Nonlinear Wave Problems that I can't Control

Onno Bokhove

School of Mathematics, University of Leeds
with Ambati, Gagarina, Van der Vegt, Zweers (Twente)

Postdoc, 2 EU PhDs: on breaking waves

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1. Introduction

Some hydrodynamics ... rogue waves are anomalously high waves defined relative to a significant wave height H_s.

- Index (Khariff et al. '09, Dysthe et al. '08):

$$AI = H_{rw}/H_s > 2 \qquad \text{or} \qquad AI = \eta_{rw}/H_s > 1.25 \quad (1)$$

- Relevance in maritime & coastal engineering —ship design & safety offshore structures
- Pyramidal rogue wave (Faulkner 2001):

Fig.1. Pyramidal wave off south Japan

Introduction

Causes rogue waves, e.g., Khariff et al. (2009) & Faulkner (2001, 2003):

- spatial wave focussing due to refraction or wave caustics, or wave focussing in coastal convergences
- crossing seas, nearly standing waves with pyramidal waves.

Introduction

Here, emphasis will lie on rogue waves & wave focussing:

- in crossing seas
- due to coastal or submarine convergences.

Moreover, (rogue) wave energy devices based on wave focussing will be introduced or discussed:

- the TapChan or tapered channel device
- the Oscillating Water Column (OWC) device with wind turbine
- a new device with a more direct energy conversion?

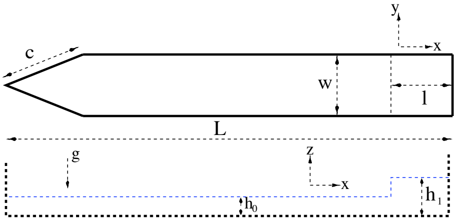
2. Three Control Challenges

Employ port-Hamiltonian methods to control:

- a wave maker to create the highest rogue wave?
- geometry and dynamo in a new rogue wave energy device?
- maximum berm formation by breaking waves on beaches?

Bore Soliton Splash

- Water channel: $L = 43.5 \pm 0.25$m, $D = 1.20$m, $w = 2 \pm 0.05$m



- Sluice compartment & gate, lifted with 2.5m/s
- V-shaped or linear convergence at other end with $c = 2.7$m
- Start at rest with water levels $h_0 \in [0.32, 0.47]$m and $h_1 \in [0.67, 1.02]$m.

Bore Soliton Splash

- 7+2 Cases of which 3 repeats (reproducibility):

#	h_0 (m) ± 0.01m	h_1 (m) ± 0.01m	H_s ± 0.05m	H_{rw} ± 0.5m	Peak #	Comments
1	0.32	0.67	-	0.6	-	bore
2	0.38	0.74	-	2.5	-	good splash
3	<u>0.41</u>	<u>0.9</u>	0.35	3.25	2nd	thin jet cf. 6 & 8
4	0.47	1.0	0.35	1	2nd	bore & low
5	0.41	1.02	0.40	1.5	1st	bore & low
6	<u>0.41</u>	<u>0.9</u>	0.35	3.5	2nd	BSS cf. 3 & 8
7	0.45	0.8	0.35	2.5	2nd	good splash
8	<u>0.41</u>	<u>0.9</u>	0.35	3.5	2nd	BSS & ++
9	0.43	0.9	0.45	1.8	1st	bubble

Phenomenology Bore Soliton Splash

- Estimated wave amplitudes $H_s \in [0.35, 0.45]$m
- After opening sluice gates three solitary waves emerge: large, medium and small
- First highest soliton often breaks into spilling breaker
- Variety of outcomes: minor/major reflections, resonances between waves, smooth waves, sheets, pyramidal waves
- $H_{rw} \in [0.6, 3.5]$m.

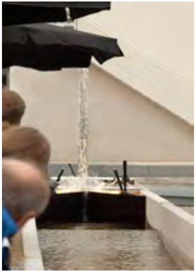
Bore-Soliton-Splash

Highest Cases 3, 6 & 8:

- $H_s = 0.35\text{m}$ & $H_{rw} = [3.25, 3.5, 3.5]\text{m}$
- *bore soliton splash*
  Wout Zweers youtube channel:
  `https://www.youtube.com/watch?v=YSXsXNX4zW0`
- Truly rogish:

$$AI = \frac{H_{rw}}{H_s} = \frac{3.5}{0.35} \approx 10.$$

Bore-Soliton-Splash

a)  b)  c) 

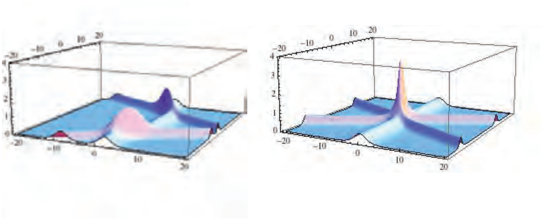
Control the Bore-Soliton-Splash?

