Complexity of a complete knot invariant

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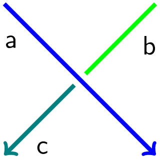
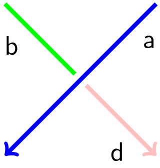
Overview

An algebraic structure associated with knots

Take an oriented knot diagram. We define an algebraic structure with two binary relations $*$ and $*'$, a generator for each arc of the diagram, and a relation for each crossing, as follows:

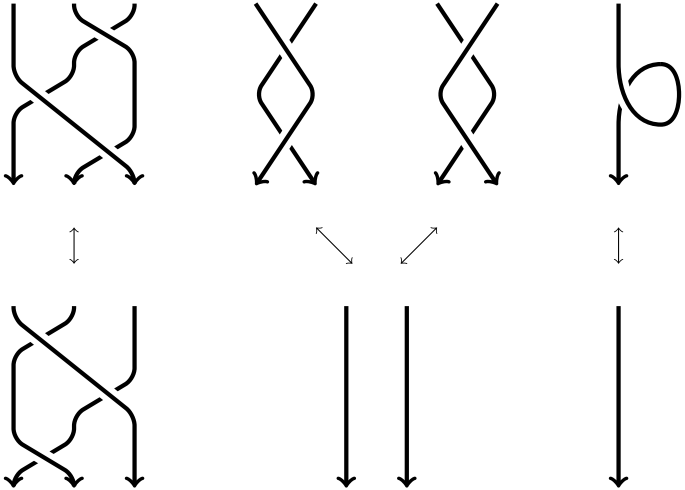


a$*$b=c

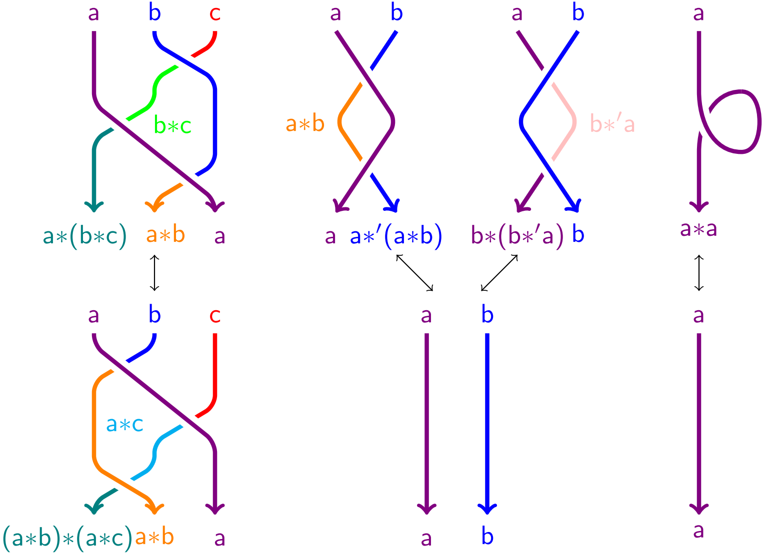


a$*'$b=d

The Reidemeister moves



Respecting the Reidemeister moves



Definitions

A *quandle* is a set with a binary operation $*$ such that

1. $\forall a \forall b \forall c[a * (b * c) = (a * b) * (a * c)]$
2. for all a, the map $b \mapsto a * b$ is a bijection
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Example

Any group with the operation of conjugation ($a * b = aba^{-1}$) is a quandle.

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Gist of Theorem (A.B.-T., S. Miller)

Isomorphism of general countable quandles is as complex as possible.

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Note

As usual, we only want to consider objects & invariants up to isomorphism, ambient isotopy, or some other such equivalence relation.

Definition

Let X and Y be Polish spaces (separable, complete-metrizable spaces), let E be an equivalence relation on X, and let F be an equivalence relation on Y. We say that E is *Borel reducible* to F, and write

$$E \leq_B F$$

if there is a Borel function $f : X \to Y$ such that

$$x_1 \; E \; x_2 \qquad \text{iff} \qquad f(x_1) \; F \; f(x_2).$$

Classes of structures

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Definition

We say such a class $\mathcal{C}$ is *Borel complete* if it is maximal: for every other first order class of countable structures $\mathcal{D}$,

$$\cong_{\mathcal{D}} \leq_B \cong_{\mathcal{C}} .$$

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Proof

We construct a mapping Q taking (directed, irreflexive) graphs to quandles such that

$$\Gamma \cong_{Graphs} \Gamma' \qquad \text{iff} \qquad Q(\Gamma) \cong_{Quandles} Q(\Gamma').$$

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Since the class of graphs is known to be Borel complete, this implies that the class of quandles is Borel complete.

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and let $\theta\colon \Omega \to \mathcal{P}(\Omega)$ be a function such that for all x in X, $[x]_\tau \in \theta([x]_\tau)$.

Then the operation $*$ on X given by

$$x * y = \begin{cases} y & \text{if } [x]_\tau \in \theta([y]_\tau) \\ \tau y & \text{if } [x]_\tau \notin \theta([y]_\tau) \end{cases}$$

makes $(X, *)$ a quandle, the *dynamical quandle derived from* (X, τ) *with respect to* θ.

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- $X = V \times 2$
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Identify $[(v, i)]_\tau$ with v, so Ω is essentially V.

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Then we define $Q(\Gamma)$ to be the dynamical quandle derived from (X, τ) with respect to θ.

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1. $\forall a \forall b \forall c [a * (b * c) = (a * b) * (a * c)]$
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Clearly if $\Gamma \cong \Gamma'$ then $Q(\Gamma) \cong Q(\Gamma')$.

Interesting part: if there is an isomorphism $f : Q(\Gamma) \to Q(\Gamma')$, why must there be an isomorphism $\Gamma \to \Gamma'$?

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Our isomorphism f need not arise from a graph isomorphism. Nevertheless, given f can we construct an isomorphism $\varphi : \Gamma \to \Gamma$?

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Then the "twinning" of (v, j) with $(v, 1 - j)$ is witnessed by the action of (u, i). So the action of $f(u, i)$ on $f(v, j)$ is nontrivial, and takes $f(v, j)$ it to *its* twin. So the first component of $f(v, j)$ is independent of $j \in \{0, 1\}$, and we take this to be $\varphi(v)$.

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These definitions of $\varphi(v)$ combine to produce a graph isomorphism from Γ to Γ'.

Open question

Is there an encoding map $Q\colon \mathrm{Graphs} \to \mathrm{Quandles}$ that is functorial?

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Our map Q fails this badly, because graph homomorphisms need not preserve non-edges.