Dedekind-finite cardinals having countable partitions

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Background

We work in set theory without the axiom of choice. Notions of finiteness were studied originally by Tarski and others, and in his MSc thesis under Fraenkel, Azriel Levy studied a number of possible definitions of ‘finiteness’. In my thesis, I extended this list, and tried to give a methodical treatment, in terms of the classes of Dedekind-finite cardinals specified by the various definitions, and further extensions were given by Supakun. A set is said to be Dedekind-finite if it has no injection to a proper subset, which is easily seen to be equivalent to saying that it has no countably infinite subset.
Definitions and notation

The only classes of Dedekind-finite cardinals considered today are the following:

\( \omega \) is the set of finite ordinals (these really are finite).
\( \Delta_4 \) is the family of cardinalities of sets having no countably infinite partition.
\( \Delta_5 \) is the family of cardinalities of sets having no surjection to a proper superset.
\( \Delta \) is the family of cardinalities of sets having no injection to a proper subset.

One verifies easily that \( \omega \subseteq \Delta_4 \subseteq \Delta_5 \subseteq \Delta \), and it is consistent that all of these are distinct.
Fraenkel–Mostowski set theory

Before Cohen’s invention of forcing, Fraenkel and Mostowski had developed a method for constructing models of $\neg$AC, in set theory with ‘atoms’, sets with no elements, but distinct from the empty set. This method is technically simpler than forcing, and makes clearer the role of the symmetry group used. Furthermore, in many cases, one can automatically transfer relative consistency results shown using FM-models to ZF-models by use of the Jech–Sochor Theorem.

In the theory FM, we have a unary predicate $U$ standing for the set of atoms, and the axioms are the usual ZF axioms, except for extensionality, which says that non-empty sets with the same members are equal and that the members of $U$ are all empty.
Fraenkel–Mostowski models

The Fraenkel–Mostowski construction starts with a ground model $\mathcal{M}$ of FM + AC, containing a set $U$ of atoms, together with a group $G$ of permutations of $U$, and a filter $\mathcal{F}$ of subgroups of $G$, closed under taking conjugates. (This last condition is required to verify the axiom of replacement in the model.)

We now allow $G$ to act on $\mathcal{M}$ by letting $g(x) = \{g(y) : y \in x\}$ (which is a valid definition by transfinite induction starting from the atoms). For any $x \in \mathcal{M}$, the pointwise and setwise stabilizers of $x$ are $G_x = \{g \in G : g$ fixes $x$ pointwise$\}$ and $G_{\{x\}} = \{g \in G : gx = x\}$. The FM model thereby determined is $\mathcal{N} = \{x \in \mathcal{M} : x \subseteq \mathcal{N} \land G_{\{x\}} \in \mathcal{F}\}$. Again this is a valid definition by transfinite induction. We usually assume that the stabilizer of each member of $U$ lies in $\mathcal{F}$, which is sufficient to guarantee that $U$ lies in the model. $x \in \mathcal{N}$ is well-orderable in $\mathcal{N} \iff G_x \in \mathcal{F}$. 
Some examples of Fraenkel–Mostowski models

In many cases we begin with a first-order structure $\mathcal{A}$ on a countably infinite set $A$, with automorphism group $G$, and we suppose that $U$ is indexed by $A$. We automatically get a corresponding group of permutations of $U$, also denoted by $G$. The model is usually ‘generated by finite supports’, meaning that the filter is generated by the pointwise stabilizers of finite sets (which is automatically closed under conjugacy).
Some examples

If $\mathcal{A}$ is the trivial structure on an infinite set, then the resulting model is Fraenkel’s, and the set $U$ is amorphous in $\mathcal{A}$, meaning that its only subsets are finite or infinite.

If $\mathcal{A}$ is a vector space of dimension $\aleph_0$ over a finite field, then $U$ is still amorphous in the model, but is definitely not the same as the first example. We say it is amorphous of projective type, the first example being strictly amorphous. This terminology derives from that for strongly minimal sets. To see that this set is not strictly amorphous, note that it admits partitions into non-singleton finite sets of equal sizes, for instance cosets of a finite subspace, but this cannot happen in a strictly amorphous set.

