Numerical Modelling of Deep and Shallow Water Waves with Variational Discontinuous Finite Element Methods

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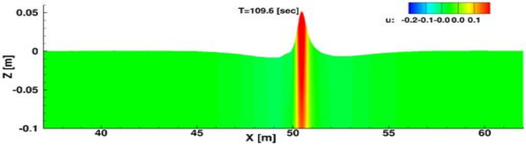

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1. Introduction

Some hydrodynamics ... rogue waves are anomalously high waves defined relative to a significant wave height H_s.

- Index (Khariff et al. '09, Dysthe et al. '08):

$$AI = H_{rw}/H_s > 2 \qquad \text{or} \qquad AI = \eta_{rw}/H_s > 1.25 \qquad (1)$$

- Relevance in maritime & coastal engineering —ship design & safety offshore structures
- Pyramidal rogue wave (Faulkner 2001):

Fig.1. Pyramidal wave off south Japan

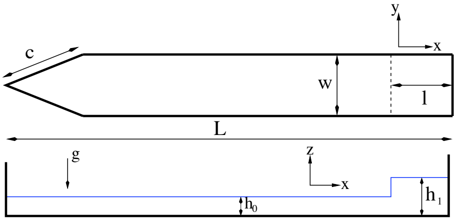
Introduction

Causes rogue waves, e.g., Khariff et al. (2009) & Faulkner (2001, 2003):

- spatial wave focussing due to refraction or wave caustics, or wave focussing in coastal convergences
- crossing seas, nearly standing waves with pyramidal waves
- emerging envelope solitons in wave train.

2. Bore Soliton Splash

- Water channel: $L = 43.5 \pm 0.25$m, $D = 1.20$m, $w = 2 \pm 0.05$m



- Sluice compartment & gate, lifted with 2.5m/s
- V-shaped or linear convergence at other end with $c = 2.7$m
- Start at rest with water levels $h_0 \in [0.32, 0.47]$m and $h_1 \in [0.67, 1.02]$m.

Bore Soliton Splash

- 7+2 Cases of which 3 repeats (reproducibility):

#	h_0 (m) $\pm$0.01m	h_1 (m) $\pm$0.01m	H_s $\pm$0.05m	H_{rw} $\pm$0.5m	Peak #	Comments
1	0.32	0.67	-	0.6	-	bore
2	0.38	0.74	-	2.5	-	good splash
3	$\underline{0.41}$	$\underline{0.9}$	0.35	3.25	2^{nd}	thin jet cf. 6 & 8
4	0.47	1.0	0.35	1	2^{nd}	bore & low
5	0.41	1.02	0.40	1.5	1^{st}	bore & low
6	$\underline{0.41}$	$\underline{0.9}$	0.35	3.5	2^{nd}	BSS cf. 3 & 8
7	0.45	0.8	0.35	2.5	2^{nd}	good splash
8	$\underline{0.41}$	$\underline{0.9}$	0.35	3.5	2^{nd}	BSS & ++
9	0.43	0.9	0.45	1.8	1^{st}	bubble

Intermezzo: Phenomenology Bore Soliton Splash

- Estimated wave amplitudes $H_s \in [0.35, 0.45]$m
- After opening sluice gates three solitary waves emerge: large, medium and small
- First highest soliton often breaks into spilling breaker
- Variety of outcomes: minor/major reflections, resonances between waves, smooth waves, sheets, pyramidal waves
- $H_{rw} \in [0.6, 3.5]$m.

Intermezzo: Bore-Soliton-Splash

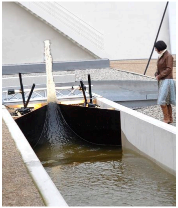
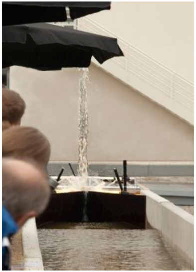
Highest Cases 3, 6 & 8:

- $H_s = 0.35$m & $H_{rw} = [3.25, 3.5, 3.5]$m
- *bore soliton splash*
  Wout Zweers youtube channel:
  https://www.youtube.com/watch?v=YSXsXNX4zW0
- Truly rogish:

$$AI = \frac{H_{rw}}{H_s} = \frac{3.5}{0.35} \approx 10.$$

Model Rogues Waves such as Bore-Soliton-Splash

- Goal: simulate rogue waves accurately.

a)  b)  c) 

3. Space-plus-Time Finite Element Water Waves

Key geometric issues:

- in the spatial discretization of Miles' variational principle for dispersive nonlinear water waves, and
- in the temporal discretization of the non-autonomous spatially discrete variational principle.

