The domination monoid in o-minimal theories

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University of Leeds

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Motivation

In [HHM08] to $\mathfrak{U}$ is associated $(\widetilde{\operatorname{Inv}}(\mathfrak{U}), \otimes) := (S^{\mathrm{inv}}(\mathfrak{U}), \otimes)/\sim_{\mathrm{D}}$.

Theorem (Haskell, Hrushovski, Macpherson)

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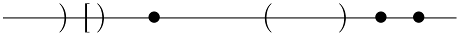

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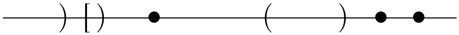

Definable subsets of $\mathfrak{U}^2$ are also quite simple. We have e.g. the set of points above the diagonal, but that is essentially as complicated as it gets.

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- DLO, by quantifier elimination.
- DOAG, by q.e. (semilinear sets, e.g. polyhedra).
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Applications in: (real) algebraic geometry, *tame topology*, number theory,...

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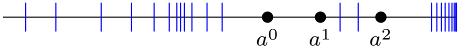

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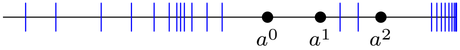

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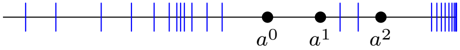

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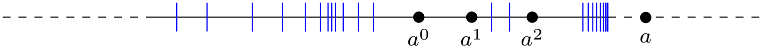

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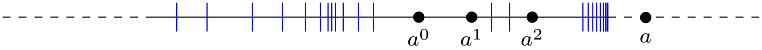

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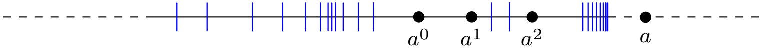

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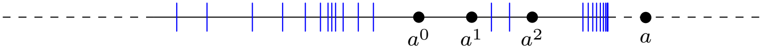

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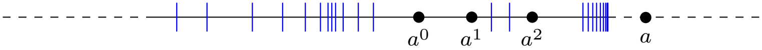

Example (A 2-type in RCF)

The element of $S_{x_0,x_1}(\mathbb{R})$ axiomatised by
$\{0 < x_1 < 1/n \mid n \in \mathbb{N}\} \cup \{0 < x_0\} \cup \{x_0^2 + x_1^2 = 1\}$.

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- We think of their realisations as living in a fixed bigger monster $\mathfrak{U}_1 \,{}^{+}\!\!\succ \mathfrak{U}$.

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Canonical extension and product

Definition ($p \in S(\mathfrak{U})$, $A \subseteq \mathfrak{U}$ small)

p A-*invariant*: whether $p(x) \vdash \varphi(x;d)$ depends only on $\varphi(x;w)$ and $\operatorname{tp}(d/A)$.

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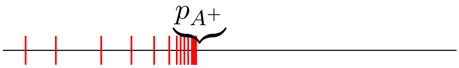

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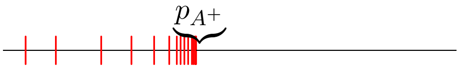

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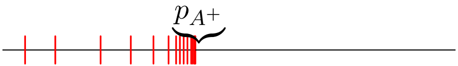

$$\varphi(x; d) \in (p \mid \mathfrak{U}B) \overset{\text{def}}{\iff} \text{for } \tilde{d} \in \mathfrak{U} \text{ such that } d \equiv_A \tilde{d}, \text{ we have } \varphi(x; \tilde{d}) \in p.$$

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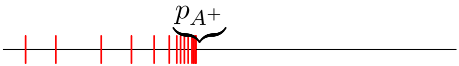
Definition ($p \in S(\mathfrak{U})$, $A \subseteq \mathfrak{U}$ small, $B \not\subseteq \mathfrak{U}$ arbitrary)

p *A-invariant*: whether $p(x) \vdash \varphi(x; d)$ depends only on $\varphi(x; w)$ and $\operatorname{tp}(d/A)$.

Say $p \in S(\mathfrak{U})$ is *invariant* iff it is A-invariant for some *small* $A \subset \mathfrak{U}$.

Example ($T = \mathrm{DLO}$, A small)

$p_{A^+}(x) \coloneqq \{x < d \mid d > A\} \cup \{x > d \mid d \not> A\}$



$$\varphi(x; d) \in (p \mid \mathfrak{U}B) \overset{\text{def}}{\iff} \text{for } \tilde{d} \in \mathfrak{U} \text{ such that } d \equiv_A \tilde{d}, \text{ we have } \varphi(x; \tilde{d}) \in p.$$

Using this, define $\varphi(x, y; d) \in p(x) \otimes q(y) \overset{\text{def}}{\iff} \varphi(x; b, d) \in p \mid \mathfrak{U}b \qquad (b \vDash q)$

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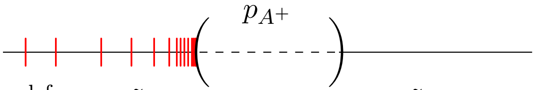

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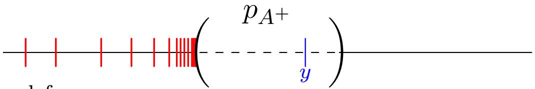

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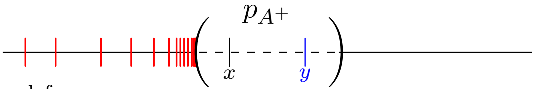

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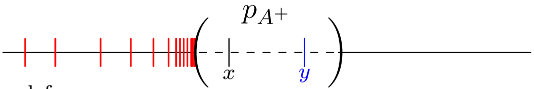

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Fact

$\otimes$ is associative. $\otimes$ commutative $\Leftrightarrow$ T stable. O-minimal theories are unstable.