If $\mathcal{A}$ is the ordered rationals, then we instead obtain Mostowski’s model. Here $U$ has a dense linear order without endpoints in the model, and is Dedekind finite, and its cardinality lies in $\Delta_4$, but it is not amorphous.
More examples

For Russell’s ‘pairs of socks’ example, let $U = \{u_{ij} : i \in \omega, j \in 2\}$. The group $G$ fixes each $\{u_{i0}, u_{i1}\}$, but can swap any pairs desired, so is the direct product of infinitely many copies of the group of order 2. We use finite supports. In the model, there is a surjection of $U$ to $\omega$, so $|U| \notin \Delta_4$. However, given any finite subset of $U$, there are pairs which do not intersect it, so whichever support we take, it cannot preserve a well-ordering of an infinite subset of $U$, so $|U| \in \Delta$ in the model (actually $|U| \in \Delta_5$). It is called ‘pairs of socks’ because there is an $\omega$-sequence of pairs, but no choice function (even for an infinite subfamily). Note that the non-$\aleph_0$-categoricity is particularly obvious in this case.

Now consider an infinite 2-branching tree with $\omega$ levels, $G$ the group of tree automorphisms, and finite supports. In the resulting model, there is no infinite branch of the resulting tree, and $|U| \notin \Delta \setminus \Delta_5$. This example is key for what follows.
Trees

To see how trees arise naturally in our present context, we give the following result:

**Lemma**

*For any set $X$, $|X| \notin \Delta_5$ if and only if there is a subset $T$ of $X$ which carries a well-founded tree structure of height $\omega$ and no leaves.*

**Sketch of proof:** Suppose $|X| \notin \Delta_5$. Then there is a surjection $f$ from $X$ to a proper superset $X \cup \{\ast\}$. Let $L_n = f^{-n}(\ast)$ for $n \in \omega$, and let $T = \bigcup_{n \in \omega} L_n$. One checks that the $L_n$ are all non-empty and form the levels of a tree under the relation $a \leq b$ if $f^n(b) = a$ for some $n$.

Conversely, if a $T$ as stated exists, then this gives rise to a surjective but not injective function from $X$ to $X$ by mapping each element of $T$ which is not the root to the immediately preceding element (which exists because it lies in a successor level, so there is a unique greatest point below it) and fixing all other points (including the root).
Trees continued

In this part of our work, we are interested in sets lying in $\Delta \setminus \Delta_4$. Two kinds of trees help here. The ones just mentioned are well-founded with $\omega$ levels, and they naturally correspond to cardinals in $\Delta \setminus \Delta_5$. So-called 2-transitive or weakly 2-transitive trees provide nice examples of cardinals in $\Delta_5 \setminus \Delta_4$. Actually the 2-transitive case is $\aleph_0$-categorical, and so corresponds rather to members of $\Delta_4$. Now a tree is a partially ordered set in which the points below any node are linearly ordered (not necessarily well-ordered) and any two nodes have a common lower bound. Droste described all the countable trees in which the automorphism group acts transitively on all 2-element chains, and on all 2-element antichains (there are countably many cases), and this was extended to the weakly 2-transitive case, where transitivity is only demanded on 2-element chains (work of Droste, Holland, and Macpherson).
2-transitive trees

Droste’s 2-transitive trees having infinite chains are listed according to how the nodes ramify, and whether the ramification points lie in the tree (positive type) or in its ‘completion’ (negative type). 2-transitivity ensures that the ramification order is constant throughout, so we have two infinite lists, positive and negative, each given by the ramification order, which can be $2, 3, 4, \ldots, \aleph_0$. These are all $\aleph_0$-categorical, so in the models we have nice examples of members of $\Delta_4$. Retaining the partial ordering in the signature, which is clearly preserved by the automorphism group, we can ‘reconstruct’ the original tree, even though the tree in the model has no well-ordering.
Weakly 2-transitive trees

In a weakly 2-transitive tree, there is no longer a requirement for the ramification order to be fixed, and if we want to get examples in $\Delta_5 \setminus \Delta_4$ it is easiest to let there be infinitely many ramification orders. There is a tricky case in the original paper, concerning special (negative) ramification points which may have a cone with a minimal element, so since we just want to produce $2^{\aleph_0}$ examples, we do not allow this, which amounts to saying that in the completion, the maximal chains are dense.
Construction using 2-transitive trees

