putting $\neg$ in front of any atomic formula, (ii) replacing $\wedge, \vee, \forall x, \exists x, (\forall x \in a), (\exists x \in a)$ by $\vee, \wedge, \exists x, \forall x, (\exists x \in a), (\forall x \in a)$, respectively, and (iii) dropping double negations.
$\vec{a}, \vec{b}, \vec{c}, \cdots$ and $\vec{x}, \vec{y}, \vec{z}, \cdots$ will be used to denote finite sequences of free and bound variables, respectively.
We use $F[a_1, \cdots, a_n]$ (by contrast with $F(a_1, \cdots, a_n)$) to denote a formula the free variables of which are among $a_1, \cdots, a_n$. We will write
$a = \{x \in b : G(x)\}$ for $(\forall x \in a)[x \in b \wedge G(x)] \wedge (\forall x \in b)[G(x) \rightarrow x \in a]$.
By $\mathsf{Tran}(a)$, $\mathsf{Ord}(a)$, $\mathsf{Lim}(a)$ we abbreviate the Δ_0–formulas expressing that a is transitive, a is an ordinal, a is a limit ordinal, respectively.
$a = b$ stands for $(\forall x \in a)(x \in b) \wedge (\forall x \in b)(x \in a)$.

2.3 Definition. (The theory T_n) T_n derives finite sets of formulas denoted by $\Gamma, \Delta, \Theta, \Xi, \cdots$. The intended meaning of Γ is the disjunction of all formulas of Γ. We use the notation Γ, A for $\Gamma \cup \{A\}$, Γ, Ξ for $\Gamma \cup \Xi$.

The *axioms of* T_n are:

Logical axioms:	$\Gamma, A, \neg A$ for every Δ_0–formula A.
Extensionality:	$\Gamma, a = b \wedge B(a) \rightarrow B(b)$ for every Δ_0-formula $B(a)$.
Pair:	$\Gamma, \exists x[a \in x \wedge b \in x]$
Union:	$\Gamma, \exists x(\forall y \in a)(\forall z \in y)(z \in x)$
Δ_0 *–Separation:*	$\Gamma, \exists y(y = \{x \in a : G(x)\})$ for every Δ_0–formula $G(b)$.
Foundation Axiom:	$\Gamma, (\exists x \in a)(x \in a) \rightarrow (\exists y \in a)(\forall z \in y)\neg(z \in a)$
Infinity:	$\Gamma, \exists x\, \mathsf{Lim}(x)$.

The *logical rules of inferences* are:

$(\wedge)$	$\vdash \Gamma, A$ and $\vdash \Gamma, B$	$\Rightarrow$	$\vdash \Gamma, A \wedge B$
$(\vee)$	$\vdash \Gamma, A_i$ for $i \in \{0,1\}$	$\Rightarrow$	$\vdash \Gamma, A_0 \vee A_1$
$(b\forall)$	$\vdash \Gamma, a \in b \rightarrow F(a)$	$\Rightarrow$	$\vdash \Gamma, (\forall x \in b)F(x)$
$(\forall)$	$\vdash \Gamma, F(a)$	$\Rightarrow$	$\vdash \Gamma, \forall x F(x)$
$(b\exists)$	$\vdash \Gamma, a \in b \wedge F(a)$	$\Rightarrow$	$\vdash \Gamma, (\exists x \in b)F(x)$
$(\exists)$	$\vdash \Gamma, F(a)$	$\Rightarrow$	$\vdash \Gamma, \exists x F(x)$
(Cut)	$\vdash \Gamma, A$ and $\vdash \Gamma, \neg A$	$\Rightarrow$	$\vdash \Gamma.$

Of course, it is demanded that in $(b\forall)$ and $(\forall)$ the free variable a is not to occur in the conclusion; a is called the *eigenvariable* of that inference.

The *non–logical rules of inferences* are:

$(\Delta_0$–COLLR) $\vdash \Gamma, (\forall x \in a)\exists y H(x,y) \quad \Rightarrow \quad \vdash \Gamma, \exists z(\forall x \in a)(\exists y \in z)H(x,y)$

for every Δ_0–formula $H(b,c)$.

$(\Pi_n$–FR) $\vdash \Gamma, \exists x \exists y \neg Q\vec{z}\,[x \in a \rightarrow H(x,y,\vec{z})],\, Q\vec{z}H(a,b,\vec{z}) \quad \Rightarrow$
$\vdash \Gamma, Q\vec{z}\,H(c,d,\vec{z}),$

where $Q\vec{z}$ stands for a string of n–1 alternating quantifiers beginning with an existential one, and $H(a,b,\vec{e})$ is a Δ_0–formula. In addition, it is demanded that a and b are different free variables neither appearing in formulas of Γ nor in $\forall x \forall y Q\vec{z}\,H(x,y,\vec{z})$.

We shall conceive of axioms as inferences with an empty set of premisses. The *minor formulas* (m.f.) of an inference are those formulas which are rendered prominently in its premises. The *principal formulas* (p.f.) of an inference are the formulas rendered prominently in its conclusion. (Cut) has no p.f. So any inference has the form

$$(*) \quad \text{For all } i < k \ \vdash \Gamma, \Xi_i \quad \Rightarrow \quad \vdash \Gamma, \Xi$$

$(0 \leq k \leq 2)$, where Ξ consists of the p.f. and Ξ_i is the set of m.f. in the i–th premise. The formulas in Γ are called *side formulas* (s.f.) of $(*)$.
Derivations of T_n are defined inductively, as usual. $\mathcal{D}$,$\mathcal{D}$',$\mathcal{D}_0, \cdots$ range as syntactic variables over T_n derivations. All this is completely standard, and we refer to [Schw] for notions like "*length of a derivation* $\mathcal{D}$" (abbreviated by $|\mathcal{D}|$), "*last inference of* $\mathcal{D}$", "*direct subderivation of* $\mathcal{D}$". We write $\mathcal{D} \vdash \Gamma$ if $\mathcal{D}$ is a derivation of Γ.

We are not going to prove Theorem 1.2*. In order to get this result, one has to adapt the techniques of this article to an infinitary system $\mathrm{KP}\omega^-_\infty$. In addition to the language of $\mathrm{KP}\omega^-$, the language of $\mathrm{KP}\omega^-_\infty$ has constants $\underline{\omega}$

and $\underline{n}$ for every $n \in \mathbb{N}$. Additional axioms of $\mathrm{KP}\omega_\infty^-$ are

$$\Gamma, \mathsf{Lim}(\underline{\omega}) \wedge (\forall x \in \underline{\omega}) \neg \mathsf{Lim}(x),$$

$$\Gamma, \underline{n} \in \underline{\omega},$$

$$\Gamma, \underline{n} \in \underline{m},$$

$$\Gamma, \underline{n'} \notin \underline{m'}$$

if $n < m$ and $n' \not< m'$.
Of course, the axioms and rules of inferences of $\mathrm{KP}\omega^-$ have to be adapted to the enriched language. Furthermore, we have in $\mathrm{KP}\omega_\infty^-$ the infinitary *ω–rule*.

$$\vdash \Gamma, F(\underline{n}) \text{ for every } n \in \mathbb{N} \ \Rightarrow \vdash \Gamma, (\forall x \in \underline{\omega}) F(x).$$

$\mathrm{KP}\omega_\infty^-$ derivations may be infinite. The ω-rule allows one to derive all instances of *IND*.

