Results of Existential Closedness of Raising to Powers Type

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Motivation

The starting point is Zilber's study of the model theory of complex exponentiation.

Recall

There is an uncountably categorical axiomatization in an infinitary logic which is conjecturally satisfied by the complex numbers.
The hard part of showing that $(\mathbb{C}, \exp)$ is indeed a model of this axiomatization is given by two statements, **Schanuel's conjecture** and **exponential-algebraic closedness**.

Schanuel's Conjecture

Suppose $z_1, \ldots, z_n \in \mathbb{C}$ *are* $\mathbb{Q}$*-linearly independent. Then*

$$\operatorname{trdeg}(z_1, \ldots, z_n, e^{z_1}, \ldots, e^{z_n}) \geq n.$$

Informally, this conjecture gives a lower bound on transcendence of exp: it is "at least as transcendental as any other group homomorphism."

Exponential Algebraic Closedness

Schanuel's Conjecture

Suppose $z_1, \ldots, z_n \in \mathbb{C}$ are $\mathbb{Q}$-linearly independent. Then

$$\mathrm{trdeg}(z_1, \ldots, z_n, e^{z_1}, \ldots, e^{z_n}) \geq n.$$

Exponential Algebraic Closedness (Informal Version)

Every system of exponential polynomial equations that does not contradict Schanuel's Conjecture has a solution in $\mathbb{C}$.

An exponential polynomial equation is an equation of the form $f(X_1, \ldots, X_n, e^{X_1}, \ldots, e^{X_n}) = 0$, where f is a polynomial in $2n$ variables. This suggests a geometric interpretation of the conjectures: Schanuel's conjecture can be seen as saying that if $\bar{z} = (z_1, \ldots, z_n)$ are $\mathbb{Q}$-linearly independent, then the point $(\bar{z}, e^{\bar{z}})$ does not lie on an algebraic variety of dimension less than n.

Geometric Exponential Algebraic Closedness

Definition

Let $V \subseteq \mathbb{C}^n \times (\mathbb{C}^\times)^n$ be an algebraic variety, π_1 and π_2 the projections on the first and second blocks of n coordinates.
V is **free** if $\pi_1(V)$ is not contained in a translate of a $\mathbb{Q}$-linear subspace and $\pi_2(V)$ is not contained in a translate of an algebraic subgroup of $(\mathbb{C}^\times)^n$, defined by equations of the form $z_1^{k_1} \cdot \ldots \cdot z_n^{k_n} = 1$ for $k_1, \ldots, k_n \in \mathbb{Z}$.
For a $\mathbb{Q}$-linear subspace $L \subseteq \mathbb{C}^n$, let π_L denote the quotient map $\mathbb{C}^n \times (\mathbb{C}^\times)^n \to \mathbb{C}^n/L \times (\mathbb{C}^\times)^n/\exp(L)$.
V is **rotund** if for every $\mathbb{Q}$-linear subspace L of $\mathbb{C}^n$, $\dim \pi_L(V) \geq n - \dim L$. In particular, with $L = \langle 0 \rangle$, $\dim V \geq n$.

Geometric EAC

Every free rotund algebraic variety intersects the graph of exp.

Examples and Known Results

As an example, consider the case $n = 1$: then $V \subseteq \mathbb{C} \times \mathbb{C}^\times$ is a variety defined by an equation $f(X, Y) = 0$, with f depending on both coordinates. The statement then is that there is $z \in \mathbb{C}$ such that $f(z, e^z) = 0$. This is proved easily using Hadamard factorization, and was well-known before Zilber's conjecture.

Using a method developed by Brownawell and Masser, based on the Newton-Kantorovich approximation theorem, D'Aquino, Fornasiero and Terzo proved that the statement holds when $\dim(\pi_1(V)) = n$.

Mantova and Masser proved that the statement holds when $\dim(\pi_1(V)) = 1$, using tools from the geometry of Riemann surfaces.

Note that the above statements completely solve the problem for subvarieties of $\mathbb{C}^2 \times (\mathbb{C}^\times)^2$, but it is still very much open for $n \geq 3$.