As mentioned, these are all $\aleph_0$-categorical, so in any model arising, $U \in \Delta_4$. From the theory of structures of members of $\Delta_4$, one can show that the well ordered tree (i.e. in the ground model) can be reconstructed from the structure in $\mathbb{N}$. I had an old method for doing this, by adding a ‘generic’ choice function, but Cherlin suggested a superior method, based directly on the first order theory. (This was developed by my student Agatha Walczak-Typke). His idea was to consider the first order theory of the given structure, then to show using the fact that $|U| \in \Delta_4$ that this theory is $\aleph_0$-categorical (satisfies the conditions of the Ryll–Nardzewski Theorem). Since the theory is a syntactic object in a countable language, one can use the usual completeness theorem to get a countable model, which is then unique up to isomorphism (uniqueness only for well-orderable models of course).
Construction using weakly 2-transitive trees

As remarked above, these are (usually) not $\aleph_0$-categorical. Since we are only really interested in constructing $2^{\aleph_0}$ distinct examples of this type, it suffices to consider those cases with infinitely many ramification orders, and also for which the maximal chains in the completion are dense (avoiding the ‘tricky’ cases). To show that in the model, $|U| \in \Delta_5$, we have to analyze possible finite supports. In the completion $T^+$ (obtained by adding ramification points to $T$), these may be assumed to be (finite) lower semilattices, where some of the points will not actually be in the structure (but the group still acts on them in a natural way). One considers components of the complement of this finite semilattice, and a straightforward argument shows membership of $\Delta_5$. Using further tricks, one can show that if the sets of ramification orders occurring in two trees are unequal, then the trees in the model are $\equiv$-inequivalent.
A notion of equivalence

At the end of the last slide I spoke about ‘equivalence’. The definition I propose is as follows. Sets $X$ and $Y$ are equivalent, written $X \equiv Y$, if for any first order structures in a countable language that one can put on one of the structures, there is an elementarily equivalent structure on the other. Notice that assuming AC, all infinite sets are equivalent by the Löwenheim–Skolem Theorems. Without choice this is very much not the case.

Example: If $X \equiv Y$ and one of $X$ and $Y$ is Dedekind finite, then so is the other. For suppose say that $X$ is Dedekind finite and $Y$ is not. Then there is a 1–1 function $f : Y \to Y$ which is not onto. Consider the structure $(Y, f)$. Since $X \equiv Y$ there is $g : X \to X$ such that $(X, g)$ is elementarily equivalent to $(Y, f)$. But saying that $f$ is 1–1 and not onto can be expressed in a first order way, so $X$ cannot be Dedekind finite after all.
Balanced trees

We try to carry out a more detailed analysis of finitely branching trees with $\omega$ levels. These were originally introduced in a study of finite versions of the axiom of dependent choice. So for instance, $DC_n$ would say that any well-founded tree in which each element has exactly $n$ immediate successors has an infinite branch. More general versions were studied in work with Thomas Forster (also looking at connections with Ramsey’s Theorem). Generalizing the uniformity of branching, we say that $T$ is balanced if every element on the same level has the same number of immediate successors. (We also desire to generalize this to the infinite branching case, but this work is still incomplete. See final slide.)
Existence of balanced subtrees

**Lemma**: Any tree \((T, \leq)\) with \(\omega\) levels, all finite, has a balanced subtree, having no leaves, and also having \(\omega\) levels.

**Sketch of the proof**: Let \(S\) be a sequence of natural numbers such that every number occurs infinitely often, say \(S = \{k_n : n \in \omega\}\). We construct a decreasing sequence \(T_n\) of subtrees of \(T\), each having \(\omega\) levels, such that for each \(n\), \(T_n\) is pruned on level \(k_n\) so that every member has the same degree (number of immediate successors) on that level. Let \(T^* = \bigcap_{n \in \omega} T_n\).