- Control the wave maker, including the resulting:
- nonlinearity
- geometry
- dispersion
- solitons
- spilling breaker or bore
- to obtain the highest bore soliton splash?

Control of Soliton-Splash with KP?

- For 3 KdV solitons with amplitude A, there is "V-shaped channel geometry" with angles θ_{max} for which $A_{max} = 9$
- Exact KP solution by Kodama (idea, Dresden '11):

$$\partial_x \underline{\left(\partial_t \eta + \frac{3}{2}\eta\partial_x\eta + \frac{1}{6}\partial_x^3\eta\right)} + \frac{1}{2}\partial_y^2\eta = 0 \qquad (2)$$



- Fine in open sea, but inconsistent in water wave channel!
- Solution: bidirectional Boussinesq model.

Rogue Wave Energy Device

Combine geometric wave focussing with linear dynamo:

- Dynamo above water on a mast attached to floater.
- More direct energy conversion!
- Mechanically robust?
- Working *prototype* (using flash lights charged by shaking).
- Link: `http://www1.maths.leeds.ac.uk/~obokhove/WaveEnergy.m4v`

3. Space-plus-Time Finite Element Water Waves

Key geometric issues:

- in the spatial discretization of Miles' variational prinicple for nonlinear water waves, and
- in the temporal discretization of the non-autonomous spatially discrete variational principle.

Verification and validation against measurements of nonlinear waves driven by a piston wave maker.

Space-plus-Time Finite Element Water Waves

Consider the (2D) potential flow water wave problem:

- gravity as restoring force with acceleration g
- velocity $\mathbf{u} = (u, w)^T = \nabla\phi$ with velocity potential $\phi = \phi(x, z, t)$
- domain $\Omega(t)$ with bottom S_B, wall $S_L : x = L$ and piston wave maker at $x = R(t)$:

$$\Omega(t) : R(t) < x < L \quad \text{and} \quad -H(x) < z < \eta(x,t) \qquad (3)$$

- free surface variables $\eta(x,t)$ & $\phi_s = \phi(x, z = \eta(x,t), t)$
- kinematic single-valued free surface $z = \eta(x,t)$:

$$\partial_t\eta + \partial_x\phi\partial_x\eta = \partial_z\phi \quad \text{at} \quad z = \eta(x,t) \qquad (4)$$

- dynamic free surface condition: continuity of pressure.

Miles' Variational Principle

- Miles 1977 (see also Cotter & B. 2010):

$$
\begin{aligned}
0 &= \delta \int_0^T \mathcal{L}[\phi, \phi_s, \eta, t]dt \\
&= \delta \int_0^T \phi_s \partial_t \eta - \mathcal{H}[\phi, \phi_s, \eta, t]dt \\
&= \delta \int_0^T \int_{R(t)}^L \phi_s \partial_t \eta - \frac{1}{2} g \eta^2 \\
&\quad - \int_{-H(x)}^{\eta(x,t)} \frac{1}{2} |\nabla \phi|^2 dz dx - \int_{-H_w}^{\eta_w} \frac{dR}{dt} \phi_w dz dt
\end{aligned}
$$

- piston wave-maker $R(t)$ with $\phi_w \equiv \phi(R(t), z, t)$, $H_w = H(R(t))$, $\eta_w = \eta(R(t), t)$
- non-autonomous due to piston wave-maker
- canonical Hamiltonian system.

Miles' Variational Principle

Variations yield, using

$$\begin{aligned} \delta\eta(x,0) = \delta\eta(x,T) &= 0 \\ \delta\phi_s &= (\delta\phi)_s + (\partial_z\phi)_s\delta\eta \\ \mathbf{n}_s &= (-\eta_x, 1)^T/\sqrt{1+(\partial_x\eta)^2}, \end{aligned}$$

dynamic/kinematic conditions and Laplace's equation, BC's:

$$\begin{aligned} \delta\eta(x,t) : \partial_t\phi_s + \frac{1}{2}(\partial_x\phi)^2 + g\eta - \frac{1}{2}(\partial_z\phi)_s^2(1+(\partial_x\eta)^2) &= 0 \\ \delta\phi_s(x,t) : \partial_t\eta + \partial_x\phi_s\partial_x\eta - (\partial_z\phi)_s(1+(\partial_x\eta)^2) &= 0 \\ \delta\phi(x,z,t) : \nabla^2\phi &= 0 \\ \delta\phi_w : dR/dt - (\partial_x\phi)_w &= 0 \\ \delta\phi_{B,L} : \mathbf{n}\cdot\boldsymbol{\nabla}\phi &= 0 \end{aligned}$$