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Definition (Domination preorder on $S^{\mathrm{inv}}_{<\omega}(\mathfrak{U})$; generalises Rudin–Keisler)

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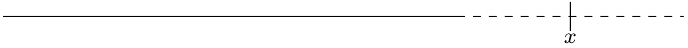

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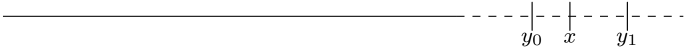

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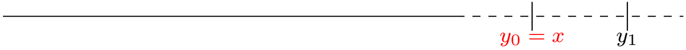

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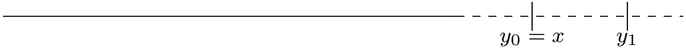

Example (Random Graph)

$p \geq_{\mathrm{D}} q \iff p \supseteq q$ after renaming/duplicating variables and ignoring realised ones.

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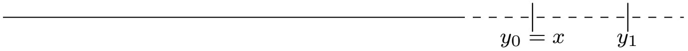

Example (Random Graph, or a set with no structure (*degenerate domination*))

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The domination monoid

Let $\widetilde{\operatorname{Inv}}(\mathfrak{U}) := S^{\mathrm{inv}}_{<\omega}(\mathfrak{U})/\sim_{\mathrm{D}}$.

Fact

If $\sim_{\mathrm{D}}$ is a congruence with respect to $\otimes$, then

- $(\widetilde{\operatorname{Inv}}(\mathfrak{U}), \otimes, \leq_{\mathrm{D}})$ is an ordered monoid, the *domination monoid*;
- the neutral element (and minimum) is the (unique) class of realised types; and
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There are some conditions ensuring compatibility, but this is a different story.

Examples

(In all of these $(\widetilde{\mathrm{Inv}}(\mathfrak{U}), \otimes)$ is well-defined)

T strongly minimal (see here)

$$(\widetilde{\mathrm{Inv}}(\mathfrak{U}), \otimes, \leq_{\mathrm{D}}) \cong (\mathbb{N}, +, \leq).$$

For T stable, $\widetilde{\mathrm{Inv}}(\mathfrak{U}) \cong \mathbb{N} \Leftrightarrow T$ is *unidimensional*, e.g. countable and $\aleph_1$-categorical, or $\mathrm{Th}(\mathbb{Z}, +)$.

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Random Graph (see here)

$\sim_{\mathrm{D}}$ is degenerate, $(\widetilde{\mathrm{Inv}}(\mathfrak{U}), \otimes)$ resembles $(S^{\mathrm{inv}}_{<\omega}(\mathfrak{U}), \otimes)$, e.g. it is noncommutative.

Weak orthogonality

I swear this is the last definition for this talk

Definition

$p(x)$ is *weakly orthogonal* to $q(y)$ iff $p(x) \cup q(y)$ is complete. Write $p \perp^{\mathrm{w}} q$.

Example

In any o-minimal T with $0 \in L$, these two are $\emptyset$-invariant 1-types:

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- $q \perp^{\mathrm{w}} p_0 \geq_{\mathrm{D}} p_1 \implies q \perp^{\mathrm{w}} p_1$. So we may expand to $(\widetilde{\operatorname{Inv}}(\mathfrak{U}), \geq_{\mathrm{D}}, \otimes, \perp^{\mathrm{w}})$.
- In particular if $q \perp^{\mathrm{w}} p \geq_{\mathrm{D}} q$ then q is realised.

Reduction to generation by 1-types

Theorem (M., T o-minimal)

If every $p \in S^{\mathrm{inv}}(\mathfrak{U})$ is $\sim_{\mathrm{D}}$ to a product of 1-types, then $\widetilde{\mathrm{Inv}}(\mathfrak{U})$ is well-defined, and $(\widetilde{\mathrm{Inv}}(\mathfrak{U}), \otimes, \geq_{\mathrm{D}}, \perp^{\mathrm{w}}) \cong (\mathscr{P}_{\mathrm{fin}}(X), \cup, \supseteq, D)$

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Hence, given an o-minimal T, to conclude the study of $\widetilde{\mathrm{Inv}}(\mathfrak{U})$ it is enough to:

1. show that invariant types are equivalent to a product of 1-types, and
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Reduction to generation by 1-types

Ok, I lied, technically there is a definition here

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Sufficient condition for 1: if c is a $\mathfrak{U}$-independent tuple, then

$$\bigcup_{f \in \mathcal{F}_T^{|x|,1}} \mathrm{tp}_{w_f}(f(c)/\mathfrak{U}) \cup \left\{ w_f = f(x) \;\middle|\; f \in \mathcal{F}_T^{|x|,1} \right\} \vdash \mathrm{tp}_x(c/\mathfrak{U}) \qquad (\dagger)$$

$\mathcal{F}_T^{|x|,1} :=$ set of $\emptyset$-definable functions of T with domain $\mathfrak{U}^{|x|}$ and codomain $\mathfrak{U}^1$.

Applications

Theorem ([HHM08])

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Here (†) holds by q.e. and the fact that e.g.

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Applications

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Corollary

In RCVF, by [EHM19] $\widetilde{\operatorname{Inv}}(\mathfrak{U}) \cong \widetilde{\operatorname{Inv}}(k) \oplus \widetilde{\operatorname{Inv}}(\Gamma)$. So $\widetilde{\operatorname{Inv}}(\mathfrak{U}) \cong \mathscr{P}_{\mathrm{fin}}(X)$, where

$$X = \{\text{invariant convex subrings of } k\} \sqcup \{\text{invariant convex subgroups of } \Gamma\}$$

The Idempotency Lemma

Lemma (M., Idempotency Lemma, T o-minimal, $M \prec^+ N \prec^+ \mathfrak{U}$)

If $b \models p \in S_1^{\mathrm{inv}}(\mathfrak{U}, M)$ then $p(\mathrm{dcl}(Nb))$ is cofinal and coinitial in $p(\mathrm{dcl}(\mathfrak{U}b))$.