Consider any \(n\). Since each member of \(\omega\) is listed infinitely often, \(\{m : k_m = n\}\) is infinite. Let \(L^m_n\) be the \(n\)th level of \(T_m\). Then \(L^0_n \supseteq L^1_n \supseteq L^2_n \ldots\) is a decreasing sequence of non-empty finite sets, so is eventually constant, and clearly this equals the \(n\)th level of \(T^*\). Furthermore every \(x\) in the \(n\)th level of \(T^*\) has at least one successor in every \(T_m\), and hence also in \(T^*\), so is not a leaf, and for the same reason, its degree is constant among nodes on that level. Hence \(T^*\) is a balanced subtree of \(T\).
We now analyze the above more carefully, and try to keep track of the discarded nodes. In fact, the idea is to retain all the original nodes, but to parcel them into pieces according to the options of what we could have done. So instead of passing to a subset of each level, we instead find an ordered partition, any one piece of which could be used for the previous construction. In the limit, everything will ‘settle down’ as before, but ultimately the original tree will be partitioned at each level $n$ into $\pi_n$, there’ll be a linear ordering associated, and the key point is that for any $X \in \pi_n$ and $Y \in \pi_{n+1}$ in the partitions of consecutive levels, there is a fixed integer $n(X, Y)$ such that for each $x \in X$, the number of successors of $x$ lying in $Y$ is equal to $n(X, Y)$. 
From the construction just given, we form a *template* \((T^*, \prec)\) from \(T\), which is also a finitely branching \(\omega\)-tree, with extra structure added. The \(n\)th level of \(T^*\) is just \(\pi_n\), and if \(X \in \pi_n\) and \(Y \in \pi_{n+1}\), then \(X \prec Y\) provided that \(n(X, Y) \neq 0\) (and \(\prec\) on the whole of \(T^*\) is the transitive closure of these individual relations). The additional structure that \(T^*\) has is the linear ordering of each level (which was defined in the above construction), and the sequence of positive integers \((n(X, Y) : Y \prec\text{-successor of } X)\) with ordering derived from that of \(\pi_{n+1}\). We note that \(x\) is a leaf of \(T\) if and only if \(n(X, Y) = 0\) for all \(Y\), where \(x \in X\).

If \(X_0 \prec X_1 \prec X_2 \prec \ldots\) is an infinite branch of \((T^*, \prec)\), then the tree induced from \(T\) on \(\bigcup_{n \in \omega} X_n\) is a balanced tree. The intuition is that \(T^*\) somehow collects together all the different possibilities for balanced subtrees of \(T\).

There is a degenerate case in which the labels on the branch are eventually all 1. In this case, the corresponding balanced subtree consists of just one branch, so this is a branch of \(T\) (even without AC).
König’s Infinite Lemma

König’s Lemma says that any infinite well-founded tree with finite levels has an infinite branch. This requires AC in its proof. We note that the ‘templates’ discussed in the preceding slide all come with ordered levels, and hence the whole can be well-ordered and so König’s Lemma is applicable. For each branch of the template, we get a corresponding balanced subtree of the original tree for which we cannot apply König’s Lemma to get a branch, unless it is the ‘degenerate’ kind with labels eventually all 1.
The general philosophy, which doesn’t always work out, is that one tries to describe more complicated Dedekind-finite sets in terms of simpler ones. For instance, the member of $\Delta_5 \setminus \Delta_4$ given by pairs of socks is a countable union of finite sets. This easily generalizes to countable unions of isomorphic $\aleph_0$-categorical structures having transitive automorphism groups. But as we have seen with the weakly 2-transitive trees example, there are many sets with cardinalities in $\Delta_5$ which cannot be expressed as countable unions of members of $\Delta_4$.

As another example, a finitely branching well-founded tree arises from the choices for the finite successor sets. However, although we know that a member of $\Delta \setminus \Delta_5$ must arise from a well-founded tree, there is no reason that I can see why the sets of successors should be finite, or indeed, even in $\Delta_4$, so the general ‘philosophy’ seems a vain hope.
A specific open question

A specific question related to this is as follows. It would be a version of finding a balanced subtree, but in the case where the successor sets may be infinite. Suppose that $T$ is a well-founded tree, and that every node has a set of immediate successors with cardinality in $\Delta_4$. We would like to show that it has a ‘balanced’ subtree. Here by ‘balanced’ I mean that any two successor sets of nodes on the same level are equivalent, in the sense defined earlier, which is the closest one can come (it seems to me) for infinite sets under these circumstances. To see why this is the right question, consider a well-founded $\omega$-tree in which all nodes on the same level have successor sets indexed by isomorphic transitive $\aleph_0$-categorical structures. Now build the corresponding FM model, and it is seen to be of precisely the kind just mentioned.