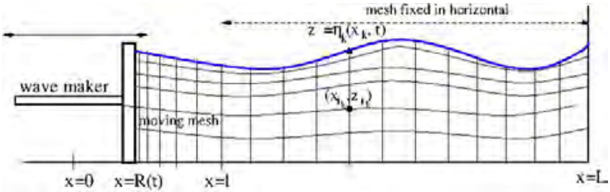
Geometric Wave Modeling: Space FEM

FEM formulation of Miles' variational principle.

- FEM test/basis functions $\tilde{\varphi}_j(x,z,t), \hat{\varphi}_k(x,t)$ with i,j in Ω
- Indices k,l,r at free surface, i',j' in interior & $m,\tilde{m}$ at wave maker.
- Substitute finite element expansions directly into VP:

$$
\begin{aligned}
\phi_h(x,z,t) &= \phi_j(t)\tilde{\varphi}_j(x,z,t)\\
\phi_{sh}(x,t) &= \phi_l(t)\varphi_l(x,t)\\
\eta_h(x,t) &= \eta_k(t)\varphi_k(x,t)
\end{aligned}
$$

Space FEM: Key Issues



- Moving mesh:
- Issue 1: realize that variations of interior nodes of mesh are (dynamically) coupled to variations of free surface nodes.
- Issue 2: find robust (variational) time integrators for (non-)autonomous system.
- Otherwise: nonlinearly unstable discretization (cf. DGFEM in 2007).

Geometric Wave Modeling: Space FEM

- The resulting spatially discrete non-autonomous VP:

$$
\begin{aligned}
0 &= \delta \int_0^T L[\phi_j, \eta_k, t] dt \\
0 &= \delta \int_0^T M_{kl}\phi_k \frac{d\eta_l}{dt} - H[\phi_j, \eta_k, t] dt \\
&= \delta \int_0^T M_{kl}\phi_k \frac{d\eta_l}{dt} - D_{kl}\phi_k\eta_l - \\
&\quad -\frac{1}{2} g M_{kl}\eta_k\eta_l - \frac{1}{2} A_{ij}\phi_i\phi_j - W_m\phi_m dt.
\end{aligned}
$$

- $M_{kl}, D_{kl}, A_{ij}, W_m$ depend on $\{\eta_k(t), t\}$: mesh and wave maker movements.

Geometric Wave Modeling: Space FEM

- The matrices and vectors depend on $\{\eta_k(t), t\}$ as follows:

$$
\begin{aligned}
M_{kl}(t) &= \int_{R(t)}^{L} \varphi_k \varphi_l dx \\
D_{kl}(t) &= -\int_{R(t)}^{L} \frac{\partial \varphi_k}{\partial t} \varphi_l dx, \\
A_{ij}(\eta_r, t) &= \int_{\Omega_h} (\nabla \tilde{\varphi}_i \cdot \nabla \tilde{\varphi}_j) dxdz \\
W_m(\eta_1, t) &= \int_{-H_w}^{\eta_1} \frac{\mathrm{d}R}{\mathrm{d}t} \tilde{\varphi}_m \big|_{x=R(t)} dz,
\end{aligned}
$$

- due to wave maker via $R(t)$ & mesh movement via $\eta(x, t)$.

Geometric Wave Modeling: Space FEM

- Issue 1: variation yields

$$
\begin{aligned}
\delta\phi_k : M_{kl}\frac{d\eta_l}{dt} - D_{kl}\eta_l - A_{ik}\phi_i - W_1\delta_{1k} &= 0\\
\delta\eta_l : \frac{d(M_{kl}\phi_k)}{dt} + D_{kl}\phi_k + M_{kl}\eta_l + \frac{1}{2}\frac{\partial A_{ij}}{\partial \eta_l}\phi_i\phi_j & \\
+\frac{\partial W_m}{\partial \eta_1}\delta_{1l}\phi_m &= 0\\
\delta\phi_i : A_{ij}\phi_i + W_{m'}\delta_{m'j'} &= 0
\end{aligned}
$$

- Hence, we can eliminate interior degree of freedom:

$$
\begin{aligned}
A_{i'j'}\phi_{i'} &= -A_{lj'}\phi_l + W_{m'}\delta_{m'j'} \Longrightarrow\\
\phi_{i'} &= -A_{lj'}A^{-1}_{j'i'}\phi_l - W_{m'}A^{-1}_{m'i'}.
\end{aligned}
$$