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If T is o-minimal and $p \in S_1^{\mathrm{inv}}(\mathfrak{U})$ then $p(y) \otimes p(z) \sim_{\mathrm{D}} p(x)$.

Proof.

A small type is enough to say e.g. "$x = z$ and $y > p(\mathrm{dcl}(Nz))$". □

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Proof idea for the Lemma: use the Monotonicity Theorem to show that, otherwise, there is $d \in \mathfrak{U}$ such that $b, f(b,d), f(f(b,d),d), \dots$ is an infinite N-independent sequence. By Steinitz exchange this is nonsense: d depends on a long enough piece of the sequence. N is used to "copy" parameters of definable functions.

Further Directions/Work in Progress

Questions:

1. In the Idempotency Lemma, can we replace N with M?
2. Can we adapt the RCF proof to, say, polynomially bounded structures?
3. Is $\widetilde{\mathrm{Inv}}(\mathfrak{U})$ generated by 1-types in every o-minimal theory? In $\mathbb{R}_{\exp}$?
4. For $T \supseteq$ RCF, can we take X to be the set of invariant T-convex subrings?
5. Can these techniques be adapted to other contexts?

E.g. weakly o-minimal theories, or other "tame" generalisations of o-minimality. Here the RCVF result is promising. Other related context: $\mathbb{Q}_p$?
More generally, the big question is:

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Thanks for listening!

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this is not a proper bibliography, it's just a list of the sources mentioned in these slides

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More examples: Branches

Example

Let T be the theory in the language $\{P_\sigma \mid \sigma \in 2^{<\omega}\}$ asserting that every point belongs to every $P_{\eta\restriction n}$ for exactly one $\eta \in 2^\omega$. Then $\widetilde{\text{Inv}}(\mathfrak{U}) \cong \bigoplus_{2^{\aleph_0}} \mathbb{N}$.

Basically, $\widetilde{\text{Inv}}(\mathfrak{U})$ here is counting how many new points are in a "branch".

More Examples: Generic Equivalence Relation

Equivalence relation E with infinitely many infinite classes (and no finite classes).

A set of generators for $\widetilde{\mathrm{Inv}}(\mathfrak{U})$ looks like this:

- a single $\sim_{\mathrm{D}}$-class $[\![0]\!]$ for realised types
- if $p_a(x) := \{E(x,a)\} \cup \{x \notin \mathfrak{U}\}$, then $[\![p_a]\!] = [\![p_b]\!]$ if and only if $\vDash E(a,b)$; corresponds to new points in an existing equivalence class
- a single $\sim_{\mathrm{D}}$-class $[\![p_g]\!]$, where $p_g := \{\neg E(x,a) \mid a \in \mathfrak{U}\}$; corresponds to new equivalence classes.

The product adds new points/new classes. So, if $\mathfrak{U}$ has κ equivalence classes,

$$\widetilde{\mathrm{Inv}}(\mathfrak{U}) \cong \mathbb{N} \oplus \bigoplus_{\kappa} \mathbb{N}$$

More Examples: Cross-cutting Equivalence Relations

$T_n :=$ n generic equivalence relations E_i; intersection of classes of different E_i always infinite. Here $(\widetilde{\mathrm{Inv}}(\mathfrak{U}), \otimes)$ is generated by:

- a single $\sim_{\mathrm{D}}$-class $[\![0]\!]$ for realised types
- if $p_a(x) := \{E_i(x,a) \mid i < n\} \cup \{x \notin \mathfrak{U}\}$, then $[\![p_a]\!] = [\![p_b]\!]$ if and only if $\vDash \bigwedge_{i<n} E_i(a,b)$; corresponds to new points in E_i-relation with a for all i
- For each $i < n$, a class $[\![p_i]\!]$ saying x is in a new E_i class, but in existing E_j-classes for $j \neq i$ (does not matter which)

So

$$\widetilde{\mathrm{Inv}}(\mathfrak{U}) \cong \prod_{i<n} \mathbb{N} \oplus \bigoplus_{\kappa} \mathbb{N}$$

Why $\prod$ instead of $\bigoplus$? If we allow, say, $\aleph_0$ equivalence relations, then

$$\widetilde{\mathrm{Inv}}(\mathfrak{U}) \cong \prod_{i<\aleph_0}^{\mathrm{bdd}} \mathbb{N} \oplus \bigoplus_{\kappa} \mathbb{N}$$

Other Notions

One can define a finer equivalence relation:

Definition

$p \equiv_{\mathrm{D}} q$ is defined as $p \sim_{\mathrm{D}} q$, but by asking the same r to work in both directions: $p \cup r \vdash q$ and $q \cup r \vdash p$.

Another notion classically studied is:

Definition

$p \geq_{\mathrm{RK}} q$ iff every model realising p realises q.

This behaves best in totally transcendental theories (because of prime models). It corresponds to $p(x) \cup \{\varphi(x,y)\} \vdash q(y)$.
But even there, modulo $\sim_{\mathrm{RK}}$ it is *not* true that every type decomposes as a product of $\geq_{\mathrm{RK}}$-minimal types (but in non-multidimensional totally transcendental theories every type decomposes as a product of strongly regular types).
A classical example where $\geq_{\mathrm{D}}$ differs from $\geq_{\mathrm{RK}}$: generic equivalence relation with a bijection s such that $\forall x\, E(x, s(x))$. Back

Hrushovski's Counterexample

Example (Hrushovski)

In DLO plus a dense-codense predicate P, $\overline{\mathrm{Inv}}(\mathfrak{U})$ is not commutative.

Proof idea.

