2. Rational functions and partial fractions

2.1. Rational functions

A **rational function** is a function of the form

$$f(x) = \frac{p(x)}{q(x)}$$

where $p(x)$ and $q(x)$ are polynomials in x with $q \equiv 0$. For example

$$\frac{x+3}{x-7}\,, \qquad \frac{x-2}{2x^3+x^2-x}\,, \qquad \frac{x^2+3x+2}{1}\,.$$

The last is the same as x^2+3x+2, so any polynomial is also a rational function.
If the numerator and denominator have a common factor, we can simplify the fraction by dividing top and bottom by that factor. For example,

$$\frac{x^2+3x+2}{x^2+2x+1} = \frac{(x+1)(x+2)}{(x+1)^2} = \frac{x+2}{x+1}\,.$$

To multiply two rational functions, their numerators are multiplied together and their denominators are multiplied together. To divide two rational functions, turn the second one upside-down and multiply. For example,

$$\begin{aligned}\left(\frac{4(x+7)}{x+1}\right) \div \left(\frac{x^2+5}{2x+2}\right) &= \left(\frac{4(x+7)}{x+1}\right) \times \left(\frac{2x+2}{x^2+5}\right)\\ &= \frac{4(x+7)(2x+2)}{(x+1)(x^2+5)} = \frac{8(x+7)}{x^2+5}\,.\end{aligned}$$

To add or subtract two rational functions, you must write them using a **common denominator**. For example,

$$\begin{aligned}\frac{1}{x+1} + \frac{2}{x+2} &= \frac{x+2}{(x+1)(x+2)} + \frac{2(x+1)}{(x+1)(x+2)}\\ &= \frac{x+2+2(x+1)}{(x+1)(x+2)} = \frac{3x+4}{(x+1)(x+2)}\,.\end{aligned}$$

Note. Often the common denominator is the product of the denominators, but sometimes you can take something smaller. For example,

$$\begin{aligned}\frac{2}{x+1} - \frac{x}{(x+1)(x+2)} &= \frac{2(x+2)}{(x+1)(x+2)} - \frac{x}{(x+1)(x+2)}\\ &= \frac{2(x+2)-x}{(x+1)(x+2)} = \frac{x+4}{(x+1)(x+2)}\,.\end{aligned}$$

2.2. Proper rational functions

A **proper** rational function is one in which the degree of the numerator is less than the degree of the denominator. Otherwise it is called **improper**.

Any rational function can be written as the sum of a polynomial and a proper rational function.

Proof. Recall that if you divide a polynomial by a divisor, then

$$\text{polynomial} = \text{divisor} \cdot \text{quotient} + \text{remainder}$$

Therefore

$$\frac{\text{polynomial}}{\text{divisor}} = \text{quotient} + \frac{\text{remainder}}{\text{divisor}} \ .$$

□

For example (using any earlier polynomial division),

$$\frac{2x^3 + 10x^2 - 3x + 1}{x + 3} = (2x^2 + 4x - 15) + \frac{46}{x + 3} \ .$$

2.3. Partial fractions

The equality

$$\frac{3x + 4}{(x + 1)(x + 2)} = \frac{1}{x + 1} + \frac{2}{x + 2}$$

expresses a complicated rational function as a sum of simple ones, a **partial fraction**. This is often useful.

Example 2.1. Consider

$$\frac{3x - 1}{(x + 1)(x - 3)} \ .$$

We try to write it as

$$\frac{3x - 1}{(x + 1)(x - 3)} = \frac{A}{x + 1} + \frac{B}{x - 3}$$

where A and B are constants. Multiplying both sides by $(x + 1)(x - 3)$ gives

$$3x - 1 = A(x - 3) + B(x + 1) \ .$$

This is an identity which is true for all x.
Putting $x = 3$, it gives $8 = 4B$, so $B = 2$.
Putting $x = -1$, it gives $-4 = -4A$, so $A = 1$.
With these values of A and B the identity does hold, for

$$3x - 1 = (x - 3) + 2(x + 1).$$

Therefore

$$\frac{3x - 1}{(x + 1)(x - 3)} = \frac{1}{x + 1} + \frac{2}{x - 3} \ .$$

Note. This method only works for proper rational functions. In general, first write the rational function as the sum of a polynomial and a proper rational function, and then convert that to partial fractions using the method above.