Verification and validation against measurements of nonlinear waves driven by a piston wave maker.

Space-plus-Time Finite Element Water Waves

Consider the (2D) potential flow water wave problem:

- gravity as restoring force with acceleration g
- velocity $\mathbf{u} = (u, w)^T = \nabla\phi$ with velocity potential $\phi = \phi(x, z, t)$
- domain $\Omega(t)$ with bottom S_B, wall $S_L : x = L$ and piston wave maker at $S_W : x = R(t)$:

$$\Omega(t) : R(t) < x < L \quad \text{and} \quad -H(x) < z < \eta(x, t) \qquad (2)$$

- free surface variables $\eta(x, t)$ & $\phi_s = \phi(x, z = \eta(x, t), t)$
- kinematic single-valued free surface $z = \eta(x, t)$:

$$\partial_t\eta + \partial_x\phi\partial_x\eta = \partial_z\phi \quad \text{at} \quad z = \eta(x, t) \qquad (3)$$

- dynamic free surface condition: continuity of pressure.

Miles' Variational Principle

- Miles 1977 (see also Cotter & B. 2010):

$$
\begin{aligned}
0 &= \delta \int_0^T \mathcal{L}[\phi, \phi_s, \eta, t] dt \\
&= \delta \int_0^T \phi_s \partial_t \eta - \mathcal{H}[\phi, \phi_s, \eta, t] dt \\
&= \delta \int_0^T \int_{R(t)}^L \phi_s \partial_t \eta - \frac{1}{2} g \eta^2 \\
&\quad - \int_{-H(x)}^{\eta(x,t)} \frac{1}{2} |\nabla \phi|^2 dz dx - \int_{-H_w}^{\eta_w} \frac{dR}{dt} \phi_w dz dt
\end{aligned}
$$

- piston wave-maker $R(t)$ with $\phi_w \equiv \phi(R(t), z, t)$, $H_w = H(R(t))$, $\eta_w = \eta(R(t), t)$
- non-autonomous due to piston wave-maker
- canonical Hamiltonian system.

Miles' Variational Principle

Variations yield dynamic/kinematic conditions and Laplace's equation, BC's:

$$
\begin{aligned}
\delta\eta(x,t) : \partial_t\phi_s + \frac{1}{2}(\partial_x\phi)^2 + g\eta - \frac{1}{2}(\partial_z\phi)_s^2(1+(\partial_x\eta)^2) &= 0\\
\delta\phi_s(x,t) : \partial_t\eta + \partial_x\phi_s\partial_x\eta - (\partial_z\phi)_s(1+(\partial_x\eta)^2) &= 0\\
\delta\phi(x,z,t) : \nabla^2\phi &= 0\\
\delta\phi_w : dR/dt - (\partial_x\phi)_w &= 0\\
\delta\phi_{B,L} : \mathbf{n}\cdot\boldsymbol{\nabla}\phi &= 0
\end{aligned}
$$

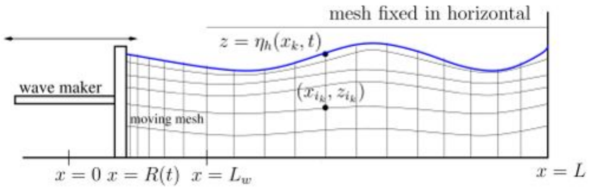
Geometric Wave Modeling: Space FEM

FEM formulation of Miles' variational principle.