The most fundamental property of sequent calculi is cut–elimination. Our sequent calculus T_n admits cut–elimination as far as it concerns cuts the cut formula of which is neither a principal formula of a non–logical rule of inference nor a principal formula of an axiom. This is a general phenomenon which will be exploited next. In order to state this fact in more precise terms, let us introduce a measure of complexity $cp(A)$ for formulas A:

Let $cp(A) = 0$ if A is Δ_0. If A is not Δ_0, then $cp(A)$ is inductively defined as follows: $cp(A) = \sup(cp(B), cp(C)) + 1$ if $A \in \{B \wedge C, B \vee C\}$; $cp(A) = cp(F(a)) + 2$ if $A \in \{(\forall x \in b) F(x), (\exists x \in b) F(x)\}$; $cp(A) = cp(F(a)) + 1$ if $A \in \{\forall x F(x), \exists x F(x)\}$.

The *cut–rank* $\rho(\mathcal{D})$ of a derivation $\mathcal{D}$ is also defined by induction:
Let $\mathcal{D}_i$, $i < k$, be the direct subderivations of $\mathcal{D}$. If the last inference of $\mathcal{D}$ is (Cut) with m.f. A and $\neg A$, let $\rho(\mathcal{D}) = \sup(cp(A) + 1, \sup\{\rho(\mathcal{D}_i) : i < k\}$. Otherwise, let $\rho(\mathcal{D}) = \sup\{\rho(\mathcal{D}_i) : i < k\}$. By $T_n \overset{k}{\underset{m}{\vdash}} \Gamma$ we mean that there is a derivation $\mathcal{D} \vdash \Gamma$ such that $|\mathcal{D}| \leq k$ and $\rho(\mathcal{D}) \leq m$.

2.4 Theorem. (Cut–elimination) *Let* $2_0^k := k$ *and* $2_{m+1}^k := 2^l$ *where* $l := 2_m^k$. *If* $n \geq 2$ *and* $T_n \overset{k}{\underset{n+m}{\vdash}} \Gamma$ *, then* $T_n \overset{p}{\underset{n}{\vdash}} \Gamma$ *where* $p := 2_m^k$.

Proof. Observe that $cp(A) < n$ holds for every p.f. A of an axiom or a non logical rule of inference. So the result can be gotten by the same proof as in [Schw]. □

One readily verifies that T_n proves every theorem of $\mathrm{KP}\omega^-$. Thus it remains to verify:

2.5 Proposition. $T_n \vdash \Pi_n$–Foundation.

Proof. Let $Q\vec{z}H(a,b,\vec{z}\,)$ be Σ_{n-1} with $H(a,b,e)$ Δ_0, $a \not\equiv b$ fresh. Then

$$T_n \vdash (\forall x \in a)\forall y Q\vec{z}H(x,y,\vec{z}\,), \exists x\exists y\neg Q\vec{z}\,[x\in a \to H(x,y,\vec{z}\,)]$$

and

$$T_n \vdash \exists y\neg Q\vec{z}H(a,y,\vec{z}\,), Q\vec{z}H(a,b,\vec{z}\,)$$

yield

$$T_n \vdash B(a), \exists x\exists y\neg Q\vec{z}\,[x\in a \to H(x,y,\vec{z}\,)],\ Q\vec{z}H(a,b,\vec{z})$$

with $B(a) \equiv (\forall x\in a)\forall y Q\vec{z}H(x,y,\vec{z}\,) \wedge \exists y\neg Q\vec{z}H(a,y,\vec{z}\,)$.
By $(\exists)$ we get

$$T_n \vdash \exists u B(u), \exists x\exists y\neg Q\vec{z}\,[x\in a \to H(x,y,\vec{z}\,)], Q\vec{z}H(a,b,\vec{z}\,).$$

Using (Π_n–FR) we obtain $T_n \vdash \exists uB(u), Q\vec{z}H(a,b,\vec{z}\,)$, thus, by $(\forall)$,

$$T_n \vdash \exists uB(u), \forall u\forall yQ\vec{z}H(u,y,\vec{z}\,).$$

Now apply $(\vee)$ twice to obtain $T_n \vdash \exists uB(u) \vee \forall u\forall yQ\vec{z}H(u,y,\vec{z}\,)$. □

In the next section we shall embed T_n into an infinitary calculus $RS(\Omega)$. To handle this with optimal ordinal bounds, we have to resort to very well behaved derivations.

2.6 Definition. A T_n derivation $\mathcal{D} \vdash \Gamma$ is said to be *n–nice* if $\rho(\mathcal{D}) \le n$ and every Σ_n–formula which is a side formula of an inference of $\mathcal{D}$ belongs to Γ. In other words, if A is Σ_n and $A \notin \Gamma$ then A can only appear in $\mathcal{D}$ as a m.f. or p.f. of an inference of $\mathcal{D}$.

Let $\exists\Sigma_n$ be the collection of formulas of the shape $\exists x\exists yA(x,y)$ with $A(a,b)\in\Pi_{n-1}$.
Let

$$\Sigma_n^* := \exists\Sigma_n \cup \bigcup_{i\le n}\Sigma_i \ \cup\ \bigcup_{j<n}\Pi_j\ \cup\ \Sigma.$$

2.7 Lemma. *Let* $\Gamma \subseteq \Sigma_n^*$, $\quad \Xi = \{\exists z_1B_1(b_1,z_1),\cdots,\exists z_rB_r(b_r,z_r)\} \subseteq \Sigma_n$, $\Theta = \{\exists y_1\exists z_1B_1(y_1,z_1),\cdots,\exists y_r\exists z_rB_r(b_r,z_r)\}$.

(i) *If* $\mathcal{D} \vdash \Gamma,\Xi$, *then we can find an n–nice* $\mathcal{D}^\star \vdash \Gamma,\Theta$.

(ii) *If* $\mathcal{D}_0 \vdash \Gamma$, *then there is an n–nice* $\mathcal{D}_0^\star \vdash \Gamma$.

Proof. (i) By 2.4 we may assume $\rho(\mathcal{D}) \le n$. We proceed by induction on $|\mathcal{D}|$. If Γ,Ξ is an axiom, then so is Γ; hence Γ,Θ is an axiom. The derivation

consisting of this axiom is n–nice. Now suppose $0 < |\mathcal{D}|$. If neither a m.f. nor a p.f. of the last inference (l.i.) of $\mathcal{D}$ is Σ_n, then the assertion follows immediately by induction hypothesis.
Now assume that a formula $\exists z A(b,z) \in \Sigma_n$ is a m.f. of the l.i. of $\mathcal{D}$. Then this must be an instance of $(\exists)$ because $\rho(\mathcal{D}) \leq n$ and $\Gamma, \Xi \subseteq \Sigma_n^\star$. So we have a p.f. $\exists y \exists z A(y,z) \in \Gamma$ and the direct subderivation $\mathcal{D}_0$ takes the form $\mathcal{D}_0 \vdash \Lambda, \Xi'$ with $\Lambda \subseteq \Gamma$ and $\Xi' = \Xi, \exists z A(b,z)$. Now apply the induction hypothesis to this situation to get an n–nice derivation $\mathcal{D}^\star \vdash \Lambda, \Theta, \exists y \exists z A(y,z)$. As $\Lambda, \Theta, \exists y \exists z A(y,z) = \Gamma$, this gives the assertion.
Finally, suppose that $\exists z B(z) \in \Sigma_n$ is the p.f. of the l.i. of $\mathcal{D}$. This must be an instance of $(\exists)$. So there is a derivation $\mathcal{D}_0 \vdash \Gamma, \Xi, B(b)$ such that $\rho(\mathcal{D}_0) \leq n$ and $|\mathcal{D}_0| < |\mathcal{D}|$. Inductively we find an n–nice derivation $\mathcal{D}_0^\star \vdash \Gamma, \Theta, B(b)$. By use of $(\exists)$, we can continue $\mathcal{D}_0^\star$ to an n–nice derivation of $\Gamma, \Theta, \exists z B(z)$. If $\exists z B(z) \in \Gamma$, we are done. Otherwise, $\exists z B(z) \in \Xi$, thus another application of $(\exists)$ gives us an n–nice derivation $\mathcal{D}^\star \vdash \Gamma, \Theta$, since then $\exists z B(z)$ does not appear as a s.f. in $\mathcal{D}_0^\star$.

