Genericity and enumeration reducibility

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Real World Computing

- Turing - importance of computer interaction
- New computational paradigms (Grid etc)
- Experimental/ analog computing
- Modelling non-linear natural phenomena
- Key role of imitation in AI
An Improbable Story . . .

- You are having trouble with a math problem, and go to your dept. for help
- You knock on Prof. A’s door
- Unfortunately Prof. A has taken early retirement - he will never again be at work
- You wait forever . . .
- Your math problem remains unsolved . . .

The real world?
Back in the real world

- You knock on Prof. A’s door
- Getting no answer, you try Prof. B (at lunch)
- You then move on to Prof. C, who is in her office . . .
- Always helpful and knowledgeable, she gives you the information you need, and you complete your project . . .
Get two different models of relative computability:

- Compute using an oracle - i.e., call deterministically on values of a total function
- Compute using emergent (or enumerated) information - i.e., call non-deterministically on values of a partial function
Enumeration reducibility

\[ b_0, b_1, \ldots \in B \text{ in some order} \]

\[ a_0, a_1, \ldots \in A \text{ enumerated in any order} \]

\[ n \in \Psi^A_i \iff \text{defn } (\exists \text{ a finite } D \subseteq A)[\langle n, D \rangle \in \Psi]. \]
The link with NT-reducibility

If \( g \) is total, we have \( f \leq_{NT} g \iff f \leq_T g \).

**THEOREM**

Let \( f, g \) be partial functions. Then

\[ f \leq_{NT} g \iff \text{Graph}(f) \leq_e \text{Graph}(g). \]
**THEOREM (Selman’s Theorem, 1971)**
For any $A, B \subseteq \mathbb{N}$

$$ A \leq_e B \iff \forall X \left[ B \text{ c.e. in } X \Rightarrow A \text{ c.e. in } X \right]. $$

**PROOF (sketch)** The left-to-right implication I will leave to you.

Conversely, assume that $A \not\leq_e B$. We will construct a $C = \bigcup_{s \geq 0} C_s$ such that $B$ is c.e. in $C$ but $A$ is not c.e. in $C$.

We satisfy “$B$ c.e. in $C$” by imposing an overall requirement

$$ \exists \langle x, y \rangle \in C \iff x \in B $$

for each $x \geq 0$. Call a finite $D \supseteq C_s$ *admissible* if it satisfies Equation (0.1) with $D$ in place of $C$, but with the right-to-left half of Equation (0.1) restricted to $x \leq s$ (so that the admissible $D$’s can be enumerated from an enumeration of $B$ and a finite amount of information about $\overline{B}$).

We satisfy $A \not= W_s^C$ (at stage $s + 1$) by looking for some admissible $D \supseteq C_s$ with $x \in W_s^D - A$. If $D$ exists, choose $C_{s+1} = D$ giving $A \not= W_s^C$. Otherwise, either $x \in A - W_s^D$ for some $x$, all admissible $D$ (so $A \not= W_s^C$ again), or

$$ \forall x \left( x \in A \iff \exists \text{ an admissible } D \text{ such that } x \in W_s^D \right), $$
giving $A \leq_e B$, a contradiction.  

\[\square\]
Define \( A \equiv_e B \iff \text{defn } A \leq_e B \& B \leq_e A \).

The \textbf{enumeration degree} — or \textit{e-degree}, written \( \deg_e(A) \) — of \( A \) is

\[
\deg_e(A) = \text{defn} \{ X \mid X \equiv_e A \}.
\]

We define \( \deg_e(A) \leq \deg_e(B) \iff \text{defn } A \leq_e B \).

We write \( \mathcal{D}_e = \text{defn} \) the set of all \( e \)-degrees with the ordering \( \leq \).

The \textbf{partial degree} of a partial function \( f \) is

\[
\deg(f) = \text{defn} \{ g \mid \text{Graph}(f) \equiv_e \text{Graph}(g) \}
= \{ g \mid f \equiv_{NT} g \}
\]

We write \( \mathcal{P} = \) the set of all partial degrees, with ordering \( \leq \) defined by

\[
\deg(f) \leq \deg(g) \iff \text{defn } \text{Graph}(f) \leq_e \text{Graph}(g) \iff f \leq_{NT} g.
\]

We say that an \( e \)-degree \( a_e \) is \textbf{total} if there is a total function \( f \) with \( \text{Graph}(f) \in a_e \).