- FEM test/basis functions $\tilde{\varphi}_j(x,z,t), \varphi_k(x,t)$ with i,j in Ω
- Indices k, l, r at free surface, i', j' in interior & $m, \tilde{m}$ at wave maker
- Substitute finite element expansions directly into VP:

$$
\begin{aligned}
\phi_h(x,z,t) &= \phi_j(t)\tilde{\varphi}_j(x,z,t) \\
\phi_{sh}(x,t) &= \phi_l(t)\varphi_l(x,t) \\
\eta_h(x,t) &= \eta_k(t)\varphi_k(x,t)
\end{aligned}
$$

Space FEM: Key Issues



- Moving mesh:
- Issue 1: realize that variations of interior nodes of mesh are (dynamically) coupled to variations of free surface nodes.
- Issue 2: find robust (variational) time integrators for (non-)autonomous system.
- Otherwise: nonlinearly unstable discretization (cf. DGFEM in 2007).

Geometric Wave Modeling: Space FEM

- The resulting spatially discrete non-autonomous VP:

$$
\begin{aligned}
0 &= \delta \int_0^T L[\phi, \tilde{\varphi}, \boldsymbol{\eta}, t] dt \\
0 &= \delta \int_0^T M_{kl}\phi_k \frac{d\eta_l}{dt} - H[\phi, \tilde{\varphi}, \boldsymbol{\eta}, t] dt \\
&= \delta \int_0^T M_{kl}\phi_k \frac{d\eta_l}{dt} - D_{kl}\phi_k\eta_l - \\
&\quad -\frac{1}{2} g M_{kl}\eta_k\eta_l - \frac{1}{2} A_{ij}\phi_i\phi_j - W_m\phi_m dt.
\end{aligned}
$$

- $M_{kl}, D_{kl}, A_{ij}, W_m$ depend on $\{\eta_k(t), t\}$: mesh and wave maker movements.

Geometric Wave Modeling: Space FEM

- The matrices and vectors depend on $\{\eta_k(t), t\}$ as follows:

$$
\begin{aligned}
M_{kl}(t) &= \int_{R(t)}^{L} \varphi_k \varphi_l dx \\
D_{kl}(t) &= -\int_{R(t)}^{L} \frac{\partial \varphi_k}{\partial t} \varphi_l dx, \\
A_{ij}(\boldsymbol{\eta}, t) &= \int_{\Omega_h} (\nabla \tilde{\varphi}_i \cdot \nabla \tilde{\varphi}_j) dxdz \\
W_m(\eta_1, t) &= \int_{-H_w}^{\eta_1} \frac{\mathrm{d}R}{\mathrm{d}t} \tilde{\varphi}_m \big|_{x=R(t)} dz,
\end{aligned}
$$

- due to wave maker via $R(t)$ & mesh movement via $\eta(x, t)$.

- Issue 1: variation yields

$$
\begin{aligned}
\delta\phi_k : M_{kl}\frac{d\eta_l}{dt} - D_{kl}\eta_l - A_{ik}\phi_i - W_1\delta_{1k} &= 0\\
\delta\eta_l : \frac{d(M_{kl}\phi_k)}{dt} + D_{kl}\phi_k + M_{kl}\eta_l + \frac{1}{2}\frac{\partial A_{ij}}{\partial\eta_l}\phi_i\phi_j & \\
+\frac{\partial W_m}{\partial\eta_1}\delta_{1l}\phi_m &= 0\\
\delta\phi_{j'} : A_{ij'}\phi_i + W_{m'}\delta_{m'j'} &= 0
\end{aligned}
$$

- Hence, we can eliminate interior degree of freedom:

$$
\begin{aligned}
A_{i'j'}\phi_{i'} &= -A_{lj'}\phi_l + W_{m'}\delta_{m'j'} \Longrightarrow\\
\phi_{i'} &= -A_{lj'}A^{-1}_{j'i'}\phi_l - W_{m'}A^{-1}_{m'i'}.
\end{aligned}
$$

Geometric Wave Modeling: Space FEM

- Eliminate $\phi_{j'}$: spatially discrete VP at free surface:

$$
\begin{aligned}
0 &= \delta \int_0^T L[\phi, \boldsymbol{\eta}, t] dt \\
0 &= \delta \int_0^T M_{kl} \phi_k \frac{d\eta_l}{dt} - H[\phi, \boldsymbol{\eta}, t] dt \\
&= \delta \int_0^T M_{kl} \phi_k \frac{d\eta_l}{dt} - D_{kl} \phi_k \eta_l - \\
&\quad -\frac{1}{2} g \eta_k \eta_l - \frac{1}{2} B_{kl} \phi_k \phi_l - C_l \phi_l - F_w dt.
\end{aligned}
$$