Let $p(x) := \{P(x)\} \cup \{x > \mathfrak{U}\}$ and $q(y) := \{\neg P(x)\} \cup \{y > \mathfrak{U}\}$. Then p, q do not commute, even modulo $\equiv_{\mathrm{D}}$ (but they do modulo $\sim_{\mathrm{D}}$).
The predicate P forbids to "glue" variables. One will be "left behind": e.g. if $r \vdash x_0 < y_0 < y_1 < x_1$, knowing that $y_1 > \mathfrak{U}$ does not imply $x_0 > \mathfrak{U}$. □

In this case, for each cut C there are generators $[\![p_{C,P}]\!]$ and $[\![p_{C,\neg P}]\!]$, with relations

- $[\![p_{C,P}]\!] \otimes [\![p_{C,P}]\!] = [\![p_{C,\neg P}]\!] \otimes [\![p_{C,P}]\!] = [\![p_{C,P}]\!]$
- (same relations swapping P and $\neg P$)
- $[\![p_{C_0,-}]\!] \otimes [\![p_{C_1,-}]\!] = [\![p_{C_1,-}]\!] \otimes [\![p_{C_0,-}]\!]$ whenever $C_0 \neq C_1$.

Back

Stable Case

In a stable theory, $\leq_{\mathrm{D}}$, $\sim_{\mathrm{D}}$ and $\equiv_{\mathrm{D}}$ can be expressed in terms of forking:

Definition

$a \rhd_E b$ iff, for all c,

$$a \mathop{\downarrow}_{E} c \Longrightarrow b \mathop{\downarrow}_{E} c$$

$p \rhd_E q$ (p *dominates* q over E) iff there are $a \vDash p$ and $b \vDash q$ such that $a \rhd_E b$

$p \bowtie_E q$ (p and q are *domination equivalent*) iff $p \rhd_E q \rhd_E p$, i.e. there are $\underbrace{a}_{\vDash p} \rhd_E \underbrace{b}_{\vDash q} \rhd_E \underbrace{c}_{\vDash p}$

$p \doteq_E q$ (p and q are *equidominant* over E) iff there are $a \vDash p$ and $b \vDash q$ such that $a \rhd_E b \rhd_E a$

These are well-behaved with non-forking extensions: we can drop $_E$.

Comparison

Proposition (T stable)

The previous definitions of $\leq_{\mathrm{D}} = \triangleleft$, $\sim_{\mathrm{D}} = \bowtie$ and $\equiv_{\mathrm{D}} = \doteq$.

Remark

The proof uses crucially stationarity of types over models.

In almost all examples we saw before, $\sim_{\mathrm{D}}$ coincides with $\equiv_{\mathrm{D}}$.

Exception: in DLO with a predicate, $(\overline{\mathrm{Inv}}(\mathfrak{U}), \otimes)$ is not commutative, while $(\widetilde{\mathrm{Inv}}(\mathfrak{U}), \otimes)$ is (in fact, it is the same as in DLO).

Fact

Even in the stable case, $\sim_{\mathrm{D}}$ and $\equiv_{\mathrm{D}}$ are generally different.

Classical Results

In the thin case (generalises superstable), this is classical:

Theorem (T thin)

$\widetilde{\mathrm{Inv}}(\mathfrak{U})$ is a direct sum of copies of $\mathbb{N}$.
If T is moreover superstable, $(\widetilde{\mathrm{Inv}}(\mathfrak{U}), \otimes)$ is generated by $\{[\![p]\!] \mid p \text{ regular}\}$.

Superstability (even just thinness) implies that $\equiv_{\mathrm{D}}$ and $\sim_{\mathrm{D}}$ coincide.

The behaviour of $\geq_{\mathrm{D}}$ in general seems related to the existence of some kind of prime models (in the stable case, “prime a-models” are the way to go).
Also, some suitable generalisation of the Omitting Types Theorem would help.

Back

(Non-multi)Dimensionality

At least in the superstable case, independence of $\widetilde{\mathrm{Inv}}(\mathfrak{U})$ on $\mathfrak{U}$ already had a name:

Definition

T is *(non-multi)dimensional* iff no type is orthogonal to (every type that does not fork over) $\emptyset$.

If $\mathfrak{U}_0 \prec^+ \mathfrak{U}_1$ one has a map $\mathfrak{e}\colon \widetilde{\mathrm{Inv}}(\mathfrak{U}_0) \to \widetilde{\mathrm{Inv}}(\mathfrak{U}_1)$.

Proposition (T thin)

$\mathfrak{e}$ surjective $\iff$ T dimensional.

Question

Is this true under stability? It boils down to the image of $\mathfrak{e}$ being downward closed. I suspect this should follow from classical results. Back

Generically Stable Part

Proposition

$q \leq_{\mathrm{D}} p$ definable/finitely satisfiable/generically stable $\implies$ so is q.

As generically stable types commute with everything, in any theory the monoid generated by their classes is well-defined. (Warning: p generically stable $\not\Rightarrow$ $p \otimes p$ generically stable)

Hope

At least in special cases, get decompositions similar to $\widetilde{\mathrm{Inv}}(\mathfrak{U}) \cong \overbrace{\widetilde{\mathrm{Inv}}(k)}^{\text{g.s. part}} \times \widetilde{\mathrm{Inv}}(\Gamma)$.

Probably one should really work in T^{eq}:

Example

In $T =$ DLO+equivalence relation with (no finite classes and infinitely many) dense classes, $\widetilde{\mathrm{Inv}}(\mathfrak{U})$ grows when passing to T^{eq}, which has more generically stable types.

Question

How can the generically stable part look like?

Interaction with Weak Orthogonality

Definition

$p(x)$ is *weakly orthogonal* to $q(y)$ iff $p \cup q$ is complete.

Remark

Weakly orthogonal types commute.

Proposition

Weak orthogonality strongly negates domination: $q \perp^{\mathrm{w}} p_0 \geq_{\mathrm{D}} p_1 \implies q \perp^{\mathrm{w}} p_1$.
In particular if $q \perp^{\mathrm{w}} p \geq_{\mathrm{D}} q$ then q is realised.