We write \( \text{TOT} = \) the set of total \( e \)-degrees.
Define $A \equiv_e B \iff \text{defn } A \leq_e B \& B \leq_e A$.

The enumeration degree — or e-degree, written $\text{deg}_e(A)$ — of $A$ is

$$\text{deg}_e(A) = \text{defn } \{X \mid X \equiv_e A\}.$$ 

We define $\text{deg}_e(A) \leq_e \text{deg}_e(B) \iff \text{defn } A \leq_e B$.

We write $\mathcal{D}_e = \text{defn}$ the set of all e-degrees with the ordering $\leq$.

The partial degree of a partial function $f$ is

$$\text{deg}(f) = \text{defn } \{g \mid \text{Graph}(f) \equiv_e \text{Graph}(g)\}$$

$$= \{g \mid f \equiv_{NT} g\}$$

We write $\mathcal{P} = \text{the set of all partial degrees, with ordering } \leq \text{ defined by } \text{deg}(f) \leq_e \text{deg}(g) \iff \text{defn } \text{Graph}(f) \leq_e \text{Graph}(g) \iff f \leq_{NT} g$.

We say that an e-degree $a_e$ is total if there is a total function $f$ with $\text{Graph}(f) \in a_e$.

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We say that an e-degree \( a_e \) is **total** if there is a total function \( f \) with \( \text{Graph}(f) \in a_e \).

We write \( \text{TOT} = \) the set of total e-degrees.
The picture we are starting to build up of the noncomputable universe will be a complex one, but one which keeps on revealing underlying form and coherence in unexpected and captivating ways.

We have extended "computability in the real world"—that is, computability relative to context—in different ways: on the one hand in introducing e-reducibility as a natural alternative to the oracle model, and on the other allowing nondeterministic computation. And now we see the structures corresponding to these approaches converging in a very satisfying and informative way:

\[ \text{TOT} \cong \mathcal{D} \]

\[ \mathcal{P} \cong \mathcal{D}_e \]

Having extended \( \mathcal{D} \), our new structures still have many of the basic properties we found earlier. Obviously part (iv) of Theorem ?? and tell us that \( \mathcal{D}_e \) is uncountable. Also:

**EXERCISE 0.3.5**

Let \( a \in \mathcal{D}_e \). Show that (i) \( a \) is countably infinite, and (ii) \( \mathcal{D}_e(\leq a) \) is countable.

**EXERCISE 0.3.6**

Show that \( \mathcal{D}_e \) has at least degree \( 0 \) consisting of all c.e. sets.

**EXERCISE 0.3.7**

Show that \( \mathcal{P} \) has at least degree \( 0 \) consisting of all p.c. functions.
Extending the Jump Operator

Let $K_A = \{x \mid x \in \Psi^A_x\}$.

Then the **e-jump** of a set $A$ is $J^A_e = \text{defn} \ A \oplus \overline{K_A}$. And the **jump** of an e-degree $a = \deg_e(A)$ is defined to be $a' = \deg_e(A \oplus \overline{K_A}) = \deg_e(J^A_e)$.

We iterate the jump in the usual way to obtain the $n^{th}$ jump $a^{(n)}$ of $a$.

**PROPOSITION 0.3.19**

The e-jump agrees with the natural embedding of the Turing jump — that is, for each $A \subseteq \mathbb{N}$ we have $\iota(\deg(A')) = \deg_e(J_e(\chi_A))$. 
Extending the Jump Operator

Let \( K_A = \{ x \mid x \in \Psi^A_x \} \).

Then the **e-jump** of a set \( A \) is \( J^A_e = \text{defn} \, A \oplus \overline{K}_A \). And the **jump** of an e-degree \( a = \deg_e(A) \) is defined to be \( a' = \deg_e(A \oplus \overline{K}_A) = \deg_e(J^A_e) \).

We iterate the jump in the usual way to obtain the \( n^{th} \) jump \( a^{(n)} \) of \( a \).

