"Symplectic" Time Discontinuous Galerkin Discretizations

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Motivation

- Obtain stable schemes for non-autonomous systems (i.e. water waves generated by a wave maker).
- Derive new geometric time stepping schemes.

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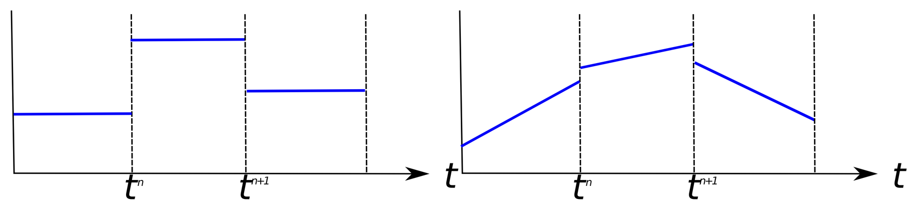
Idea

- Discretize variational principle in time.
- Order of approximations defines order of scheme.
- Flux inspired by nonconservative products theory [2].



Piecewise continuous approximation in time.

Variational Principle

The dynamics of harmonic oscillator, e.g. pendulum, is embedded in the functional

$$L(p,q) := \int_0^T \left(p\frac{\mathrm{d}\,q}{\mathrm{d}\,t} - H(p,q)\right)\mathrm{d}t$$

- angle $q(t)$;
- momentum $p(t)$;
- Hamiltonian $H(p,q)$, total energy of the system;
- for pendulum $H(p,q) := \omega^2(1-\cos(p)) + q^2/2$.

Applying the variational principle $\delta L(p,q) = 0$, integrating by parts, using the end point conditions $\delta q(0) = \delta q(T) = 0$ and the arbitrariness of the variations; the dynamics of harmonic oscillator emerges as follows:

$$\frac{\mathrm{d}\,q}{\mathrm{d}\,t} - \frac{\delta H}{\delta p} = 0 \quad \text{and} \quad \frac{\mathrm{d}\,p}{\mathrm{d}\,t} + \frac{\delta H}{\delta q} = 0$$

with initial conditions $p(t=0) = p_0$ and $q(t=0) = q_0$.

Discrete Variational Principle

- Divide the time domain into N finite time intervals $I_n = [t_n, t_{n+1}]$, $n = 0, \dots, N-1$.
- Each time interval I_n is then related to a fixed interval $\hat{I} = \zeta \in [-1,1]$ through the mapping F_n defined as

$$\hat{I} \to I_n : \zeta \mapsto t = \frac{1}{2}\Big(t_n(1-\zeta) + t_{n+1}(1+\zeta)\Big).$$

- Functions (p,q) are approximated as polynomial expansions (p^τ, q^τ) per time interval I_n.
- (p,q) are discontinuous across the time edges, the following jump $[[\cdot]]$ and average $\{\{\cdot\}\}_{\alpha,\beta}$ operators are defined

$$[[f]]\Big|_{t_{n+1}} = f^{n+1,-} - f^{n+1,+},$$
$$\{\{f\}\}\Big|_{t_{n+1}} = \alpha f^{n+1,-} + \beta f^{n+1,+},$$

(α,β) are the arbitrary, $\alpha+\beta = 1$; $f^{n+1,+} := \lim_{\epsilon\downarrow 0} f(t_{n+1}+\epsilon)$ and $\lim_{\epsilon\downarrow 0} f^{n+1,-} := f(t_{n+1}-\epsilon)$.

The discrete functional for the harmonic oscillator, analogous to continuous VP, can now defined as

$$L^\tau(p^\tau,q^\tau) := \sum_{n=0}^{N-1}\int_{t_n}^{t_{n+1}}\left(p^\tau\frac{\mathrm{d}\,q^\tau}{\mathrm{d}\,t} - H(p^\tau,q^\tau)\right)\mathrm{d}t - \sum_{n=-1}^{N-1}[[q^\tau]]\{\{p^\tau\}\}_\alpha^\beta\Big|_{t_{n+1}}.$$