(ii) follows from (i) with $\Xi = \emptyset$. □

3 Upper Bounds

The reader would be advised to acquaint himself with [P Part II]. In this Section we adopt the calculus $RS(\Omega)$ and the terminology of [P].
The derivability relation $RS(\Omega) \vert\frac{\alpha}{\rho}\, \Gamma$ embodies the notion of $RS(\Omega)$–derivation. We shall write $\mathcal{D}_\Omega \vert\frac{\alpha}{\rho}\, \Gamma$ if $\mathcal{D}_\Omega$ is a proof tree witnessing $RS(\Omega) \vert\frac{\alpha}{\rho}\, \Gamma$. We shall use $\mathcal{D}_\Omega, \mathcal{D}_\Omega^n, \cdots$ as syntactic variables for $RS(\Omega)$ derivations.
An $\mathcal{L}_{RS}$–formula is $\Sigma_n(\mathsf{L}_\alpha)$ $(\Pi_n(\mathsf{L}_n))$ if it is of the form $A(s_1, \cdots, s_n)^{\mathsf{L}_\alpha}$ for a Σ_n–formula (Π_n–formula) $A(a_1, \cdots, a_n)$ and RS–terms $s_1, \cdots, s_n$ being members of RS_α. Likewise, an $\mathcal{L}_{RS}$–formula is $\Sigma_n^*(\mathsf{L}_\alpha)$ if it is of the form $A(s_1, \cdots, s_n)^{\mathsf{L}_\alpha}$ with $A(a_1, \cdots, a_n)$ being Σ_n^* and $s_1, \cdots, s_n$ being members of RS_α.
Analogous with T_n, we say that a $RS(\Omega)$ derivation $\mathcal{D}_\Omega \vert\frac{\alpha}{\rho}\, \Gamma$ is *n–nice* if $\rho < \Omega + n$ and every $\Sigma_n(\mathsf{L}_\Omega)$–formula appearing as a side formula of an inference of $\mathcal{D}_\Omega$ belongs to Γ.
If Γ is a set of $\mathcal{L}_{RS}$–formulas, we mean by $\Gamma \ll \alpha$ that $A \ll \alpha$ holds for every member A of Γ, where $A \ll \alpha$ means that, for every RS–term L_η occuring in A, we have $\eta \ll \alpha$. $\alpha \# \beta$ stands for the natural sum of α and β (cf. [P Lemma 23]).

The nice thing about n–nice derivation is that they allow us to improve on the Reduction Lemma ([P Lemma 38]). But beforehand, we have to consider two simple transformations which lead from n–nice $RS(\Omega)$ derivations to n–nice $RS(\Omega)$ derivations.

3.1 Lemma. (Inversion and Weakening) *Let* $E \equiv (\forall y \in \mathsf{L}_\Omega)G(y)$. *Let* $s \in RS_\Omega$.

(i) If $\mathcal{D}_\Omega \overset{\beta}{\underset{\rho}{\vdash}} \Lambda, E$ *is an* n*–nice* $RS(\Omega)$ *derivation and* $|\, s \,| \ll \gamma$, *then there is an* n*–nice derivation* $\mathcal{D}_\Omega^\Box$ *satisfying*

$$\mathcal{D}_\Omega^\Box \overset{\gamma\#\beta}{\underset{\rho}{\vdash}} \Lambda, G(s)\,.$$

(ii) If $\mathcal{D}_\Omega \overset{\beta}{\underset{\rho}{\vdash}} \Gamma$ *is* n*–nice, then we can find an* n*–nice*

$$\mathcal{D}_\Omega^\Diamond \overset{\beta}{\underset{\rho}{\vdash}} \Gamma, \Lambda\,.$$

Proof. Both of the assertions are to be proved by induction on β. (ii) is a triviality. As to (i), note that $E \notin \Sigma_n(\mathsf{L}_\Omega)$; thus cancelling E in a derivation does not affect its n–niceness. The additional parameter γ comes in when the last inference of $\mathcal{D}$ was an instance of $(\bigwedge)$ with principal formula E. In this situation we have a function f with $dom(f) = \mathcal{O}(E)$, $f \ll \beta$ and n–nice derivations $\mathcal{D}_\Omega^{G(t)} \overset{f(G(t))}{\underset{\rho}{\vdash}} \Lambda, E, G(t)$ for $t \in RS_\Omega$. Inductively we obtain an n–nice

$$\mathcal{D}_\Omega^\triangle \overset{\gamma\#f(G(s))}{\underset{\rho}{\vdash}} \Lambda, G(s)\,.$$

Now make use of $f \ll \beta$ and $|\, s \,| \ll \gamma$ to compute that $\gamma\#f(G(s)) \ll \gamma\#\beta$. Hence we get the desired derivation. $\Box$

3.2 Refined Reduction Lemma. *We identify* 0 *with* L_0. *Let* $B \equiv (\exists y \in \mathsf{L}_\Omega)(\exists z \in \mathsf{L}_\Omega)A(y,z)$ *where* $A(0,0)$ *is* $\Pi_{n-1}(\mathsf{L}_\Omega)$. *Let* $\rho = \Omega + (n-1)$. *Let* $\mathcal{D}_\Omega \overset{\alpha}{\underset{\rho}{\vdash}} \Gamma, B$ *as well as* $\mathcal{D}'_\Omega \overset{\beta}{\underset{\rho}{\vdash}} \Lambda, \neg B$ *be* n*–nice* $RS(\Omega)$ *derivations such that* $\Gamma, \Lambda \subseteq \Sigma_n^*(\mathsf{L}_\Omega)$. *Then we can find an* n*–nice derivation*

$$\mathcal{D}_\Omega^\star \overset{\alpha\#\beta}{\underset{\rho}{\vdash}} \Gamma, \Lambda\,.$$

Proof. By induction on α we construct a derivation $\mathcal{D}_\Omega^\star \overset{\alpha\#\beta}{\underset{\rho}{\vdash}} \Gamma, \Lambda$ such that every $\Sigma_n(\mathsf{L}_\Omega)$–formula appearing as a side formula in $\mathcal{D}_\Omega^\star$ also appears as a side formula in $\mathcal{D}_\Omega$ or $\mathcal{D}'_\Omega$. Hence, $\mathcal{D}_\Omega^\star$ will be automatically n–nice. We may assume that B is the *p.f.* of the *l.i.* of $\mathcal{D}_\Omega$ for otherwise the assertion follows immediately by induction hypothesis (*i.h.*). So the direct subderivation (*d.s.*) of $\mathcal{D}_\Omega$ has the form $\mathcal{D}_\Omega^0 \overset{\alpha_0}{\underset{\rho}{\vdash}} \Gamma_0, C$ where $C \equiv (\exists z \in \mathsf{L}_\Omega)A(s,z)$,
$\Gamma_0 \subseteq \Gamma, B$, $\quad C \ll \alpha$, and $\alpha_0 \ll \alpha$. If $C \in \Gamma$, then the *i.h.* gives us an n–nice derivation

$$\mathcal{D}_\Omega^+ \overset{\alpha_0\#\beta}{\underset{\rho}{\vdash}} \Gamma_0 \setminus \{B\}, \Lambda, C,$$

so we get a derivation $\mathcal{D}_\Omega^\star \overset{\alpha\#\beta}{\underset{\rho}{\vdash}} \Gamma, \Lambda$ by Weakening. If $C \notin \Gamma$, then n–niceness of $\mathcal{D}_\Omega$ implies that the l.i. of $\mathcal{D}_\Omega^0$ is $(\bigvee)$ with p.f. C, and the d.s. $\mathcal{D}_\Omega^1$ of $\mathcal{D}_\Omega^0$

