Some questions concerning Rival-Sands for graphs and posets

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Joint work with Marta Fiori Carones, Alberto Marcone and Paul Shafer

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We recall that a poset has width κ, written $w(P) = \kappa$, if κ is minimal such that P does not have antichains of size κ.

Related principles: RSg

RSg is stronger than RT^2_2 (it is equivalent to ACA_0 over RCA_0). But a slight modification of it turns out to be equivalent to RT^2_2.

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- wRSg: for every countable graph G, there exists an infinite set $H \subseteq G$ such that every point of H is adjacent to 0, 1, or infinitely many points of G.
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The proof of Rival and Sands actually yields more than what RSpo states.

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- sRSpo: for every countable poset P of finite width, there is an infinite chain C such that every point p of P is comparable with 0 or *cofinitely* many elements of C.

Even more is true: we do not actually need any bound on the size of the antichains.

- RSpo^+: for every countable poset P without infinite antichains, there is an infinite chain C such that every point p of P is comparable with 0 or cofinitely many elements of C.

proof

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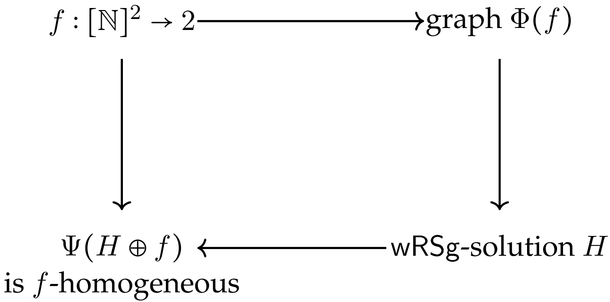

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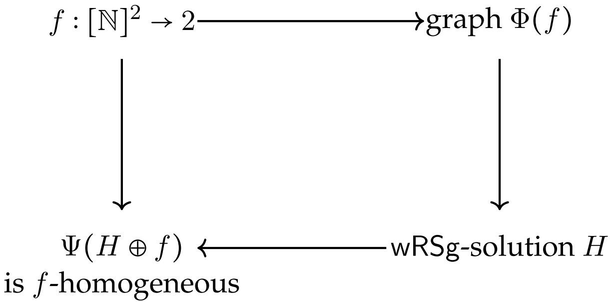

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Weihrauch reducibility

We see principles as (multivalued) functions, mapping instances to (set of) solutions for that instance. For example, RT^2_2 takes as input a binary coloring f of $[\mathbb{N}]^2$ and gives as output an infinite homogeneous set for H.

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Given two principles P and Q, we say that P is *Weihrauch reducible* to Q, and we write $\mathsf{P} \leq_W \mathsf{Q}$, if there are two Turing functionals Φ, Ψ such that, for every instance I_P of P, $\Phi(I_P)$ is an instance of Q such that, for every Q-solution S_Q of Q, $\Psi(S_Q \oplus I_P)$ is a P-solution to I_P.

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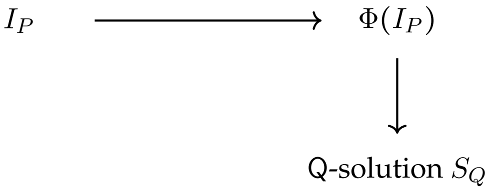

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$I_P \longrightarrow \Phi(I_P)$

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$\Psi(S_Q \oplus I_P)$ is a P-solution $\longleftarrow$ Q-solution S_Q

Comparison with classical principles

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Lemma

CADS $\leq_W$ wRSg, SADS $\nleq_W$ wRSgr.

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$\mathsf{SRT}^2_2 <_W \mathsf{LPO} * \mathsf{wRSgr}$, $\mathsf{SRT}^2_2 <_W (\mathsf{LPO} \times \mathsf{LPO}) * \mathsf{wRSg}$

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Can we do better than this?
Is it true that $\mathsf{wRSgr} \leq_W \mathsf{wRSg}$*?*

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Lemma (Gavalec-Vojtas)

If κ is an infinite regular cardinal and G is a graph such that $|G| = \kappa$, then there is $H \subseteq G$ such that $|H| = \kappa$ and every element g of G is adjacent to 0, 1 or κ many elements of H. Moreover, every $h \in H$ is adjacent to 0 or κ-many elements of H.

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If κ is singular and G is a graph with $|G| = \kappa$, then for every $\lambda < \kappa$ there is $H_\lambda \subseteq G$ with $|H_\lambda| = \kappa$ and every g in G is adjacent to 0, 1 or at least λ many elements of H_λ. *Moreover, every $h \in H$ is adjacent to 0 or at least λ many elements of H.*

Things go quite differently for RSpo:

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Let κ be an infinite regular cardinal, and P be a poset of finite width with $|P| = \kappa$. Then, there is a chain C such that $|C| = \kappa$ and for every $p \in P$, p is comparable with 0 or κ many elements of C.