Question

Under which conditions if $p \not\perp^{\mathrm{w}} q$ then they dominate a common nonzero class?

Known:

- Superstable (or *thin*) is enough. See here
- Fails in the Random Graph.

Action on Type Space

$f \in \mathrm{Aut}(\mathfrak{U})$ acts on $p \in S(\mathfrak{U})$ by changing parameters in formulas:

$$f \cdot p := \{\varphi(x, f(d)) \mid \varphi(x, d) \in p\}$$

Consider this action restricted to $\mathrm{Aut}(\mathfrak{U}/A)$.

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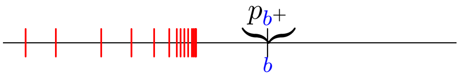
Example

$T = \mathsf{DLO}$, consider $p_{b^+}(x) := \{x < d \mid d > b\} \cup \{x > d \mid d \leq b\}$



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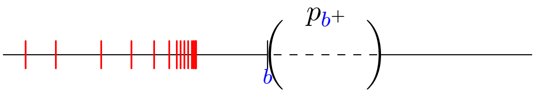

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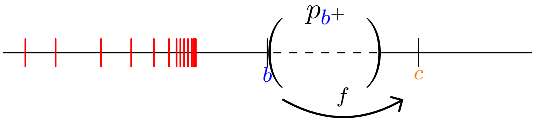

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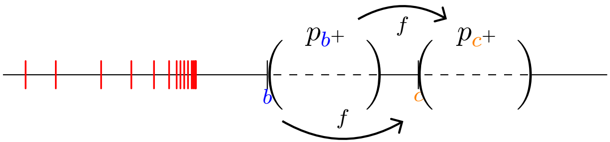

Back

Invariant Extension

How to canonically extend an invariant type to bigger sets

Recall: $p \in S_x^{\text{inv}}(\mathfrak{U}, A) \iff$ whether $p(x) \vdash \varphi(x; d)$ or not depends only on $\text{tp}(d/A)$

Fact (B arbitrary, A small)

Every $p \in S_x^{\text{inv}}(\mathfrak{U}, A)$ has a unique extension $(p \mid \mathfrak{U}B) \in S_x^{\text{inv}}(\mathfrak{U}B, A)$

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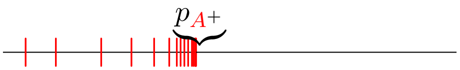

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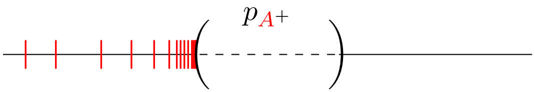

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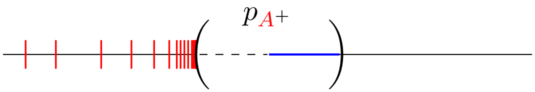

Invariant Extension

How to canonically extend an invariant type to bigger sets

Recall: $p \in S^{\mathrm{inv}}_x(\mathfrak{U}, A) \iff$ whether $p(x) \vdash \varphi(x; d)$ or not depends only on $\mathrm{tp}(d/A)$

Fact (B arbitrary, A small)

Every $p \in S^{\mathrm{inv}}_x(\mathfrak{U}, A)$ has a unique extension $(p \mid \mathfrak{U}B) \in S^{\mathrm{inv}}_x(\mathfrak{U}B, A)$: for tuples d from $\mathfrak{U}B$

$$\varphi(x; d) \in (p \mid \mathfrak{U}B) \stackrel{\text{def}}{\iff} \text{for } \tilde{d} \in \mathfrak{U} \text{ such that } d \equiv_A \tilde{d}, \text{ we have } \varphi(x; \tilde{d}) \in p.$$

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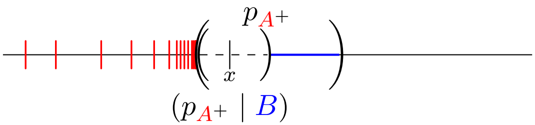
$p_{A^+}(x) := \{x < d \mid d > A\} \cup \{x > d \mid d \not> A\}$ " $=$ " $(p_{A^+} \mid \mathfrak{U}B)(x)$ (now $d \in \mathfrak{U}B$)



Product of Invariant Types

Definition (p invariant)

$$\varphi(x, y; d) \in p(x) \otimes q(y) \overset{\text{def}}{\iff} \varphi(x; b, d) \in p \mid \mathfrak{U}b \qquad (b \vDash q)$$

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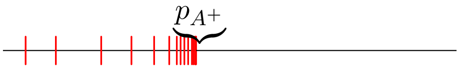

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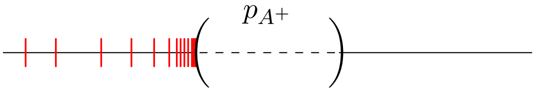

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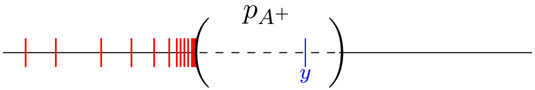

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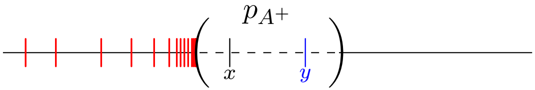