Geometric Wave Modeling: Space FEM

- Eliminate $\phi_{j'}$: spatially discrete VP at free surface:

$$
\begin{aligned}
0 &= \delta \int_0^T L[\phi_k, \eta_k, t] dt \\
0 &= \delta \int_0^T M_{kl}\phi_k \frac{d\eta_l}{dt} - H[\phi_k, \eta_k, t] dt \\
&= \delta \int_0^T M_{kl}\phi_k \frac{d\eta_l}{dt} - D_{kl}\phi_k\eta_l - \\
&\quad -\frac{1}{2} g\eta_k\eta_l - \frac{1}{2} B_{kl}\phi_k\phi_l - C_l\phi_l - F_w dt.
\end{aligned}
$$

- Cf. discrete Boundary Element Method
- Function $F_w(t) = -\frac{1}{2} W_{\tilde{m}'} A^{-1}_{\tilde{m}'m'} W_{m'}$ & Schur complement

$$
\begin{aligned}
B_{kl}(\eta_r, t) &= (A_{kl} - A_{ki'} A^{-1}_{i'j'} A_{j'l}) \\
C_l(\eta_r, t) &= W_1\delta_{1l} - A_{lj'} A^{-1}_{j'm'} W_{m'}.
\end{aligned}
$$

Geometric Wave Modeling: Time FEM

- Substitute a discontinuous Galerkin finite element expansion into the VP in each time slab $[t_n, t_{n+1}]$:

$$\begin{aligned} \eta_l &= \eta_l^{n,+}\left(\frac{t_{n+1}-t}{\Delta t_n}\right) + \eta_l^{n+1,-}\left(\frac{t-t_n}{\Delta t_n}\right) \\ \phi_k &= \phi_k^{n+1/2}\frac{2(t-t_n)}{\Delta t_n} + \phi_k^{n,+}\left(\frac{t_n+t_{n+1}-2t}{\Delta t_n}\right). \end{aligned}$$

- Define the delta function due to $M_{kl}\phi_k \mathrm{d}\eta_l/dt$–term by splitting node n in a narrow element with C^0-connection, and calculating its contribution in the limit of zero width:

$$\begin{aligned} p = M_{kl}\phi_k &= p_L + (p_R - p_L)(4\tau - 3\tau^2) \\ q = \eta_l &= q_L + (q_R - q_L)\tau \\ \int_0^1 p\frac{dq}{dt}d\tau &= p_R(q_R - q_L), \end{aligned}$$

- Issue 2: algebraic VP with "extended Hamiltonian" H:

$$
\begin{aligned}
0 &= \delta \sum_{n=0}^{N} L[\phi_k^{n,+}, \phi_k^{n+1/2}, \eta_k^{n,+}, \eta_k^{n+1,-}]dt \\
&= \delta \sum_{n=0}^{N} M_{kl}^{n+1/2} \phi_k^{n+1/2} (\eta_l^{n+1,-} - \eta_l^{n,+}) \\
&- \frac{\Delta t}{2} \big(H^{n+1/2}[\phi_k^{n+1/2}, \eta_k^{n,+}] + H^{n+1/2}[\phi_k^{n+1/2}, \eta_k^{n+1,-}] \big) \\
&\quad + M_{kl}^{n+1,+} \phi_k^{n+1,+} (\eta_l^{n+1,+} - \eta_l^{n+1,-})
\end{aligned}
$$

Geometric Wave Modeling: Time FEM

- Variations (evaluation $\dagger = n + 1/2$):

$$\delta\phi_k^{n+1,+} : \eta_l^{n+1,+} = \eta_l^{n+1,-} \quad \text{continuous!}$$

$$\delta\eta_l^{n,+} : M_{kl}^{n+1/2}\phi_k^{n+1/2} = M_{kl}^{n}\phi_k^{n,+} - \frac{1}{2}\Delta t \frac{\partial H^{n+1/2}[\phi_k^{n+1/2},\eta_k^{n,+}]}{\partial\eta_l^{n,+}}$$

$$\delta\phi_k^{n+1/2} : M_{kl}^{n+1/2}\eta_l^{n+1,-} = M_{kl}^{n+1/2}\eta_l^{n,+} + \frac{1}{2}\Delta t\left(\frac{\partial H^{\dagger}[\phi_k^{n+1/2},\eta_k^{n,+}]}{\partial\phi_k^{n+1/2}} + \frac{\partial H^{\dagger}[\phi_k^{n+1/2},\eta_k^{n+1,-}]}{\partial\phi_k^{n+1/2}}\right)$$

