Products of CW complexes:
the full story

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Notation

For $n \in \mathbb{N}$, let

- D^n denote the closed ball of radius 1 about the origin in $\mathbb{R}^n$ (the *n-disc*),
- $\mathring{D}^n$ its interior (the open ball of radius 1 about the origin), and
- S^{n-1} its boundary (the *$n-1$-sphere*).

CW complexes

Definition

A Hausdorff space X is a *CW complex* if there is a set of continuous functions $\varphi^n_\alpha : D^n \to X$, for α in an arbitrary index set and $n \in \mathbb{N}$ a function of α, such that:

1. $\varphi^n_\alpha \restriction \mathring{D}^n$ is a homeomorphism to its image, and X is the disjoint union as α varies of these homeomorphic images $\varphi^n_\alpha[\mathring{D}^n]$ ("cells," denoted e^n_α).

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2. *Closure-finiteness:* For each φ_α^n, $\varphi_\alpha^n[S^{n-1}]$ is contained in finitely many cells all of dimension less than n.
3. *Weak topology:* A set is closed if and only if its intersection with each closed cell $\varphi_\alpha^n[D^n]$ is closed.

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Proof

Identify each edge with the unit interval, with x_0 at 0. For every $f : \mathbb{N} \to \mathbb{N}$, consider the open neighbourhood $U(x_0; f)$ of x_0 whose intersection with $e^1_{X,n}$ is the interval $[\,0, 1/(f(n) + 1)\,)$.

These form a neighbourhood base, but for any countably many f_i, there is a g that eventually dominates each of them, so $U(x_0; g)$ does not contain any of the $U(x_0; f_i)$. □

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Convention

In this talk, $X \times Y$ is always taken to have the product topology, so "$X \times Y$ is a CW complex" means "the product topology on $X \times Y$ is the same as the weak topology".

Example (Dowker, 1952)

Let X be the "star" with central vertex x_0 and countably many edges $e^1_{X,n}$ ($n \in \mathbb{N}$) emanating from it (and the countably many "other end" vertices of those edges).

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Consider the subset of $X \times Y$

$$H = \left\{ \left(\frac{1}{f(n)+1}, \frac{1}{f(n)+1} \right) \in e^1_{X,n} \times e^1_{Y,f} : n \in \mathbb{N}, f \in \mathbb{N}^{\mathbb{N}} \right\}$$

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Then $\left(\frac{1}{g(k)+1}, \frac{1}{g(k)+1}\right) \in U \times V \cap H$. So in the product topology, $(x_0, y_0) \in \bar{H}$.

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A *subcomplex* A of a CW complex X is what you would expect.

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A *subcomplex* A of a CW complex X is a subspace which is a union of cells of X, such that if $e^n_\alpha \subseteq A$ then its closure $\bar{e^n_\alpha} = \varphi^n_\alpha[D^n]$ is contained in A.

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Definition

Let κ be a cardinal. We say that a CW complex X is *locally less than* κ if for all x in X there is a subcomplex A of X with fewer than κ many cells such that x is in the interior of A. We write *locally finite* for locally less than $\aleph_0$, and *locally countable* for locally less than $\aleph_1$.

Proposition

If κ is a regular uncountable cardinal, then a CW complex W is locally less than κ if and only if every connected component of W has fewer than κ many cells.

Proof sketch.

$\Leftarrow$ is trivial. For $\Rightarrow$, given any point w, recursively fill out to get an open (hence clopen) subcomplex containing w with fewer than κ many cells, using the fact that the cells are compact to control the number of cells along the way. □

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Theorem (J. Milnor, 1956)

If X and Y are both (locally) countable, then $X \times Y$ is a CW complex.

Theorem (Y. Tanaka, 1982)

If neither X nor Y is locally countable, then $X \times Y$ is not a CW complex.

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So Cantor's Theorem can be expressed as $2^{\aleph_0} \geq \aleph_1$.

The *Continuum Hypothesis* (CH) is the assertion that $2^{\aleph_0} = \aleph_1$.
It is *independent* of the standard ZFC axioms for set theory:

- There are models of ZFC in which CH is true.
- There are models of ZFC in which CH is false.

What was known, beyond ZFC

Theorem (Liu Y.-M., 1978)

Assuming CH, $X \times Y$ is a CW complex if and only if one of them is locally finite, or both are locally countable.

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Theorem (Y. Tanaka, 1982)

Assuming $\mathfrak{b} = \aleph_1$, $X \times Y$ is a CW complex if and only if one of them is locally finite, or both are locally countable.

Can we do better?

Question

Can we show, without assuming any extra set-theoretic axioms, that the product $X \times Y$ of CW complexes X and Y is a CW complex if and only if either

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Answer (follows from Tanaka's work)

No.

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Answer (A.B.-T.)

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Pushing Dowker's example harder

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For this we need to talk more about eventual domination.

A definition

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The bounding number $\mathfrak{b}$ is the least cardinality of a set of functions that is unbounded with respect to $\leq^*$, i.e. such that no one g is $\geq^*$ them all, i.e.,

$$\mathfrak{b} = \min\{|\mathcal{F}| : \mathcal{F} \subseteq \mathbb{N}^{\mathbb{N}} \wedge \forall g \in \mathbb{N}^{\mathbb{N}} \exists f \in \mathcal{F} \neg(f \leq^* g)\}.$$

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is consistent with ZFC (ie, is true in some model of ZFC).

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Consider the subset of $X \times Y$

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where we have identified each edge with the unit interval, with 0 at the centre vertex.

