INACCESSIBLE SET AXIOMS MAY HAVE LITTLE CONSISTENCY STRENGTH

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Abstract. The paper investigates inaccessible set axioms and their consistency strength in constructive set theory. In ZFC inaccessible sets are of the form $V_\kappa$ where $\kappa$ is a strongly inaccessible cardinal and $V_\kappa$ denotes the $\kappa$-th level of the von Neumann hierarchy. Inaccessible sets figure prominently in category theory as Grothendieck universes and are related to universes in type theory. The objective of this paper is to show that the consistency strength of inaccessible set axioms heavily depends on the context in which they are embedded. The context here will be the theory CZF$^-$ of constructive Zermelo Fraenkel set theory but without $\in$ - Induction (foundation). Let INAC be the statement that for every set there is an inaccessible set containing it. CZF$^- +$ INAC is a mathematically rich theory in which one can easily formalize Bishop style constructive mathematics and a great deal of category theory. CZF$^- +$ INAC also has a realizability interpretation in type theory which gives its theorems a direct computational meaning. The main result presented here is that the proof theoretic ordinal of CZF$^- +$ INAC is a small ordinal known as the Feferman - Schütte ordinal $\Gamma_0$.

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1. Introduction

Constructive Set Theory originated from the work of Myhill [17] and was introduced as a natural formalism which relates to Bishop’s constructive mathematics [6] as ZFC relates to classical Cantorian mathematics. It is characterised by the

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use of the same standard first order language as classical set theory and it makes no explicit use of the notions of construction or constructive object. Therefore the conventions, practice and ideas of the set theoretical presentation of ordinary mathematics may be used in the development of constructive mathematics as well. Moreover, its theorems are theorems of classical set theory too.

Various systems of intuitionistic and constructive set theory have been introduced in the literature so far. We shall be concerned with Constructive Zermelo Fraenkel set theory, $\text{CZF} [1, 2, 3]$, as it has a natural interpretation in Martin L"of Type Theory by means of which its notion of set receives a constructive meaning.

The paper investigates the consistency strength of inaccessible set axioms in $\text{CZF}^-$, i.e. $\text{CZF}$ minus $\in$-Induction. In $\text{ZFC}$ inaccessible sets are of the form $V_\kappa$ where $\kappa$ is a strongly inaccessible cardinal and $V_\kappa$ denotes the $\kappa$-th level of the von Neumann hierarchy. Inaccessible sets figure prominently in category theory as Grothendieck universes and are related to universes in type theory. We show that the consistency strength of inaccessible set axioms heavily depends on the context in which they are embedded.

The context here will be $\text{CZF}^-$. Let $\text{INAC}$ be the statement that for every set there is an inaccessible set containing it. Then $\text{CZF}^- + \text{INAC}$ is a mathematically rich theory in which one can easily formalize Bishop style constructive mathematics and a great deal of category theory. $\text{CZF}^- + \text{INAC}$ also has a realizability interpretation in Martin L"of Type Theory which gives its theorems a direct computational meaning. The main result presented here is that the proof theoretic ordinal of $\text{CZF}^- + \text{INAC}$ is the Feferman - Sch"utte ordinal $\Gamma_0$.

To put this result into perspective we shall prove that $\text{CZF}^- + \text{INAC}$ plus the principle of excluded middle, $\text{EM}$, has the strength of $\text{ZFC} + \forall \alpha \exists k \left( \alpha < k \wedge k \text{ is a strong inaccessible cardinal} \right)$. It is also worth noting that the strength of $\text{CZF} + \text{INAC}$ exceeds the strength of all systems considered in Reverse Mathematics. In fact, the techniques of [18] can be used to show that the strength of
CZF + INAC is the same as that of the fragment of second order arithmetic known as $\Delta^1_2 - CA + BI$.

In our investigations we have been led by an analogy with important results obtained in the classical case of Kripke Platek set theory, KP. Research by Jäger and Pohlers [13] has supplied a proof theoretic analysis for an extension of KP, denoted KPi, which is obtained from Kripke Platek set theory by adding an axiom stating the existence of many admissible sets. The upshot is that KPi has the same proof theoretic strength as $\Delta^1_2 - CA + BI$. Jäger [11] has furthermore shown that if restrictions are imposed on the principle of induction (i.e. foundation), then the proof theoretic strength of the theories strongly decreases. In particular, if foundation is completely omitted from KPi then the proof theoretic strength of the system obtained is measured by the ordinal $\Gamma_0$.

We have therefore started to investigate the system CZF looking for similar outcomes in this constructive context. The results for KPi referred to above have been obtained by use of partial cut elimination. Unfortunately, this technique is not suitable for CZF and restrictions. In fact, even when completely omitting the principle of $\in$ - Induction from CZF, we are still left with the axioms of Collection which have an unrestricted quantifier complexity. Therefore a different approach is needed in this case.

The strategy adopted here is to introduce a realizability interpretation for systems of constructive set theory into suitable classical theories for which the proof theoretic strength may be determined by use of cut elimination techniques. The proof theoretic strength of the classical theory will then provide an upper bound for the constructive set theory. Regarding the realizability interpretation, we have to depart substantially from Aczel’s interpretation of constructive set theory in MLTT ([3]). That interpretation, in fact, makes use of the $W$ type in order to interpret the Regular Extension Axiom, REA, of which INAC is a strengthening, and would therefore require a far too strong theory. We shall instead introduce
a theory of iterated fixed point definitions with ordinals, called $\mathbf{\hat{ID}}^\ast$, whose proof theoretic strength is shown to be $\Gamma_0$. In $\mathbf{\hat{ID}}^\ast$ we shall build up two hierarchies: $(U_\alpha)_\alpha$ of universes and $(V_\alpha)_\alpha$ of types of iterative sets. In addition, we shall define a bisimulation relation on $V_\alpha$ in the style of [15], and interpret equality between sets as maximum bisimulations among types in $V_\alpha$, giving rise to a non-well-founded membership relation. This will allow us to define a realizability interpretation for $\mathbf{CZF}^- + \mathbf{INAC}$ in $\mathbf{\hat{ID}}^\ast$, providing an upper bound for $\mathbf{CZF}^- + \mathbf{INAC}$.

The lower bound will be obtained by interpreting $\mathbf{ATR}_0^i$, an intuitionistic version of the subsystem $\mathbf{ATR}_0$ of second order arithmetic, in $\mathbf{CZF}^- + \mathbf{INAC}$ and by showing that $\Gamma_0$ is a lower bound for $\mathbf{ATR}_0^i$. We will rely on the literature for a well ordering proof in such a theory, in a similar way as in [20].

This will enable us to conclude that $\Gamma_0$ is a measure for the strength of $\mathbf{CZF}^- + \mathbf{INAC}$.

2. The system $\mathbf{CZF}$

The language of $\mathbf{CZF}$ is the first order language of Zermelo Fraenkel set theory, $LST$, with the non logical primitive symbol $\in$. We assume that $LST$ has also a constant, $\omega$, for the set of the natural numbers. The logical symbols include all the intuitionistic operators $\bot$, $\land$, $\lor$, $\rightarrow$, $\forall$, $\exists$.

**Definition 2.1 (Axioms of $\mathbf{CZF}$).** The axioms of $\mathbf{CZF}$ include the axioms and rules of intuitionistic logic with equality and the following set theoretic axioms.

1. **Extensionality**
   \[ \forall a \forall b (\forall y (y \in a \iff y \in b) \rightarrow a = b). \]

2. **Pair**
   \[ \forall a \forall b \exists x \forall y (y \in x \iff y = a \lor y = b). \]

3. **Union**
   \[ \forall a \exists x \forall y (y \in x \iff \exists z \in a y \in z). \]
(4) $\Delta_0$ - Separation scheme

For every bounded formula $\varphi(y)$,

$$\forall a \exists x \forall y (y \in x \leftrightarrow y \in a \land \varphi(y)),$$

where a formula $\varphi(x)$ is bounded, or $\Delta_0$, if all the quantifiers occurring in it are bounded, i.e. of the form $\forall x \in b$ or $\exists x \in b$.

(5) Subset Collection scheme

For every formula $\varphi(x, y, u)$,

$$\forall a \forall b \exists c \forall u (\forall x \in a \exists y \in b \varphi(x, y, u) \rightarrow$$

$$\exists d \in c (\forall x \in a \exists y \in d \varphi(x, y, u) \land \forall y \in d \exists x \in a \varphi(x, y, u))).$$

(6) Strong Collection scheme

For every formula $\varphi(x, y)$,

$$\forall a (\forall x \in a \exists y \varphi(x, y) \rightarrow$$

$$\exists b (\forall x \in a \exists y \in b \varphi(x, y) \land \forall y \in b \exists x \in a \varphi(x, y))).$$

(7) Infinity

$$(\omega 1) \quad 0 \in \omega \land \forall y (y \in \omega \rightarrow y + 1 \in \omega),$$

$$(\omega 2) \quad \forall x (0 \in x \land \forall y (y \in x \rightarrow y + 1 \in x) \rightarrow \omega \subseteq x),$$

where $y + 1$ is $y \cup \{y\}$, and 0 is the empty set, defined in the obvious way.

(8) $\in$ - Induction scheme

For every formula $\varphi(a)$,

$$(IND_{\in}) \quad \forall a (\forall x \in a \varphi(x) \rightarrow \varphi(a)) \rightarrow \forall a \varphi(a),$$

In the following we shall be concerned with a subsystem of CZF with restricted induction.

Definition 2.2. Let $\text{CZF}^-$ be the system $\text{CZF}$ without the $\in$ - Induction scheme.
We shall consider an extension of CZF$^-$ with an axiom proposed by Aczel in [1], called the ‘Regular Extension Axiom’ (REA). In that context REA was introduced in order to accommodate inductive definitions in CZF. Aczel [3] also showed that we can extend the interpretation of CZF in MLTT to CZF + REA by making use of a stronger theory of types, i.e. the one with the so called W type [16].

**Definition 2.3.** A set $c$ is said to be regular if it is transitive, inhabited (i.e. $\exists u\ u \in c$) and for any $u \in c$ and set $R \subseteq u \times c$ if $\forall x \in u\ \exists y\ \langle x, y \rangle \in R$ then there is a set $v \in c$ such that

$$\forall x \in u\ \exists y \in v\ \langle x, y \rangle \in R \land \forall y \in v\ \exists x \in u\ \langle x, y \rangle \in R.$$  

We write $\text{Reg}(a)$ for ‘$a$ is regular’.

**Definition 2.4.** Let (REA) be the principle

$$\forall x \exists y\ (x \in y \land \text{Reg}(y)).$$

A stronger version of REA is INAC, defined as follows.$^1$

**Definition 2.5.** Let (INAC) be the principle

$$\forall x \exists y\ (x \in y \land \text{Reg}(y))\text{ and } y\text{ is a model of CZF}^-,$$

i.e. the structure $\langle y, \in \rangle (y \times y)$ is a model of CZF$^-$.  

We say that a set is inaccessible if it is regular and a model of CZF$^-$ and we write INAC($y$) for ‘$y$ is inaccessible’.

$^1$In this paper we shall determine the strength of CZF$^- + \text{INAC}$ but not that of CZF$^- + \text{REA}$. The strength of CZF$^- + \text{REA}$ is still unknown. We will briefly address the problem of establishing a lower bound for CZF$^- + \text{REA}$ in Remark 9.15.
**Remark 2.6.** In order to give a precise characterization of the notion of inaccessibility, we need to clarify the meaning of \( \langle y, \in \upharpoonright (y \times y) \rangle = \text{CZF}^- \), in particular with regard to the axiom schemes of \( \text{CZF}^- \).

One possibility would be to formalize the notion of satisfaction for formulas of \( \text{CZF} \).

Alternatively, one can select the universal closure of finitely many axioms of \( \text{CZF}^- \), say \( \varphi_1, \ldots, \varphi_n \) and formalize \( \text{INAC}(y) \) by \( \text{Reg}(y) \land \varphi_1^y \land \cdots \land \varphi_n^y \), due to the following observations. Regularity of a set \( y \) implies that \( y \) is a model of Strong Collection. In addition, \( \text{CZF}^- \) minus Strong Collection may be equivalently formalized by means of finitely many axioms. In fact, the Subset Collection scheme may be replaced by a single axiom as shown in [1] and [19] and we may replace the schema of Restricted Separation by a finite number of special cases as in [17], Appendix A. This will enable us to completely formalize the notion of Inaccessibility.

In the following we shall prove that the proof theoretic ordinal of \( \text{CZF}^- + \text{INAC} \) is \( \Gamma_0 \). This result may be fully appreciated if we compare it with the classical case. Given two theories \( S \) and \( T \), let us write \( S \equiv T \) to denote that \( S \) and \( T \) have the same proof theoretic strength. Let \( \text{EM} \) denote the principle of excluded middle. We shall show that

\[
\text{CZF}^- + \text{INAC} + \text{EM} \equiv \text{ZFC} + \forall \alpha \exists \kappa (\alpha < \kappa \land \kappa \text{ is a strong inaccessible cardinal}).
\]

Let us note first of all that Collection implies Replacement, and classically Full Separation may be obtained as a consequence of Replacement. In addition, \( \in \)-Induction is equivalent to Foundation. Finally, Subset Collection implies what is known as Myhill’s Exponentiation axiom. This is the statement for which given two sets \( a \) and \( b \), the collection of all the functions from \( a \) to \( b \) is a set. Classically, the Exponentiation axiom is easily seen to be equivalent to Powerset. Therefore we are able to state the following Lemma (see e.g. [1] for a proof).
Lemma 2.7.

\[ \text{CZF} + \text{EM} = \text{ZF}. \]

Let \( \text{ZF}^- \) denote the subsystem of \( \text{ZF} \) obtained by removing the axiom of Foundation. Then the proof of Lemma 2.7 also yields

\[ \text{CZF}^- + \text{EM} = \text{ZF}^- \]

In the following we denote with \( \text{WF} \) the class of well founded sets as defined e.g. in Kunen ([14], p. 95).

Lemma 2.8. \( \text{ZF}^- + \text{INAC} \equiv \text{ZF} + \text{INAC} \).

Proof. We shall give an interpretation of \( \text{ZF} + \text{INAC} \) in \( \text{ZF}^- + \text{INAC} \), by interpreting a formula \( \varphi(\vec{u}) \) by \( \vec{u} \in \text{WF} \rightarrow \varphi(\vec{u})^{WF} \). We can then show that if \( \varphi(\vec{u}) \) holds in \( \text{ZF} + \text{INAC} \), then \( \vec{u} \in \text{WF} \rightarrow \varphi(\vec{u})^{WF} \) holds in \( \text{ZF}^- + \text{INAC} \).

We refer to [14], § 4.4, Theorem 4.1, for the proof that the axioms of \( \text{ZF} \) relativized to \( \text{WF} \) hold true in \( \text{ZF}^- \). We only show that \text{INAC} relativized to \( \text{WF} \) holds true in \( \text{ZF}^- + \text{INAC} \). That is, given a set \( a \in \text{WF} \), we want to find a set \( d \in \text{WF} \) such that \( a \in d \) and \( d \) is inaccessible.