has the form $\mathcal{D}^1_\Omega\overset{\alpha_1}{\underset{\rho}{\vdash}}\Gamma_1, A(s,t)$ where $\Gamma_1 \subseteq \Gamma_0$ and $A(s,t), \alpha_1 \ll \alpha_0$ Using the i.h. and $\alpha_1\#\beta \ll \alpha_0\#\beta$ we find a derivation

$$\mathcal{D}^2_\Omega\overset{\alpha_0\#\beta}{\underset{\rho}{\vdash}}\Gamma_1 \setminus \{B\}, \Lambda, A(s,t)$$

with all the required properties. From $A(s,t) \ll \alpha_0$ it follows $|\, s\,|, |\, t\,| \ll \alpha_0$, provided that both of s and t occur in $A(s,t)$. But if, for instance, s would not occur in $A(s,t)$, then $A(s,t) \equiv A(0,t)$, so we would be able to replace s by 0. Therefore we may assume $|\, s\,|, |\, t\,| \ll \alpha_0$. Consequently, by the use of Inversion and Weakening (3.1), $\mathcal{D}'_\Omega$ yields an n–nice derivation

$$\mathcal{D}^3_\Omega\overset{\alpha_0\#\beta}{\underset{\rho}{\vdash}}\Lambda, \neg A(s,t).$$

Now continue $\mathcal{D}^2_\Omega$ and $\mathcal{D}^3_\Omega$ via (cut) to get a derivation $\mathcal{D}^\star_\Omega\overset{\alpha\#\beta}{\underset{\rho}{\vdash}}\Gamma, \Lambda$ such that every $\Sigma_n(\mathsf{L}_\Omega)$ $s.f.$ of $\mathcal{D}^\star_\Omega$ is among the side formulas of $\mathcal{D}_\Omega$ or $\mathcal{D}'_\Omega$. To ensure this, note that $A(s,t) \ll \alpha_0 \ll \alpha\#\beta$. □

3.3 Theorem. (T_n Embedding Theorem) *Let $\vec{e}$ denote the string $e_1, \cdots, e_j$. Let $\Gamma[\vec{e}] = \{A_1[\vec{e}]\}, \cdots, A_l[\vec{e}]\}$ be a set of Σ^*_n–formulas. Suppose $T_n \vdash \Gamma[\vec{e}]$. Then we can find an integer $k > 0$ such that for all $RS(\Omega)$ terms $s_1, \cdots, s_j$ with stages $< \Omega$, there exists an n–nice $RS(\Omega)$ derivation*

$$\mathcal{D}_\Omega\overset{\alpha}{\underset{\rho}{\vdash}}\Gamma[\vec{s}]^{\mathsf{L}_\Omega}$$

where $\alpha = \Omega^k\#|s_1|\#\cdots\#|s_j|$ and $\rho = \Omega + (n-1)$.

Proof. By 2.7 there is an n–nice T_n derivation $\mathcal{D} \vdash \Gamma[\vec{e}]$. We proceed by induction on $|\, \mathcal{D}\,|$. If the l.i. of $\mathcal{D}$ is (Cut) with cut formula $A[\vec{e}]$ then $\mathrm{cp}(A[\vec{e}]) < n$. This implies $\mathsf{rk}(A[\vec{s}]^{\mathsf{L}_\Omega}) < \rho$. So the assertion follows from the i.h. via (cut).
Here we would like to refer to [P] for an embedding of $\mathrm{KP}\omega^-$ into $RS(\Omega)$ which takes account of the various axioms of $\mathrm{KP}\omega^-$ with precise ordinal bounds. Unfortunately, [P] does not supply the necessary information. Fortunately, there is another article in this volume that does. For most of the embedding we shall rely on [Bu].

We now restrict our attention to the situation in which (Π_n–FR) is the last inference of $\mathcal{D}$. Then the d.s. of $\mathcal{D}$ has the form

$$\mathcal{D}^0 \vdash \Lambda[\vec{e}], \exists x \exists y \neg Q\vec{z}\,(x \in a \rightarrow H[x,y,\vec{z},\vec{e}]), Q\vec{z}\,H[a,b,\vec{z},\vec{e}]$$

with $\Lambda[\vec{e}], Q\vec{z}\,H[a,b,\vec{z},\vec{e}] \subseteq \Gamma[\vec{e}]$. Inductively there exists an integer $k_0 > 0$ such that for all $RS(\Omega)$ terms $\vec{s}, r, t$ with stages $< \Omega$, there is an n–nice $RS(\Omega)$–derivation.

$$(1) \qquad \mathcal{D}^1_\Omega \overset{\beta}{\underset{\rho}{|}} X, D(r), E(r,t)$$

where $\beta = \Omega^{k_0} \# \, |\vec{s}| \, \# \, |r| \, \# \, |t|$, $X = \Lambda[\vec{s}]^{\mathsf{L}_\Omega}$,

$$D(r) \equiv (\exists x \exists y \neg Q\vec{z}\,(x \in r \to H[x,y,\vec{z},\vec{s}]))^{\mathsf{L}_\Omega},$$

and

$$E(r,t) \equiv (Q\vec{z}\,H[r,t,\vec{z},\vec{s}])^{\mathsf{L}_\Omega}.$$

Letting $f(r,t) := (\Omega^{k_0} \cdot \omega^{|r|+1}) \# \, |\,\vec{s}\,| \, \# \, |\,t\,|$ we want to show that there exists an n–nice derivation

$$(\star) \qquad \mathcal{D}^{r,t}_\Omega \overset{f(r,t)}{\underset{\rho}{|}} X, E(r,t).$$

To verify $(\star)$ we induct on $|\,r\,|$. By assumption, for all r', q satisfying $|\,r'\,| < |\,r\,|$, there exists an n–nice derivation

$$\mathcal{D}^{r',q}_\Omega \overset{f(r',q)}{\underset{\rho}{|}} X, E(r',q)\,.$$

For every p there is an n–nice (cf. [Bu Lemma 2.7])

$$\mathcal{D}^p_\Omega \overset{\eta}{\underset{0}{|}} p \neq r', \neg E(r',q), E(p,q)$$

with $\eta = f(r',q) \# \, |\,p\,|$.
Using (*cut*) we obtain for every $|\,r'\,| < |\,r\,|$ an n–nice

$$\mathcal{D}^\star_\Omega \overset{\eta+1}{\underset{\rho}{|}} p \neq r', E(p,q), X$$

since $\mathsf{rk}(E(r',q)) < \rho$. By applying $(\bigwedge)$, we get an n–nice

$$\mathcal{D}^+_\Omega \overset{\gamma}{\underset{\rho}{|}} p \notin r, E(p,q), X$$

with $\gamma := (\Omega^{k_0} \cdot \omega^{|r|}) \# \, |\,\vec{s}| \, \# \, |\,q\,| \, \# \, |\,p\,|$.
Letting

$$B(p,q,r) \equiv (Q\vec{z}\,(p \in r \to H[p,q,\vec{z},\vec{s}]))^{\mathsf{L}_\Omega},$$

there are n–nice derivations

$$\mathcal{D}^\diamondsuit_\Omega \overset{\gamma}{\underset{0}{|}} p \in r, B(p,q,r) \ \text{ and } \ \mathcal{D}^\heartsuit_\Omega \overset{\gamma}{\underset{0}{|}} \neg E(p,q), B(p,q,r)\,.$$