References

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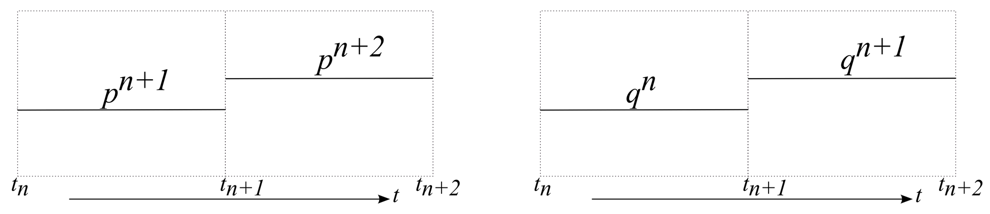
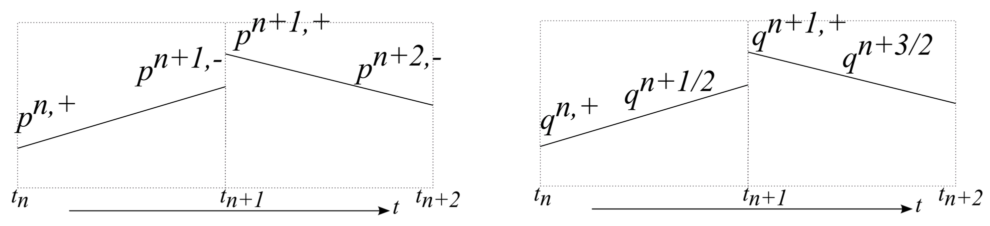
Known schemes

Symplectic Euler



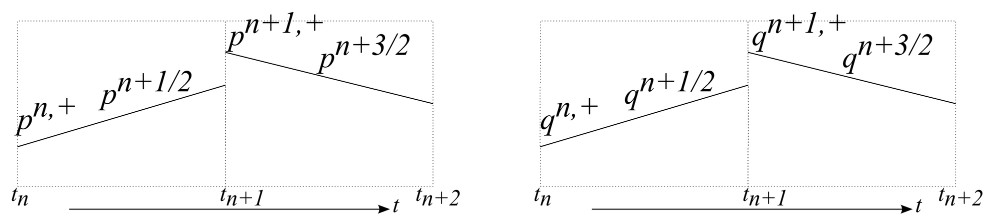
Piecewise constant approximation in time.

Störmer-Verlet



Piecewise linear approximation in time.

Modified symplectic midpoint

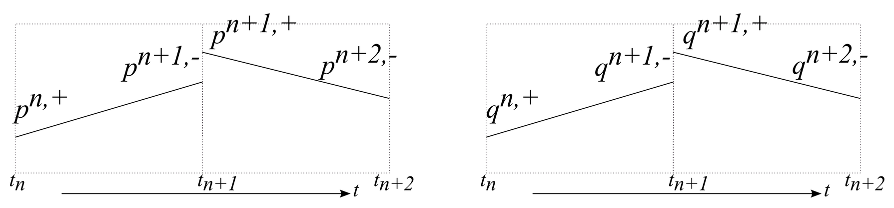


Piecewise linear approximation in time.

New explicit scheme

- Approximations:

$$p^\tau = p^{n,+}(1-\zeta)/2 + p^{n+1,-}(1+\zeta)/2,$$
$$q^\tau = q^{n,+}(1-\zeta)/2 + q^{n+1,-}(1+\zeta)/2;$$



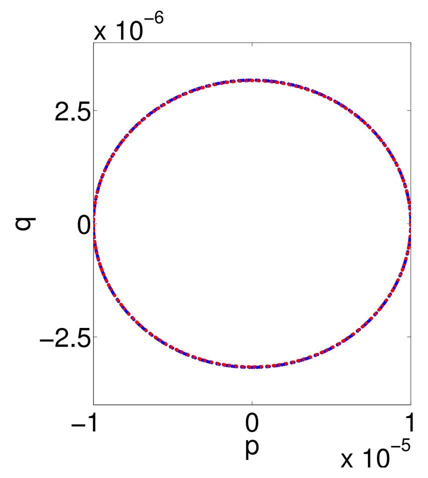
Piecewise linear approximation in time.