Let then \( a \in \text{WF} \). By \text{INAC}, we obtain an inaccessible set \( c \) such that \( a \in c \). By use of Separation, let

\[ b := \{ y \in c : y \in \text{WF} \}. \]

Then \( b \in \text{WF} \). We clearly have \( a \in b \), as \( a \) is well founded. Suppose \( u \in b \), \( R \subseteq u \times b \) and \( \forall x \in u \exists y \in b \langle x, y \rangle \in R \). Since all the elements of \( b \) are in \( c \) and \( c \) is regular, there exists \( v \in c \) such that \( \forall x \in u \exists y \in v \langle x, y \rangle \in R \) and \( \forall y \in v \exists x \in u \langle x, y \rangle \in R \). We observe that \( v \subseteq b \), therefore \( v \in \text{WF} \); also \( v \in c \), and hence \( v \in b \). This shows that \( b \) is regular.

Finally, \( b \) is a model of \( \text{CZF}^- \), since \( c \) is. \( \Box \)

Lemma 2.9. \( \text{ZF}^- + \text{INAC} \equiv \text{ZFC} + \text{INAC} \).
Proof. In view of the previous Lemmas, we only need to note that we can interpret \( \text{ZFC} + \text{INAC} \) in \( \text{ZF} + \text{INAC} \) by means of the constructible universe \( L \).

Lemma 2.10. \( \text{CZF}^- + \text{INAC} + \text{EM} \equiv \text{ZFC} + \forall \alpha \exists \kappa (\alpha < \kappa \land \kappa \text{ is a strong inaccessible cardinal}) \).

Proof. As a consequence of the previous Lemmas we only need to show that \( \text{INAC} \) implies \( \forall \alpha \exists \kappa (\alpha < \kappa \land \kappa \text{ is a strong inaccessible cardinal}) \), on the basis of \( \text{ZFC} \).

Let \( \alpha \) be an ordinal. By \( \text{INAC} \) let us take a set \( a \) such that \( \alpha \in a \) and \( a \) is inaccessible. Let \( \kappa \) be the least ordinal not in \( a \). We need to prove that \( \kappa \) is an uncountable regular cardinal and that it is a strong limit, i.e. for any \( \lambda < \kappa \), \(|2^\lambda| < \kappa\), where \(|2^\lambda|\) denotes cardinal exponentiation.

We note first of all that as \( a \models \text{Infinity} \), we must have \( \kappa > \omega \).

For a contradiction suppose that \( \kappa \) is not a regular cardinal. Then there is \( \gamma < \kappa \) and a function \( f : \gamma \to \kappa \) such that \( \forall \eta < \kappa \exists \beta < \gamma (\eta < f(\beta)) \). Since \( f : \gamma \to a \) and \( \gamma \in a \), by regularity of \( a \) we find a set \( v \in a \) such that \( \forall \beta \in \gamma \exists x \in v (x = f(\beta)) \) and \( \forall x \in v \exists \beta \in \gamma (x = f(\beta)) \). Hence \( v = \text{ran}(f) \). Let \( w = \bigcup v \). Then the assumptions on \( \kappa \) give us \( w = \kappa \). We also have \( w \in a \), since \( v \in a \) and \( a \models \text{Union} \), contradicting \( \kappa \notin a \). Therefore \( \kappa \) is a regular cardinal.

We finally want to show that if \( \lambda < \kappa \) then \(|2^\lambda| < \kappa\). Since \( \kappa \) is the least ordinal not in \( a \), we have \( \lambda \in a \). We note that any inaccessible set is closed under taking power set (as shown in [19], Lemma 2.6), so that \( 2^\lambda \in a \). Using the Axiom of Choice, let us take an ordinal \( \pi \) and a bijection \( f : 2^\lambda \to \pi \).

For a contradiction assume that \( \kappa \leq \pi \). Let

\[ b = \{ g \in 2^\lambda : f(g) < \kappa \}. \]

Then clearly \( b \subseteq 2^\lambda \) and since \( a \) is closed under taking power set, \( b \in a \).

Then \( f \upharpoonright b : b \to \kappa \) gives a counterexample to the regularity of \( a \). Hence \(|2^\lambda| < \kappa \). □
3. The theory $\widehat{\text{ID}}^*$

We shall now introduce a theory of positive fixed point definitions, $\widehat{\text{ID}}^*$, and show that its proof theoretic strength is measured by the ordinal $\Gamma_0$.

We shall subsequently give a realizability interpretation for $\text{CZF}^- + \text{INAC}$ in $\widehat{\text{ID}}^*$, thus obtaining an upper bound for the proof theoretic strength of the constructive set theory.

**Definition 3.1** (The language $\mathbf{L}_0$). Let $\mathbf{L}_0$ be a two sorted language with general variables ranging over the entire universe of discourse, denoted by $x, y, z, \ldots$, and ordinal variables, denoted by $\alpha, \beta, \gamma, \ldots$. The language $\mathbf{L}_0$ also includes individual constants: $\mathbf{k}, \mathbf{s}$ (combinators), $\mathbf{p}, \mathbf{p}_0, \mathbf{p}_1$ (pairing and projections), $\mathbf{0}$ (zero), $\mathbf{S}_\mathbf{N}$ (successor on the natural numbers), $\mathbf{p}_\mathbf{N}$ (predecessor on the natural numbers), $\mathbf{d}_\mathbf{N}$ (definition by numerical cases), $\mathbf{S}_{\Omega}$ (successor on the ordinals).

The terms of $\mathbf{L}_0$ are the variables and the constants only.

The relation symbols include a unary predicate symbol $\mathbf{N}$ (natural numbers) and a binary predicate symbol, $<$, on the ordinals. In addition, a ternary relation symbol $\text{App}$ (for application).

The notion of formula is defined in the usual way.

In order to simplify the formulation of the axioms of $\widehat{\text{ID}}^*$ we consider a definitional extension of $\mathbf{L}_0$ with application terms, defined inductively as follows.

(i) Each variable and constant is an application term.

(ii) If $t, s$ are application terms then $ts$ is an application term.

Application terms will be used in conjunction with the following abbreviations.

(i) $t \simeq x$ for $t = x$ when $t$ is a variable or constant.

(ii) $ts \simeq x$ for $\exists y \exists z \,(t \simeq y \land s \simeq z \land \text{App}(y, z, x))$.

(iii) $t \downarrow$ for $\exists x \,(t \simeq x)$.

(iv) $t \simeq s$ for $\forall x \,(t \simeq x \leftrightarrow s \simeq x)$.

(v) $\varphi(t, \ldots)$ for $\exists x \,(t \simeq x \land \varphi(x, \ldots))$. 
(vi) $t_1 t_2 \ldots t_n$ for $(\ldots(t_1 t_2)\ldots)t_n$.

**Definition 3.2.** (i) A formula is $\Delta^\Omega_0$ if all the ordinal quantifiers occurring in it are bounded, i.e. of the form $\forall \alpha < \beta$ or $\exists \alpha < \beta$.

(ii) The class of $\Sigma^\Omega$ formulas is defined as the smallest class containing all the $\Delta^\Omega_0$ formulas and closed under $\land$, $\lor$, $\forall x$, $\exists x$, $\forall \alpha < \beta$, $\exists \alpha < \beta$, $\exists \beta$.

**Definition 3.3** (The language $L^*$). Let $L_0(Q, R)$ be an extension of $L_0$ by means of two new unary predicate symbols $Q$ and $R$. The notion of formula is modified accordingly. Let us denote by $\varphi(Q^+, R, x, \alpha)$ a formula in which at most $x, \alpha$ occur free and $Q$ occurs only positively. The language $L^*$ of $\hat{\text{ID}}^*$ results from $L_0$ by adding a binary predicate symbol $P_\varphi$ for each $\Delta^\Omega_0$ formula $\varphi(Q^+, R, x, \alpha)$ of $L_0(Q, R)$.

We shall use the abbreviations:

$$P^\alpha_\varphi(s) := P_\varphi(\alpha, s),$$

$$P^{\leq \alpha}_\varphi(s) := (\exists \beta < \alpha) P^\beta_\varphi(s).$$

**Definition 3.4** (Axioms and rules of $\hat{\text{ID}}^*$). The theory $\hat{\text{ID}}^*$ is the $L^*$ theory whose axioms and rules are the usual axioms and rules of first order classical logic and in addition the following principles (1) - (15).

**Applicative structure**

(1) $kab = a$.

(2) $sab \downarrow \land sabc \simeq ac(bc)$.

(3) $k \neq s$.

(4) $p_{ab} \downarrow \land p_0p_{ab} = a \land p_1p_{ab} = b$.

(5) $N(0) \land \forall a (N(a) \rightarrow N(SNa))$.

(6) $\forall a (N(a) \rightarrow (\neg SNa = 0 \land p_N(SNa) = a))$.

(7) $\forall a (N(a) \rightarrow (\neg a = 0 \rightarrow N(p_Na) \land S_N(p_Na) = a))$.

(8) $N(a) \land N(b) \land a = b \rightarrow d_Nabcd = c$. 
(9) \( N(a) \wedge N(b) \wedge \neg a = b \rightarrow d_{Nabcd} = d \).

(10) \( \exists \alpha (S_\Omega \beta = \alpha \wedge \beta < \alpha \wedge \forall \gamma < \alpha (\gamma = \beta \vee \gamma < \beta)) \).

Fixed point definitions

(11) Fixed points
\[ \forall x (\varphi(P^x_\phi, P^{\leq x}_\varphi, x, \alpha) \leftrightarrow P^\alpha_\varphi(x)). \]

(12) Linearity
\[ \alpha \not< \alpha \wedge (\alpha < \beta \wedge \beta < \gamma \rightarrow \alpha < \gamma) \wedge (\alpha < \beta \vee \alpha = \beta \vee \beta < \alpha). \]

(13) \( \Sigma^\Omega \) - Reflection
For every \( \Sigma^\Omega \) formula \( \varphi \),
\[ \varphi \rightarrow \exists \alpha \varphi^\alpha. \]

(14) \( \Delta^\Omega_0 \) - Induction on the natural numbers
For all \( \Delta^\Omega_0 \) formulas \( \varphi(x) \),
\[ (\Delta^\Omega_0 - \text{IND}_N) \varphi(0) \wedge \forall x (N(x) \rightarrow (\varphi(x) \rightarrow \varphi(x + 1))) \rightarrow \forall x (N(x) \rightarrow \varphi(x)). \]

(15) \( \Delta^\Omega_0 \) - Induction on the ordinals
For all \( \Delta^\Omega_0 \) formulas \( \varphi(\alpha) \),
\[ (\Delta^\Omega_0 - \text{IND}_\Omega) \forall \alpha (\forall \beta < \alpha \varphi(\beta) \rightarrow \varphi(\alpha)) \rightarrow \forall \alpha \varphi(\alpha). \]

We write \( \langle a, b \rangle \) for \( pab \), \( a_0 \) for \( p_0a \) and \( a_1 \) for \( p_1a \). In addition, we write \( \langle a, \langle b, c \rangle \rangle \), etc.

Let \( \alpha + 1 \) stand for \( S_\Omega \alpha \).

4. Proof theoretic strength of \( \widehat{\text{ID}^*} \)

In this Section we shall determine the proof theoretic strength of \( \widehat{\text{ID}^*} \) by means of recursively saturated models.

For this purpose we shall consider the theories \( \widehat{\text{ID}_n} \) of \( n \) - iterated fixed point definitions and show that any recursively saturated model of their union, \( \bigcup_{n \in \omega} \widehat{\text{ID}_n} \), may be expanded.
to a model of $\hat{\text{ID}}^*$.

The theories $\hat{\text{ID}}_n$ were introduced in [9] to prove Hancock’s conjecture and their proof theoretic ordinals were there shown to be $\alpha_n$, where $\alpha_0 := \epsilon_0$
and $\alpha_{n+1} := \varphi\alpha_n.0$. Therefore

$$\bigcup_{n\in\omega} \hat{\text{ID}}_n = \Gamma_0$$

(where $|T|$ denotes the proof theoretic ordinal of the theory $T$).

Consequently, by proving that $\hat{\text{ID}}^*$ is conservative over $\bigcup_{n\in\omega} \hat{\text{ID}}_n$,
we are able to conclude that $|\hat{\text{ID}}^*| = \Gamma_0$.

Let us first of all formulate the theory

$$\hat{\text{ID}}_{<\omega} = \bigcup_{n\in\omega} \hat{\text{ID}}_n.$$  

4.1. **The theory $\hat{\text{ID}}_{<\omega}$.** The language of $\hat{\text{ID}}_{<\omega}$ is the language of Peano arithmetic augmented by two fresh unary predicate symbols $Q$ and $R$. We denote this new language by $\hat{\mathbb{L}}$. As in Section 3 we shall write $\varphi(Q^+, R, u, v)$ for a formula $\varphi$ in which only the variables $u$ and $v$ occur free and $Q$ occurs only positively. We assume that $\hat{\mathbb{L}}$ contains a new predicate $\mathbb{I}_\varphi(u, v)$ for each formula $\varphi(Q^+, R, u, v)$.

For this Section only we shall be using the notation $\cdot <'_{\Omega}$ for the less relation on the ordinals as introduced in Section 3, while we shall use $\cdot <$ to denote the usual less relation on the natural numbers.

We adopt the following conventions

$$\mathbb{I}_\varphi^t(s) := \mathbb{I}_\varphi(t, s),$$

$$\mathbb{I}_{\varphi^t}(s) := \exists y < t \mathbb{I}_\varphi(y, s).$$
The axioms of $\hat{\text{ID}}_{<\omega}$ include the axioms of Peano arithmetic, with the Induction scheme extended to all formulas of $\hat{\mathcal{L}}$. In addition, we have the following, for each $n \in \mathbb{N}$

\[ (\text{FPN}) \quad \forall u \leq \bar{n} \forall x \left( \varphi(I_{x}^{u}, I_{x}^{<u}, x, u) \leftrightarrow I_{x}^{u}(x) \right), \]

where as usual $\bar{n}$ denotes the numeral corresponding to the natural number $n$.

We have that $\hat{\text{ID}}_{<\omega} = \bigcup_{n \in \omega} \hat{\text{ID}}_{n}$ and as already mentioned it is known, (cf. [9]), that $|\bigcup_{n \in \omega} \hat{\text{ID}}_{n}| = \Gamma_{0}$.

4.2. Conservativity of $\hat{\text{ID}}^*$ over $\hat{\text{ID}}_{<\omega}$. In the following we prove that $\hat{\text{ID}}^*$ is conservative over $\hat{\text{ID}}_{<\omega}$ for arithmetic sentences.

Let us first of all recall the notion of a recursively saturated model (see e.g. [7]). We denote by $\Phi(x)$ a set of formulas $\varphi(x)$ each with at most the variable $x$ free. We say that $\Phi(x)$ is satisfiable in $\mathcal{M}$ if there is an element $m \in \mathcal{M}$ which simultaneously satisfies each $\varphi(x) \in \Phi(x)$. In addition $\Phi(x)$ is finitely satisfiable in $\mathcal{M}$ if and only if every finite subset of $\Phi(x)$ is satisfiable in $\mathcal{M}$.

**Definition 4.1.** Let $\mathcal{L}$ be a recursive language. A model $\mathcal{M}$ of $\mathcal{L}$ is recursively saturated if for every finite set $\{c_1, \ldots, c_n\}$ of new constant symbols, every recursive set $\Phi(x)$ of formulas of $\mathcal{L} \cup \{c_1, \ldots, c_n\}$ and every $n$-tuple $a_1, \ldots, a_n$ of elements of $\mathcal{M}$, if $\Phi(x)$ is finitely satisfiable in $\langle \mathcal{M}, a_1, \ldots, a_n \rangle$ then $\Phi(x)$ is satisfiable in $\langle \mathcal{M}, a_1, \ldots, a_n \rangle$.