So using cuts we arrive at an n–nice derivation $\mathcal{D}^\triangle_\Omega \overset{\gamma+3}{\underset{\rho}{|}} B(p,q,r), X$.
Applying two $(\bigwedge)$–inferences gives us an nice

$$(2) \qquad \mathcal{D}^2_\Omega \overset{\delta}{\underset{\rho}{|}} X, \neg D(r)$$

where $\delta := (\Omega^{k_0} \cdot \omega^{|r|}) \# \, |\,\vec{s}| \, \# \Omega \# \Omega$. Now 3.2 applied to $\mathcal{D}^1_\Omega$ and $\mathcal{D}^2_\Omega$ ((1),(2)) yields the desired n–nice

$$\mathcal{D}^{r,t}_\Omega \overset{f(r,t)}{\underset{\rho}{|}} X, E(r,t)\,.$$

This finishes the proof of $(\star)$. From $(\star)$ the assertion follows with $k := k_0+1$. □

3.4 Theorem. *Set $\Omega^\alpha(0) := \alpha$ and $\Omega^\alpha(k+1) := \Omega^{\Omega^\alpha(k)}$. Let A be a Σ_1 sentence. Put $B \equiv A^{\mathsf{L}_\Omega}$. If $n \geq 2$ and* KP$\omega^- + \Pi_n$–Foundation $\vdash A$, *then $RS(\Omega)\overset{\beta}{\underset{\beta}{|}} B$ holds for some $\beta < \vartheta_{\Omega^\omega(n-1)}(0)$.*

Proof. The assumption implies $T_n \vdash A$ by 2.5. So we can employ 3.3 to get for some k, $RS(\Omega)\overset{\Omega^k}{\underset{\rho}{|}} B$ with $\rho = \Omega + (n-1)$. Then cut elimination ([P Theorem 3.9]) yields $RS(\Omega)\overset{\delta}{\underset{\Omega+1}{|}} B$ where $\delta = \Omega^{\Omega^k(n-2)} = \Omega^k(n-1)$.
By the use of the Collapsing Lemma [P Corollary 42] this becomes $RS(\Omega)\overset{\vartheta_\delta(0)}{\underset{\vartheta_\delta(0)}{|}} B$. As $\vartheta_\delta(0) < \vartheta_{\Omega^\omega(n-1)}(0)$, this proves our theorem. □

3.4 will provide an upper bound for the minimal Σ_1–model of KP$\omega^- + \Pi_n$–Foundation if we can show that such derivations are sound with respect to the constructible hierarchy.
Let $\mathcal{T}$ be the collection of terms of $RS(\Omega)$ with stages $< \Omega$. In order to state the next result, we need to differentiate between the RS-term $\check{\mathsf{L}}_\alpha$ and the α–th level of the constructible hierarchy, L_α. For $t \in \mathcal{T}$ we define $l(t)$ as follows:

$$l(t) = \mathsf{L}_\alpha \text{ if } t \equiv \check{\mathsf{L}}_\alpha,$$
$$l(t) = \{x \in \mathsf{L}_\alpha : \mathsf{L}_\alpha \models F[x, l(t_1), \cdots, l(t_k)]\}$$
$$\text{if } t \equiv \{u \in \check{\mathsf{L}}_\alpha : F[u, t_1, \cdots, t_k]^{\check{\mathsf{L}}_\alpha}\}.$$

3.5 Soundness Theorem. *Let $\Gamma[a_1, \cdots, a_n]$ be a set of Σ–formulas; let $t_1, \cdots, t_k \in \mathcal{T}$ with stages $< \beta$. If $RS(\Omega)\overset{\beta}{\underset{\Omega}{|}} \Gamma[t_1, \cdots, t_k]^{\underline{L}_\Omega}$ and $\beta < \Omega$, then $L_\beta \models \Gamma[l(t_1), \cdots, l(t_k)]$, where the last formula denotes the disjunction of the formulas $\Gamma[l(t_1), \cdots l(t_k)]$.*

Proof by induction on β. Note that no inference (Cl_Ω) appears. As to the inferences $(\bigwedge)$, one needs to verify that $l(t) \subseteq \{l(s) : s \in \mathcal{T}, |s| < |t|\}$. Since $l(t) \subseteq \mathsf{L}_{|t|}$, it suffices to show

$$\mathsf{L}_{|t|} = \{l(s) : s \in \mathcal{T}, |s| < |t|\}.$$

This is easily done by induction on $|t|$. □

3.6 Theorem. *For $n \geq 2$ we have*

$$|\,\mathrm{KP}\omega^- + \Pi_n\text{–Foundation}\,|_{\Pi_2} \leq \vartheta_{\Omega^\omega(n-1)}(0).$$

Proof. By 3.4, 3.5, 2.1. □

4 Lower Bounds

We are left with the task to show that the upper bounds for minimal Π_2 models established in the previous section are best possible. To this end, we shall first define a relation $\lhd$ which is Δ_1–definable in $KP^- + \Sigma_1$–Foundation. In order to illuminate the meaning of $\lhd$, let ω_1^{CK} be the least admissible ordinal above ω, and let $\mathcal{A}$ be $\mathsf{L}_{\omega_1^{CK}}$, i.e. the least admissible set containing ω (cf.[Ba II]). Then $\mathcal{A}$ is a model of KP. Finally, let δ be the least ordinal so that $(\omega_1^{CK})^\delta = \delta$. If $\lhd_{\mathcal{A}}$ denotes the relation on $\mathcal{A}$ induced by $\lhd$, then $\lhd_{\mathcal{A}}$ can be easily visualized if one considers it to be obtained by projecting (cf.[Ba V.5]) the order relation of the ordinals below δ into $\mathcal{A}$. Observe also that $\delta = \varepsilon_{\omega_1^{CK}+1}$, where $\varepsilon_{\omega_1^{CK}+1}$ stands for the first ordinal $\beta > \omega_1^{CK}$ satisfying $\omega^\beta = \beta$. This is because $(\omega_1^{CK})^\alpha = \omega^{\omega_1^{CK}\cdot\alpha}$ for all α.

We define, as usual, the ordered pair $\langle x, y\rangle$ of x, y by $\langle x, y\rangle = \{\{x\}, \{x, y\}\}$ and prove that $\langle x, y\rangle = \langle z, w\rangle$ iff $x = z$ and $y = w$. This gives rise to define ordered n–tuples for $n > 2$, as follows, by induction on n:

$$\langle x_1, \cdots, x_n\rangle = \langle x_1, \langle x_2, \cdots, x_n\rangle\rangle.$$

4.1 Definition. (KP$^-$+Σ_1–Foundation) We simultaneously define a class of ordinal representations OR along with a binary relation $\lhd$ on OR as follows:

(1) $0 \in$ OR.

(2) If $1 \leq n, \alpha_1, \cdots, \alpha_n \in \mathrm{Ord}\setminus\{0\}, s_1, \cdots, s_n \in \mathrm{OR}$ and $s_1 \rhd \cdots \rhd s_n$, then

$$\hat{\Omega}^{s_1}\alpha_1 \oplus \cdots \oplus \hat{\Omega}^{s_n}\alpha_n := \langle n, \langle s_1, \alpha_1\rangle, \cdots, \langle s_n, \alpha_n\rangle\rangle \ \in \mathrm{OR}.$$

(3) If $s \in \mathrm{OR}\setminus\{0\}$, then $0 \lhd s$.

(4) If $s = \hat{\Omega}^{s_n}\alpha_1 \oplus \cdots \oplus \hat{\Omega}^{s_n}\alpha_n \in \mathrm{OR}$ and $t = \hat{\Omega}^{t_1}\beta_1 \oplus \cdots \oplus \hat{\Omega}^{t_k}\beta_k \in \mathrm{OR}$, then
$s \lhd t$ if one of the following holds:
(i) $n < k$ and $s_i = t_i$ as well as $\alpha_i = \beta_i$ for $1 \leq i \leq n$.
(ii) There exists $m \leq n, k$ such that
  (a) $s_i = t_i$ and $\alpha_i = \beta_i$ for $1 \leq i < m$,
  (b) Either $s_m \lhd t_m$ or $s_m = t_m$ and $\alpha_m < \beta_m$.