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Let $g\colon \mathbb{N} \to \mathbb{N}^+$ be an increasing function such that $[0, 1/g(n)) \subset e^1_{X,n} \cap U$ for every $n \in \mathbb{N}$. Take $f \in \mathcal{F}$ such that $f \not\leq^* g$.

Consider the edge $e^1_{Y,f}$ of Y:

Let $k \in \mathbb{N}$ be such that $\frac{1}{f(k)+1} \in e^1_{Y,f} \cap V$ and $f(k) > g(k)$.

Then $\left(\frac{1}{f(k)+1}, \frac{1}{f(k)+1}\right) \in U \times V \cap H$. So in the product topology, $(x_0, y_0) \in \bar{H}$.

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A complete characterisation

Theorem (A.B.-T.)

Let X and Y be CW complexes. Then $X \times Y$ is a CW complex if and only if one of the following holds:

1. *X or Y is locally finite.*
2. *One of X and Y is locally countable, and the other is locally less than $\mathfrak{b}$.*

Proof

The forward direction was actually done by Tanaka (1982).

So it remains to show that if X and Y are CW complexes such that X is locally countable and Y is locally less than $\mathfrak{b}$, then $X \times Y$ is a CW complex.

By the Proposition earlier, we may assume that X has countably many cells and Y has fewer than $\mathfrak{b}$ many cells.

Topologies

Any compact subset of a CW complex X is contained in finitely many cells, and each closed cell $\bar{e}^n_\alpha$ is compact. So requiring X to have the weak topology is equivalent to requiring that the topology be *compactly generated*: a set is closed if and only if its intersection with every compact set is closed.

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Any sequential space is compactly generated. Since D^n is sequential for every n, we have that CW complexes are sequential.

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So suppose

- $H \subset X \times Y$ is sequentially closed, and
- $(x_0, y_0) \in X \times Y \setminus H$.

We want to construct open neighbourhoods U of x_0 in X and V of y_0 in Y such that $(U \times V) \cap H = \emptyset$.

Constructing neighbourhoods

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Lemma

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Wrinkle in proof.

Use compactness of cells. □

Back to the proof of the Theorem

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Basic idea

The construction is essentially by simultaneous induction on cell number on the X side (after enumerating the cells of X in a reasonable order) and dimension on the Y side.

For each new cell e_α that you consider on the Y side, you get a function $f_\alpha : \mathbb{N} \to \mathbb{N}$ defining an open set on the X side avoiding H. Since there are fewer than $\mathfrak{b}$ many α, they can be eventually dominated by a single function f, with respect to which the e_α part of the neighbourhood can be chosen.

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Naïvely implemented, that doesn't work ($f_\alpha \leq^* f$ isn't good enough).

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If $f_\alpha(n) \leq f(n)$ for all n, then $U(x; f_\alpha) \supseteq U(x; f)$.

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Solution

Mix $\leq$, $\leq^*$, and promises about the growth rate of f.

Making it work

Lemma (Adding one cell to finite subcomplexes)

Suppose

- W *and* Z *are CW complexes,*
- W' *is a finite subcomplex of* W*,*
- Z' *is a finite subcomplex of* Z*,*
- U *is a subset of* W' *that is open in* W'*,*
- V *is a subset of* Z' *that is open in* Z'*, and*
- H *is a sequentially closed subset of* $W \times Z$ *such that the closure of* $U \times V$ *is disjoint from* H*.*

Let e *be a cell of* Z *whose boundary is contained in* Z'*. Then there is a* $p \in \mathbb{N}$ *such that, if* $V^{e,p}$ *is* V *extended by the width* $1/(p+1)$ *collar in* e*, then* $U \times V^{e,p}$ *has closure disjoint from* H*.*

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Proof sketch.

Use compactness, normality and sequentiality of $W' \times (Z' \cup e)$. □

Making it work

Lemma (Adding a Y-side cell, fitting X-side promises)

Let

- Y' *be a finite subcomplex of* Y *containing* y_0,
- F *be a function from* $\mathbb{N}$ *to* $\mathbb{N}$,
- s *be a function from the indices of* Y' *to* $\mathbb{N}$ *such that* $U(x_0; F) \times U(y_0; s) \subseteq X \times Y'$ *has closure disjoint from* H,
- i *be a natural number, and*
- Y'' *be a subcomplex of* Y *that is a one-cell extension of* Y', $Y'' = Y' \cup e_\alpha$.

Then there is a function $f : \mathbb{N} \to \mathbb{N}$ *such that*

1. $f(n) \geq F(n)$ *for all* n *in* $\mathbb{N}$, *and* $f(n) = F(N)$ *for all* $n < i$,
2. *for every* $f' : \mathbb{N} \to \mathbb{N}$ *such that* $f' \geq^* f$ *and* $f' \geq F$, *there is a* $q \in \mathbb{N}$ *such that* $U(x_0; f') \times U(y_0; s \cup \{(\alpha, q)\})$ *has closure disjoint from* H.

Making it work

The construction is actually by simultaneous induction on cell number on the X side, *with promises of lower bounds on the eventual X side function f*, and dimension on the Y side.

For each new cell e_α^{k+1} that you consider on the Y side, using f_k as an X-side promise (F in the previous Lemma), you get a function $f_\alpha : \mathbb{N} \to \mathbb{N}$ defining an open set on the X side avoiding H. Since there are fewer than $\mathfrak{b}$ many α, they can be eventually dominated by a single function f_{k+1}, with respect to which the e_α part of the neighbourhood can be chosen *using the previous Lemma*.

□