**Remark 4.2.** It is known (see e.g. [7], § 2.4) that given any countable model $\mathfrak{N}_0$ of a theory $T$, there is an elementary extension $\mathfrak{N}_0 \prec \mathcal{M}$, which is countable and recursively saturated. In addition, by the Löwenheim Skolem theorem, if a theory $T$ is formulated in a countable language, then given any model $\mathcal{M}$ of $T$ there is a countable model $\mathfrak{N}_0 \prec \mathcal{M}$ of $T$.

In order to prove that $\hat{\text{ID}}^*$ is conservative over $\hat{\text{ID}}_{<\omega}$ (for arithmetic sentences), we consider a recursively saturated model $\mathcal{M}$ of $\hat{\text{ID}}_{<\omega}$ and define an expansion $\mathcal{M}^*$
of $\mathcal{M}$. This expansion will provide us with an interpretation for the symbols of $\widehat{\mathbf{ID}}^*$ which are not in $\mathcal{L}$, in particular for the ordinal variables. We than prove that $\mathcal{M}^*$ is a model of $\widehat{\mathbf{ID}}^*$.

**Definition 4.3.** Let $\mathcal{M}$ be a model of $\widehat{\mathbf{ID}}_{<\omega}$. We define an expansion $\mathcal{M}^*$ of $\mathcal{M}$ to the language $\mathcal{L}^*$ as follows.

(i) The interpretation of the ordinal variables $\alpha, \beta, \ldots$ in $\mathcal{M}^*$ is given by elements of the standard part of $\mathcal{M}$, i.e. elements of $ST(\mathcal{M}) := \{\bar{n}^M : n \in \mathbb{N}\}$, where as usual $\bar{n}^M$ denotes the interpretation in $\mathcal{M}$ of the numeral corresponding to the natural number $n$. For example, $\mathcal{M}^* \models \forall \alpha \psi(\alpha)$ iff for all $x \in ST(\mathcal{M})$, $\mathcal{M}^* \models \psi[x/\alpha]$.

(ii) The interpretation of a predicate of the form $P_\varphi(\alpha, t)$ is $(I_\varphi(\bar{n}, t))^M$ where $\bar{n}^M \in ST(\mathcal{M})$ and $\alpha = \bar{n}^M$.

(iii) The predicate $<_\Omega$ is interpreted as the predicate $<$, which represents the usual less relation on the natural numbers. Therefore $\alpha <_\Omega \beta$ is interpreted as $\bar{n}^M < \bar{m}^M$ with $\bar{n}^M$ the interpretation in $\mathcal{M}^*$ of $\alpha$ and $\bar{m}^M$ the interpretation of $\beta$.

(iv) Applicative structure. The relation $\text{App}(t, s, y)$ is interpreted by $\{t\}(s) \simeq y$, the latter expressing that the partial recursive function with code $t$ applied to $s$ yields $y$.

In the following we shall relax the notation omitting the distinction between $\alpha, \beta, \ldots$ and their interpretations in $\mathcal{M}^*$, as well as between the numerals $\bar{n}, \bar{m}, \ldots$ and $n, m, \ldots$.

**Proposition 4.4.** Let $\mathcal{M} \models \widehat{\mathbf{ID}}_{<\omega}$, and suppose $\mathcal{M}$ is recursively saturated. Then

$$\mathcal{M}^* \models \widehat{\mathbf{ID}}^*,$$

where $\mathcal{M}^*$ is an expansion of $\mathcal{M}$ as in Definition 4.3.
Proof. Let $\mathcal{M} \models \widehat{\text{ID}}_{<\omega}$, for $\mathcal{M}$ recursively saturated. We need to prove that $\mathcal{M}^*$ is a model of each axiom of $\widehat{\text{ID}}^*$. 

The axioms of the applicative structure hold as a consequence of Definition 4.3 (iv). Let us consider the remaining axioms.

Axiom (12): The linearity of the $<_{\Omega}$ relation on the ordinals clearly holds since ordinals are interpreted as natural numbers and the less relation on the ordinals is interpreted as the usual less relation on the natural numbers.

Axiom (11): We need to prove that

$$\mathcal{M}^* \models \forall \alpha \forall x \left( \varphi(P^\alpha_\varphi, P^{<\alpha}_\varphi, x, \alpha) \iff P^\alpha_\varphi(x) \right).$$

Let $\alpha^{\mathcal{M}} = m^{\mathcal{M}} \in ST(\mathcal{M})$. Let $n > m$, then as

$$\mathcal{M} \models \forall u < \bar{n} \forall x \left( \varphi(I^u_\varphi, I^{<u}_\varphi, x, u) \iff I^u_\varphi(x) \right),$$

we obtain

$$\mathcal{M} \models \forall x \left( \varphi(I^m_\varphi, I^{<m}_\varphi, x, m) \iff I^m_\varphi(x) \right).$$

Hence

$$\mathcal{M}^* \models \forall x \left( \varphi(P^\alpha_\varphi, P^{<\alpha}_\varphi, x, \alpha) \iff P^\alpha_\varphi(x) \right),$$

as required.

Axiom (14): We need to prove that

$$\mathcal{M}^* \models \varphi(0) \land \forall x \left( N(x) \rightarrow (\varphi(x) \rightarrow \varphi(x + 1)) \right) \rightarrow \forall x \left( N(x) \rightarrow \varphi(x) \right),$$

with $\varphi(x)$ a $\Delta^\Omega_0$ formula. We note that as $\varphi$ is $\Delta^\Omega_0$, all the ordinal quantifiers which occur in it are bounded, i.e. of the form $\forall \beta < \alpha$ or $\exists \beta < \alpha$ so that we may replace them by $\forall z < \bar{n}$ and $\exists z < \bar{n}$ respectively. Therefore the claim follows from the Induction principle in $\widehat{\text{ID}}_{<\omega}$.

Axiom (15) holds as a consequence of the principle of induction in $\widehat{\text{ID}}_{<\omega}$, again because of the given interpretation of the ordinals.

We observe that the quantifier bound on $\varphi$ does not play any role in this case. This shows that we could have had a principle of full induction on the ordinals in
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\[ \text{without this fact producing an increase in the proof theoretic strength of the theory itself.} \]

Axiom (13): Finally, let us show that for every \( \Sigma^\Omega \) formula \( \varphi \),

\[ \mathcal{M}^* \models \varphi \rightarrow \exists \alpha \varphi^\alpha. \]

We proceed by induction on the formula \( \varphi \). The atomic case is trivial. If \( \varphi \) is of either form \( \varphi_0 \land \varphi_1 \), \( \varphi_0 \lor \varphi_1 \), \( \exists x \psi(x) \), or \( \exists \alpha \psi(\alpha) \) the assertion follows readily from the induction hypothesis.

(i) Let \( \varphi \) be \( \forall x \psi(x) \). Then \( \mathcal{M}^* \models \psi(m) \) for every \( m \in \mathcal{M}^* \), so that by induction hypothesis \( \mathcal{M}^* \models \exists \beta \psi(m)^\beta \) for any \( m \), hence \( \mathcal{M}^* \models \forall x \exists \beta \psi(x)^\beta \). We argue by way of contradiction, assuming that

\[ (+) \quad \mathcal{M}^* \models \forall \alpha \exists x \forall \beta < \alpha \neg \psi(x)^\beta. \]

Let \( \theta_\alpha(x) := \forall \beta < \alpha \neg \psi(x)^\beta \) and \( \Phi(x) := \{ \theta_\delta(x) : \delta \in ST(\mathcal{M}) \} \).

Take \( \theta_{\delta_1}, \ldots, \theta_{\delta_r} \). Without loss of generality we may assume that

\[ \delta_1 \leq \delta_2 \leq \cdots \leq \delta_r. \]

By \( (+) \) there exists \( m \in \mathcal{M}^* \) such that

\[ \mathcal{M}^* \models \forall \beta < \delta_r \neg \psi(m)^\beta, \]

so that \( \mathcal{M} \models \theta_{\delta_1}(m) \land \ldots \land \theta_{\delta_r}(m) \). Since \( \mathcal{M} \) is recursively saturated, there exists \( m^* \in \mathcal{M} \) such that

\[ \mathcal{M} \models \theta_\delta(m^*), \]

for all \( \delta \). Hence for all \( \delta \) \( \mathcal{M}^* \models \forall \beta < \delta \neg \psi(m^*)^\beta \). Thus

\[ \mathcal{M}^* \models \forall \beta \neg \psi(m^*)^\beta. \]

The latter contradicts the assumption \( \mathcal{M}^* \models \forall x \exists \beta \psi(x)^\beta \). Therefore \( \mathcal{M}^* \models \exists \alpha \forall x \exists \beta < \alpha \psi(x)^\beta \), and hence \( \mathcal{M}^* \models \exists \alpha \forall x \psi(x)^\alpha \).

\( \overline{\text{D}}^* \)
Finally let us consider the case in which \( \phi \) is \( \forall \gamma < \delta \psi(\gamma) \). By induction hypothesis, for any \( \gamma < \delta \), \( \mathcal{M}^* \models \exists \beta \psi(\gamma)^\beta \). As \( \delta \) corresponds to a standard number, there are only finitely many \( \gamma < \delta \). We may take \( \alpha > \beta \), for every \( \gamma < \delta \), and conclude \( \mathcal{M}^* \models \exists \alpha \forall \gamma < \delta \psi(\gamma)^\alpha \).

\[ \square \]

**Theorem 4.5.** The theory \( \widehat{\text{iD}}^* \) is conservative over \( \widehat{\text{iD}}_{<\omega} \) for arithmetic sentences.

**Proof.** Suppose that \( \widehat{\text{iD}}^* \vdash \varphi \), for any arithmetic sentence \( \varphi \). Let \( \mathcal{M} \models \widehat{\text{iD}}_{<\omega} \) and \( \mathcal{M} \) countable. Then by Remark 4.2 we have a countable model, \( \mathcal{M} \prec \mathcal{N} \), of \( \widehat{\text{iD}}_{<\omega} \) which is recursively saturated. By Proposition 4.4, there is an expansion \( \mathcal{M}^* \) of \( \mathcal{M} \) which is a model of \( \widehat{\text{iD}}^* \). Therefore \( \mathcal{M}^* \models \varphi \), hence \( \mathcal{M} \models \varphi \), so that \( \mathcal{N} \models \varphi \). By the completeness theorem we conclude that \( \widehat{\text{iD}}_{<\omega} \vdash \varphi \).

\[ \square \]

**Remark 4.6.** It would also be possible to give a proof theoretic proof of Theorem 4.5 by use of partial cut elimination and asymmetric interpretation.

5. Modelling the universe

The objective of this Section is to emulate a system of Martin-Löf type theory in \( \widehat{\text{iD}}^* \) via codes for types. To further the understanding of this construction, we shall give a description of the pertaining type theory, \( \text{MT}^* \), before launching its codification in \( \widehat{\text{iD}}^* \). We will, however, assume a prior knowledge of Martin-Löf type theory as for instance given in [16]. The ingredients of our type theory \( \text{MT}^* \) are the following. The basic types are the type of natural numbers \( \mathbb{N} \) and the finite types \( \mathbb{N}_k \) of \( k \) elements. If \( A \) and \( B \) are types then \( A + B \) is a type. If \( A \) is a type and \( B(x) \) is a type for every \( x \in A \), then \( \Pi(A, B) \) and \( \Sigma(A, B) \) are types. If \( A \) is a type and \( a, b \in A \), then \( I(A, a, b) \) is a type. Moreover, we assume that we are given a set of ordinals \( \Omega \) (which is external to \( \text{MT}^* \)). For each \( \alpha \in \Omega \), \( \text{MT}^* \) contains types \( U_\alpha \) and \( V_\alpha \). Each \( U_\alpha \) is a universe of small types (in the sense of [16]) which contains \( \mathbb{N}, \mathbb{N}_k \) as elements, is closed under the ordinary type constructors \( +, \Pi, \Sigma, I \), and in addition contains \( U_\beta \) and \( V_\beta \) as elements, for \( \beta < \alpha \).
Each type $V_\alpha$ resembles the type of iterative sets over a universe of small types which was used by Aczel (cf. [1]) to interpret constructive Zermelo-Fraenkel set theory in Martin-Löf type theory. In natural deduction style the introduction rule for $V_\alpha$ is

$$(V_\alpha\text{-introduction}) \quad \frac{A \in U_\beta \quad f \in A \to V_\alpha}{\sup(A, f) \in V_\alpha}$$

for every $\beta < \alpha$. The crucial difference between Aczel’s type of iterative sets $V$ and $V_\alpha$ is that the latter is not equipped with an elimination rule which would allow one to carry out structural induction and recursion over $V_\alpha$. The omission of the elimination rules for the $V_\alpha$ is responsible for the proof-theoretic weakness of $\text{MT}^\ast$.

Henceforth we shall work in $\hat{\text{ID}}^\ast$. To emulate $\text{MT}^\ast$ in $\hat{\text{ID}}^\ast$ we build two hierarchies, $(U_\alpha)_\alpha$ and $(V_\alpha)_\alpha$, of sets of codes for types. $(U_\alpha)_\alpha$ is a set of codes for small types and therefore will be called a universe. Each $V_\alpha$ emulates a type of iterative sets over the universes $(U_\beta)_\beta$, for $\beta < \alpha$. In particular, $U_0$ is a universe of small types closed under the usual type constructors $\Pi, \Sigma, +, I, N, N_k$ and $V_1$ is a type of iterative sets over $U_0$ but without elimination rules (see Definition 5.1, clauses (i) - (vi) and (x), (xi)).

New levels in the $U$-hierarchy are introduced by reflecting on previous ones, i.e. each universe $U_\alpha$ contains for all $\beta < \alpha$ all the objects of the earlier universe $U_\beta$, plus a code, $u_\beta$, for that universe (see clauses (vii) and (viii)). Every $U_\alpha$ also contains a code, $v_\beta$, for each type $V_\beta$, where $\beta < \alpha$ (clause (ix)) and it is closed under the usual type constructors.

Every type $V_\alpha$ contains all the elements of any previous type in the $V$-hierarchy (clause (xii)). We shall prove later on that each $V_{\alpha+2}$ contains an object, $\hat{v}_\alpha$, representing the type $V_\alpha$ (see Definition 5.7 and Remark 5.8).

Finally, clause (c) of Definition 5.1 is introduced to ensure the complementarity of the relations $R_2^\alpha$ and $R_3^\alpha$ (this solution is due to Aczel in another context).
The types of iterative sets $\mathbb{V}_\alpha$ will be endowed with an equivalence relation, $\equiv_\alpha$, and an elementhood relation, $\in_\alpha$, to allow for a realizability interpretation of CZF$^-$. However, unlike in the case of Aczel’s interpretation of CZF in Martin Löf type theory via the type $\mathbb{V}$ ([1]), the iterative sets $\mathbb{V}_\alpha$ are not generated inductively and therefore $\equiv_\alpha$ can not be derived from an inductive structure as in the case of $\mathbb{V}$. The remedy will be to define $\equiv_\alpha$ as the maximum bisimulation on $\mathbb{V}_\alpha$, this being the topic of Section 6.