- Cf. discrete Boundary Element Method
- Function $F_w(t) = -\frac{1}{2} W_{\tilde{m}'} A^{-1}_{\tilde{m}' m'} W_{m'}$ & Schur complement

$$
\begin{aligned}
B_{kl}(\boldsymbol{\eta}, t) &= (A_{kl} - A_{ki'} A^{-1}_{i'j'} A_{j'l}) \\
C_l(\boldsymbol{\eta}, t) &= W_1 \delta_{1l} - A_{lj'} A^{-1}_{j'm'} W_{m'}.
\end{aligned}
$$

Geometric Wave Modeling: Time FEM

- Substitute a discontinuous Galerkin finite element expansion into the VP in each time slab $[t_n, t_{n+1}]$:

$$\begin{aligned} \eta_l &= \eta_l^{n,+}\left(\frac{t_{n+1}-t}{\Delta t_n}\right) + \eta_l^{n+1,-}\left(\frac{t-t_n}{\Delta t_n}\right) \\ \phi_k &= \phi_k^{n+\frac{1}{2}}\frac{2(t-t_n)}{\Delta t_n} + \phi_k^{n,+}\left(\frac{t_n+t_{n+1}-2t}{\Delta t_n}\right). \end{aligned}$$

- Define the delta function due to $M_{kl}\phi_k \mathrm{d}\eta_l/dt$–term by splitting node n in a narrow element with C^0-connection, and calculating its contribution in the limit of zero width:

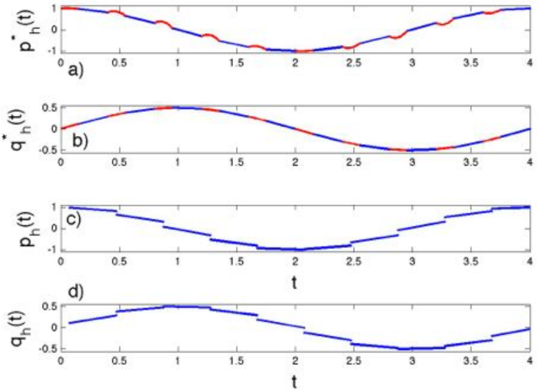
$$\begin{aligned} p = M_{kl}\phi_k &= p_L + (p_R - p_L)(4\tau - 3\tau^2) \\ q = \eta_l &= q_L + (q_R - q_L)\tau \end{aligned}$$

$$\int_0^1 p\frac{dq}{d\tau}d\tau = p_R(q_R - q_L) = M_{kl}^{n+1,+}\phi_k^{n+1,+}(\eta_l^{n+1,+} - \eta_l^{n+1,-}).$$

Limit Continuous to Discontinuous Galerkin FEM

- The integral over $p_h \mathrm{d}q_h/\mathrm{d}t$ over an even element becomes

$$\int_0^1 p_h \frac{\mathrm{d}q_h}{\mathrm{d}\tau}\mathrm{d}\tau = p_R^*(q_R^* - q_L^*). \tag{4}$$



Geometric Wave Modeling: Time FEM

- Issue 2: algebraic VP with "extended Hamiltonian" H:

$$0 = \delta \sum_{n=0}^{N} L[\boldsymbol{\phi}^{n,+}, \boldsymbol{\phi}^{n+1/2}, \boldsymbol{\eta}^{n,+}, \boldsymbol{\eta}^{n+1,-}]dt$$

$$= \delta \sum_{n=0}^{N} M_{kl}^{n+1/2} \phi_k^{n+1/2} (\eta_l^{n+1,-} - \eta_l^{n,+})$$

$$- \frac{\Delta t}{2} \left(H^{n+1/2}[\boldsymbol{\phi}^{n+1/2}, \boldsymbol{\eta}^{n,+}] + H^{n+1/2}[\boldsymbol{\phi}^{n+1/2}, \boldsymbol{\eta}^{n+1,-}] \right)$$

$$+ \delta \sum_{n=-1}^{N} M_{kl}^{n+1,+} \phi_k^{n+1,+} (\eta_l^{n+1,+} - \eta_l^{n+1,-})$$