**PROPOSITION**

The e-jump agrees with the natural embedding of the Turing jump — that is, for each \( A \subseteq \mathbb{N} \) we have \( \iota(\deg(A')) = \deg_e(J_e(\chi_A)) \).
Local information content

**THEOREM**
For each $A \subseteq \mathbb{N}, n \geq 0$ we have $\deg_e(A) \leq 0_e^{(n)} \iff A \in \Sigma_{n+1}$.

- The most important case is $n = 1$, telling us that:

  $$\mathcal{D}_e(\leq 0'_e) = \text{the set of all } \Sigma_2 \text{ e-degrees}$$
Local forcing

**DEFINITION**

We say \( A \subseteq \mathbb{N} \) is **1-generic** if for every c.e. set \( X \) of strings, either

(a) \( (\exists \tau \subseteq A)[\tau \in X] \), or

(b) \( (\exists \tau \subseteq A)(\forall \sigma \supseteq \tau)[\sigma \notin X] \).

We say \( A \) — and any \( \tau' \supseteq \) such a \( \tau \) — **forces** \( X \). We write \( A \models X \) — or \( \tau' \models X \) — as appropriate.
Forcing and Category lxi can avoid being 1-generic. To do this it must avoid strings satisfying clause (a) of Definition 0.9.1, but without ever giving up (a) as an option, and ending in clause (b).

This next basic result corresponding to Feferman's Theorem 0.7.2 gives us a bound on the 1-generic set.

**THEOREM 0.9.4 (The Existence Theorem for 1-Generic Sets)**

There exists a 1-generic set $A \leq T'$. 

**PROOF**

We construct strings $\sigma_0 \subset \sigma_1 \subset \ldots \subset A = \bigcup_{i \geq 0} \sigma_i$ so that for each $i \geq 0$ we have

\[
\sigma_{i+1} \text{ grabs a string in } W_i \text{ if it possibly can — and if it cannot it blames } \sigma_i \text{ for already having satisfied clause (b) of the definition of forcing } W_i.
\]
n-generics are easy to build using an oracle

**THEOREM** (The Existence Theorem for 1-Generic Sets)
There exists a 1-generic set $A \leq_T \emptyset'$.

**PROOF** We construct strings $\sigma_0 \subset \sigma_1 \subset \ldots \subset A = \bigcup_{i \geq 0} \sigma_i$ so that for each $i \geq 0$ we have $\sigma_{i+1} \Vdash W_i$.

The idea is that at stage $i + 1$ of the construction $\sigma_{i+1}$ grabs a string in $W_i$ if it possibly can — and if it cannot it blames $\sigma_i$ for already having satisfied clause (b) of the definition of forcing $W_i$. 
A simple example

Show that no 1-generic $A$ is computable.

**SOLUTION**  Define $X = \{ \sigma \mid \varphi_i \text{ and } \sigma \text{ disagree on some argument } y \}$. Then

$$\sigma \in X \iff (\exists s, y)[y < |\sigma| \& \varphi_{i,s}(y) \downarrow \neq \sigma(y)],$$

computable relation

so $X \in \Sigma_1$ and so is c.e.

This means that if $A$ is 1-generic $A \Vdash X$. So by Definition 0.9.1 there is a $\tau \subset A$ such that $\tau \Vdash X$.

**Case (a).**  $\tau \in X$ — then $\tau(y) \neq \varphi_i(y)$ some $y < |\tau|$. So $A \neq \varphi_i$ since $\tau \subset A$.

**Case (b).**  For all $\sigma \supset \tau$ we have $\sigma \notin X$. But then we cannot have $\varphi(|\tau|) \downarrow$, since otherwise

$$\sigma = \tau \upharpoonright (1 \cdot \varphi_i(|\tau|)) \in X.$$

So $\varphi$ is not total and so cannot be the characteristic function of $A$. □
Forcing in the Turing and enumeration degrees

- If a Turing degree $a$ contains a $1$-generic set, we say $a$ is $1$-generic.

- If an enumeration degree $a$ contains a $1$-generic set, we say $a$ is set $1$-generic.
Forcing partial functions
(mainly Case 1971, and Copestake 1988)

A partial string \( \sigma \) is a finite sequence of symbols from \( \mathbb{N} \cup \{\uparrow\} \). We write \( S^* = (\mathbb{N} \cup \{\uparrow\})^{<\omega} \) for the set of all partial strings.