In the following we shall assume that the variables $e, f, \ldots, n$ range over the natural numbers.

**Definition 5.1.** Using a simultaneous positive fixed point definition we define eight relations $R^\alpha_1, \ldots, R^\alpha_8$, for each ordinal $\alpha$.

To increase intelligibility, we write

\[
\begin{align*}
\mathbb{U}_\alpha \models^w n & \text{ set } \quad \text{for } R^\alpha_1(n) \\
\mathbb{U}_\alpha \models^w n \in m & \quad \text{for } R^\alpha_2(n, m) \\
\mathbb{U}_\alpha \models^w n \notin m & \quad \text{for } R^\alpha_3(n, m) \\
\mathbb{U}_\alpha \models^w n = m \in k & \quad \text{for } R^\alpha_4(n, m, k) \\
\mathbb{U}_\alpha \models^w n = m \notin k & \quad \text{for } R^\alpha_5(n, m, k) \\
\mathbb{U}_\alpha \models^w n = m & \quad \text{for } R^\alpha_6(n, m) \\
\mathbb{V}_\beta \models n \text{ set} & \quad \text{for } R^\alpha_7(n) \\
\mathbb{V}_\beta \models n = m & \quad \text{for } R^\alpha_8(n, m).
\end{align*}
\]

We shall also use the following conventions.
(a)

\[ \pi(n, m) = \langle 0, n, m \rangle \]
\[ \sigma(n, m) = \langle 1, n, m \rangle \]
\[ \text{pl}(n, m) = \langle 2, n, m \rangle \]
\[ i(n, m, k) = \langle 3, n, m, k \rangle \]
\[ \hat{N} = \langle 4, 0 \rangle \]
\[ \hat{N}_k = \langle 4, k + 1 \rangle \]
\[ \text{sup}(n, m) = \langle 5, n, m \rangle \]
\[ u_\alpha = \langle 6, \alpha \rangle \]
\[ v_\alpha = \langle 7, \alpha \rangle . \]

(b)

\[ U_{<\alpha} \models \ldots := \exists \beta < \alpha \ U_\beta \models \ldots . \]
\[ V_{<\alpha} \models \ldots := \exists \beta < \alpha \ V_\beta \models \ldots . \]

(c)

\[ U_\alpha \models n \ \text{set} \quad := \quad U_\alpha \models^w n \ \text{set} \quad \text{and} \quad \forall m \left( U_\alpha \models^w m \in n \ \text{iff} \ \neg U_\alpha \models^w m \notin n \right) \quad \text{and} \quad \forall m, m' \left( U_\alpha \models^w m = m' \in n \ \text{iff} \ \neg U_\alpha \models^w m = m' \notin n \right) . \]
\[ U_\alpha \models m \in n \quad := \quad U_\alpha \models n \ \text{set} \quad \text{and} \quad U_\alpha \models^w m \in n . \]
\[ U_\alpha \models m \notin n \quad := \quad U_\alpha \models n \ \text{set} \quad \text{and} \quad U_\alpha \models^w m \notin n . \]
\[ U_\alpha \models m = m' \in n \quad := \quad U_\alpha \models n \ \text{set} \quad \text{and} \quad U_\alpha \models^w m = m' \in n . \]
\[ U_\alpha \models m = m' \notin n \quad := \quad U_\alpha \models n \ \text{set} \quad \text{and} \quad U_\alpha \models^w m = m' \notin n . \]
\[ U_\alpha \models n = n' \quad := \quad U_\alpha \models n \ \text{set} \quad \text{and} \quad U_\alpha \models n' \ \text{set} \quad \text{and} \quad \forall m \left( U_\alpha \models^w m \in n \ \text{iff} \ U_\alpha \models^w m \in n' \right) . \]
(d)\
\[ \mathbb{U}_\alpha \models^w \text{Fam}(k, e) := \mathbb{U}_\alpha \models^w \text{set}, \quad \forall j \left( \mathbb{U}_\alpha \models^w j \notin k \lor \mathbb{U}_\alpha \models^w ej \text{ set} \right) \]
\[ \text{and } \forall j, i \left( \mathbb{U}_\alpha \models^w i = j \notin k \lor \mathbb{U}_\alpha \models^w ei = ej \right), \]

(e)\
\[ \mathbb{V}_\alpha \models \text{Fam}(k, e) := \mathbb{U}_{<\alpha} \models k \text{ set}, \quad \forall j \left( \mathbb{U}_{<\alpha} \models j \notin k \lor \mathbb{V}_\alpha \models ej \text{ set} \right) \]
\[ \text{and } \forall j, i \left( \mathbb{U}_{<\alpha} \models i = j \notin k \lor \mathbb{V}_\alpha \models ei = ej \right). \]

where \text{Fam}(k, e) is spelled out as “\(e\) is a family of types over \(k\)”. The clauses of the iterated fixed point definition are the following.

(i)\
\[ \mathbb{U}_\alpha \models^w \langle 4, j \rangle \text{ set} \quad \text{if } j \in \mathbb{N}, \]
\[ \mathbb{U}_\alpha \models^w n \in \langle 4, j \rangle \quad \text{if } j = 0 \lor m + 1 < j, \]
\[ \mathbb{U}_\alpha \models^w n \notin \langle 4, j \rangle \quad \text{if } j \neq 0 \land m + 1 \geq j, \]
\[ \mathbb{U}_\alpha \models^w n = m \in \langle 4, j \rangle \quad \text{if } n = m \land (j = 0 \lor m + 1 < j), \]
\[ \mathbb{U}_\alpha \models^w n = m \notin \langle 4, j \rangle \quad \text{if } n \neq m \lor (j \neq 0 \land m + 1 \geq j). \]
(ii) If \( U_\alpha \models^w \text{Fam}(k, e) \) and \( \neg U_\alpha \models^\prec \pi(k, e) \text{ set} \), then \( U_\alpha \models^w \pi(k, e) \text{ set} \) and

\[
U_\alpha \models^w n \in \pi(k, e) \quad \text{if} \quad \forall j \left( U_\alpha \models^w j \notin k \lor U_\alpha \models^w nj \in e(j) \right)
\]

\[
\text{and} \quad \forall j, i \left( U_\alpha \models^w i = j \lor U_\alpha \models^w ni = nj \in e(j) \right),
\]

\[
U_\alpha \models^w n \notin \pi(k, e) \quad \text{if} \quad \exists j \left( U_\alpha \models^w j \in k \land U_\alpha \models^w nj \notin e(j) \right)
\]

\[
\text{or} \quad \exists j, i \left( U_\alpha \models^w i = j \land U_\alpha \models^w ni = nj \notin e(j) \right),
\]

\[
U_\alpha \models^w n = m \in \pi(k, e) \quad \text{if} \quad U_\alpha \models^w n \in \pi(k, e) \text{ and } U_\alpha \models^w m \in \pi(k, e)
\]

\[
\text{and} \quad \forall j \left( U_\alpha \models^w j \notin k \lor U_\alpha \models^w nj = mj \in e(j) \right),
\]

\[
U_\alpha \models^w n = m \notin \pi(k, e) \quad \text{if} \quad U_\alpha \models^w n \notin \pi(k, e) \text{ or } U_\alpha \models^w m \notin \pi(k, e)
\]

\[
\text{or} \quad \exists j \left( U_\alpha \models^w j \in k \land U_\alpha \models^w nj = mj \notin e(j) \right).
\]

(iii) If \( U_\alpha \models^w \text{Fam}(k, e) \) and \( \neg U_\alpha \models^\prec \sigma(k, e) \text{ set} \), then \( U_\alpha \models^w \sigma(k, e) \text{ set} \) and

\[
U_\alpha \models^w n \in \sigma(k, e) \quad \text{if} \quad U_\alpha \models^w n_0 \in k \text{ and } U_\alpha \models^w n_1 \in e(n_0),
\]

\[
U_\alpha \models^w n \notin \sigma(k, e) \quad \text{if} \quad U_\alpha \models^w n_0 \notin k \text{ or } U_\alpha \models^w n_1 \notin e(n_0),
\]

\[
U_\alpha \models^w n = m \in \sigma(k, e) \quad \text{if} \quad U_\alpha \models^w n \in \sigma(k, e) \text{ and } U_\alpha \models^w m \in \sigma(k, e)
\]

\[
\text{and} \quad U_\alpha \models^w n_0 = m_0 \in k
\]

\[
\text{and} \quad U_\alpha \models^w n_1 = m_1 \in e(n_0),
\]

\[
U_\alpha \models^w n = m \notin \sigma(k, e) \quad \text{if} \quad U_\alpha \models^w n \notin \sigma(k, e) \text{ or } U_\alpha \models^w m \notin \sigma(k, e)
\]

\[
\text{or} \quad U_\alpha \models^w n_0 = m_0 \notin k
\]

\[
\text{or} \quad U_\alpha \models^w n_1 = m_1 \notin e(n_0).
\]
(iv) If $\mathbb{U}_\alpha \models^w \text{set}$, $\mathbb{U}_\alpha \models^w \text{m set}$ and $\neg \mathbb{U}_\alpha \models \text{pl}(n, m) \text{ set}$, then $\mathbb{U}_\alpha \models^w \text{pl}(n, m) \text{ set}$, and

\[
\mathbb{U}_\alpha \models^w i \in \text{pl}(n, m) \quad \text{if} \quad (i_0 = 0 \text{ and } \mathbb{U}_\alpha \models^w i_1 \in n)
\]

or $i_0 = 1$ and $\mathbb{U}_\alpha \models^w i_1 \in m$, $\mathbb{U}_\alpha \models^w i \notin \text{pl}(n, m)$ if $i_0 \neq 0$ and $i_0 \neq 1$

or $i_0 = 0$ and $\mathbb{U}_\alpha \models^w i_1 \notin n$

or $i_0 = 1$ and $\mathbb{U}_\alpha \models^w i_1 \notin m$, $\mathbb{U}_\alpha \models^w i = j \in \text{pl}(n, m)$ if $i_0 = j_0 = 0$ and $\mathbb{U}_\alpha \models^w i_1 = j_1 \in n$

or $i_0 = j_0 = 1$ and $\mathbb{U}_\alpha \models^w i_1 = j_1 \in m$, $\mathbb{U}_\alpha \models^w i = j \notin \text{pl}(n, m)$ if $i_0 = j_0 = 0$ and $\mathbb{U}_\alpha \models^w i_1 = j_1 \notin n$

or $i_0 = j_0 = 1$ and $\mathbb{U}_\alpha \models^w i_1 = j_1 \notin m$.

(v) If $\mathbb{U}_\alpha \models^w \text{set}$ and $\neg \mathbb{U}_\alpha \models \text{i}(n, m, k) \text{ set}$, then $\mathbb{U}_\alpha \models^w \text{i}(n, m, k) \text{ set}$ and

\[
\mathbb{U}_\alpha \models^w j \in \text{i}(n, m, k) \quad \text{if} \quad j = 0 \text{ and } \mathbb{U}_\alpha \models^w m = k \in n,
\]

\[
\mathbb{U}_\alpha \models^w j \notin \text{i}(n, m, k) \quad \text{if} \quad j \neq 0 \text{ or } \mathbb{U}_\alpha \models^w m = k \notin n,
\]

\[
\mathbb{U}_\alpha \models^w j = j' \in \text{i}(n, m, k) \quad \text{if} \quad j = j' = 0 \text{ and } \mathbb{U}_\alpha \models^w m = k \in n,
\]

\[
\mathbb{U}_\alpha \models^w j = j' \notin \text{i}(n, m, k) \quad \text{if} \quad \text{not } j = j' = 0 \text{ or } \mathbb{U}_\alpha \models^w m = k \notin n.
\]
(vi)

\[ \mathbb{U}_\alpha \models^w e = f \quad \text{if} \quad \mathbb{U}_\alpha \models^w e \text{ set and } \mathbb{U}_\alpha \models^w f \text{ set} \]

and \( \forall j (\mathbb{U}_\alpha \models^w j \notin e \lor \mathbb{U}_\alpha \models^w j \in f) \)

and \( \forall j (\mathbb{U}_\alpha \models^w j \in e \lor \mathbb{U}_\alpha \models^w j \notin f) \)

and \( \forall j, j' (\mathbb{U}_\alpha \models^w j = j' \notin e \lor \mathbb{U}_\alpha \models^w j = j' \in f) \)

and \( \forall j, j' (\mathbb{U}_\alpha \models^w j = j' \in e \lor \mathbb{U}_\alpha \models^w j = j' \notin f) \).

(vii) If \( \beta < \alpha \), then

\[ \mathbb{U}_\alpha \models^w u_\beta \text{ set and} \]

\[ \mathbb{U}_\alpha \models^w m \in u_\beta \quad \text{if} \quad \mathbb{U}_\beta \models m \text{ set,} \]

\[ \mathbb{U}_\alpha \models^w m \notin u_\beta \quad \text{if} \quad \neg \mathbb{U}_\beta \models m \text{ set,} \]

\[ \mathbb{U}_\alpha \models^w m = m' \in u_\beta \quad \text{if} \quad \mathbb{U}_\beta \models m = m', \]

\[ \mathbb{U}_\alpha \models^w m = m' \notin u_\beta \quad \text{if} \quad \neg \mathbb{U}_\beta \models m = m'. \]

(viii) If \( \mathbb{U}_{<\alpha} \models n \text{ set} \) then

\[ \mathbb{U}_\alpha \models^w n \text{ set and} \]

\[ \mathbb{U}_\alpha \models^w m \in n \quad \text{if} \quad \mathbb{U}_{<\alpha} \models m \in n, \]

\[ \mathbb{U}_\alpha \models^w m \notin n \quad \text{if} \quad \mathbb{U}_{<\alpha} \models m \notin n, \]

\[ \mathbb{U}_\alpha \models^w m = m' \in n \quad \text{if} \quad \mathbb{U}_{<\alpha} \models m = m' \in n, \]

\[ \mathbb{U}_\alpha \models^w m = m' \notin n \quad \text{if} \quad \mathbb{U}_{<\alpha} \models m = m' \notin n. \]
(ix) If $\beta < \alpha$, then

\[ U_\alpha \models^w \nu_\beta \text{ set and} \]

\[ U_\alpha \models^w n \in \nu_\beta \quad \text{if} \quad \forall \beta \models n \text{ set}, \]

\[ U_\alpha \models^w n \notin \nu_\beta \quad \text{if} \quad \neg \forall \beta \models n \text{ set}, \]

\[ U_\alpha \models^w m = m' \in \nu_\beta \quad \text{if} \quad \forall \beta \models m = m', \]

\[ U_\alpha \models^w m = m' \notin \nu_\beta \quad \text{if} \quad \neg \forall \beta \models m = m'. \]

(x) If $\forall \alpha \models \text{Fam}(k, e)$ and $\neg \forall < \alpha \models \text{sup}(k, e)$ set then

\[ \forall \alpha \models \text{sup}(k, e) \text{ set}. \]

(xi) If $\forall \alpha \models \text{Fam}(k, e)$, $\forall \alpha \models \text{Fam}(k', e')$, $\forall < \alpha \models k = k' \quad \text{and} \quad \forall n, m \left( \forall < \alpha \models n = m \notin k \quad \lor \quad \forall \alpha \models en = e'm \right), \quad \text{then} \]

\[ \forall \alpha \models \text{sup}(k, e) = \text{sup}(k', e'). \]

(xii) If $\forall < \alpha \models n \text{ set}$, then

\[ \forall \alpha \models n \text{ set}. \]

If $\forall < \alpha \models m = m'$, then

\[ \forall \alpha \models m = m'. \]

We shall sometimes write $n \in U_\alpha$, for $U_\alpha \models n \text{ set}$ and similarly $n \in \forall \alpha$, for $\forall \alpha \models n \text{ set}$. We shall preferably use the first letters of the alphabet $a, b, c, \ldots$ and also $u, v, \ldots$ to denote iterative sets i.e. those $a$ such that $\forall \alpha \models a \text{ set}$, for some $\alpha$.