OR as well as $\lhd$ are defined by a Σ_+ inductive definition with closure ordinal

ω (cf.[Ba VI]). We are obliged to show that these definitions can be given within $\mathrm{KP}^- + \Sigma_1$–Foundation. Let $\chi_{OR}(s) = 1$ if $s \in \mathrm{OR}$, 0 otherwise. Let $\chi_\lhd(\langle s,t\rangle) = 1$ if $s \lhd t$, 0 otherwise. Then both functions can be defined by a recursion on a certain well–founded relation. To see this, define for $a \in \mathsf{L}$, $\mathsf{rank}_\mathsf{L}(a) = \mathsf{min}\{\alpha : a \in \mathsf{L}_\alpha\}$, and define for sets $x, y, u, v \in \mathsf{L}$,

$$\langle x,y\rangle \prec \langle u,v\rangle \quad \text{iff} \quad \mathsf{rank}_\mathsf{L}(x)\#\mathsf{rank}_\mathsf{L}(y) < \mathsf{rank}_\mathsf{L}(u)\#\mathsf{rank}_\mathsf{L}(v).$$

Then $\prec$ is easily seen to be a well–founded relation. If we now put,

$$G(\langle u,v\rangle) = \langle \chi_{OR}(u), \chi_\lhd(\langle u,v\rangle)\rangle,$$

then for some class function H, being Σ_1–definable in $\mathrm{KP}^- + \Sigma_1$–Foundation,

$$G(\langle u,v\rangle) = H(\langle u, v, G\,|\,\{\langle x,y\rangle : \langle x,y\rangle \prec \langle u,v\rangle\}\rangle).$$

This is a form of recursion which does not lead outside $\mathrm{KP}^- + \Sigma_1$–Foundation as $\{b : b \prec x\}$ is always a set and induction along $\prec$ with regard to Σ_1–formulas is implied by Σ_1–Foundation. Thus χ_{OR} and $\chi_\lhd$ are Σ_1–definable in the latter theory, hence OR and $\lhd$ are Δ_1–definable in this theory.

4.2 Definition. (i) Let $\hat{1} := \hat{\Omega}^0 1, \hat{\Omega} := \hat{\Omega}^{\hat{1}} 1$.

(ii) We define $s + b$ for $s, t \in \mathrm{OR}$ by the following recursion:
1. $s + 0 := s$ and $0 + s := s$.
2. Let $s = \hat{\Omega}^{s_1}\alpha_1 \oplus \cdots \oplus \hat{\Omega}^{s_k}\alpha_k$ and $t = \hat{\Omega}^{t_1}\beta_1 \oplus \cdots \oplus \hat{\Omega}^{t_m}\beta_m$ $(1 \leq k, m)$. If $s_1 \lhd t_1$ put, $s + t := t$. Otherwise, look for the largest index n $(1 \leq n \leq k)$ such that $t_1 \unlhd s_n$, and set:
2.1 $s + t := \hat{\Omega}^{s_1}\alpha_1 \oplus \cdots \oplus \hat{\Omega}^{s_n-1}\alpha_{n-1} \oplus \hat{\Omega}^{t_1}(\alpha_n + \beta_1) \oplus \hat{\Omega}^{t_2}\beta_2 \oplus \cdots \oplus \hat{\Omega}^{t_m}\beta_m$
if $s_n = t_1$;
2.2 $s + t := \hat{\Omega}^{s_1}\alpha_1 \oplus \cdots \oplus \hat{\Omega}^{s_n}\alpha_n \oplus \hat{\Omega}^{t_1}\beta_1 \oplus \cdots \oplus \hat{\Omega}^{t_m}\beta_m$
if $t_1 \lhd s_n$.

(iii) Let $\hat{\Omega}^s 0 := 0$. The mapping $\alpha \mapsto \hat{\alpha} := \hat{\Omega}^0\alpha$ constitutes a bijection between the ordinals and $\{s \in \mathrm{OR} \mid s \lhd \hat{\Omega}\}$ which also preserves the respective orderings; i.e. $\alpha < \beta$ implies $\hat{\alpha} \lhd \hat{\beta}$.

In the sequel we use the following abbreviations: $\hat{\Omega}^s := \hat{\Omega}^s 1$, $\hat{\Omega}^\alpha := \hat{\Omega}^{\hat{\alpha}}$, $s + \alpha := s + \hat{\alpha}$, where $s \in \mathrm{OR}$ and $\alpha \in Ord$.
p, q, r, s, t are supposed to range over elements of OR.

4.3 Definition. For a formula $A(x)$ let

$$Prog(\lhd, A) := \forall s[(\forall t \lhd s)A(t) \rightarrow A(s)].$$

For a collection $\mathcal{H}$ of set-theoretic formulas we mean by $TI(s, \mathcal{H})$ the schema:

$$Prog(\lhd, A) \rightarrow (\forall t \lhd s)A(t)$$

for all $\mathcal{H}$–formulas $A(x)$.

4.4 Lemma. (i) $\mathrm{KP}^- + \Sigma_1$–Foundation $\vdash \lhd$ *is a linear ordering.*
(ii)*Let* $n \geq 2$. *Suppose* $A(x)$ *is* Π_n. *Let* A_k *be the formula* $\forall s[(\forall t \lhd s A(t)) \to (\forall t \lhd s + \hat{\Omega}^k) A(t)]$. *Then*

$$\mathrm{KP}^- + \Pi_n\text{–Foundation} \vdash Prog(\lhd, A) \to A_k.$$

Proof. (i) follows from the definition of $\lhd$.
(ii) We proceed by outer induction on k. Assume $Prog(\lhd, A)$ and $(\forall t \lhd s) A(t)$. Then also $(\forall t \unlhd s) A(t)$. For $k = 0$ this gives the assertion, since $t \lhd s + \hat{\Omega}^0$ implies $t \unlhd s$. Now let $k = m + 1$. So we get A_m by the inductive assumption. Let $B(\alpha)$ be the formula $(\forall t \lhd s + \hat{\Omega}^m \alpha) A(t)$. Then $B(\alpha)$ is provably equivalent (in $\mathrm{KP}^- + \Sigma_1$–Foundation) to a Π_n–formula. Suppose $(\forall \beta < \alpha) B(\beta)$. Clearly, $B(0)$ holds. If α is a limit, then for $t \lhd s + \hat{\Omega}^m \alpha$ there exists $\beta < \alpha$ such that $t \lhd s + \hat{\Omega}^m \beta$ (this follows from the definition of $\lhd$), hence $B(\alpha)$ holds. Now let α be a successor $\gamma + 1$. Then $(\forall t \lhd s + \hat{\Omega}^m \gamma) A(t)$. Using A_m, this implies $(\forall t \lhd s + \hat{\Omega}^m \gamma + \hat{\Omega}^m) A(t)$, thus $(\forall t \lhd s + \hat{\Omega}^m \alpha) A(t)$, hence $B(\alpha)$. By the above considerations, we have $\forall \alpha[(\forall \beta < \alpha) B(\beta) \to B(\alpha)]$, hence $\forall \alpha B(\alpha)$ via Π_n–Foundation. In view of the definition of $\lhd$, it becomes clear that for every $t \lhd s + \hat{\Omega}^k$ there is a δ such that $t \lhd s + \hat{\Omega}^m \delta$. Therefore $(\forall t \lhd s + \hat{\Omega}^k) A(t)$ follows from $\forall \alpha B(\alpha)$. □