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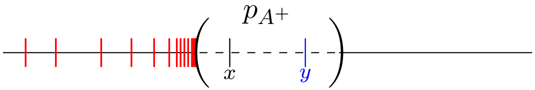
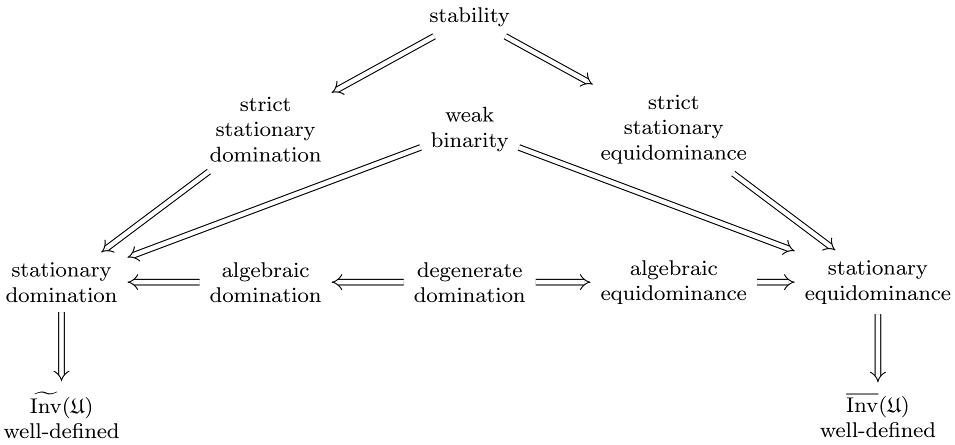


Fact

$\otimes$ is associative. It is commutative if and only if T is stable.

Map of Sufficient Conditions



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Sufficient Conditions

Proposition

$q_0 \geq_{\mathrm{D}} q_1 \implies p \otimes q_0 \geq_{\mathrm{D}} p \otimes q_1$ is implied by any of the following:

- q_1 algebraic over q_0: every $c \models q_1$ is algebraic over some $b \models q_0$. E.g. $q_1 = f_* q_0$ for some definable function f. Reason: $\{c \mid (b,c) \models r\}$ does not grow with $\mathfrak{U}$.
- Or even *weakly binary*: $\mathrm{tp}(a/\mathfrak{U}) \cup \mathrm{tp}(b/\mathfrak{U}) \cup \mathrm{tp}(ab/M) \models \mathrm{tp}(ab/\mathfrak{U})$: few questions about $a \models p$ and $c \models q_1$.
- T is stable.

Any condition in the Proposition implies that if there is some $r \in S_{yz}(M)$ witnessing $q_0(y) \geq_{\mathrm{D}} q_1(z)$, then there is one such that, in addition, if

- $b, c \in \mathfrak{U}_1 \ {}^{+}\!\succ \mathfrak{U}$ are such that $(b,c) \models q_0 \cup r$,
- $p \in S^{\mathrm{inv}}(\mathfrak{U}, M)$ and $a \models p(x) \mid \mathfrak{U}_1$,
- $r[p] := \mathrm{tp}_{xyz}(abc/M) \cup \{x = w\}$.

then $p \otimes q_0 \cup r[p] \vdash p \otimes q_1$. We call this *stationary domination*. Back

A Counterexample

(with SOP and IP_2)

Idea: DLO

Back

A Counterexample

(with SOP and IP_2)

Idea: 2-coloured DLO

Back

A Counterexample

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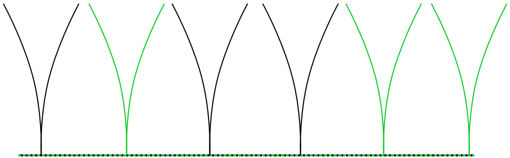
Idea: fiber over a 2-coloured DLO



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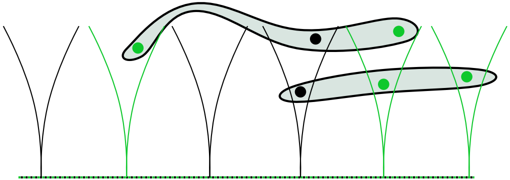

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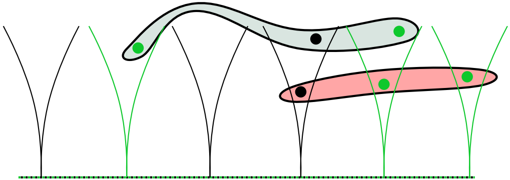
Idea: fiber over a 2-coloured DLO; put a generic tripartite 3-hypergraph on **some** triples of fibers: $R_3(x, z, w) \to \big(G(\pi x) < \neg G(\pi z) < G(\pi w)\big)$



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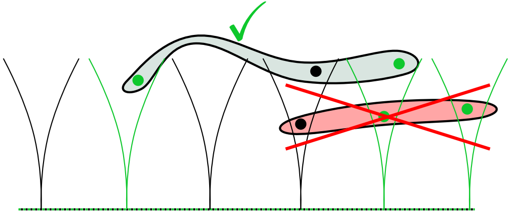

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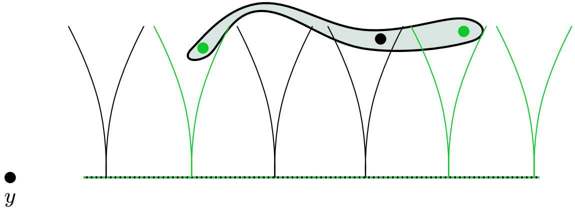

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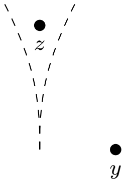

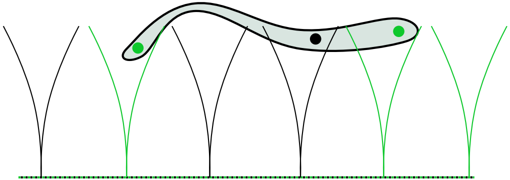

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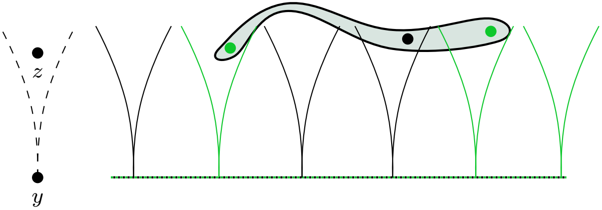