**Remark 5.2.** Note that by Definition 5.1 (c), for any $n \in U_\beta$,

\[ U_\beta \models m \in n \quad \text{iff} \quad \neg U_\beta \models m \notin n, \]

\[ U_\beta \models m = m' \in n \quad \text{iff} \quad \neg U_\beta \models m = m' \notin n. \]
Lemma 5.3. Let $\beta \leq \alpha$. Then if $\mathbb{U}_\beta \models n \text{ set }$ and $\mathbb{U}_\beta \models n' \text{ set }$, the following hold for any $m, m'$,

(i) $\mathbb{U}_\alpha \models n \text{ set }$,

(ii) $\mathbb{U}_\beta \models m \in n$ iff $\mathbb{U}_\alpha \models m \in n$,

(iii) $\mathbb{U}_\beta \models m \notin n$ iff $\mathbb{U}_\alpha \models m \notin n$,

(iv) $\mathbb{U}_\beta \models m = m' \in n$ iff $\mathbb{U}_\alpha \models m = m' \in n$,

(v) $\mathbb{U}_\beta \models m = m' \notin n$ iff $\mathbb{U}_\alpha \models m = m' \notin n$,

(vi) $\mathbb{U}_\beta \models n = n'$ iff $\mathbb{U}_\alpha \models n = n'$.

In addition if $\mathbb{V}_\beta \models a \text{ set }$ and

$\mathbb{V}_\beta \models a' \text{ set }$, then

(vii) $\mathbb{V}_\alpha \models a \text{ set }$,

(viii) $\mathbb{V}_\beta \models a = a'$ iff $\mathbb{V}_\alpha \models a = a'$.

Proof. The proof is by induction, following the clauses in Definition 5.1. We show only (i) and (ii), the other cases being similar.

(i) Suppose $\mathbb{U}_\beta \models n \text{ set }$ and $\beta \leq \alpha$.

Then by Definition 5.1 (c)

$\mathbb{U}_\beta \models^w n \text{ set }$ and

$\forall m (\mathbb{U}_\beta \models^w m \in n \iff \neg \mathbb{U}_\beta \models^w m \notin n)$ and

$\forall m, m' (\mathbb{U}_\beta \models^w m = m' \in n \iff \neg \mathbb{U}_\beta \models^w m = m' \notin n)$.

From $\mathbb{U}_\beta \models^w n \text{ set }$ we have $\mathbb{U}_\alpha \models^w n \text{ set }$, by Definition 5.1 (viii). From the same definition we also obtain

if $\mathbb{U}_\beta \models^w m \in n$ then $\mathbb{U}_\alpha \models^w m \in n$ and
if $U_\beta \models^w m \not\in n$ then $U_\alpha \models^w m \not\in n$.

In addition,

if $U_\beta \models^w m = m' \in n$ then $U_\alpha \models^w m = m' \in n$ and

if $U_\beta \models^w m = m' \not\in n$ then $U_\alpha \models^w m = m' \not\in n$.

Therefore

$$U_\alpha \models n \text{ set}.$$ (ii) We want to show that if $U_\beta \models n \text{ set}$,

then

$$U_\beta \models m \in n \text{ iff } U_\alpha \models m \in n.$$

By Definition 5.1 (c),

$$U_\beta \models m \in n \text{ iff } U_\beta \models n \text{ set and } U_\beta \models^w m \in n.$$

By Definition 5.1 (viii) and by the same argument as above we obtain,

if $U_\beta \models^w m \in n$ then $U_\alpha \models^w m \in n$,

while in (i) we have proved that $U_\alpha \models n \text{ set}$.

Therefore,

if $U_\beta \models m \in n$ then $U_\alpha \models m \in n$.

Suppose now that $U_\beta \models n \text{ set}$ and $U_\alpha \models m \in n$. Since $U_\beta \models n \text{ set}$, then

$$\exists \gamma \leq \beta U_\gamma \models n \text{ set}.$$
Therefore condition (ii) of Definition 5.1 can not be applied, and \( \mathbb{U}_\alpha \models m \in n \) can only hold provided that \( \exists \gamma \leq \beta \mathbb{U}_\gamma \models m \in n \). So that \( \mathbb{U}_\beta \models m \in n \). \( \square \)

**Definition 5.4.** Let

\[
\mathbb{U}_\alpha \models \text{Fam}(k, e) := \mathbb{U}_\alpha \models k \text{ set, } \forall j (\mathbb{U}_\alpha \models j \notin k \lor \mathbb{U}_\alpha \models e_j \text{ set})
\]

and \( \forall j, i (\mathbb{U}_\alpha \models i = j \notin k \lor \mathbb{U}_\alpha \models e_i = e_j) \).

As a consequence of Definition 5.1 (c) we have the following Lemma.

**Lemma 5.5.** Clauses (i) - (ix) of Definition 5.1 hold true with “\( \mathbb{U}_\alpha \models w \)” being replaced by “\( \mathbb{U}_\alpha \models \)”.

**Proof.** By a simple induction following the clauses in Definition 5.1 and by use of Remark 5.2. We show one example only. It is clear that the modified version of clause (i) holds.

Clause (ii): Suppose that \( \mathbb{U}_\alpha \models \text{Fam}(k, e) \) and \( \neg \mathbb{U}_{<\alpha} \models \pi(k, e) \text{ set} \). This is

\[
\neg \mathbb{U}_{<\alpha} \models \pi(k, e) \text{ set} \quad \text{and}
\]

\[
\mathbb{U}_\alpha \models k \text{ set, } \forall j (\mathbb{U}_\alpha \models j \notin k \lor \mathbb{U}_\alpha \models e_j \text{ set}),
\]

\[
\forall j, i (\mathbb{U}_\alpha \models i = j \notin k \lor \mathbb{U}_\alpha \models e_i = e_j).
\]

We want to show that

\[
\mathbb{U}_\alpha \models \pi(k, e) \text{ set}.
\]

By Definition 5.1 (c) this holds if and only if

\[
\mathbb{U}_\alpha \models^w \pi(k, e) \text{ set} \quad \text{and} \quad \forall m (\mathbb{U}_\alpha \models^w m \in \pi(k, e) \iff \neg \mathbb{U}_\alpha \models^w m \notin \pi(k, e))
\]

and \( \forall m, m' (\mathbb{U}_\alpha \models^w m = m' \in \pi(k, e) \iff \neg \mathbb{U}_\alpha \models^w m = m' \notin \pi(k, e)) \).

We obviously have

\[
\mathbb{U}_\alpha \models^w \pi(k, e) \text{ set}.
\]
By Remark 5.2, \( \forall m \ (U_{\alpha} \models m \in \pi(k,e) \iff \neg U_{\alpha} \models m \notin \pi(k,e)) \). Hence
\[
\forall m \ (U_{\alpha} \models^{w} m \in \pi(k,e) \iff \neg U_{\alpha} \models^{w} m \notin \pi(k,e)).
\]
Similarly we can show
\[
\forall m, m' \ (U_{\alpha} \models^{w} m = m' \in \pi(k,e) \iff \neg U_{\alpha} \models^{w} m = m' \notin \pi(k,e)).
\]
so that
\[
U_{\alpha} \models \pi(k,e) \text{ set}.
\]
The other cases are similar. \(\square\)

**Definition 5.6.** We shall use the following abbreviations
\[
U \models n \text{ set} := \exists \alpha \ U_{\alpha} \models n \text{ set},
\]
\[
U \models m \in n := \exists \alpha \ U_{\alpha} \models m \in n,
\]
\[
U \models m \notin n := \exists \alpha \ U_{\alpha} \models m \notin n,
\]
\[
U \models m = m' \in n := \exists \alpha \ U_{\alpha} \models m = m' \in n,
\]
\[
U \models m = m' \notin n := \exists \alpha \ U_{\alpha} \models m = m' \notin n,
\]
\[
U \models m = m' := \exists \alpha \ U_{\alpha} \models m = m'.
\]
Similarly we shall write
\[
V \models n \text{ set} := \exists \alpha \ V_{\alpha} \models n \text{ set},
\]
\[
V \models n = m := \exists \alpha \ V_{\alpha} \models n = m.
\]
Finally let
\[
U \models \text{Fam}(k,e) := U \models k \text{ set}, \ \forall j \ (U \models j \in k \implies U \models e_j \text{ set})
\]
and \(\forall j, i \ (U \models i = j \implies U \models e_i = e_j)\).
\[
V \models \text{Fam}(k,e) := U \models k \text{ set}, \ \forall j \ (U \models j \in k \implies V \models e_j \text{ set})
\]
and \(\forall j, i \ (U \models i = j \implies V \models e_i = e_j)\).
We shall also write \( n \in U \) for \( U \models n \text{ set} \), and \( a \in V \) for \( V \models a \text{ set} \).

**Definition 5.7.** Let \( \hat{v}_\alpha = \sup(v_\alpha, h) \), with \( h \) a canonical index for the identity function.

**Remark 5.8.** (i) Note that \( V_{\alpha+2} \models \hat{v}_\alpha \text{ set} \), i.e. \( V_{\alpha+2} \models \sup(v_\alpha, h) \text{ set} \). In fact, this is the case iff

\[
\neg V_{\alpha+2} \models \sup(v_\alpha, h) \text{ set} \quad \text{and} \quad U_{\alpha+2} \models v_\alpha \text{ set}, \quad \forall j \ (U_{\alpha+2} \models j \not\in v_\alpha \lor V_{\alpha+2} \models hj \text{ set}),
\]

\[
\forall j, i \ (U_{\alpha+2} \models i = j \not\in v_\alpha \lor V_{\alpha+2} \models hi = hj).
\]

By Definition 5.1 (ix) and Lemma 5.5,

\[
U_{\alpha+1} \models v_\alpha \text{ set}.
\]

Hence clearly

\[
\neg V_{\alpha+2} \models \sup(v_\alpha, h) \text{ set}.
\]

In addition for any \( j \),

\[
U_{\alpha+1} \models j \in v_\alpha \text{ iff } V_\alpha \models j \text{ set}.
\]

For any \( j \), if \( V_\alpha \models j \text{ set} \) then \( V_\alpha \models hj \text{ set} \), as \( h \) is the identity function. Hence by Definition 5.1 (xii), \( V_{\alpha+2} \models hj \text{ set} \).

Similarly, for any \( j, i \), if \( U_{\alpha+1} \models i = j \in v_\alpha \) then \( V_{\alpha+2} \models hi = hj \).

Hence we conclude

\[
V_{\alpha+2} \models \hat{v}_\alpha \text{ set}.
\]

(ii) Let \( V_\alpha \models a \text{ set} \). Then, inductively, there is \( \beta \leq \alpha \) such that \( a \) is of the form \( \sup(k, e) \), with \( k \) and \( e \) such that \( U_{\beta} \models k \text{ set}, \forall j \ (U_{\beta} \models j \not\in k \lor V_\beta \models ej \text{ set}), \forall j, i \ (U_{\beta} \models i = j \not\in k \lor V_\beta \models ei = ej) \) and \( \neg V_{\beta} \models \sup(k, e) \text{ set} \).
By injectivity of the coding functions, and because of the requirement
\( \neg \forall_{< \beta} \models \sup(k, e) \text{ set} \), such \( k \) and \( e \) are unique, so that we may denote \( k \) with \( \bar{a} \) and \( e \) with \( \tilde{a} \).

**Lemma 5.9.** Clauses (i) - (viii), of Definition 5.1 hold true when we replace "\( U_{\alpha} \models \)" by "\( U \models \)"; provided we also remove condition "\( \neg U_{\alpha} \models \pi(k, e) \text{ set} \)" from (ii) and similar conditions from (iii), (iv) and (v). In addition we replace "\( U < \alpha \models \)" by "\( U \beta \models \)" and take \( \beta \) arbitrary in clauses (vii) and (viii).

Clauses (x) and (xi) hold true when we replace "\( U < \alpha \models \)" and "\( V_{\alpha} \models \)" by respectively "\( U \models \)" and "\( V_{\beta} \models \)" and removal condition "\( \neg V_{\alpha} \models \sup(k, e) \text{ set} \)" from (x).

Clauses (xii) and (ix) hold true when we replace "\( V_{\alpha} \models \)" by "\( V_{\beta} \models \)" and "\( U_{\alpha} \models w \)" by "\( U \models w \)" and let \( \beta \) be arbitrary.

**Proof.** Following the clauses in Definition 5.1, using Lemmas 5.3 and 5.5 as well as the principle of \( \Sigma^\Omega \) - Reflection. We show one example only.

(ii) Suppose that \( U \models \text{Fam}(k, e) \). Then we need to prove that \( U \models \pi(k, e) \text{ set} \).

By Definition 5.6, \( U \models \text{Fam}(k, e) \) if

\[
U \models k \text{ set}, \quad \forall j \left( U \models j \in k \rightarrow U \models e_j \text{ set} \right) \quad \text{and} \\
\forall j, i \left( U \models i = j \in k \rightarrow U \models e_i = e_j \right).
\]

That is

\[
\exists \alpha U_{\alpha} \models k \text{ set}, \quad \forall j \left( U_{\alpha} \models j \in k \rightarrow \exists \beta U_{\beta} \models e_j \text{ set} \right) \quad \text{and} \\
\forall j, i \left( U_{\alpha} \models i = j \in k \rightarrow \exists \beta U_{\beta} \models e_i = e_j \right).
\]

By use of \( \Sigma^\Omega \) - Reflection, we find a \( \delta \geq \alpha \) such that

\[(\exists \beta \leq \delta) \forall j \left( U_{\alpha} \models j \in k \rightarrow U_{\beta} \models e_j \text{ set} \right) \]

and similarly

\[(\exists \beta \leq \delta) \forall j, i \left( U_{\alpha} \models i = j \in k \rightarrow U_{\beta} \models e_i = e_j \right) \]
Hence by Lemma 5.3 we obtain
\[ \forall j \left( \mathbb{U}_\delta \models j \not\in k \lor \mathbb{U}_\delta \models e \text{ set} \right) \quad \text{and} \]
\[ \forall j, i \left( \mathbb{U}_\delta \models i = j \not\in k \lor \mathbb{U}_\delta \models e \text{ set} = e \right), \]
so that by Lemma 5.5
\[ \mathbb{U}_\delta \models \pi(k, e) \text{ set}. \]
Hence
\[ \mathbb{U} \models \pi(k, e) \text{ set}. \]
The other cases are similar. \(\square\)

6. Bisimulations

Central to this Section will be the notion of *bisimulation*. The idea of bisimulation can be traced at least as far back as the early days of automata theory. Let \(\text{Act}\) be a set of atomic actions. We want to model the notion of an automaton which has a state at each moment, and which changes its state according to evolution by the atomic states. Whether these changes of state are due to factors inside or outside of the automaton is not captured in this model. Two automata \(A\) and \(A'\) are called *bisimilar* if an observer who watches their actions would not be able to tell the difference between \(A\) and \(A'\).