Intermezzo: Time FEM

- Variations (evaluation $\dagger = n + 1/2$):

$$\delta\phi_k^{n+1,+} : \eta_l^{n+1,+} = \eta_l^{n+1,-} \quad \text{continuous!}$$

$$\delta\eta_l^{n,+} : M_{kl}^{n+1/2}\phi_k^{n+1/2} = M_{kl}^{n}\phi_k^{n,+} - \frac{1}{2}\Delta t \frac{\partial H^{n+1/2}[\boldsymbol{\phi}^{n+1/2}, \boldsymbol{\eta}^{n,+}]}{\partial \eta_l^{n,+}}$$

$$\delta\phi_k^{n+1/2} : M_{kl}^{n+1/2}\eta_l^{n+1,-} = M_{kl}^{n+1/2}\eta_l^{n,+} + \frac{1}{2}\Delta t \left(\frac{\partial H^{\dagger}[\boldsymbol{\phi}^{n+1/2}, \boldsymbol{\eta}^{n,+}]}{\partial \phi_k^{n+1/2}} + \frac{\partial H^{\dagger}[\boldsymbol{\phi}^{n+1/2}, \boldsymbol{\eta}^{n+1,-}]}{\partial \phi_k^{n+1/2}} \right)$$

$$\delta\eta_l^{n+1,-} : M_{kl}^{n+1,+}\phi_k^{n+1,+} = M_{kl}^{n+1/2}\phi_k^{n+1/2} - \frac{1}{2}\Delta t \frac{\partial H^{n+1/2}[\boldsymbol{\phi}^{n+1/2}, \boldsymbol{\eta}^{n+1,-}]}{\partial \eta_l^{n+1,-}}$$

Geometric Wave Modeling: Unfolding

- Reduces to 2nd-order symplectic Störmer-Verlet time integrator in the autonomous case
- Reduces to 2nd-order symplectic Störmer-Verlet time integrator "Kamiltonian" non- autonomous case
- "Unfold" the kinetic energy term again to avoid direct inversion of the matrix $A_{i'j'}$
- This defines the nature of the time discretization of $\phi_{i'}$ in the interior
- Use Newton iteration and Petc linear algebra routines.

Intermezzo: Unfolding

- "Unfold" the kinetic energy term again:

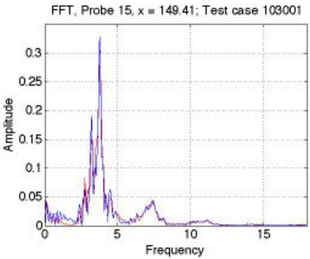
$$\sum_{n=0}^{N}\frac{1}{2}\Delta t^n\Big(K^\dagger(\tilde{\phi}^{n,+},\phi^\dagger,\boldsymbol{\eta}^{n,+})+K^\dagger(\tilde{\phi}^{n+1,-},\phi^\dagger,\boldsymbol{\eta}^{n+1,-})\Big)\equiv$$

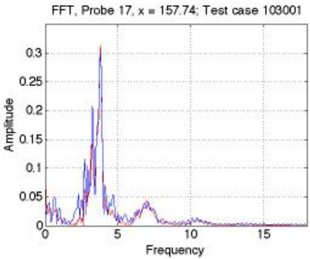
$$\sum_{n=0}^{N}\frac{\Delta t^n}{4}\Big(B^\dagger_{kl}(\boldsymbol{\eta}^{n,+})+B^\dagger_{kl}(\boldsymbol{\eta}^{n+1,-})\Big)\phi_k^{n+1/2}\phi_l^\dagger=$$