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If κ is singular, then there is a P of width 3 containing no chain C of size κ such that for every $p \in P$, p is comparable with 0 or κ many elements of C.

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What about sRSpo?

sRSpo: for every countable poset P of finite width, there is an infinite chain C such that every point p of P is comparable with 0 or *cofinitely* many elements of C.

Lemma

If $\kappa > \omega$, then there is a poset P of width 3 such that for every chain C of size κ such that for every $p \in P$, p is comparable with 0 or κ many elements of C there is p_C comparable with κ-many elements of C and non-comparable with κ many elements of C.

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Consider $\kappa \times \omega$.

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Lemma (Gavalec-Vojtas)

If T is a tree such that $|T| = \kappa$, $w(T) = \lambda < \kappa$ such that for every $\nu < \lambda$ $2^\nu < \kappa$, then sRSpo *holds for T.*

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In this framework, RSpo^+ behaves interestingly.

Lemma (Gavalec-Vojtas)

If κ is an infinite regular cardinal, then RSpo^+ holds if and only if there is no κ-Suslin tree.

What happens if we consider countable posets of finite *height* instead of finite width?

Question

Suppose that P is a countable poset such that every chain has size bounded by a certain k. Can we find an infinite antichain A such that every $p \in P$ is comparable with 0, 1 or infinitely many elements of A?

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What about higher cardinalities?

Sketch of a proof of RSpo^+.

RSpo^+: for every countable poset P without infinite antichains, there is an infinite chain C such that every point p of P is comparable with 0 or cofinitely many elements of C.

- We suppose that P contains a chain of order type ω (in case it does not, then it contains a chain of order type ω^*, so we can consider $(P, >_P)$ instead of $(P, <_P)$ and the same proof works).

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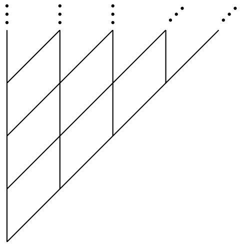
- Let us consider

$$\mathscr{B} := \{B \subseteq P : \forall b, b' \in B \exists c \in B (c >_P b \wedge c >_P b')\},$$

and let $M = \{m_0, m_1, \dots\}$ be $\subseteq$-maximal for $\mathscr{B}$.



- We define the chain $C = \{c_0, c_1, \dots\}$ as

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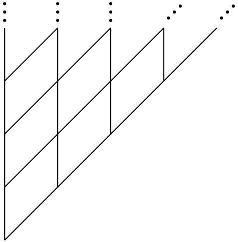

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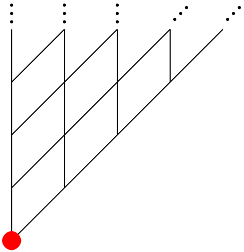

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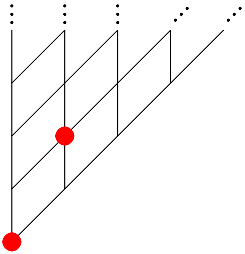

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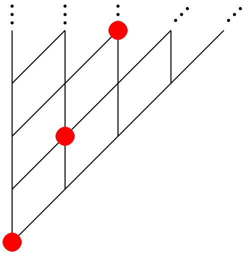

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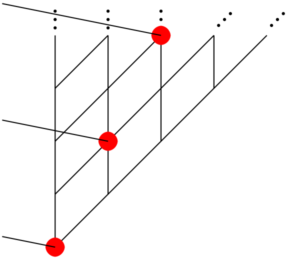

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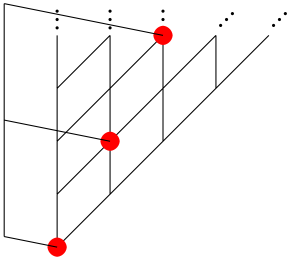

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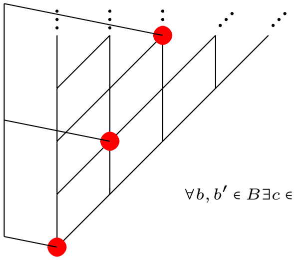

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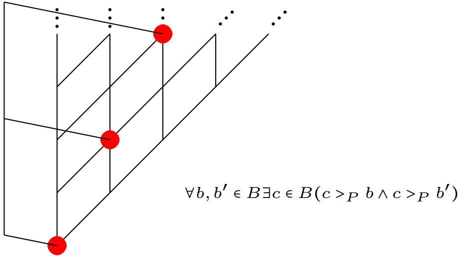
$$c_0 := m_0, \quad c_{i+1} := m_{\min\{j : m_j >_P c_0, \dots, m_j >_P c_i\}}$$

- C is clearly of order type ω and a solution: if it was not, then we could use any counterexample chain to enlarge M.



Question

Is there a "more constructive" proof?

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