Later on we'll see another known result, due to Zilber himself, for varieties of a particular form.

Variants of Schanuel's Conjecture: Abelian Varieties

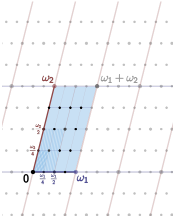
Schanuel's conjecture fits in a broad framework in transcendental number theory (a very big conjecture by Grothendieck, the *Period Conjecture*, would imply it). Similar statements appear in the literature in different settings, and it's natural to try to transfer the EAC problem.

Recall

An **abelian variety** is a connected, projective algebraic group, i.e. a projective algebraic variety that has a regular group structure.
The most well-known class of abelian varieties is the class of **elliptic curves**, i.e. 1-dimensional abelian varieties.

Every complex abelian variety A of dimension g is isomorphic to a group of the form $\mathbb{C}^g/\Lambda$, where Λ is a lattice in $\mathbb{C}^g$ (a discrete subgroup of rank $2g$). The surjection $\exp : \mathbb{C}^g \to A$ with kernel Λ is known as the **exponential** of A.

Variants of Schanuel's Conjecture: Abelian Varieties



picture from Wikipedia

Abelian Schanuel's Conjecture

Let A be an abelian variety, $\overline{x} = x_1, \ldots, x_n \in \mathbb{C}^g$ such that $\exp(\overline{x})$ is not contained in any proper **abelian** *subvariety of A. Then $(\overline{x}, \exp(\overline{x}))$ does not lie in any* **algebraic** *subvariety of $\mathbb{C}^g \times A$.*

Abelian Exponential Algebraic Closedness (Bays, Kirby, Zilber...)

Every free rotund subvariety $V \subseteq \mathbb{C}^g \times A$ intersects the graph of exp.

Variants of Schanuel's Conjecture: the j-Function

Let $\mathbb{H}$ denote the upper half plane (complex numbers with positive imaginary part). Consider the action of $\mathrm{SL}_2(\mathbb{R})$ on $\mathbb{H}$ by Möbius transformations, i.e. $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \cdot z = \frac{az+b}{cz+d}$.

The modular j invariant is a function $j : \mathbb{H} \to \mathbb{C}$ which classifies elliptic curves.

1. j is $\mathrm{SL}_2(\mathbb{Z})$-invariant: $j(gz) = j(z)$ for all $g \in \mathrm{SL}_2(\mathbb{Z})$ and $z \in \mathbb{H}$.
2. There are polynomials $\Phi_N(X, Y)$ for every $N \in \mathbb{N}^{>0}$ such that for every $w_1, w_2 \in \mathbb{C}$, $\Phi_N(w_1, w_2) = 0$ if and only if there are $z_1, z_2 \in \mathbb{H}$ and $g \in \mathrm{GL}_2(\mathbb{Q})$ with coprime integer entries such that $\det(g) = N$ and $z_2 = gz_1$.

These are known as **modular polynomials**, and their importance lies in the fact that they represent the only algebraic relations that are preserved under j.

Variants of Schanuel's Conjecture: the j-Function

Modular Schanuel's Conjecture

Suppose $z_1, \ldots, z_n \in \mathbb{H}$ lie in different $\mathrm{GL}_2(\mathbb{Q})$-orbits, and none of them is fixed by a matrix in $\mathrm{GL}_2(\mathbb{Q})$. Then $\mathrm{trdeg}(z_1, \ldots, z_n, j(z_1), \ldots, j(z_n)) \geq n$.

In this context it's harder to see what the analogues of freeness and rotundity are.
V is free if it is not contained in a variety defined by modular conditions.
The analogue of rotundity is broadness, another condition on dimensions.

j-Algebraic Closedenss (Aslanyan 2018)

Every free and broad algebraic variety $V \subseteq \mathbb{H}^n \times \mathbb{C}^n$ intersects the graph of j.

