Sequential and distributive forcing without choice

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STUK 7
January 14, 2022

joint with A. Karagila
Forcing over models of ZFC is very well understood.

Moreover, research of the past decades has shown that they are not merely curiosities but are vital in understanding ZFC itself (e.g., large cardinals, structure of the reals).

Of particular importance are models of DC.

Forcing over models of ZF is not well understood, with or without DC.

Recall:

Definition

Dependent choice, DC, says that every tree without maximal nodes has an infinite branch. DC $\prec_\kappa$, says that for every $\alpha < \kappa$ and an $\alpha$-closed tree $T$ without maximal nodes, $T$ has a branch of length $\alpha$.
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But there is nothing stopping us from forcing over models of ZF + ¬AC.

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**Definition**

Dependent choice, DC, says that every tree without maximal nodes has an infinite branch. DC$_{<\kappa}$, says that for every $\alpha < \kappa$ and an $\alpha$-closed tree $T$ without maximal nodes, $T$ has a branch of length $\alpha$. 
Who loves AC a lot?

Theorem

Let $M$ be $L(R)$ of an $Add(\omega, \omega_1)$-extension of $V$ and $\kappa$ be a regular uncountable cardinal, then $DC$ holds in $M$ and for each of the following there is a forcing $P$ not adding reals:

▶ $M_P$ is an $Add(\omega, \kappa)$-extension of $V$.
▶ $M_P$ is a $Add(\omega, \omega_1)^*S\omega_2$ extension of $V$.
▶ $M_P$ is a $Add(\omega, \omega_1)^*SI\omega_2$ extension of $V$.

If $M$ is richer, e.g. the Solovay model, we can obtain many more models like this.

Question

Is there any result where it is easier to force over $M$? Can we add a dominating family of size $\omega_1$ while not introducing a mad family over the Solovay model? (Roitman’s problem: $d = \omega_1 < a$)
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- $M^\mathbb{P}$ is an $Add(\omega, \kappa)$-extension of $V$.
- $M^\mathbb{P}$ is a $Add(\omega, \omega_1) \ast S_{\omega_2}$ extension of $V$.
- $M^\mathbb{P}$ is a $Add(\omega, \omega_1) \ast S\mathbb{I}_{\omega_2}$ extension of $V$.

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Preserving DC

We established that models of DC play an important role. So the following question is natural:

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Which forcings preserve DC?
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**Theorem (Asperó-Karagila)**

Proper forcing preserves DC.
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**Theorem (Asperó-Karagila)**
Proper forcing preserves DC.

**Corollary**
$\sigma$-closed forcing preserves DC.

A close relative to $\sigma$-closure is $\sigma$-distributivity. Typically applications of $\sigma$-closure only require $\sigma$-distributivity.

**Question**
Does $\sigma$-distributive forcing preserve DC?
Distributive and sequential

**Definition**

\( \mathbb{P} \) is \( \leq \left| X \right| - \)distributive if for any family \( \langle D_x : x \in X \rangle \) of dense open subsets of \( \mathbb{P} \), \( \bigcap_{x \in X} D_x \) is dense.

In ZFC, distributivity and sequentiality are equivalent (for non-trivial forcings). What happens in ZF?

Sequential and distributive forcing without choice
**Definition**

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Note: Whenever $\mathbb{P}$ is $\leq |X|$-distributive and $Y \leq^* X$ ($X$ surjects onto $Y$), then $\mathbb{P}$ is $\leq |Y|$-distributive.
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**Definition**

$\mathbb{P}$ is $\leq |X|$-sequential if for any $\mathbb{P}$-generic $G$ over $V$ and any $f : X \rightarrow V$ in $V[G]$, $f \in V$.

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Lemma

Suppose $\mathbb{P}$ is $\leq |X|$-distributive, then $\mathbb{P}$ is $\leq |X|$-sequential.
Distributive and sequential

**Lemma**

Suppose $\mathbb{P}$ is $\leq |X|$-distributive, then $\mathbb{P}$ is $\leq |X|$-sequential.