Example 2.2. Write $\dfrac{x^3}{(x+1)(x+2)}$ in partial fractions.
This is not a proper rational function. Note that $(x+1)(x+2) = x^2+3x+2$.
Divide x^3 by x^2+3x+2. It gives quotient $x-3$ and remainder $7x+6$.
Therefore

$$\frac{x^3}{(x+1)(x+2)} = x-3+\frac{7x+6}{(x+1)(x+2)} .$$

Now using the usual method,

$$\frac{7x+6}{(x+1)(x+2)} = \frac{A}{x+1}+\frac{B}{x+2} .$$

Therefore

$$7x+6 = A(x+2)+B(x+1) .$$

Putting $x=-2$ gives $-8=-B$, so $B=8$.
Putting $x=-1$ gives $-1=A$.
Therefore

$$\frac{7x+6}{(x+1)(x+2)} = -\frac{1}{x+1}+\frac{8}{x+2}$$

so

$$\frac{x^3}{(x+1)(x+2)} = x-3-\frac{1}{x+1}+\frac{8}{x+2} .$$

2.4. Quadratic factors

If one of the factors is quadratic, one has to allow the corresponding numerator to be linear (i.e. of degree 1).

Example 2.3. Write $\dfrac{5x+7}{(x-1)(x^2+x+2)}$ in partial fractions.
Write

$$\frac{5x+7}{(x-1)(x^2+x+2)} = \frac{A}{x-1}+\frac{Bx+C}{x^2+x+2} .$$

Multiply both sides by the denominator.

$$5x+7 = A(x^2+x+2)+(Bx+C)(x-1) .$$

Putting $x=1$ gives $12=4A$, so $A=3$. Therefore

$$5x+7 = 3(x^2+x+2)+(Bx+C)(x-1) = 3x^2+3x+6+Bx^2+Cx-Bx-C .$$

Since this is true for all x, we can compare coefficients. Therefore

$$\begin{aligned} x^2 &: 0 = 3 + B \\ x &: 5 = 3 + C - B \\ \text{constant terms} &: 7 = 6 - C \end{aligned}$$

Therefore $B = -3$ and $C = -1$, and

$$\frac{5x+7}{(x-1)(x^2+x+2)} = \frac{3}{x-1} - \frac{3x+1}{x^2+x+2} \ .$$

2.5. **Repeated factors**

So far each factor has occurred just once. If the denominator includes a factor like $(x-a)^2$, we include partial fractions of the form

$$\frac{A}{x-a} + \frac{B}{(x-a)^2} \ .$$

EXAMPLE 2.4. Write $\dfrac{3x+5}{(x-2)^2}$ in partial fractions.
Write

$$\frac{3x+5}{(x-2)^2} = \frac{A}{x-2} + \frac{B}{(x-2)^2} \ .$$

As usual, multiply the denominator to get

$$3x + 5 = A(x-2) + B.$$

Comparing coefficients gives $A = 3$ and $B = 11$, and so

$$\frac{3x+5}{(x-2)^2} = \frac{3}{x-2} + \frac{11}{(x-2)^2} \ .$$

EXAMPLE 2.5. Write $\dfrac{x^2-17x-8}{(x-3)(x+2)^2}$ in partial fractions.
Write

$$\frac{x^2-17x-8}{(x-3)(x+2)^2} = \frac{A}{x-3} + \frac{B}{x+2} + \frac{C}{(x+2)^2} \ .$$

Multiply the denominator to get

$$\begin{aligned} x^2 - 17x - 8 =& A(x+2)^2 + B(x-3)(x+2) + C(x-3) \\ =& A(x^2+4x+4) + B(x^2-x+6) + C(x-3) \ . \end{aligned}$$

Put $x = 3$ to get $9 - 17 \times 3 - 8 = 25A$, so $A = -2$.
Put $x = -2$ to get $4 + 17 \times 2 - 8 = -5C$, so $C = -6$.
Compare coefficients of x^2 to get $1 = A + B$, so $B = 3$.

Therefore
$$\frac{x^2-17x-8}{(x-3)(x+2)^2} = -\frac{2}{x-3} + \frac{3}{x+2} - \frac{6}{(x+2)^2} \ .$$

2.6. **Worked examples**

EXAMPLE 2.6. Simplify the following as much as possible:
$$\frac{(x+3)^2}{(x^2+1)(x+3)} \ , \qquad \frac{(x^2-2x-8)}{(x+2)} \ .$$

EXAMPLE 2.7. Express the following as a single fraction, simplifying as much as possible:
$$\frac{1}{(x-2)} + \frac{1}{(x-3)} \ , \qquad \frac{(x+2)}{(x-5)} - \frac{(x-4)}{(x+2)} \ , \qquad 1 + \frac{1}{(x-1)} \ .$$

EXAMPLE 2.8. Write the following as the sum of a polynomial and a proper rational function:
$$\frac{(x^3+4x-3)}{(x-2)} \ , \qquad \frac{(x^2-1)}{(x^2+1)} \ .$$

EXAMPLE 2.9. Write as partial fractions:
$$\frac{2}{(x+1)(x-1)} \ , \quad \frac{(2x+1)}{x(x+1)} \ , \quad \frac{(3+2x-x^2)}{(x-1)(2x^2+x+1)} \ , \quad \frac{(3x^2+x+4)}{(x+1)^2(x-1)} \ , \quad \frac{(2x^2+2x-17)}{(x+3)(x+2)} \ .$$