4.5 Lemma. *Let* OT *be the smallest subset of* OR *containing* 0 *and* $\hat{1}$ *closed under the rule:*
If $s = \hat{\Omega}^{s_1} \alpha_1 \oplus \cdots \oplus \hat{\Omega}^{s_k} \alpha_k$ *and* $s_1, \cdots, s_k, \hat{\alpha}_1, \cdots \hat{\alpha}_k \in \mathrm{OT}$, *then* $s \in \mathrm{OT}$. *Every* $p \in \mathrm{OT}$ *is* Δ_1*–definable in* $\mathrm{KP}^- + \Sigma_1$–Foundation,*and so we may assume that the language of this theory contains a constant for every* $p \in \mathrm{OT}$.
Let $K_n := \mathrm{KP}^- + \Pi_n$–Foundation. *For all* $n \geq 2$ *and* $p \in \mathrm{OT}$

$$K_n + TI(p, \Pi_{n+1}) \vdash TI(\hat{\Omega}^p, \Pi_n).$$

Proof. Suppose $A(t) \in \Pi_n$ and let

$$B(s) := \forall r[(\forall t \lhd r) A(t) \to (\forall t \lhd r + \hat{\Omega}^s) A(t)].$$

Note that $B(s)$ is equivalent to a Π_{n+1}–formula. We will show

Claim $\qquad K_n \vdash Prog(\lhd, A) \to Prog(\lhd, B).$

To see that the Claim implies the lemma, note that from the claim we can conclude

$$K_n + TI(p, \Pi_{n+1}) \vdash Prog(\lhd, A) \to (\forall t \lhd p) B(t).$$

Hence, $K_n + TI(p, \Pi_{n+1}) \vdash Prog(\lhd, A) \to B(p)$. Setting $r = 0$ in $B(p)$, we get

$$K_n + TI(p, \Pi_{n+1}) \vdash Prog(\lhd, A) \to (\forall t \lhd \hat{\Omega}^p)A(t),$$

which is what we needed for the lemma; hence we only need to prove the Claim.
We will work in K_n. Assume $Prog(\lhd, A)$ and $(\forall s' \lhd s)B(s')$; we want to conclude $B(s)$. By cases we have:

Case 1: $s = 0$. $B(0) \equiv \forall r[(\forall t \lhd r)A(t) \to (\forall t \lhd r+1)A(t)]$, which is immediate from $Prog(\lhd, A)$.

Case 2: $s = s' + 1$. Suppose $(\forall t \lhd r)A(t)$. We want to show $(\forall t \lhd r + \hat{\Omega}^s)A(t)$. Let $C(\alpha) := (\forall t \lhd r + \hat{\Omega}^{s'}\alpha)A(t)$. Then $C(0)$. By the use of $B(s'), C(\beta)$ implies $C(\beta+1)$. If α is a limit and $t' \lhd r + \hat{\Omega}^{s'}\alpha$, then there is a $\beta < \alpha$ such that $t' \lhd r + \hat{\Omega}^{s'}\beta$. This shows $(\forall \beta < \alpha)C(\beta) \to C(\alpha)$. And hence, using Π_n–Foundation, $\forall \alpha C(\alpha)$. Since for $t \lhd r + \hat{\Omega}^s$ there is an ordinal α such that $t \lhd r + \hat{\Omega}^{s'}\alpha$, we obtain $(\forall t \lhd r + \hat{\Omega}^s)A(t)$.

Case 3: s is of the shape $\hat{\Omega}^{s_1}\alpha_1 + \cdots + \hat{\Omega}^{s_j}\alpha_j$ where $s_j \neq 0$ or α_j is a limit. For $t \lhd r + \hat{\Omega}^s$, we then find a $s' \lhd s$ such that $t \lhd r + \hat{\Omega}^{s'}$. Therefore $B(s)$ is implied by $(\forall s' \lhd s)B(s')$. This completes the proof of Case 3, and hence the proof of the Claim. □

4.6 Proposition. *Let* $\hat{\Omega}^k(0) := k$, $\hat{\Omega}^k(m+1) := \hat{\Omega}^{\hat{\Omega}^k(m)}$. *Then for every* $n \geq 2$ *and* k

$$\mathrm{KP}^- + \Pi_n\text{–Foundation} \vdash TI(\hat{\Omega}^k(n-1), \Pi_2).$$

*Proof.*4.4(ii) yields (setting $s = 0$)

$$\mathrm{KP}^- + \Pi_n\text{–Foundation} \vdash TI(\hat{\Omega}^k(1), \Pi_n).$$

Hence, using 4.5, $\mathrm{KP}^- + \Pi_n$–Foundation $\vdash TI(\hat{\Omega}^k(n-1), \Pi_2)$. □

The reason for the invention of $\lhd$ is that we want to mimic for certain $\alpha > \Omega$ the definition of ϑ_α within fragments of KPω. How far this is possible is foreshadowed by 4.6.

4.7 Definition. Let k and $n \geq 2$ be fixed. We intend to simultaneously define sets $\bar{C}(s, \beta) \subseteq \mathrm{OR}$ and functions $\bar{\vartheta}_s : Ord \to Ord$ for all $s \lhd \hat{\Omega}^k(n)$ and all ordinals β by recursion on s (with respect to $\lhd$). Thereby 4.6 will be employed to guarantee that this type of recursion is actually available in $\mathrm{KP}\omega^- + \Pi_n$ –Foundation.
It should also be recognized that the relations $t \in \bar{C}(s, \beta)$ and $\bar{\vartheta}_s(\alpha) = \beta$ are

Σ_1–definable in the above theory.
Suppose that $\bar{C}(t,\gamma)$ and $\bar{\vartheta}_t(\gamma)$ are defined for all $t \lhd s$ and all $\gamma \in Ord$.
By Σ_1–Recursion on $i < \omega$, we then define sets $\bar{C}_i(s,\gamma)$ as follows:

$\bar{C}_0(s,\gamma) = \{0, \hat{1}\} \cup \{\hat{\alpha} \mid \alpha < \gamma\}$,

$\bar{C}_{i+1}(s,\gamma) = \bar{C}_i(s,\gamma) \cup \{t + t' \mid t, t \in \bar{C}_i(s,\gamma)\} \cup \{\hat{\Omega}^t \beta \mid t, \hat{\beta} \in \bar{C}_i(s,\gamma)\}$

$\cup \{\bar{\vartheta}_t(\beta) \mid t \lhd s \wedge t, \hat{\beta} \in \bar{C}_i(s,\gamma)\}$.

Let $\bar{C}(s,\gamma) := \bigcup\limits_{i<\omega} \bar{C}_i(s,\gamma)$.

Then $\bar{C}_i(s,\gamma)$ is a set by Σ–Collection and Infinity; thus $\bar{C}(s,\gamma)$ is a set. So $\bar{C}(s,\gamma)$ is defined for every ordinal γ.
Before we can give a definition of $\bar{\vartheta}_s$, we have to observe two facts:

Fact 1: For every $t \unlhd s$ one can find an ordinal δ such that $t \in \bar{C}(t,\delta)$.

Fact 2: For every ξ there exists $\eta > \xi$ so that $\hat{\eta} \notin \bar{C}(s,\eta)$.