Raising to Powers

Theorem (Zilber 2002)

Exponential algebraic closedness holds for free rotund varieties of the form $L \times W$, where L is a $\mathbb{R}$-linear vector subspace of $\mathbb{C}^n$ and W is an algebraic subvariety of $(\mathbb{C}^{\times})^n$.

Zilber's proof used tools from enumerative geometry (Newton polytopes). How does this translate to the abelian and modular settings?

Abelian Varieties - Density

Lemma

Let A be an abelian variety of dimension g, $L \subseteq \mathbb{C}^g$ a sufficiently generic linear space. Then $\exp(L)$ is dense in A.

Proof idea.

As we know, $A \cong \mathbb{C}^g/\Lambda$ for some lattice of rank $2g$, so as a real Lie group it is isomorphic to $\mathbb{R}^{2g}/\mathbb{Z}^{2g}$. A linear subspace of $\mathbb{R}^{2g}$ is dense in this group if and only if it is not contained in any $\mathbb{Q}$-linear space, so we identify the counterparts of $\mathbb{Q}$-linear spaces in $\mathbb{C}^g$ and obtain the lemma. □

Abelian Varieties - Intersections

Theorem

Let A be an abelian variety of dimension g, $L \subseteq \mathbb{C}^g$ be a sufficiently generic linear space. Then for every algebraic variety $W \subseteq A$ s.t. the variety $L \times W$ is free and rotund, $L \times W$ intersects the graph of exp.

Proof idea.

This amounts to finding intersections between $\exp(L)$ and W. This is given by the density of $\exp(L)$ once we make sure that there is a "sufficiently big" subset of W that is not "parallel" to L. This set turns out to be Zariski open, by a combination of the fibre dimension theorem and the Ax-Schanuel theorem (the rotundity assumption is crucial here). □

j Function - Density

The analogue of linear spaces in this setting is represented by *Möbius varieties*.

Definition

A **Möbius subvariety** of $\mathbb{H}^n$ is a subvariety of $\mathbb{H}^n$ defined by equations of the form $z_i = gz_j$ for $g \in \mathrm{SL}_2(\mathbb{R})$ and $z_i = c$ for $c \in \mathbb{H}$.

For example, $\left\{(z_1, z_2) \in \mathbb{H}^2 | z_2 = z_1 + \sqrt{2}\right\}$ is a Möbius variety, defined by the matrix $g = \begin{pmatrix} 1 & \sqrt{2} \\ 0 & 1 \end{pmatrix}$.

Lemma

Let L be a free Möbius subvariety of $\mathbb{H}^n$ (so none of its coordinates is constant, and no pair of its coordinates satisfies a relation in $\mathrm{GL}_2(\mathbb{Q})$). Then $j(L)$ is dense in $\mathbb{C}^n$.

This is proved using Ratner's Orbit Closure Theorem, an important result from ergodic theory.

j Function - Intersections

Theorem

Let L be a free Möbius subvariety of $\mathbb{H}^n$. Then for every W algebraic subvariety of $\mathbb{C}^n$ such that $L \times W$ is broad, $L \times W$ intersects the graph of j.

Proof idea.

Again this is about finding intersections in $j(L) \cap W$. If L has dimension 1, and W has dimension $n-1$, then we need to study a single function: L is given by points of the form $(z, g_2 z, \dots, g_n z)$ and W is defined by a polynomial $f(Y_1, \dots, Y_n)$, so we look for zeros of the function $z \mapsto f(j(z), j(g_2 z), \dots, j(g_n z))$. This can be done using the particular form of the density result (we show that we can approximate f with functions that are very small with respect to their derivatives and use Newton's approximation method).

In higher dimension we use induction. □

Future Work

Do any of these methods generalize to slightly more general forms of the conjecture? For example, it is known (Peterzil-Starchenko) that if A is an abelian variety of dimension g and $V \subseteq \mathbb{C}^g$ is an algebraic variety, then the closure of $\exp(V)$ in A can be described in terms of finitely many subgroups (closures of images of linear spaces).
We're trying to answer this question in joint work with Aslanyan, Kirby, and Mantova.

Thank you!