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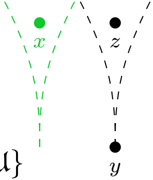

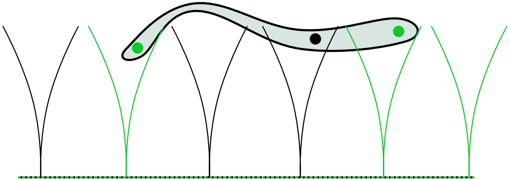

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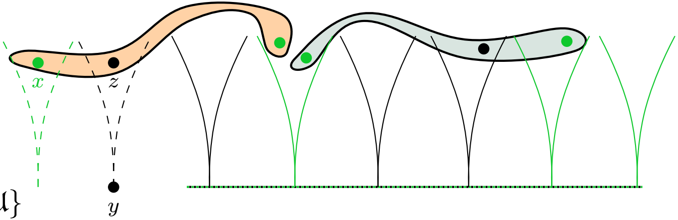

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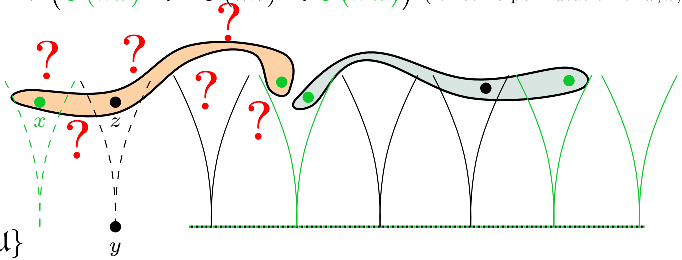

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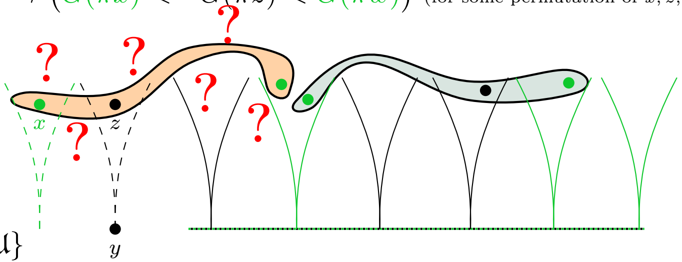

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Supersimple version here. Also works for a number of variations of $\sim_{\mathrm{D}}$.

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Another Counterexample

Ternary, supersimple, ω-categorical, can be tweaked to have degenerate algebraic closure

Replacing the densely coloured DLO with a random graph R_2 yields a supersimple counterexample of SU-rank 2; forking is $a \underset{C}{\downarrow} b \iff (a \cap b \subseteq C) \wedge (\pi a \cap \pi b \subseteq \pi C)$.

$$R_3(x_0, x_1, x_2) \to \bigvee_{\sigma \in S_3} \left(R_2(\pi x_{\sigma 0}, \pi x_{\sigma 1}) \wedge R_2(\pi x_{\sigma 0}, \pi x_{\sigma 2}) \wedge \neg R_2(\pi x_{\sigma 1}, \pi x_{\sigma 2})\right)$$

(exactly two edges between $\pi x_0, \pi x_1, \pi x_2$)

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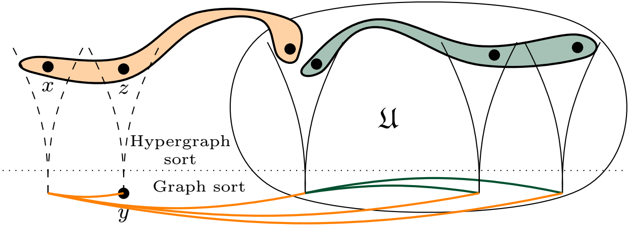

$q_0 \cup r \vdash q_1$: no hyperedges to decide. Same problem: $p \otimes q_0(x, y) \not\geq_{\mathrm{D}} p \otimes q_1(t, z)$.

Strongly Minimal Theories

$(\widetilde{\mathrm{Inv}}(\mathfrak{U}), \otimes)$ **well-defined by stability**

Example

If T is strongly minimal, $(\widetilde{\mathrm{Inv}}(\mathfrak{U}), \otimes, \leq_{\mathrm{D}}) \cong (\mathbb{N}, +, \leq)$.

(for T stable, $\widetilde{\mathrm{Inv}}(\mathfrak{U}) \cong \mathbb{N} \Leftrightarrow T$ is *unidimensional*, e.g. countable and $\aleph_1$-categorical, or $\mathrm{Th}(\mathbb{Z}, +)$)

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In this case, $\widetilde{\mathrm{Inv}}(\mathfrak{U})$ is basically "counting the dimension". E.g.: in ACF_0 we have $p(x_1, \ldots, x_n) \sim_{\mathrm{D}} q(y_1, \ldots, y_m) \iff \operatorname{tr}\deg(x/\mathfrak{U}) = \operatorname{tr}\deg(y/\mathfrak{U})$.
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Taking products corresponds to adding dimensions: if $(a, b) \vDash p \otimes q$, then $\dim(a/\mathfrak{U}b) = \dim(a/\mathfrak{U})$, and in strongly minimal theories

$$\dim(ab/\mathfrak{U}) = \dim(b/\mathfrak{U}) + \dim(a/\mathfrak{U}b)$$

More generally, in superstable theories (or even *thin* theories), by classical results $\widetilde{\mathrm{Inv}}(\mathfrak{U}) \cong \bigoplus_{i<\lambda}(\mathbb{N}, +, \leq)$, for some λ.