**Proof.**

Let $\dot{f}$ be a $\mathbb{P}$-name for $f$. For every $x \in X$, we let

$$D_x := \{ p \in \mathbb{P} : \exists y \in V (p \models \dot{f}(\check{x}) = \check{y}) \}.$$
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If $p \in \bigcap_{x \in X} D_x \cap G$, then $f(x) = y$, where $y$ is unique such that $p \models \dot{f}(\check{x}) = \check{y}$.

$\square$
Minor positive results

**Theorem**

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Let $\mathbb{P}$ be $\leq |X|$-distributive, then $AC_X$ is preserved.

**Definition**

$AC_X$ says that any family $\langle A_x : x \in X \rangle$ of non-empty sets has a choice function $f : X \to \bigcup_{x \in X} A_x$. 
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$\text{AC}_X$ says that any family $\langle A_x : x \in X \rangle$ of non-empty sets has a choice function $f : X \to \bigcup_{x \in X} A_x$.

**Proof.**

Let $p \in \mathbb{P}$, $\dot{F}$ be a $\mathbb{P}$-name such that $p \vDash \forall x \in \check{X}(\dot{F}(x) \neq \emptyset)$. Let

$$D_x := \{q \in \mathbb{P} : \exists \tau (q \vDash \tau \in F(\check{x}))\}.$$
**Minor positive results**

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$$D_x := \{ q \in \mathbb{P} : \exists \tau (q \vdash \tau \in F(\check{x})) \}.$$

If $q \in \bigcup_{x \in X} D_x$, use $AC_X$ to pick a name $\tau_x$ for every $x$ such that $q \vdash \tau_x \in F(\check{x})$. Combine the $\tau_x$ to form a name.
Now what about DC?

Recall that proper forcing preserves DC.

Definition

\( \mathbb{P} \) is *quasiproper* iff for every \( p \in \mathbb{P} \) and a set \( X \), there is a countable elementary submodel \( M \) of \( V_\alpha \), for some large \( \alpha \), such that \( p, X, \mathbb{P} \in M \) and there is an \( M \)-generic condition \( q \leq p \).

The difference with properness is that \( M \) strongly depends on the choice of \( p \), whereas in properness, any \( M \) works.

E.g. club shooting through a stationary, co-stationary set is quasiproper, but of course far from being proper. But even there, there is a stationary set of \( M \)'s that work.
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**Theorem**

*Quasiproper forcing preserves DC. Moreover if $\mathbb{P}$ is $\sigma$-sequential and $\mathbb{P}$ preserves DC, then $\mathbb{P}$ is quasiproper.*
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**Theorem**

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If $\dot{T}$ is a name for a tree without maximal nodes, $\dot{T} \in M$ and $G \cap M$ is generic over $M$, then $\dot{T}^G \cap M[G \cap M]$ is a subtree of $\dot{T}^G$ with the same properties.
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In $V[G]$, construct an elementary chain $\langle M_n : n \in \omega \rangle$ of countable elementary submodels of $(V_\alpha)^V$ such that for every $n$ and $D \in M_n$ dense,

$$M_{n+1} \cap D \cap G \neq \emptyset.$$
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$M := \bigcup_{n \in \omega} M_n \in V$ is elementary in $(V_\alpha)^V$ and $G \cap M \cap D \neq \emptyset$ for every $D \in M$ dense.
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$M := \bigcup_{n \in \omega} M_n \in V$ is elementary in $(V_\alpha)^V$ and $G \cap M \cap D \neq \emptyset$ for every $D \in M$ dense. But then some condition in $G$ forces this and it is a generic condition.
Main result

Theorem

Let $\kappa$ be an infinite cardinal. It is consistent with $\text{ZF} + \text{DC}_{<\kappa}$ that

1. there is a $\kappa$-distributive forcing that violates $\text{DC}$,
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Let $\kappa$ be an infinite cardinal. It is consistent with $\text{ZF} + \text{DC}_{<\kappa}$ that

1. there is a $\kappa$-distributive forcing that violates DC,
2. there is a $\kappa$-sequential forcing that violates $\text{AC}_\omega$.