The concept of bisimulation was discovered several times in different places. A version of modal logic was proposed by van Benthem in 1976 and one for processes by Park in 1981. For more information on bisimulation see [4].

In this Section we shall be working in the \(\tilde{\mathcal{ID}}^*\) analogue of type theory as introduced in Section 5 and define a notion of system and bisimulation for systems as in ([15]). A type theoretic notion of system was introduced by Hallnäs ([10]) and Lindström ([15]) in order to allow for an extension of Aczel’s interpretation of \(\text{CZF}^-\) in \(\text{MLTT}\) which would accommodate non well founded sets.
The primordial example of a system is Aczel’s type $V$ of iterative sets. Its maximum bisimulation coincides with the relation $\bar{\varepsilon}$ which Aczel used for the interpretation of $\text{CZF}^-$ in $\text{MLTT}$. The relation $\bar{\varepsilon}$ on $V \times V$ defined by

$$\alpha \bar{\varepsilon} \beta : \iff (\exists x \in \bar{\beta}(\alpha \equiv \beta x))$$

is well founded in the latter case since $V$ has an inductive definition, this being pivotal for interpreting set induction. Hallnäs ([10]) and Lindström ([15]), however, showed that if one is given a type $S$ which just satisfies the closure properties of $V$ (but is not necessarily the least such type) it is still possible to define a relation $\varepsilon_S$ on $S \times S$ which, in general, is not well founded, but allows one to interpret $\text{CZF}$ with the exception of $\varepsilon$-induction.

For the readers convenience we will recall the notions and results of [15], but without giving proofs. Since the motivations for the constructions of [15] are to a large extent revealed in the proofs, the reader is ultimately referred to [15] for more details.

In the following we show that each universe $V_\alpha$ is a system in the sense of [15] and we define a maximum bisimulation for each $V_\alpha$. We also prove that the bisimulations enjoy preservation properties along the hierarchy.

To increase readability we shall make use of the so called Curry - Howard isomorphism, and freely interchange types with their corresponding logical connectives and quantifiers, i.e., when convenient, we use the logical notations $\forall, \exists, \lor, \land$ for the type operations $\pi, \sigma, \oplus, \otimes$, respectively.

**Definition 6.1** (System). A *system* over $U_\alpha$ consists of a type $S$ together with an assignment of $\bar{a} \in U_\alpha$ and $\bar{a} : \bar{a} \rightarrow S$ to each $a \in S$.

Given a relation $R$, we shall write $R(a, b)$ for $\langle a, b \rangle \in R$.

**Definition 6.2** (Bisimulation). A binary relation $R$ on a system $S$ over $U_\alpha$ is a *bisimulation* on $S$, if given $a, b \in S$

$$R(a, b) \rightarrow \forall x \in \bar{a} \exists y \in \bar{b} R(\bar{a}x, \bar{by}) \land \forall y \in \bar{b} \exists x \in \bar{a} R(\bar{ax}, \bar{by}).$$
Lindström ([15]) has shown that given these notions of system and bisimulation we can prove that there is a maximum bisimulation, \( \equiv_S \), for each system \( S \).

We note that for every ordinal \( \alpha \), the type \( V_{\alpha+1} \) is a system over \( U_\alpha \) in the sense of Definition 6.1. Therefore following the proof of Proposition 2.2 in [15], we can show that there is a maximum bisimulation, \( \equiv_\alpha \), for each \( V_\alpha \), \( \alpha > 0 \).

For \( a, b \in V_\alpha \), we shall define the relation \( a \equiv_\alpha b \) in stages, defining first \( a \equiv_n b \), for each natural number \( n \). Intuitively, \( (a \equiv_\alpha b) \) holds if and only if \( (a \equiv_n b) \) holds for each \( n \in \mathbb{N} \) and also whenever \( n > m \), the proof of \( (a \equiv_n b) \) is an extension of the proof of \( (a \equiv_m b) \).

**Definition 6.3.** For \( \alpha > 0 \), for \( a, b \in V_\alpha \), define \( a \equiv_n b \) by recursion as follows

\[
\begin{align*}
(a \equiv_0 b) &= \hat{N}_1; \\
(a \equiv_{n+1} b) &= \sigma(\pi(\tilde{a}, \Lambda x. \sigma(\tilde{b}, \Lambda y. (\tilde{a}x \equiv_n \tilde{b}y))), \\
&\quad \Lambda w. \pi(\tilde{b}, \Lambda y. \sigma(\tilde{a}, \Lambda x. (\tilde{a}x \equiv_n \tilde{b}y))).
\end{align*}
\]

For each natural number \( n \), we define projection functions, \( h_n \), as follows

\[
\begin{align*}
h_0(a, b)(f, g) &= \hat{0}_1, & \text{for } (f, g) \in (a \equiv_1 b), \\
h_{n+1}(a, b)(f, g) &= (f', g'), & \text{for } (f, g) \in (a \equiv_{n+2} b),
\end{align*}
\]

where \( \mathbb{U}_0 \models \hat{0}_1 \in \hat{N}_1 \), i.e. \( \hat{0}_1 \) is the only element of \( \hat{N}_1 \), and

\[
f' = \Lambda x. ((fx)_0, h_n(\tilde{a}x, \tilde{b}(fx)_0)(fx)_1),
g' = \Lambda y. ((gy)_0, h_n(\tilde{a}(gy)_0, \tilde{b}y)(gy)_1).
\]

Further, let

\[
(a \equiv_\infty b) = \pi(\hat{N}, \Lambda n. (a \equiv_n b)).
\]

Finally, let

\[
(a \equiv_\alpha b) = \sigma((a \equiv_\infty b), \Lambda z. \pi(\hat{N}, \Lambda n. i((a \equiv_n b), h_n(a, b)z(n + 1), z(n)))�)
\]

**Proposition 6.4.** For any ordinal \( \alpha > 0 \), the relation \( \equiv_\alpha \) on \( V_\alpha \) of Definition 6.3 is the maximum bisimulation on \( V_\alpha \). In particular, for \( a, b \in V_\alpha \),
(i) \( (a \equiv_{\alpha} b) \in \mathbb{U}_{<\alpha} \).

(ii) \( (a \equiv_{\alpha} b) \rightarrow \forall x \in \tilde{a} \exists y \in \tilde{b} (\tilde{ax} \equiv_{\alpha} \tilde{by}) \land \forall y \in \tilde{b} \exists x \in \tilde{a} (\tilde{ax} \equiv_{\alpha} \tilde{by}) \).

(iii) If \( R \) is a relation on \( V_\alpha \) such that

\[
R(a, b) \rightarrow \forall x \in \tilde{a} \exists y \in \tilde{b} R(\tilde{ax}, \tilde{by}) \land \forall y \in \tilde{b} \exists x \in \tilde{a} R(\tilde{ax}, \tilde{by}),
\]

then \( R(a, b) \rightarrow (a \equiv_{\alpha} b) \).

**Proof.** It can be easily seen that \( (a \equiv_{\alpha} b) \in \mathbb{U}_{<\alpha} \). We refer to [15], Proposition 2.2, for a proof of (ii) and (iii). \( \square \)

**Corollary 6.5.** For any ordinal \( \alpha > 0 \), the relation \( \equiv_{\alpha} \) on \( V_\alpha \) defined above is an equivalence relation such that

\[
a \equiv_{\alpha} b \iff \forall x \in \tilde{a} \exists y \in \tilde{b} (\tilde{ax} \equiv_{\alpha} \tilde{by}) \land \forall y \in \tilde{b} \exists x \in \tilde{a} (\tilde{ax} \equiv_{\alpha} \tilde{by}).
\]

**Proof.** See [15]. \( \square \)

**Lemma 6.6.** Let \( a, b \in V_\beta \) and let \( 0 < \beta \leq \alpha \). Then

\[
\exists x \mathbb{U}_{<\beta} \models x \in (a \equiv_{\beta} b) \iff \exists x \mathbb{U}_{<\alpha} \models x \in (a \equiv_{\alpha} b).
\]

**Proof.** The claim clearly follows from Definition 6.3. \( \square \)

**Definition 6.7.** Let \( V_\alpha \models a, b \text{ set} \). Then

\[
(b \in_{\alpha} a) := \sigma(\tilde{a}, \Lambda x. b \equiv_{\alpha} \tilde{ax}).
\]

**Definition 6.8.** For \( V \models a, b \text{ set} \), let

\[
(a \equiv_V b) := \exists \alpha (a \in V_\alpha \land b \in V_\alpha \land a \equiv_{\alpha} b).
\]

\[
b \in_V a := \exists \alpha (a \in V_\alpha \land b \in_{\alpha} a).
\]
Lemma 6.9. Let $\mathcal{V} \models a, b$ set. Then

$$(a \equiv \mathcal{V} b) \iff \forall x \in \bar{a} \exists y \in \bar{b} (\bar{a}x \equiv \mathcal{V} \bar{by}) \land \forall y \in \bar{b} \exists x \in \bar{a} (\bar{a}x \equiv \mathcal{V} \bar{by}).$$

Proof. The claim follows from Lemma 6.6. □

7. Realizability

We shall now approach the final stage of the interpretation of $\text{CZF}^- + \text{INAC}$ in $\hat{\text{ID}}^*$, and proceed to define a notion of realizability for each universe $\mathcal{V}_\alpha$. We inductively associate to each formula $\varphi$ of $\text{LST}$ a new formula $e \models_\alpha \varphi$, having the same free variables as $\varphi$ plus a fresh number number variable $e$.

We shall subsequently define a notion of realizability, $\models$, for the universe $\mathcal{V}$.

Definition 7.1 (Realizability in $\mathcal{V}_\alpha$). For each formula $\varphi(x_1, \ldots, x_n)$ of $\text{LST}$ containing at most $x_1, \ldots, x_n$ free and with no occurrence of $\omega$, we define

$$e \models_\alpha \varphi(x_1, \ldots, x_n),$$

for $e$ a natural number, as follows

$$e \models_\alpha \bot := \mathcal{U}_{\leq} = e \in \hat{\mathcal{N}}_0;$$

$$e \models_\alpha (x = y) := \mathcal{U}_{\leq} = e \in (x \equiv_\alpha y) \land \mathcal{V}_\alpha = x, y \text{ set};$$

$$e \models_\alpha (x \in y) := \mathcal{U}_{\leq} = e \in (x \in_\alpha y) \land \mathcal{V}_\alpha = x, y \text{ set};$$

$$e \models_\alpha \psi \land \chi := e_0 \models_\alpha \psi \land e_1 \models_\alpha \chi;$$

$$e \models_\alpha \psi \lor \chi := (e_0 = 0 \rightarrow e_1 \models_\alpha \psi) \lor (e_0 \neq 0 \rightarrow e_1 \models_\alpha \chi);$$

$$e \models_\alpha \psi \rightarrow \chi := \forall q (q \models_\alpha \psi \rightarrow eq \models_\alpha \chi);$$

$$e \models_\alpha \exists x \in a \psi(x) := \mathcal{U}_{<} = e_0 \in \bar{a} \land e_1 \models_\alpha \psi(\bar{a}(e_0));$$

$$e \models_\alpha \forall x \in a \psi(x) := \forall i (\mathcal{U}_{<} = i \notin \bar{a} \land e_i \models_\alpha \psi(\bar{a}i));$$

$$e \models_\alpha \exists x \psi(x) := e_1 \models_\alpha \psi(e_0) \land \mathcal{V}_\alpha = e_0 \text{ set};$$

$$e \models_\alpha \forall x \psi(x) := \forall u \in \mathcal{V}_\alpha (eu \models_\alpha \psi(u)).$$
Definition 7.2. Let
\[ \hat{\emptyset} = \sup(\hat{\mathcal{N}}_0, h_0), \]
where \( \hat{\mathcal{N}}_0 \) is the empty type and \( h_0 \) is (an index for) the empty function.

If \( b \in \mathbb{V} \), let
\[ \text{suc}(b) = \sup(\text{pl}(\tilde{b}, \hat{\mathcal{N}}_1), \Lambda x.g(x, \tilde{b}, \Lambda y.b)), \]
where \( g \) is a function such that
\[ g(n, \tilde{b}, \Lambda y.b) = \begin{cases} \tilde{b}((n)_1), & \text{if } (n)_0 = 0, \\ b & \text{otherwise}. \end{cases} \]

By recursion over \( \mathbb{N} \), let \( \Delta(n) \) be defined as follows:
\[ \Delta(0) = \hat{\emptyset}, \]
\[ \Delta(n + 1) = \text{suc}(\Delta(n)). \]

Let now
\[ \hat{\omega} = \sup(\hat{\mathcal{N}}, \Lambda n.\Delta(n)). \]

Remark 7.3. Note that by a \( \Delta^0_0 \)-Induction on the natural numbers \( \Delta(n) \in \mathbb{V}_1 \) and hence \( \hat{\omega} \in \mathbb{V}_1 \). (In fact, \( \hat{\emptyset} \in \mathbb{V}_1 \), since \( \hat{\mathcal{N}}_0 \in \mathbb{U}_0 \). In addition, if \( \Delta(n) \in \mathbb{V}_\alpha \), for \( \alpha > 0 \), then \( \Delta(n + 1) \in \mathbb{V}_\alpha \). Hence \( \hat{\omega} \in \mathbb{V}_1 \), as \( \hat{\mathcal{N}} \in \mathbb{U}_0 \).)

Definition 7.4. If \( \varphi(x_1, \ldots, x_n) \) is a formula of \( \text{LST} \) possibly with some occurrences of \( \omega \), let \( \psi(x_1, \ldots, x_n, \omega) \) be the formula \( \varphi(x_1, \ldots, x_n) \) where the occurrences of \( \omega \) are made explicit. We define realizability for \( \varphi(x_1, \ldots, x_n) \) as follows
\[ e \models_\alpha \varphi(x_1, \ldots, x_n) := e \models_\alpha \psi(x_1, \ldots, x_n, \hat{\omega}). \]

We say that a formula \( \varphi \) of \( \text{LST} \) is realizable in \( \mathbb{V}_\alpha \) (provably in \( \hat{\text{ID}}^* \)) if there is a natural number \( e \) such that \( \hat{\text{ID}}^* \vdash (e \models_\alpha \varphi) \).
Definition 7.5 (Realizability in $\mathbb{V}$). For each formula $\varphi(x_1,\ldots,x_n)$ of $LST$ containing at most $x_1,\ldots,x_n$ free and with no occurrence of $\omega$, we define $e \vDash \varphi(x_1,\ldots,x_n)$, for $e$ a natural number, by replacing in Definition 7.1 “$e \vdash_{\alpha} \ldots$” by “$e \vDash \ldots$”, also “$\mathbb{U}_{<\alpha}$”, “$\mathbb{V}_{\alpha}$” by “$\mathbb{U}$”, “$\mathbb{V}$”, respectively, and “$\equiv_{\alpha}$”, “$\in_{\alpha}$”, by “$\equiv_{\mathbb{V}}$” and “$\in_{\mathbb{V}}$”, respectively.