- Quadrature:

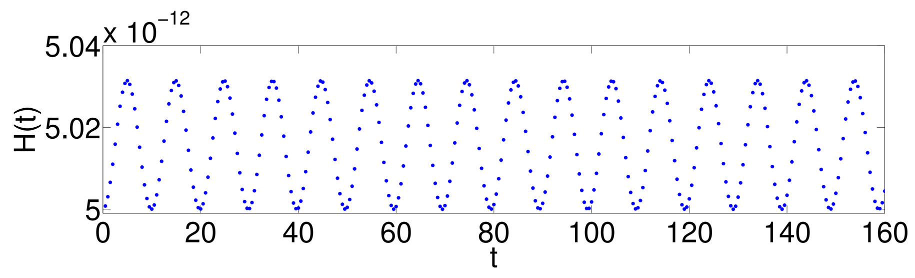
$$\int_{t_n}^{t_{n+1}} H(p^\tau,q^\tau)\mathrm{d}t = \frac{\Delta t}{2}\Big(H(p^{n,+},q^{n,+}) + H(p^{n+1,-},q^{n+1,-})\Big);$$

- Scheme:

$$q^{n+1,-} = q^{n,-} + \Delta t\frac{\partial H(p^{n,+},q^{n,+})}{\partial p^{n,+}},$$
$$p^{n+1,-} = p^{n,-} - \Delta t\frac{\partial H(p^{n,+},q^{n,+})}{\partial q^{n,+}},$$
$$q^{n+1,+} = q^{n,+} + \Delta t\frac{\partial H(p^{n+1,-},q^{n+1,-})}{\partial p^{n+1,-}},$$
$$p^{n+1,+} = p^{n,+} - \Delta t\frac{\partial H(p^{n+1,-},q^{n+1,-})}{\partial q^{n+1,-}}.$$



a) Phase space conservation in (theory) and (numerically).



b) Energy conservation.

Properties:

- Second order;
- Fully explicit;
- "Symplectic" – follows from spectral analysis in extended sense;
- Same linear stability requirement as for Störmer-Verlet scheme – following a special analysis;
- Same dispersion error as for Störmer-Verlet scheme – following a special analysis.

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Damped oscillator

- Continuous VP, with energy $H(p,q) = \frac{1}{2}p^2 + \frac{1}{2}q^2$

$$0 = \delta\int_0^T \tilde{L}(p,q,t)\mathrm{d}t = \delta\int_0^T\left(p\frac{\mathrm{d}\,q}{\mathrm{d}\,t} - H(p,q)\right)e^{\gamma t}\mathrm{d}t$$

- Equations

$$\delta(pe^{\gamma t}): \quad \frac{\mathrm{d}\,q}{\mathrm{d}\,t} = \frac{\partial H}{\partial p} = p,$$
$$\delta q: \quad \frac{\mathrm{d}\,p}{\mathrm{d}\,t} + \gamma p = -\frac{\partial H}{\partial q} = -q.$$

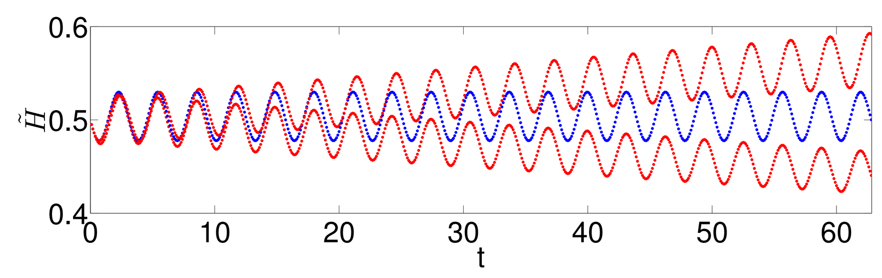
- Define a coordinate transformation $q = Q\exp(-\gamma t/2)$ and $p = P\exp(-\gamma t/2)$ such that the VP becomes

$$0 = \delta\int_0^T \tilde{L}(P,Q,t)\mathrm{d}t = \delta\int_0^T\left(P\frac{\mathrm{d}\,Q}{\mathrm{d}\,t} - \tilde{H}(P,Q,t)\right)\mathrm{d}t$$

with modified Hamiltonian $\tilde{H}(P,Q,t) = \frac{P^2}{2} + \frac{Q^2}{2} + \frac{\gamma PQ}{2}$.