Corollary

It is consistent, that $\kappa$-sequential does not imply $\kappa$-distributive.

It is consistent, that $\sigma$-distributive forcing does not preserve DC.
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Let $\kappa$ be an infinite cardinal. It is consistent with $\text{ZF} + \text{DC}_{<\kappa}$ that
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It is consistent, that $\sigma$-distributive forcing does not preserve $\text{DC}$. 

The \(\kappa\)-Cohen model

Let \(\kappa\) be a regular cardinal and assume GCH. The \(\kappa\)-Cohen model is constructed as follows:

Consider \(\mathbb{P} = \text{Add}(\kappa, \kappa)\) which consists of \(p : \text{dom} \; p \to 2^{<\kappa}\), \(\text{dom} \; p \in [\kappa]^{<\kappa}\). This adds \(\kappa\) many \(\kappa\)-Cohen reals \(\langle \dot{c}_\alpha^G : \alpha < \kappa \rangle\).

We let \(G\) be the group of automorphisms of \(\mathbb{P}\) induced by permutations of \(\kappa\):

\[
\pi(p)(\alpha) = p(\pi^{-1}(\alpha)).
\]

\(\pi\) further extends to \(\mathbb{P}\)-names by recursion on their rank:

\[
\pi(\dot{x}) = \{(\pi(p), \pi(\dot{y})) : (p, \dot{y}) \in \dot{x}\}.
\]

Note: \(\pi(\dot{c}_\alpha) = \dot{c}_{\pi(\alpha)}\).
The $\kappa$-Cohen model

For any $E \in [\kappa]^{<\kappa}$, we let $\text{fix}(E) := \{\pi \in G : \pi \restriction E = \text{id}\}$.

A $\mathbb{P}$-name $\dot{x}$ is symmetric if there is $E \in [\kappa]^{<\kappa}$ so that $\pi(\dot{x}) = \dot{x}$ for every $\pi \in \text{fix}(E)$.

A $\mathbb{P}$-name $\dot{x}$ is hereditarily symmetric if it is symmetric and all names appearing in it are hereditarily symmetric. The class of hereditarily symmetric names is called HS.

Definition
Let $G$ be $\mathbb{P}$-generic over $V$. The $\kappa$-Cohen model is $M := \{\dot{x}^G : \dot{x} \in \text{HS}\}$.

Theorem
$V \subseteq M \subseteq V[G]$ and $M \models \text{ZF + DC}_{<\kappa}$. $A := \{\dot{c}_\alpha^G : \alpha < \kappa\} \in M$ and every well-orderable subset of $A$ is contained in $\{\dot{c}_\alpha^G : \alpha < \beta\}$ for some $\beta < \kappa$. 
Main result

**Theorem**

Let $\kappa$ be an infinite cardinal. It is consistent with $\text{ZF} + \text{DC}_{<\kappa}$ that

1. there is a $\kappa$-distributive forcing that violates DC,
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Let $\kappa$ be an infinite cardinal. It is consistent with $\text{ZF} + \text{DC}_{<\kappa}$ that

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The first forcing $\mathbb{Q}_0$ is very natural:
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1. there is a $\kappa$-distributive forcing that violates $\text{DC}$,
2. there is a $\kappa$-sequential forcing that violates $\text{AC}_{\omega}$.

The first forcing $\mathbb{Q}_0$ is very natural:

$\mathbb{Q}_0$ consists of (reverse) well-founded trees $(t, \trianglelefteq)$, ordered by end-extension, such that $t \subseteq A$ is well-orderable. It adds a generic tree on $A$. 
Lemma

$\mathcal{Q}_0$ is $\kappa$-distributive.
Lemma
\( \mathcal{Q}_0 \) is \( \kappa \)-distributive.