Back

Dense Linear Orders

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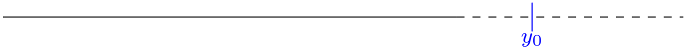

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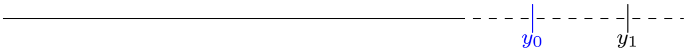

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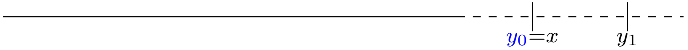
- Every element is idempotent: e.g. if $p(x) = \mathrm{tp}(x > \mathfrak{U})$, then $p(x) \sim_{\mathrm{D}} p(y_1) \otimes p(y_0)$ (seen before: glue x and y_0):



Dense Linear Orders

$(\widetilde{\mathrm{Inv}}(\mathfrak{U}), \otimes)$ well-defined by binarity

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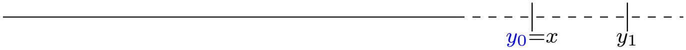

$\widetilde{\mathrm{Inv}}(\mathfrak{U})$ is the free idempotent commutative monoid generated by the invariant cuts:

$$(\widetilde{\mathrm{Inv}}(\mathfrak{U}), \otimes, \leq_{\mathrm{D}}) \cong (\mathscr{P}_{\mathrm{fin}}(\{\text{invariant cuts}\}), \cup, \subseteq)$$

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Random Graph

$(\widetilde{\mathrm{Inv}}(\mathfrak{U}), \otimes)$ well-defined by binarity

In the Random Graph, $\sim_{\mathrm{D}}$ is degenerate and $(\widetilde{\mathrm{Inv}}(\mathfrak{U}), \otimes)$ resembles closely $(S^{\mathrm{inv}}_{<\omega}(\mathfrak{U}), \otimes)$. For instance, it is not commutative:

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These types do not commute, even modulo $\sim_{\mathrm{D}}$:

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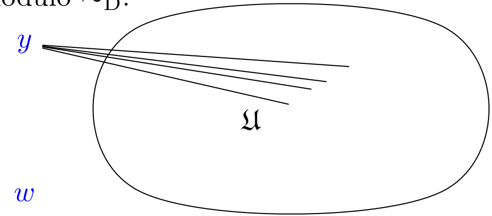

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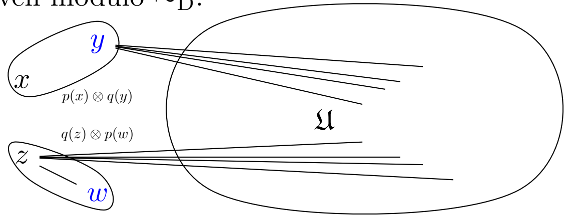

Proof Idea.

As $p_x \otimes q_y \vdash \neg E(x,y)$ and $q_z \otimes p_w \vdash E(z,w)$, gluing cannot work. But in the random graph domination is degenerate and there is not much more one can do. □

More examples here · Back to examples · Back to counterexamples

Properties Preserved by Domination

Domination equivalence is quite coarse; for instance it does not preserve Morley rank (generic equivalence relation), nor dp-rank (DLO).

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- Definability (over *some* small set, not necessarily the same as q)
- Finite satisfiability (in *some* small set, not necessarily the same as q)
- Generic stability (over *some* small set, not necessarily the same as q)
- Weak orthogonality to a fixed type

Generic stability is particularly interesting:

- It is possible to have $\widetilde{\mathrm{Inv}}(\mathfrak{U}) \neq \widetilde{\mathrm{Inv}}(\mathfrak{U}^{\mathrm{eq}})$ (more g.s. types, e.g. DLO+dense eq. rel.).
- Using [Tan15], strongly regular g.s. types are $\leq_{\mathrm{D}}$-minimal (among the nonrealised ones).
- $(\widetilde{\mathrm{Inv}}^{\mathrm{gs}}(\mathfrak{U}), \otimes, \leq_{\mathrm{D}})$ makes sense in any theory (can be trivial).

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You asked for it

Let T be o-minimal. Let $p(x) \in S^{\mathrm{inv}}(\mathfrak{U}, M_0)$, let $c \vDash p$ be $\mathfrak{U}$-independent.

1. There is a tuple $b \in \operatorname{dcl}(\mathfrak{U}c)$ of maximal length among those satisfying a product of nonrealised invariant 1-types.
2. Let b be as above, and let $q := \operatorname{tp}(b/\mathfrak{U}) = q_0 \otimes \ldots \otimes q_n$, where $q_i \in S_1^{\mathrm{inv}}(\mathfrak{U})$. Up to replacing q_i with $\tilde{q}_i \sim_{\mathrm{D}} q_i$, we may assume that either $q_i \perp^{\mathrm{w}} q_j$ or $q_i = q_j$.

Let b, q as above, $q_i \in S^{\mathrm{inv}}(\mathfrak{U}, M)$ and $M_0 \preceq M \prec^+ N \prec^+ N_1 \prec^+ \mathfrak{U}$.

3. Up to replacing b with another $\tilde{b} \vDash q$, we may assume $b \in \operatorname{dcl}(Nc)$.
4. Let b, q be as above, $r := \operatorname{tp}_{xy}(cb/N_1)$, and $\mathcal{F}^{m,1}_{T(M)}$ the set of $T(M)$-definable functions with domain $\mathfrak{U}^m$ and codomain $\mathfrak{U}^1$. Then $p(x) \cup r(x,y) \vdash q(y)$ and

$$q(y) \cup r(x,y) \vdash \pi_M(x) := \bigcup_{f \in \mathcal{F}^{|x|,1}_{T(M)}} \operatorname{tp}_{w_f}(f(c)/\mathfrak{U}) \cup \left\{ w_f = f(x) \;\middle|\; f \in \mathcal{F}^{|x|,1}_{T(M)} \right\}$$

Using this and some valuation theory, in RCF, it can be shown that $q \cup r \vdash p$.

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