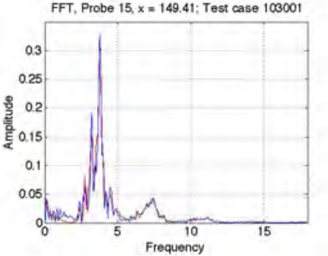
$$\delta\eta_l^{n+1,-} : M_{kl}^{n+1/2}\phi_k^{n+1,+} = M_{kl}^{n}\phi_k^{n+1/2} - \frac{1}{2}\Delta t \frac{\partial H^{n+1/2}[\phi_k^{n+1/2},\eta_k^{n+1,-}]}{\partial\eta_l^{n+1,-}}$$

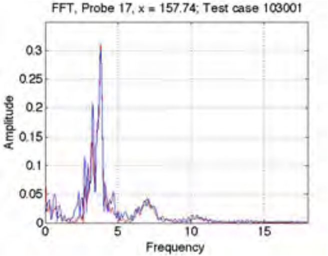
Geometric Wave Modeling: Unfolding

- Reduces to 2nd-order symplectic Störmer-Verlet time integrator in the autonomous case
- "Unfold" the kinetic energy term again to avoid direct inversion of the matrix $A_{i'j'}$
- This defines the nature of the time discretization of $\phi_{i'}$ in the interior
- Use Newton iteration and Petc linear algebra routines.

Geometric Wave Modeling: Validation

- *Driven wave focusing* in MARIN's wave tank `http://www1.maths.leeds.ac.uk/~obokhove/202002_zoom_splash_2.avi`
- Entire *wave tank* MARIN with false vertical wall instead of *beach*.
- Comparison with measured data MARIN is good:





4. Conclusion

- Posed control challenges: bore soliton splash, rogue wae energy device.
- Key issues in variational nonlinear water wave modelling.
- Discontinuous Galerkin variational time approach generalizes to new pseudo-symplectic integrators.
- All extends to 3D & more general mesh movement.
- UK project in preparation: flood control coupled to ensemble predictions of local precipitation and water flow in uplands & river systems.

References

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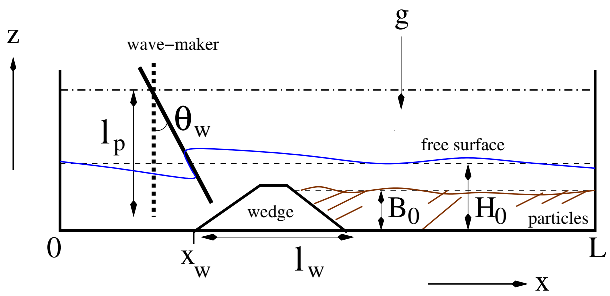
Ad. 2a. Maximize Shingle Beach Formation

Hele-Shaw beach dynamics

- Sketch set-up Hele-Shaw cell:



- All breaking *wave types* observed.
- *Beaches, berms & sand bars* emerge in min/hr by waves with wave forcing $T \approx 1$s.

Ad. 2b. Smooth Case 9: No Rogue Wave

Take $h_0 = 0.43$m instead of $h_0 = 0.41$m, same $h_1 = 0.9$m:

Smooth Case 9: No Rogue Wave

Smooth Case 9: No Rogue Wave

Smooth Case 9: No Rogue Wave

Smooth Case 9: No Rogue Wave

Smooth Case 9: No Rogue Wave

Smooth Case 9: No Rogue Wave

Smooth Case 9: No Rogue Wave

Ad. 2c. Bore-Soliton-Splash with KP?

- Kadomtsev-Petviashvili equation is unidirectional dispersive wave equation with weak lateral or y–dependence (Kodama 2010):

$$\partial_x\left(\underline{\partial_t\eta + \frac{3}{2}\eta\partial_x\eta + \frac{1}{6}\partial_x^3\eta}\right) + \frac{1}{2}\partial_y^2\eta = 0 \qquad (5)$$

- Free surface deviation $\eta(x, t)$ (scaled).

Bore-Soliton-Splash with KP?

- When 2 KdV sech^2–solitons with $A = 2c$ & phase speed c:

$$\eta(x,t) = 2c\,\text{sech}^2\left(\sqrt{3c/2}(x - x_0 - ct)\right) \qquad (6)$$

approach each other under an angle 2θ (or one soliton grazes a wall with slant θ): then there is an angle θ_{max} for which $A_{max} = 4$ (Yeh & Kodama JFM 2011):