Proof.
Work in \( V \). Let \( \dot{F} \in \text{HS} \) and \( p \in \mathbb{P} \) such that \( p \models \dot{F}(\alpha) \subseteq \mathcal{Q}_0 \) is dense open for every \( \alpha < \beta, \beta < \kappa \).
Lemma
\( \mathcal{Q}_0 \) is \( \kappa \)-distributive.

Proof.
Work in \( V \). Let \( \dot{F} \in HS \) and \( p \in \mathbb{P} \) such that \( p \Vdash \dot{F}(\alpha) \subseteq \dot{Q}_0 \) is dense open for every \( \alpha < \beta, \beta < \kappa \). Say \( \text{fix}(E) \) fixes \( \dot{F} \).
Generic trees

Lemma

$Q_0$ is $\kappa$-distributive.

Proof.

Work in $V$. Let $\dot{F} \in HS$ and $p \in P$ such that $p \Vdash \dot{F}(\alpha) \subseteq \dot{Q}_0$ is dense open for every $\alpha < \beta$, $\beta < \kappa$. Say fix$(E)$ fixes $\dot{F}$. Also let $\dot{t}, \dot{\triangle} \in HS$ and wlog we may assume that $p \Vdash_p (\dot{t}, \dot{\triangle}) \in \dot{Q}_0 \land \dot{t} = \dot{A} \upharpoonright E$ and dom $p = E$. 

Sequential and distributive forcing without choice
Lemma

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Proof.

Work in $V$. Let $\dot{F} \in HS$ and $p \in P$ such that $p \Vdash \dot{F}(\alpha) \subseteq \dot{Q}_0$ is dense open for every $\alpha < \beta$, $\beta < \kappa$. Say $\text{fix}(E)$ fixes $\dot{F}$. Also let $\dot{t}, \dot{\triangleleft} \in HS$ and wlog we may assume that $p \Vdash P(\dot{t}, \dot{\triangleleft}) \in \dot{Q}_0 \land t = \dot{A} \upharpoonright E$ and $\text{dom} \ p = E$. In $M$, by $\text{DC}_{<\kappa}$ there are $(t_\alpha, \trianglelefteq_\alpha) \in Q_0$,

$$(t_\alpha, \trianglelefteq_\alpha) \leq (t, \triangleleft) \land (t_\alpha, \trianglelefteq_\alpha) \in F(\alpha).$$
Lemma
$\mathcal{Q}_0$ is $\kappa$-distributive.

Proof.
Work in $V$. Let $\dot{F} \in \text{HS}$ and $p \in \mathbb{P}$ such that $p \Vdash \dot{F}(\alpha) \subseteq \dot{\mathcal{Q}}_0$ is dense open for every $\alpha < \beta$, $\beta < \kappa$. Say $\text{fix}(E)$ fixes $\dot{F}$. Also let $\dot{t}, \dot{\triangleleft} \in \text{HS}$ and wlog we may assume that $p \Vdash_{\mathbb{P}} (\dot{t}, \dot{\triangleleft}) \in \dot{\mathcal{Q}}_0 \land \dot{t} = \dot{A} \upharpoonright E$ and $\text{dom } p = E$. In $M$, by $\text{DC}_{<\kappa}$ there are $(t_\alpha, \triangleleft_\alpha) \in \mathcal{Q}_0$,

$$(t_\alpha, \triangleleft_\alpha) \leq (t, \triangleleft) \land (t_\alpha, \triangleleft_\alpha) \in \mathcal{F}(\alpha).$$

In $V$, by $\kappa$-closure of $\mathbb{P}$ we find $p_\alpha \leq p$, $t_\alpha, \triangleleft_\alpha \in \text{HS}$ and $E_\alpha \in [\kappa]^{<\kappa}$ so that

$$\text{dom } p_\alpha = E_\alpha \land p_\alpha \Vdash \cdots \land t_\alpha = \dot{A} \upharpoonright E_\alpha.$$
Lemma

\( \mathcal{Q}_0 \) is \( \kappa \)-distributive.