- The scheme

$$p^{n+1} = p^n - \Delta t\frac{\partial H(p^{n+1},q^n)}{\partial q^n} - (1-e^{-\gamma\Delta t})p^n,$$
$$q^{n+1} = q^n + \Delta t\frac{\partial H(p^{n+1},q^n)}{\partial p^{n+1}}.$$



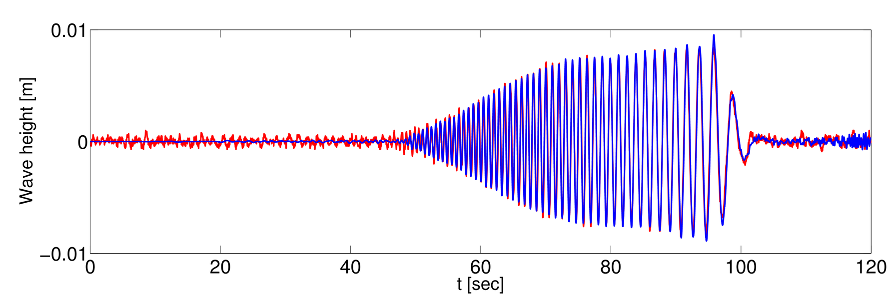
Modified total energy $\tilde{H}$. Our approximation versus other approximations with straightforward implementation of γ-term.

Water waves in a basin

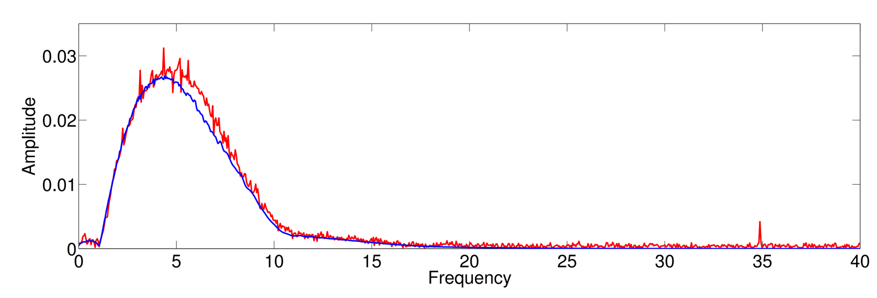
For Ω with solid wall boundaries and a flat bottom, consider the variational principle from [5]:

$$\mathcal{L}(\phi,\eta,\phi_s) := \int_0^T\left(\int_{r(t)}^L \phi_s\frac{\partial\eta}{\partial t}\mathrm{d}x - \int_{r(t)}^L\frac{1}{2}g(\eta^2 - H^2)\mathrm{d}x - \int_{-H(x)}^{\eta(x,t)}\int_{r(t)}^L\frac{1}{2}|\nabla\phi|^2\mathrm{d}z\mathrm{d}x - \int_{-H(r(t))}^{\eta(x,t)}\frac{\mathrm{d}\,r}{\mathrm{d}\,t}\phi_w\mathrm{d}z\right)\mathrm{d}t,$$

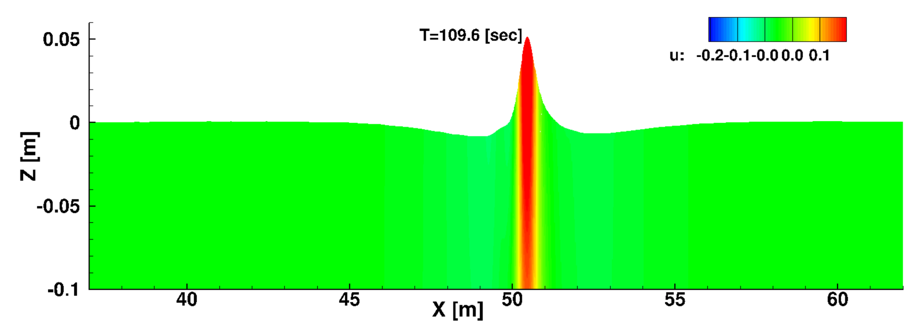
- free surface $z = \eta(x,t)$; topography $z = -H(x)$;
- velocity potential $\phi = \phi(x,z,t)$; $\phi_s = \phi(x,z=h,t)$;
- wave maker movement $x = r(t)$.



a) Comparison of wave heights in experiments (red) and numerical simulations (blue) for a wave probe $x = 20$ [3].



b) Spectra of wave heights in experiments (red) and numerical simulations (blue) for a wave probe $x = 50$ [3].



c) Nonlinear potential flow waves produced by a wavemaker. Test case 202002 with flat bottom and a splashing wave [3].

Discussion

- A new variational approach to derive time discretization schemes was explored, which resulted into known schemes for Hamiltonian systems.
- Stable schemes for non-autonomous systems are also obtained in a systematic way.
- New time integration schemes are obtained using a novel techniques to analyse the time-discontinuous schemes [4].
- Future plans include higher order time integration methods.