The notion of realizability in $\mathbb{V}$ for a formula of $LST$ possibly with occurrences of $\omega$ is given in the obvious way.

We say that a formula $\varphi$ of $LST$ is realizable in $\mathbb{V}$ (provably in $\hat{\text{ID}}^\ast$) if there is a natural number $e$ such that $\hat{\text{ID}}^\ast \vdash (e \vDash \varphi)$.

8. Soundness of realizability

Theorem 8.1. For every ordinal $\alpha > 0$, $\mathbb{V}_{\alpha}$ is a realizability model for $\text{CZF}^\neg$.

Proof. The proof is essentially the same as in Aczel [3], as was observed in Lindström [15], Proposition 2.4. Details of the proof with the exact realizers may be found in [8].

Theorem 8.2. $\mathbb{V}$ is a realizability model for $\text{CZF}^\neg$.

Proof. Using Lemma 5.9, the proof being similar to that of Theorem 8.1.

Remark 8.3. Let $a \in \mathbb{V}$. Then $a \in \mathbb{V}_{\alpha}$ for some $\alpha$. We note that we can determine the ordinal $\alpha$ from the code $a$ by looking for pairs of the form $\langle 6, \gamma \rangle$ or $\langle 7, \gamma \rangle$ in $a$. Since $a$ is a finite string of codes, this process is effective and we can compare the second component of all these pairs and take the maximum $\gamma$. We then let $\alpha = \gamma + 1$. If there are no pairs whose first component is either 6 or 7, we let $\alpha = 0$.

For $a \in \mathbb{V}$, we shall write $v_a$ for the code of the universe $\mathbb{V}_{\alpha}$, i.e. for $\langle 7, \alpha \rangle$, where $\alpha$ is obtained from $a$ as described above. We shall also write $\hat{v}_a$ for $\sup(v_a, h)$.

Theorem 8.4. $\mathbb{V}$ is a realizability model for the axiom $\text{REA}$. 
Proof. We want to show that there is a natural number $e$ such that

$$e \iff \forall x \exists y (x \in y \land \text{Reg}(y)).$$

This is the case if and only if

$$\forall u \in \mathcal{V} (eu \iff \exists y (u \in y \land \text{Reg}(y)),$$

i.e. if and only if

$$\forall u \in \mathcal{V} ((eu) \iff (u \in (eu)_0 \land \text{Reg}((eu)_0)) \land \mathcal{V} \models (eu)_0 \text{ set}).$$

Let $a \in \mathcal{V}$. Then there is an ordinal $\alpha$ such that $a \in \mathcal{V}_\alpha$.

Let us fix, by Remark 8.3,

$$(ea)_0 = \check{v}_a.$$

Clearly $\mathcal{V} \models (ea)_0 \text{ set}$, since by Remark 5.8 (i), $\mathcal{V}_{\alpha+2} \models \check{v}_a \text{ set}.$

We need to prove that we can find $s, t$ such that $(ea)_1$ is of the form $\langle s, t \rangle$, with $s \iff a \in \check{v}_a$ and $t \iff \text{Reg}(\check{v}_a)$.

This will enable us to conclude that for any $a \in \mathcal{V}_\alpha$,

$$ea = \langle \check{v}_a, \langle s, t \rangle \rangle,$$

hence

$$e = \Lambda x. \langle \check{v}_x, \langle s, t \rangle \rangle.$$

Let us first of all prove that there is $s$ such that $s \iff a \in \check{v}_a$. This is the case if and only if

$$\mathcal{U} \models s \in (a \in \check{v}_a),$$

i.e. if and only if

$$\mathcal{U} \models s_0 \in \check{v}_a \land s_1 \iff a = hs_0.$$

Since $h$ is the identity function, we can take $s_0$ to be $a$ itself and $s_1$ to be $f(a)$, where $f$ denotes the realizer of the identity axiom as defined e.g. in [5], XII, 1.5. Hence

$$s = \langle a, f(a) \rangle.$$
We need to determine $t$ in such a way that

$$
t \models Trans(\hat{\alpha}) \land \forall z \forall R \left( z \in \hat{\alpha} \land R \subseteq z \times \hat{\alpha} \land \forall x \in z \exists y R(x, y) \rightarrow \exists v \in \hat{\alpha} \left( \forall x \in z \exists y \in v R(x, y) \land \forall y \in v \exists x \in z R(x, y) \right) \right).$$

Clearly $t$ has to be a pair, and we shall call $f$ its left projection. We want to show that

$$f \models Trans(\hat{\alpha}).$$

This holds if and only if

$$f \models \forall x \in \hat{\alpha} \forall y \in x \left( y \in \hat{\alpha} \right),$$

i.e. if and only if

$$\forall i \left( \mathbb{U} \models i \notin \alpha \lor f_i \models \forall y \in h_i \left( y \in \hat{\alpha} \right) \right),$$

that is

$$\forall i \left( \mathbb{U} \models i \notin \alpha \lor \forall j \left( \mathbb{U} \models j \notin \overline{i} \lor f_{ij} \models \overline{h}_{ij} \in \hat{\alpha} \right) \right)$$

i.e.

$$\forall i \left( \mathbb{U} \models i \notin \alpha \lor \forall j \left( \mathbb{U} \models j \notin \overline{i} \lor (f_{ij})_0 \in \alpha \land (f_{ij})_1 \models \overline{h}_{ij} = h(f_{ij})_0) \right) \right).$$

We note that $h_i = i$ and $h(f_{ij})_0 = (f_{ij})_0$. Therefore we obtain

$$\forall i \left( \mathbb{U} \models i \notin \alpha \lor \forall j \left( \mathbb{U} \models j \notin \overline{i} \lor (f_{ij})_0 \in \alpha \land (f_{ij})_1 \models \overline{i} j = (f_{ij})_0) \right) \right).$$

Hence

$$\forall i \left( \mathbb{V}_\alpha \models i \text{ set} \rightarrow \forall j \left( \mathbb{U} \models j \in \overline{i} \rightarrow (\mathbb{V}_\alpha \models (f_{ij})_0 \text{ set} \land (f_{ij})_1 \models \overline{i} j = (f_{ij})_0) \right) \right).$$

For $\mathbb{V}_\alpha \models i \text{ set}$ and $\mathbb{U} \models j \in \overline{i}$, we can take $(f_{ij})_0$ to be $\overline{i} j$ and $(f_{ij})_1$ to be $f(\overline{i} j)$, giving

$$f_{ij} = (\overline{i} j, f(\overline{i} j)).$$

Finally,

$$f = \Lambda x \Lambda y. (\overline{x} y, f(\overline{x} y)).$$
Let us now determine $t_1$ such that
\[
t_1 \vdash \forall z \forall R \left( z \in \hat{v}_a \land R \subseteq z \times \hat{v}_a \land \forall x \in z \exists y R(x, y) \right) \\
\rightarrow \exists v \in \hat{v}_a \left( \forall x \in z \exists y \in v R(x, y) \land \forall y \in v \exists x \in z R(x, y) \right).
\]
Suppose that for $b, R \in V$,
\[
r \vdash b \in \hat{v}_a \land R \subseteq b \times \hat{v}_a \land \forall x \in b \exists y R(x, y).
\]
Then $r = \langle \langle r_0 \rangle \rangle, r_1 \rangle$. We shall call $q$ the realizer $r_1$ and assume that
\[
q \vdash \forall x \in b \exists y R(x, y).
\]
This holds if and only if
\[
\forall i \left( U \models i \not\in \bar{b} \lor q_i \vdash \exists y R(\tilde{b}i, y) \right),
\]
that is
\[
\forall i \left( U \models i \not\in \bar{b} \lor (qi)_1 \vdash R(\tilde{b}i, (qi)_0) \right).
\]
In addition,
\[
(r_0)_0 \vdash b \in \hat{v}_a
\]
if and only if
\[
U \models (r_0)_0 \in \nu_a \quad \text{and} \quad (r_0)_1 \vdash b = h((r_0)_0).
\]
which is
\[
\forall \alpha \models (r_0)_0 \text{ set} \quad \text{and} \quad (r_0)_1 \vdash b = (r_0)_0.
\]
We want to find $g$ such that
\[
g \vdash \exists v \in \hat{v}_a \left( \forall x \in b \exists y \in v R(x, y) \land \forall y \in v \exists x \in b R(x, y) \right).
\]
This is the case iff
\[
U \models g_0 \in \nu_a
\]
and
\[
g_1 \vdash \forall x \in b \exists y \in h g_0 R(x, y) \land \forall y \in h g_0 \exists x \in b R(x, y).
\]
This is

\[ \forall \alpha \models g_0 \text{ set}, \]

\[ (g_1)_0 \models \forall x \in b \exists y \in g_0 R(x, y) \quad \text{and} \quad (g_1)_1 \models \forall y \in g_0 \exists x \in b R(x, y). \]

Therefore, \( \forall \alpha \models g_0 \text{ set} \) and

\[ \forall n (U \models n \not\in \bar{b} \lor (g_1)_0 n \models \exists y \in g_0 R(\bar{b}n, y)) \]

and

\[ \forall m (U \models m \not\in \bar{g}_0 \lor (g_1)_1 m \models \exists x \in b R(x, \bar{g}_0 m)). \]

This is the same as \( \forall \alpha \models g_0 \text{ set} \) and

\[ \forall n (U \models n \not\in \bar{b} \lor (U \models ((g_1)_0 n)_0 \in \bar{g}_0 \land ((g_1)_0 n)_1 \models R(\bar{b}m, \bar{g}_0 ((g_1)_0 n)_0)))) \]

and

\[ \forall m (U \models m \not\in \bar{g}_0 \lor (U \models ((g_1)_1 m)_0 \in \bar{b} \land ((g_1)_1 m)_1 \models R(\bar{b}(g_1)_1 m)_0, \bar{g}_0 m))). \]

Let

\[ g_0 = \sup(((r_0)_0)_0, \Lambda w. h(qw)_0), \]

and

\[ g_1 = \langle \Lambda w. \langle w, (qw)_1 \rangle, \Lambda w. \langle w, (qw)_1 \rangle \rangle. \]

This yields

\[ t = \langle \Lambda x \Lambda y. \langle \hat{x}y, f(\hat{x}y) \rangle, \Lambda x \Lambda y \Lambda z. \langle \sup(((z)_0)_0), \Lambda w. h(z_1 w)_0, \Lambda w. \langle w, (z_1 w)_1 \rangle, \Lambda w. \langle w, (z_1 w)_1 \rangle \rangle \rangle. \]

Finally

\[ e = \Lambda u. \langle \hat{v}_u, \langle \langle u, f(u) \rangle, t \rangle \rangle, \]

with \( t \) as above. \( \square \)
Theorem 8.5. For every formula $\varphi(x_1, \ldots, x_n)$ there exist primitive recursive functions $f_\varphi$ and $g_\varphi$ such that for $a_1, \ldots, a_n \in \mathbb{V}_\alpha$ and $e, r \in \mathbb{N}$, 

- if $e \models_\alpha \varphi(a_1, \ldots, a_n)$ then $f_\varphi(a_1, \ldots, a_n, e) \models (\varphi(a_1, \ldots, a_n))^{\hat{v}_\alpha}$, and
- if $r \models (\varphi(a_1, \ldots, a_n))^{\hat{v}_\alpha}$ then $g_\varphi(a_1, \ldots, a_n, r) \models_\alpha \varphi(a_1, \ldots, a_n)$.

Proof. The proof is by induction on $\varphi$.

Let $\varphi$ be $(a = b)$, with $a, b \in \mathbb{V}_\alpha$. Then by Definition 7.1,

$$e \models_\alpha (a = b) \text{ iff } U_{<\alpha} \models e \in (a \equiv_\alpha b).$$

Since $a, b \in \mathbb{V}_\alpha$, by Lemma 5.9, and Definition 6.8,

$$U_{<\alpha} \models e \in (a \equiv_\alpha b) \text{ iff } U \models e \in (a \equiv_\forall b).$$

Therefore

$$e \models (a = b)^{\hat{v}_\alpha}.$$

If $\varphi$ is $(a \in b)$, then the proof is similar, and also in this case $f_\varphi$ and $g_\varphi$ can be taken to be the identity function. If $\varphi$ is $(\psi \land \chi)$, $(\psi \lor \chi)$, $(\psi \rightarrow \chi)$ or $\exists x \psi(x, a_1, \ldots, a_n)$, then we simply use the induction hypothesis.

It is worth noting that for $\Sigma$ formulas, $f_\varphi$ and $g_\varphi$ may be taken to be the identity function.

The only interesting case is the universal quantifier.

Let $\varphi$ be $\forall x \psi(x, a_1, \ldots, a_n)$ and suppose that

$$e \models_\alpha \forall x \psi(x, a_1, \ldots, a_n).$$

Then

$$\forall u \in \mathbb{V}_\alpha (eu \models_\alpha \psi(u, a_1, \ldots, a_n)).$$

Suppose $U_{\alpha+1} \models i \in \mathbb{V}_\alpha$, then $\mathbb{V}_\alpha \models i \text{ set}$, so that

$$ei \models_\alpha \psi(i, a_1, \ldots, a_n).$$
By induction hypothesis there is $f_\psi$ such that if $\forall \alpha \models i \text{ set}$, then

$$f_\psi(a_1, \ldots, a_n, ei) \models [\psi(i, a_1, \ldots, a_n)]^{\hat{v}_\alpha}.$$

This is

$$\forall i (\mathcal{U}_{\alpha+1} \models i \in v_\alpha \rightarrow f_\psi(a_1, \ldots, a_n, ei) \models [\psi(i, a_1, \ldots, a_n)]^{\hat{v}_\alpha})$$

or, equivalently,

$$\forall i (\mathcal{U}_{\alpha+1} \models i \in v_\alpha \rightarrow f_\psi(a_1, \ldots, a_n, ei) \models [\psi(h_{hi}, a_1, \ldots, a_n)]^{\hat{v}_\alpha}).$$

Therefore

$$\Lambda y. f_\psi(a_1, \ldots, a_n, ey) \models \forall x \in \hat{v}_\alpha [\psi(x, a_1, \ldots, a_n)]^{\hat{v}_\alpha}.$$

On the other hand, suppose that

$$r \models \forall x \in \hat{v}_\alpha [\psi(x, a_1, \ldots, a_n)]^{\hat{v}_\alpha}.$$

Then by induction hypothesis if $\forall \alpha \models i \text{ set}$, then $g_\psi(a_1, \ldots, a_n, ri) \models_\alpha \psi(h_{hi}, a_1, \ldots, a_n)$ hence

$$\Lambda y. g_\psi(a_1, \ldots, a_n, ry) \models_\alpha \forall x \psi(x, a_1, \ldots, a_n).$$

\[\square\]

**Corollary 8.6.** $\mathcal{V}$ is a realizability model for $\text{CZF}^- + \text{INAC}$. 

**Proof.** By Theorem 8.2, we only need to show that there is a realizer for $\text{INAC}$. 

Let $a \in \mathcal{V}$, then $a \in \mathcal{V}_\alpha$ for some $\alpha$. We have shown in Theorem 8.4 that $\hat{v}_\alpha$ is a regular set and that there is a realizer $e$ such that $(ea)_1 \models a \in \hat{v}_\alpha \land \text{Reg}(\hat{v}_\alpha)$.