Proof.

Work in \( V \). Let \( \dot{F} \in \text{HS} \) and \( p \in \mathbb{P} \) such that \( p \Vdash \dot{F}(\alpha) \subseteq \dot{Q}_0 \) is dense open for every \( \alpha < \beta, \beta < \kappa \). Say \( \text{fix}(E) \) fixes \( \dot{F} \). Also let \( \dot{t}, \dot{\triangle} \in \text{HS} \) and wlog we may assume that \( p \Vdash \mathbb{P}(\dot{t}, \dot{\triangle}) \in \dot{Q}_0 \land \dot{t} = \dot{A} \upharpoonright E \) and \( \text{dom } p = E \).

In \( M \), by \( \text{DC}_{<\kappa} \) there are \( (t_\alpha, \triangle_\alpha) \in \mathcal{Q}_0 \),

\[
(t_\alpha, \triangle_\alpha) \leq (t, \triangle) \land (t_\alpha, \triangle_\alpha) \in F(\alpha).
\]

In \( V \), by \( \kappa \)-closure of \( \mathbb{P} \) we find \( p_\alpha \leq p, \dot{t}_\alpha, \dot{\triangle}_\alpha \in \text{HS} \) and \( E_\alpha \in [\kappa]^{<\kappa} \) so that

\[
\text{dom } p_\alpha = E_\alpha \land p_\alpha \Vdash \cdots \land \dot{t}_\alpha = \dot{A} \upharpoonright E_\alpha.
\]

The trick is to find permutations \( \pi_\alpha \in \mathcal{G} \) fixing \( E \), such that for every \( \alpha < \gamma < \beta \),

\[
\pi''_\alpha E_\alpha \cap \pi''_\gamma E_\gamma = E.
\]
Generic trees

Then

\[ \pi_\alpha(p_\alpha) \models \pi_\alpha(t_\alpha, \trianglelefteq_\alpha) \leq \pi_\alpha(t, \trianglelefteq) \land \pi_\alpha(t_\alpha, \trianglelefteq_\alpha) \in \pi_\alpha(\dot{F}(\check{\alpha})) \]
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\[ \pi_\alpha(p_\alpha) \vdash \pi_\alpha(\dot{t}_\alpha, \dot{\sqsubseteq}_\alpha) \leq \pi_\alpha(\dot{t}, \dot{\sqsubseteq}) \land \pi_\alpha(\dot{t}_\alpha, \dot{\sqsubseteq}_\alpha) \in \pi_\alpha(\dot{F}(\dot{\alpha})) \]

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Also \( p' = \bigcup_{\alpha < \beta} \pi_\alpha(p_\alpha) \leq p \) and if \( \dot{t}' := \bigcup_{\alpha < \beta} \pi_\alpha(\dot{t}_\alpha), \)
\( \dot{\sqsubseteq}' := \bigcup_{\alpha < \beta} \pi_\alpha(\dot{\sqsubseteq}_\alpha), \) then

\[ p' \vdash (\dot{t}', \dot{\sqsubseteq}') \leq (\dot{t}, \dot{\sqsubseteq}) \land (\dot{t}', \dot{\sqsubseteq}') \in \bigcap_{\alpha < \beta} \dot{F}(\alpha). \]

The names are fixed by \( \text{fix}(E'), \) \( E' = \bigcup_{\alpha < \beta} \pi''_\alpha E_\alpha \in [\kappa]^{<\kappa}. \)
Generic trees

**Lemma**

If $G$ is $\mathbb{Q}_0$-generic over $V$, then $T = \bigcup G \in V[G]$ is a tree without maximal nodes and no infinite branch.
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Suppose $(t, \sqsubseteq) \models a \in t \land a$ is maximal, then pick $b \in A \setminus t$ and extend $t$ by putting $b$ above $a$. 
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then there is $n > 0$ such that $a_n \notin t$ as $t$ is well-founded.
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then there is $n > 0$ such that $a_n \notin t$ as $t$ is well-founded. Extend $t$ by adding $a_n$ as a minimal element. \qed
Main result

Theorem
Let $\kappa$ be an infinite cardinal. It is consistent with $\text{ZF} + \text{DC}_{<\kappa}$ that
1. there is a $\kappa$-distributive forcing that violates $\text{DC}$,
2. there is a $\kappa$-sequential forcing that violates $\text{AC}_\omega$. 