$$\sum_{n=0}^{N}\frac{\Delta t^n}{4}\Big(A^\dagger_{kl}(\boldsymbol{\eta}^{n,+})\phi_k^\dagger\phi_l^\dagger+A^\dagger_{kl}(\boldsymbol{\eta}^{n+1,-})\phi_k^\dagger\phi_l^\dagger$$
$$+A^\dagger_{kj'}(\boldsymbol{\eta}^{n,+})\phi_k^\dagger\phi_{j'}^{n,+}+A^\dagger_{kj'}(\boldsymbol{\eta}^{n+1,-})\phi_k^\dagger\phi_{j'}^{n+1,-}$$
$$+A^\dagger_{i'l}(\boldsymbol{\eta}^{n,+})\phi_{i'}^{n,+}\phi_l^\dagger+A^\dagger_{i'l}(\boldsymbol{\eta}^{n+1,-})\phi_{i'}^{n+1,-}\phi_l^\dagger$$
$$+A^\dagger_{i'j'}(\boldsymbol{\eta}^{n,+})\phi_{i'}^{n,+}\phi_{j'}^{n,+}+A^\dagger_{i'j'}(\boldsymbol{\eta}^{n+1,-})\phi_{i'}^{n+1,-}\phi_{j'}^{n+1,-}\Big)$$

Geometric Wave Modeling: Validation

- *Driven wave focusing* in MARIN's wave tank `http://www1.maths.leeds.ac.uk/~obokhove/202002_zoom_splash_2.avi`
- Entire *wave tank* MARIN with false vertical wall instead of *beach* (Kristina, B. Van Groesen 2014 run-up tsunami's).
- Comparison with measured data MARIN is good:





4. Conclusion

- Posed simulation challenges: rogue waves.
- Key issues in variational nonlinear water wave modelling.
- Discontinuous Galerkin variational time approach generalizes to new pseudo-symplectic integrators.
- All extends to 3D & more general mesh movement.
- Extend numerics to hybrid geometric model with potential flow limit & depth-averaged shallow water limit (Cotter & Bokhove 2010, Gagarina et al. 2013).
- Extend numerics to mixture theoretic 2-phase model with intermittent breaking waves, & geometric potential flow model as limit (Bokhove 2014).
- . . . *break through* experiment.
- Funding: NWO, STW & SFFF.

Bore Soliton Splash & Tohoku Tsunami

- "Critical run-up factors and impact . . . Tohoku Tsunami" by Lekkas et al. 2011.
- Vertical run-up Tohoku tsunami highest in Onagawa Bay: 42m for a rough estimate of 7.5m high incoming waves.
- Ratio run-up/wave height: $AI \approx 5.5$ (shallow water rogue waves Nikolkina & Didenkulova 2011).
- Recall BSS/incoming wave height is $AI \approx 10$.

References

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Kamiltonian: Time FEM

- Reformulate non-autonomous Hamiltonian system; change $t = \tilde{t} \to \tau$ & then make $\tau = \tau(t)$:

$$
\begin{aligned}
0 &= \delta \int_0^T M_{kl}(\tau)\phi_k \frac{d\eta_l}{d\tau} - H[\phi, \boldsymbol{\eta}, \tau] d\tau \Longrightarrow \\
0 &= \delta \int_0^T M_{kl}(\tau)\phi_k \frac{d\eta_l}{dt} + p\frac{d\tau}{dt} - H[\phi, \boldsymbol{\eta}, \tau] - p dt
\end{aligned}
$$

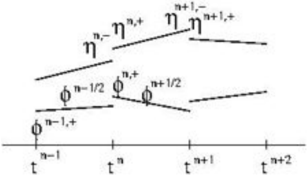
- Dynamics in new conjugate variables $\{M\phi, \boldsymbol{\eta}\}, \{p, \tau\}$ with autonomous Kamiltonian $K(\phi, \boldsymbol{\eta}, \tau, p) = H(\phi, \boldsymbol{\eta}, \tau) + p$:

$$
\begin{aligned}
\delta p : d\tau/dt &= \partial K/\partial p = 1, \quad \delta\tau : \ldots \\
\delta \phi_k : M_{kl} d\eta_l/dt &= \partial H/\partial \phi_k \\
\delta \eta_l : d(M_{kl}\phi_k)/dt &= -\partial H/\partial \eta_l
\end{aligned}
$$

- Equation for p decoupled from rest.