**Definition**

A partial function \( \psi \in (\mathbb{N} \cup \{\uparrow\})^\omega \) is 1-generic if for every c.e. set \( X \subseteq S^* \), either

(a) \((\exists \sigma \subseteq \psi)[\sigma \in X]\), or

(b) \((\exists \sigma \subseteq \psi)(\forall \tau \in S^*)[\sigma \subseteq \tau \Rightarrow \tau \notin X] \).

A partial degree, or e-degree, \( a \) is 1-generic if there is a 1-generic \( \psi \in a \).
Does set 1-generic = generic for e-degrees?

First notice

- if the partial function $\psi$ is 1-generic, then it has no p.c. extension.
  - Ask $\psi$ to force each set $X_i \subseteq S^*$ given by
    \[ \tau \in X \iff (\exists x < |\tau|) [ \varphi_i(x) \downarrow \neq \tau(x) ]. \]

So - no 1-generic function can be p.c.

In fact ...

- if $\psi$ is $n$-generic then $\text{Graph}(\psi)$ does not contain an infinite $\Sigma_n$ subset.
  - there is no $(n + 1)$-generic e-degree below $0_e^{(n)}$. 
If \( \psi \) is 1-generic then \( \text{Dom}(\psi) <_e \psi \).

1. To show \( \psi \neq \Psi_i^{\text{Dom}(\psi)} \) for any \( i \), ask \( \psi \) to force each \( X_i \subseteq S^* \) given by

\[
\tau \in X_i \iff \exists x < |\tau|, y \] \( \tau(x) \neq y \land \langle x, y \rangle \in \Psi_i^{\text{Dom}(\tau)} \)

2. If \( A \) is a 1-generic set and \( \psi <_e A \), then \( \psi \) has a p.c. extension and so is not 1-generic.

3. Ask \( A \) to force the set \( X_i \) of binary strings given by

\[
\tau \in X_i \iff \exists x, y, z \] \( \langle x, y \rangle, \langle x, z \rangle \in \Psi_i^{\tau^+} \land y \neq z \)
And for some n-generic function $\psi$.

- $(\Leftarrow)$ Assume $A$ does not force some $\Sigma_n$ set $X \in 2^\omega$, and show $\psi$ does not force $\hat{X} = \{\tau \in S^* \mid \text{Dom}(\tau) \in X\}$, where $\text{Dom}(\tau) = \chi_{\text{Dom}(\tau)}|\tau|$.

- $(\Rightarrow)$ Given $A$ n-generic, build $\psi$ as the union of partial strings $\psi_i$ forcing the $i^{\text{th}} \Sigma_n$ set of partial strings, with $A = \text{Dom}(\psi)$.

So... set 1-generic e-degrees are never generic!

**COROLLARY** (Copestake, 1988)

Every 1-generic e-degree $a$ bounds a set 1-generic $b < a$, where no $c \leq b$ is 1-generic.
Copiestake’s basics

(1) There exists an $n$-generic e-degree below $0^{(n)}$ for each $n \geq 1$.
(2) There is no $(n + 1)$-generic e-degree below $0^{(n)}$, any $n \geq 1$.
(3) Every 1-generic e-degree is quasi-minimal.
(4) Every 2-generic e-degree bounds a minimal pair of e-degrees.
(5) If $a$ is a 1-generic e-degree then every r.e. partial ordering can be embedded below $a$ (in the 1-generic degrees below $a$).

And all results hold with ‘set 1-generic’ in place of ‘1-generic’
Comparing 1-generic Turing and e-degrees

For the Turing case

Every 1-generic is is generalised low. In particular, all the 1-generic degrees \( \leq 0' \) are low.

But (C, Li, Sorbi and Yang/ Copestake) there exists a set 1-generic e-degree which bounds no minimal pair . . . and hence is not low.
Questions

- Characterise the jumps of the (set) 1-generic e-degrees below $0'$

- Are e-degrees of 1-generic sets (below $0'$) closed downwards?

- Are the e-degrees of 1-generic sets (below $0'$) definable?
Thank you!