Q1 consist of partitions $e$ of a well-orderable subset of $A$ into finitely many pieces. When extending a condition $e$, we may add new pieces and extend already present ones, but without combining them. The generic adds a partition of $A$. 

Sequential and distributive forcing without choice University of East Anglia
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$Q_1$ is not $\sigma$-distributive ($D_n = \{ e \in Q_1 : |e| > n \}$), but:

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**Proof.**

In $M$, $\dot{f}$ is a $\mathbb{Q}_1$ name. In $V$, $[\dot{f}] \in HS$, fix($E$) fixes $[\dot{f}]$, $a$ is a partition of $E$, dom $p = E$ and

\[ p \models P \dot{A} \upharpoonright a \models \mathbb{Q}_1 [\dot{f}] : \gamma \to \dot{M}. \]
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In \( M \), \( \dot{f} \) is a \( Q_1 \) name. In \( V \), \([\dot{f}] \in HS\), \( \text{fix}(E) \) fixes \([\dot{f}]\), \( a \) is a partition of \( E \), \( \text{dom } p = E \) and

\[ p \models P \dot{A} \upharpoonright a \models Q_1 [\dot{f}]: \gamma \rightarrow \dot{M}. \]

We claim that in \( M \), \( \dot{A} \upharpoonright a \), decides all values of \( f \). Otherwise

\[ p' \models \dot{A} \upharpoonright a_0 \models Q_1 [\dot{f}](\alpha) = \dot{x} \land \dot{A} \upharpoonright a_1 \models Q_1 [\dot{f}](\alpha) = \dot{y}. \]

\( a_0 \) and \( a_1 \) are partitions of \( E' \supseteq E \) extending \( a \), \( \text{dom } p' = E' \).
**Generic partitions**

Let $\pi$ fix $E$ and map $E' \setminus E$ away from itself. Then $q := \pi(p') \cup p' \leq p$, $q \models \dot{A} \upharpoonright a_0 \parallel \dot{A} \upharpoonright \pi(a_1)$, so

$$q \models \dot{x} = \pi(\dot{y}).$$

But also $q \models \dot{A} \upharpoonright a_1 \parallel \dot{A} \upharpoonright \pi(a_1)$ so

$$q \models \dot{y} = \pi(\dot{y}).$$

Ultimately

$$q \models \dot{x} = \dot{y}.$$
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Suppose we can pick $a_n \in [X]^n$ for every $n \in \omega$. Let

$$k_n > \sum_{i < n} k_i.$$  

Then suppose we can pick $x_n \in a_{k_n} \setminus \bigcup_{i < n} a_{k_i}$ for every $n \in \omega$. Then \{x_n : n \in \omega\} is a countably infinite subset of $X$, so $X$ is not amorphous.
Open questions

**Question**
Assume $\neg\text{AC}$. Is there a $\sigma$-sequential forcing that is not $\sigma$-sequential? Is the equivalence some known choice principle?

**Theorem**
If there is an infinite set $A$, such that $\mathcal{P}(A) < \omega$ is Dedekind-finite (no countable subset), then $\text{Add}(A, 1)$ is $\sigma$-sequential but not $\sigma$-distributive.

**Theorem**
In the Gitik model, every $\sigma$-sequential forcing is trivial. In particular, $\sigma$-distributive $\iff$ $\sigma$-sequential.

In the Gitik model, every cardinal is singular of cofinality $\omega$. It uses a proper class of strongly compact cardinals.
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Thank you!