Intermezzo: Time FEM

- Introduce time slabs & expand variables discontinuously:



- Use in VP with jump terms at time nodes:

$$0 = \delta \sum_{n=0}^{N} \int_{t^n}^{t^{n+1}} M_{kl}(\tau)\phi_k \frac{d\eta_l}{dt} + p\frac{d\tau}{dt} - H[\phi, \boldsymbol{\eta}, \tau] - p\ dt +$$

$$\sum_{n=-1}^{N} M_{kl}(\tau^{n+1,+})(\eta_l^{n+1,+} - \eta_l^{n+1,-}) + p^{n+1,+}(\tau^{n+1,+} - \tau^{n+1,-})$$

Intermezzo: Time FEM

- Use quadrature to get algebraic VP:

$$
\begin{aligned}
0 = \delta \sum_{n=0}^{N} M_{kl}^{n+1/2}\phi_k^{n+1/2}(\eta_l^{n+1,-} - \eta_l^{n,+})\\
+p^{n+1/2}(\tau^{n+1,-} - \tau^{n,+})\\
-\frac{\Delta t}{2}\left(H[\phi^{n+1/2}, \boldsymbol{\eta}^{n,+}, \tau^{*}] + H[\phi^{n+1/2}, \boldsymbol{\eta}^{n+1,-}, \tau^{**}]\right)\\
-\Delta t p^{n+1/2} + \delta \sum_{n=-1}^{N} \left(M_{kl}^{n+1,+}\phi_k^{n+1,+}(\eta_l^{n+1,+} - \eta_l^{n+1,-})\right.\\
\left. +p^{n+1,+}(\tau^{n+1,+} - \tau^{n+1,-1})\right)
\end{aligned}
$$

- Time τ becomes continuous: $\delta p^{n+1,+} : \tau^{n+1,+} = \tau^{n+1,-}$!
- Previously used: $\tau^* = \tau^{**} = \tau^{n+1/2}$.
  Alternative: $\tau^* = \tau^{n,+}, \tau^{**} = \tau^{n+1,-}$.

Time DGFEM: Stormer-Verlet

- Discrete Lagrangian ($\gamma = 0$):

$$
\begin{aligned}
L_h(p_h, q_h) = &\sum_{n=-1}^{N} (q^{n+1} + q^n - 2q^{n+1/2}) p_r^{n+1} \\
&+ \sum_{n=0}^{N-1} (p_l^{n+1} + p_r^{n+1})(q^{n+1/2} - q^n) \\
&- \frac{\Delta t_n}{2} \big(H(p_l^{n+1}, q^{n+1/2}) + H(p_r^{n+1}, q^{n+1/2}) \big)
\end{aligned}
$$

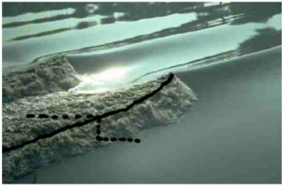
- Stormer-Verlet with $p_l^{n+1} = p_r^n$ continuous & q remains DG:

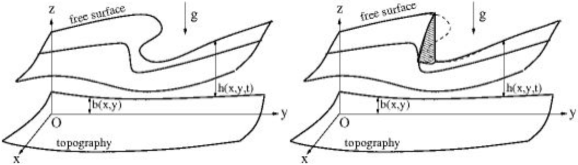
$$
\begin{aligned}
\delta p_l^{n+1} &: q^{n+1/2} = q^n + \Delta t p_l^{n+1}/2 \\
\delta q^{n+1/2} &: p_r^{n+1} = p_l^{n+1} - \frac{\Delta t}{2} \big((q^{n+1/2})^3 + (q^{n+1/2})^3 \big) \\
\delta p_r^{n+1} &: q^{n+1} = q^{n+1/2} + \Delta t p_r^{n+1}/2
\end{aligned}
$$

Bores in Boussinesq Models

Derived a new water wave model (Cotter & B. 2010, Gagarina et al. 2013c):

- full water wave dispersion & potential flow as limit
- depth-averaged shallow water equations as other limit
- bore relations (if any) derived variationally (Clebsch):