Fact 1 can be easily shown by induction on TC(t). For Fact 2, let $\xi_0 = \sup\{\alpha + 1 \mid \hat{\alpha} \in \bar{C}(s, \xi + 1)\}$, $\xi_{j+1} = \sup\{\alpha + 1 \mid \hat{\alpha} \in \bar{C}(s, \xi_j + 1)\}$. Put $\eta := \sup_{j<\omega} \xi_j$. Then $\xi_j < \eta$ for all $j < \omega$. Using induction on $i < \omega$, one shows $\bar{C}_i(s,\eta) \subseteq \bigcup_{j<\omega} \bar{C}(s,\xi_j)$. Consequently, $\hat{\eta} \notin \bar{C}(s,\eta)$, for otherwise we could find a $j < \omega$ such that $\hat{\eta} \in \bar{C}(s,\xi_j)$, which would yield the contradiction $\eta < \xi_{j+1}$. By virtue of Fact 1 and Fact 2, we define $\bar{\vartheta}_s(\xi)$ by recursion on ξ as follows

$$\bar{\vartheta}_s(\xi) = \text{ least } \eta \text{ such that: } [\hat{\eta} \notin \bar{C}(s,\eta) \wedge s \in \bar{C}(s,\eta) \wedge (\forall \varsigma < \xi)(\bar{\vartheta}_s(\varsigma) < \eta)].$$

4.8 Proposition. *For fixed k and $n \geq 2$,* $\mathrm{KP}\omega^- + \Pi_n$–Foundation *proves the assertion:*

For all $s \lhd \hat{\Omega}^k(n-1)$, the function $\bar{\vartheta}_s$ is totally defined on the ordinals.

Proof. By the above, if one assumes the totality of all the functions $\bar{\vartheta}_t$ for $t \lhd s$, then the totality of $\bar{\vartheta}_s$ only needs tools from $\mathrm{KP}\omega^- + \Sigma_1$–Foundation. Now, the totality of $\bar{\vartheta}_s$ can be expressed by a Π_2–formula. Therefore the assertion follows by $TI(\hat{\Omega}^k(n-1), \Pi_2)$. If we were to give a more rigorous proof, we would have to invoke the Second Recursion Theorem for KP (cf.[Ba V.2]). By glancing over the proof of [Ba V.2.3] it turns out that the Second Recursion Theorem is already provable in $\mathrm{KP}^- + \Sigma_1$–Foundation. The Second Recursion Theorem gives us a Σ–formula A such that

$A(s,\xi,\eta)$ iff $s \in \mathrm{OR} \wedge \xi, \eta \in \mathrm{Ord} \wedge$

$\exists f \, \exists g \Big[\mathrm{fun}(f) \wedge \mathrm{fun}(g) \wedge \mathrm{dom}(g) = (\eta + 1) \times \omega \wedge$

$\mathrm{dom}(f) = (\{t \in \bigcup \mathrm{rng}(g) \mid t \lhd s\} \times \{\beta \mid \hat{\beta} \in \bigcup \mathrm{rng}(g)\}) \cup (\{s\} \times \{\varsigma \mid \varsigma < \eta\}) \wedge$

$(\forall\delta \le \eta)\,(g_\delta(0) = \{0, \hat{1}\} \cup \{\hat{\alpha} \mid \alpha < \delta\} \wedge$

$(\forall i < \omega)[g_\delta(i+1) = g_\delta(i) \cup \{t + t' \mid t, t' \in g_\delta(i)\} \cup \{\hat{\Omega}^t\beta \mid t, \hat{\beta} \in g_\delta(i)\} \cup$

$\{\hat{\alpha} \mid \exists t\, \exists\hat{\beta}\ (t, \hat{\beta} \in g_\delta(i) \wedge t \lhd s \wedge \alpha = f_t(\beta))\}]) \wedge$

$\hat{\eta} \notin \mathrm{rng}(g_\eta) \wedge s \in \mathrm{rng}(g_\eta) \wedge (\forall\varsigma < \xi)(f_s(\varsigma) < \eta) \wedge$

$(\forall\delta < \eta)[\hat{\delta} \in \mathrm{rng}(g_\delta) \vee s \notin \mathrm{rng}(g_\delta) \vee (\exists\varsigma < \xi)(\delta \le f_s(\varsigma))] \wedge$

$\forall t\, \forall\beta\ (\langle t, \beta\rangle \in \mathrm{dom}(f) \to A(t, \beta, f_t(\beta)))\Big].$

In the formulation of $A(s, \xi, \eta)$, $f_s(\beta)$ abbreviates $f(\langle s, \beta\rangle)$; $\mathrm{fun}(f)$ means that f is a function, and $\mathrm{dom}(g)$ and $\mathrm{rng}(g)$ denote the domain of g and the range of g, respectively.
Setting $B(s) := \forall\xi\exists\eta A(s, \xi, \eta)$, we can apply (by 4.6) $TI(\hat{\Omega}^k(n), \Pi_2)$ on s to get (along with what we have been reflecting on in 4.7) $(\forall s \lhd \hat{\Omega}^k(n))B(s)$.
By induction on s one also verifies in the outer world

$$\forall\xi\forall n(A(s, \xi, \eta) \longleftrightarrow \bar{\vartheta}_s(\xi) = \eta). \qquad \Box$$

Let $\varepsilon_{\Omega+1}$ denote $\vartheta_0(\Omega + 1)$ and let $C_{\varepsilon_{\Omega+1}}(0)$ be the set defined in [P Definition 26]. Note that $\Omega^\alpha = \omega^{\Omega\cdot\alpha}$.
The mapping

$$e : C_{\varepsilon_{\Omega+1}}(0) \to \mathrm{OR}$$

is defined to be $e(\alpha) = \hat{\alpha}$ for $\alpha < \Omega$,

$$e(\Omega^{\beta_1} \cdot \alpha_1 + \cdots + \Omega^{\beta_k} \cdot \alpha_k) = \hat{\Omega}^{e(\beta_1)}\alpha_1 \oplus \cdots \oplus \hat{\Omega}^{e(\beta_k)}\alpha_k$$

if $\beta_1 > 0$, $\beta_1 > \cdots > \beta_k$, and $0 < \alpha_i < \Omega$ $(1 \le i \le k)$. Here we have used the fact that any ordinal $\Omega \le \beta < \varepsilon_{\Omega+1}$ can uniquely be represented in such a way.
e is order preserving with respect to $<$ and $\lhd$: By induction on $\beta \in C_{\varepsilon_{\Omega+1}}(0)$ one establishes

$$(+) \qquad (\forall\alpha < \Omega)[\vartheta_\beta(\alpha) = \bar{\vartheta}_{e(\beta)}(\alpha)].$$

The inductive assumption then implies that ϑ_β and $\bar{\vartheta}_{e(\beta)}$ obey the same recursive definition, and thus they are equal.

4.9 Theorem *Let $n \ge 2$. Let ϱ_n be $\vartheta_{\Omega^\omega(n-1)}(0)$. L_{ϱ_n} is the minimal Σ_1 and Π_2 model of* $\mathrm{KP}\omega^- + \Pi_n$–Foundation.

Proof. Note that $\sup_{k<\omega} \Omega^k(n-1) = \Omega^\omega(n-1)$ and thus

$$\sup_{k<\omega} \vartheta_{\Omega^k(n-1)}(0) = \vartheta_{\Omega^\omega}(n-1)(0) = \varrho_n.$$

Hence, the assertion follows from 2.1, 3.6, 4.8 and the last (+). □

For proving Theorem 1.2 from 4.9 we need to verify that $\vartheta_{\Omega^\omega(m)}(0) = \theta_{\Omega^\omega(m)}(0)$. Here we have to invoke [Schü,IX]. From [Schü,IX.24] it follows that $\bar{\theta}_{\Omega^\omega(m)}$ is the fixed point free version of $\vartheta_{\Omega^\omega(m)}$; thus $\vartheta_{\Omega^\omega(m)}(0) = \bar{\theta}_{\Omega^\omega(m)}(0)$. As $\mu(\Omega^\omega(m)) = 0$ (cf.[Schü,IX.24]), this gives us $\vartheta_{\Omega^\omega(m)}(0) = \theta_{\Omega^\omega(m)}(0)$.

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