In addition, by Theorem 8.1, there is a realizer uniform in $\alpha$ for the conjunction $\varphi_1 \land \ldots \land \varphi_n$ of the axioms of $\text{CZF}^-$ selected in Remark 2.6.

Let $\chi$ be $\varphi_1 \land \ldots \land \varphi_n$ and $r$ be such that $r \models_\alpha \chi$. By Theorem 8.5, there is a primitive recursive function $f_\chi$ such that

$$f_\chi(r) \models_\alpha \chi^{\hat{v}_\alpha}.$$
Therefore $\langle (ea)_1, f_\chi(r) \rangle \vdash a \in \hat{\nu}_\alpha \land \text{Reg}(\hat{\nu}_\alpha) \land \chi^{\hat{\nu}_\alpha}$.

Finally,

$$\text{Ax.}\langle \hat{\nu}_x, \langle (ex)_1, f_\chi(r) \rangle \rangle \vdash \text{INAC}.$$ 

□

9. The lower bound

In order to determine the lower bound for $\text{CZF}^- + \text{INAC}$ we shall give an interpretation of an intuitionistic version of the subsystem $\text{ATR}_0$ of second order arithmetic, $\text{ATR}_1^0$, in the constructive set theory. The system $\text{ATR}_0$ was introduced by Friedman and Simpson in their programme of Reverse Mathematics and is characterized by a comprehension axiom restricted to arithmetic formulas, a set induction axiom, and the axiom of arithmetical transfinite recursion, which gives it its name. Following [20], we shall rely on the literature for a well ordering proof in $\text{ATR}_1^0$, by means of which the ordinal $\Gamma_0$ will be shown to be a measure for the proof theoretic strength of the theory.

**Definition 9.1** (Intuitionistic second order arithmetic, $\mathbb{Z}_2^i$). The language of intuitionistic second order arithmetic, $\mathbb{L}_2$, is a two sorted language, with number variables $i, j, k, n, m, \ldots$ for natural numbers and set variables $X, Y, Z, \ldots$ for sets of natural numbers. It has constant symbols 0 and 1 as well as symbols for the operations of multiplication and addition, $\cdot$ and $+$. Numerical terms are defined in the obvious way from the number variables, 0, 1, $\cdot$ and $+$. Predicates include equality, $=$, a symbol for the standard order relation on the natural numbers, $<$, and a symbol for the membership relation of a natural number to a set, $\in$. Atomic formulas are of the form $s = t$, $s < t$, $s \in X$, for $s, t$ numerical terms and $X$ a set variable. Compound formulas are built up in the usual way, by means of the connectives and quantifiers (both for set and number variables).

The axioms of $\mathbb{Z}_2^i$ consist of the axioms of first order intuitionistic logic plus the following.
(i) Number theoretic axioms
   (a) $n + 1 \neq 0$,
   (b) $m + 1 = n + 1 \rightarrow m = n$,
   (c) $m + 0 = m$,
   (d) $m + (n + 1) = (m + n) + 1$,
   (e) $m \cdot 0 = 0$,
   (f) $m \cdot (n + 1) = (m \cdot n) + m$,
   (g) $\neg m < 0$,
   (h) $m < n + 1 \leftrightarrow (m < n \lor m = n)$.

(ii) Induction axiom

   \[(0 \in X \land \forall n \ (n \in X \rightarrow n + 1 \in X)) \rightarrow \forall n \ (n \in X).\]

(iii) Comprehension axiom

   \[\exists X \forall n \ (n \in X \leftrightarrow \varphi(n)),\]
   for $\varphi(n)$ any formula of $L_2$ and $X$ not free in $\varphi(n)$.

**Definition 9.2** (Arithmetical formula). A formula of $L_2$ is said to be *arithmetical* if it contains no set quantifiers.

**Definition 9.3** ($\text{ACA}_0^i$). Let $\text{ACA}_0^i$ be the subsystem of $Z_2^i$ defined by means of the following axioms

(i) Number theoretic axioms.

(ii) Induction axiom.

(iii) Arithmetical Comprehension, i.e. the axiom of Definition 9.1 (iii) restricted to arithmetical formulas only.

**Definition 9.4.** Let $\prec$ be a binary relation on $\mathbb{N}$. We define

\[\text{WF}(\prec) := \forall X \ (orall j \ (\forall i \ i \prec j i \in X \rightarrow j \in X) \rightarrow \forall j \ j \in X).\]
**Lemma 9.5** (Arithmetical Transfinite Induction). For $\prec$ a binary relation and $\varphi(j)$ an arithmetical formula, the following principle is provable in $\text{ACA}_0^1$:

$$
(\text{WF}(\prec) \land \forall j (\forall i (i < j \rightarrow \varphi(i)) \rightarrow \varphi(j))) \rightarrow \forall j \varphi(j).
$$

**Proof.** This is a consequence of the definition of $\text{WF}(\prec)$ and of the axiom of Arithmetical Comprehension. \qed

**Definition 9.6** (The principle $(ATR)$). Let $(ATR)$ be the following scheme

$$
\text{WF}(\prec) \rightarrow \exists Y \forall j Y_j = \{n : \theta(n, Y_{\prec j})\},
$$

where $\prec$ is a primitive recursive binary relation, $\theta(n, Y)$ is an arithmetical formula and we use the following conventions

$$
Y_j := \{m : 2^j \cdot 3^m \in Y\},
$$

$$
Y_{\prec j} := \{2^i \cdot 3^m : i \prec j \land m \in Y_i\}.
$$

**Definition 9.7** (The system $\text{ATR}_0^1$). Let $\text{ATR}_0^1$ be the subsystem of $\text{Z}_2^1$ obtained by adding the scheme $(ATR)$ to the system $\text{ACA}_0^1$.

**Remark 9.8.** The principle $(ATR)$ as presented in the previous Definition is more restrictive than in Friedman’s original formulation which allowed for relations of the form $n \prec_X m \iff 2^n3^m \in X$, where $X$ ranges over arbitrary sets of numbers. From Corollary 9.13, however, it will follow that our version of $\text{ATR}_0^1$ and Friedman’s have the same proof-theoretic strength.

We now give an obvious interpretation of $\text{ATR}_0^1$ in $\text{CZF}^- + \text{INAC}$, and show that all the axioms of $\text{ATR}_0^1$ hold under this interpretation.

Variables on the natural numbers are interpreted as elements of $\omega$, while set variables are interpreted as subsets of $\omega$.

Before going to the main result, we shall prove a theorem of $\text{CZF}^- + \text{INAC}$ which will allow us to show that the axiom $(ATR)$ holds under the interpretation in the constructive set theory.
In the following we write $a$ for $a_1, \ldots, a_k$. We also write $m \not< n$ for $m < n \lor m = n$.

Finally, by $\text{WF}(<)$ we denote the $LST$ formula obtained from Definition 9.4 by means of the above interpretation.

**Lemma 9.9.** For any binary relation $<$ and for any $\Delta_0$-formula $\varphi(x, \vec{a})$, we have in $\text{CZF}^-$

$$(\text{WF}(<) \land \forall x \in \omega \ (\forall y \in \omega \ (y < x \rightarrow \varphi(y, \vec{a})) \rightarrow \varphi(x, \vec{a}))) \rightarrow \forall x \in \omega \ \varphi(x, \vec{a}).$$

**Proof.** The proof follows from the definition of $\text{WF}(<)$ and from $\Delta_0$-Separation. \hfill $\square$

**Theorem 9.10.** Let $\theta(u, v, \vec{a})$ be a $\Delta_0$-formula of $\text{CZF}^- + \text{INAC}$. Let $<$ be a decidable binary relation on $\omega$, i.e. such that $\forall m, n \in \omega \ (m < n \lor \neg m < n)$.

Suppose also that $\text{WF}(<)$. Then there is $Z \subseteq \omega$ such that

$$\forall j \in \omega \ Z_j = \{m \in \omega : \theta(m, Z_{<j}, \vec{a})\}.$$ 

**Proof.** Let $A$ be an inaccessible set such that $\vec{a} \in A$.

For any $j \in \omega$ and $Y \subseteq \omega$, let

$$\psi(j, Y) := \forall i \in \omega \ (i \not< j \rightarrow Y_i = \{n \in \omega : \theta(n, Y_{<i}, \vec{a})\}).$$

Note that $\psi(u, v)$ is a $\Delta_0$ formula, since $\theta(u, v, \vec{a})$ is $\Delta_0$.

We prove first of all that for any $j \in \omega$ and $Y \subseteq \omega$,

(i) $\psi(j, Y) \land i < j \rightarrow \psi(i, Y)$,

(ii) $\psi(j, Y) \land \psi(j, U) \rightarrow \forall i \in \omega \ (i \not< j \rightarrow Y_i = U_i)$.

(i) clearly holds by definition.

We use Lemma 9.9 to show that (ii) holds. Suppose that $\forall k \in \omega$ such that $k < j$
we have

$$\psi(k, Y) \land \psi(k, U) \rightarrow \forall i \in \omega \ (i \not< k \rightarrow Y_i = U_i).$$
Suppose also that $\psi(j, Y)$ and $\psi(j, U)$. By (i) we simply need to prove that $Y_j = U_j$, i.e.

$$\{ n \in \omega : \theta(n, Y_{\prec j}, \vec{a}) \} = \{ n \in \omega : \theta(n, U_{\prec j}, \vec{a}) \}.$$ 

By definition we have

$$Y_{\prec j} = \{ 2^i \cdot 3^m \in \omega : i \prec j \land m \in Y_i \},$$

and similarly

$$U_{\prec j} = \{ 2^i \cdot 3^m \in \omega : i \prec j \land m \in U_i \}.$$ 

As for $i \prec j$, $Y_i = U_i$ we can conclude

$$Y_j = U_j.$$ 

Hence (ii) holds.

Going back to the main goal, we show first of all that

$$\forall j \in \omega \exists Y \in A \psi(j, Y).$$

Let us fix $j \in \omega$ and suppose that

$$\forall i \in \omega \ (i \prec j \rightarrow \exists U \in A \psi(i, U)).$$

As $\prec$ is decidable we obtain

$$\forall i \in \omega \exists U \in A \ (i \prec j \rightarrow \psi(i, U)).$$

By regularity of $A$, there is $B \in A$ such that

$$\forall i \in \omega \exists U \in B \ (i \prec j \rightarrow \psi(i, U)).$$

Let

$$V := \{ 2^i \cdot 3^x \in \omega : i \prec j \land \exists U \in B(\psi(i, U) \land x \in U_i) \}.$$ 

Then $V$ is a set and in addition $V \in A$, as $A$ is inaccessible and hence it models $\Delta_0$ - Separation. By (ii) we obtain

$$\forall i \prec j \ V_i = \{ n \in \omega : \theta(n, V_{\prec i}, \vec{a}) \}.$$
Let

\[ W := \{ n \in \omega : \theta(n, V, \vec{a}) \} \].

Then if we let

\[ Y := V \cup \{ 2^j \cdot 3^x \in \omega : x \in W \}, \]

we can conclude \( Y \in A \) and \( \psi(j, Y) \).

Hence we have shown that

\[ \forall j \in \omega \exists Y \in A \psi(j, Y). \]

By regularity of \( A \) there exists \( D \in A \) such that

\[ \forall j \in \omega \exists Y \in D \psi(j, Y). \]

Let

\[ Z := \{ 2^j \cdot 3^x : \exists Y \in D (\psi(j, Y) \land x \in Y) \}. \]

Then \( Z \in A \) and by \((ii)\),

\[ \forall j \in \omega Z_j = \{ n \in \omega : \theta(n, Z_{\prec j}, \vec{a}) \}. \]

\[ \Box \]

**Theorem 9.11.** The axioms of \( \text{ATR}_0^i \) hold under the given interpretation in \( \text{CZF}^- + \text{INAC} \).

**Proof.** The validity of the number theoretic axioms is straightforward.

Regarding the axiom of induction, we note that if \( X \subseteq \omega \), then

\[ (0 \in X \land \forall n \in \omega (n \in X \rightarrow n + 1 \in X)) \rightarrow \forall n \in \omega (n \in X), \]

holds in \( \text{CZF}^- + \text{INAC} \) as a consequence of axiom \((\omega 2)\).

In order to prove Arithmetical Comprehension, we observe that arithmetical formulas of \( \text{L}_2 \) are interpreted by \( \Delta_0 \) formulas of \( \text{LST} \). Hence we need to show that

\[ \exists X \subseteq \omega \forall n \in \omega (n \in X \leftrightarrow \varphi(n, \vec{a})), \]
holds in $\text{CZF}^- + \text{INAC}$ for $\varphi(n, \bar{a})$ a $\Delta_0$ formula, which is a consequence of $\Delta_0$ Separation.

Finally, $(\text{ATR})$ holds in $\text{CZF}^- + \text{INAC}$ as a consequence of Theorem 9.10. □

In view of Theorem 9.11, we only need to show that $\Gamma_0 \leq |\text{ATR}_0|$ in order to conclude that $\Gamma_0$ is a lower bound for the proof theoretic strength of $\text{CZF}^- + \text{INAC}$.

**Lemma 9.12.** Let $\text{OT}$ be the ordinal representation system for $\Gamma_0$ as given in Schütte ([22], VIII.5). For $\alpha \in \text{OT}$ let $WF(\alpha)$ be the statement that the ordering relation on $\text{OT}$ restricted to ordinals below $\alpha$ is well founded. Then we have the following

$$\text{ATR}_0 \vdash \alpha \in \text{OT} \land WF(\alpha) \rightarrow WF(\varphi_00).$$

**Proof.** This result was shown for $\text{ATR}^1$ in [20], Lemma 4.11. Examining the proof of [20], Lemma 4.11 and the parts of [22], VIII.5 on which it draws, one sees that the proof requires only the induction axiom. □

**Corollary 9.13.** Let $\sigma_0 = \varphi_00$, $\sigma_{n+1} = \varphi_\sigma_00$.

(i) For all (meta) $n$, $\text{ATR}_0 \vdash WF(\sigma_n)$,

(ii) $\Gamma_0 \leq |\text{ATR}_0|$.

**Proof.** (i) follows from Lemma 9.12 by meta induction on $n$. (ii) follows from (i) as $\Gamma_0 = \sup_n \sigma_n$. □

**Corollary 9.14.** $|\text{CZF}^- + \text{INAC}| = \Gamma_0$.

**Proof.** $|\text{CZF}^- + \text{INAC}| \geq \Gamma_0$ follows from Theorem 9.11 and Corollary 9.13 (ii).

As to $|\text{CZF}^- + \text{INAC}| \leq \Gamma_0$, we have $|\text{CZF}^- + \text{INAC}| \leq |\hat{\text{ID}}^*|$ by the results of Sections 5 and 8. By Section 4, we know that $|\hat{\text{ID}}^*| = |\hat{\text{ID}}_{<\omega}| = \Gamma_0$. □

**Remark 9.15.** As mentioned earlier, the proof-theoretic strength of $\text{CZF}^- + \text{REA}$ is presently unknown. The main tool in establishing a lower bound for $\text{CZF}^- +$
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INAC was the embedding of $\text{ATR}_0$ into it. Inspection of that proof shows that it would not work with REA in place of INAC.

Remark 9.16. It is also possible to add the anti-foundation axiom to $\text{CZF}^- + \text{INAC}$ without increasing the proof-theoretic strength as is shown in [21].

References


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