New Water Wave Model

- Dynamics of surface variables $\mathbf{u}^*(x,y,t), h(x,y,t)$:

$$\partial_t h + \nabla \cdot \left(h\bar{\mathbf{u}}\right) = 0 \qquad (5a)$$

$$\partial_t \mathbf{u}^* + \nabla B + q h \bar{\mathbf{u}}^\perp = 0, \qquad (5b)$$

where $h\bar{\mathbf{u}} = h\mathbf{u}^* + \underline{\int_b^{b+h} \boldsymbol{\nabla}_H \varphi \mathrm{d}z}$, PV $q = (\partial_x v^* - \partial_y u^*)/h$ & $B = \frac{1}{2}|\mathbf{u}^*|^2 + g\,(h+b) - \underline{\frac{1}{2}((\partial_z\varphi)_s^2(1+|\boldsymbol{\nabla}_H(h+b)|^2)}$

- Interior dynamics velocity potential difference φ:

$$\nabla^2 \varphi + \nabla \cdot \mathbf{u}^* = 0 \qquad (6)$$

- At the free surface $z = h(x,y,t) + b(x,y)$, the potential is $\varphi = 0$.

Single-Phase Mixture Theory

- Dispersive bores difficult to model (numerically)?
- Hence, consider instead incompressible two-phase (here air & water) system.
- Bulk density $\rho = \rho_l + \rho_g \equiv \rho_{0l}\varphi + \rho_{0g}(1-\varphi)$.
- Bulk velocity $\rho \mathbf{u} = \rho_l \mathbf{u}_l + \rho_g \mathbf{u}_g$.
- Equations of motion with a very simple closure are:

$$
\begin{aligned}
\partial_t \rho_l + \nabla \cdot (\rho_l \mathbf{u}_l) &= 0\\
\rho_l(\partial_t \mathbf{u}_l + \mathbf{u}_l \cdot \nabla \mathbf{u}_l) &= -\varphi \nabla p + \rho_l \mathbf{g} - \rho_l c(\mathbf{u}_l - \mathbf{u})\\
\partial_t \rho_g + \nabla \cdot (\rho_g \mathbf{u}_g) &= 0\\
\rho_g(\partial_t \mathbf{u}_g + \mathbf{u}_g \cdot \nabla \mathbf{u}_g) &= -(1-\varphi)\nabla p + \rho_g \mathbf{g} - \rho_g c(\mathbf{u}_g - \mathbf{u}).
\end{aligned}
$$

Single-Phase Mixture Equations

- Scale species hydrostatically in local pancakes of wave breaking.
- Horizontal velocity equals bulk velocity.
- Vertical velocity has segregation terms:

$$\begin{aligned} w_l - w &= && -K(1-\varphi) \quad \text{and} \\ w_g - w &= && K\varphi/\tilde{\rho} \quad \text{with} \quad K = g(1-\tilde{\rho})/c. \end{aligned} \tag{7}$$

- Density ratio $\tilde{\rho} = \rho_{0g}/\rho_{0l}$.

Single-Phase Mixture Equations

- Now add non-hydrostatic terms to bulk flow only in vertical momentum equation.
- Mixed phases unmix hydrostatically:

$$
\begin{aligned}
\partial_t(\rho \mathbf{u}_H) + \nabla \cdot (\rho \mathbf{u} \mathbf{u}_H + p) &= 0\\
\partial_t(\rho w) + \nabla \cdot (\rho \mathbf{u} w) &= -\partial_z p - \rho g\\
\partial_t \varphi + \nabla \cdot (\mathbf{u}\varphi) - \partial_z(K\varphi(1-\varphi)) &= 0\\
\nabla_H \cdot \mathbf{u}_H + \partial_z(w + K\varphi(1-\varphi)(1-\tilde{\rho})/\tilde{\rho}) &= 0\\
\rho = \rho_{0g}(1-\varphi) + \rho_{0l}\varphi &\\
K = g(1-\tilde{\rho})/c \quad \text{with} \quad \tilde{\rho} &= \rho_{0g}/\rho_{0l}.
\end{aligned}
$$

- When phases separated then **potential-flow limit** is enclosed.