The Topological μ-Calculus

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Modal logic and Kripke semantics

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$$p \mid \bot \mid \rightarrow \mid \Diamond\varphi$$

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- $\varphi \vee \psi := \neg\varphi \rightarrow \psi$
- $\Box\varphi := \neg\Diamond\neg\varphi$

Kripke semantics:

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- $[\![\varphi \to \psi]\!] = (W \setminus [\![\varphi]\!]) \cup [\![\psi]\!]$
- $[\![\Diamond\varphi]\!] = R^{-1}\,[\![\varphi]\!]$

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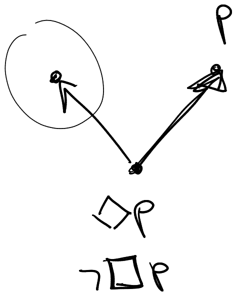
$w \in [\![\Diamond\varphi]\!] \iff \exists v\ (wRv \ \& \ v \in [\![\varphi]\!])$

Models: Triples $\mathcal{M} = (W, R, [\![\cdot]\!])$

A Kripke model



Axiomatization for modal logic

The basic modal logic is called K.

Axioms

- All classical tautologies
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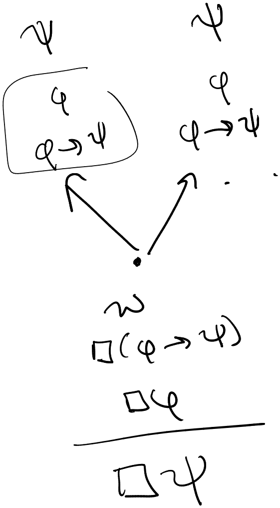
Theorem

A formula is valid over the class of Kripke models iff it is derivable in K.

Proof (Soundness)

$$\Box(\varphi \to \psi) \to (\Box\varphi \to \Box\psi)$$



Proof (Completeness)

Canonical model: $\mathcal{M}_c = (\underline{W_c}, R_c, [\![\cdot]\!]_c)$

W_c = set of "

Proof (Completeness)

Lemma (Truth lemma)

If $T \in W_c$ and φ is any formula, $T \in [\![\varphi]\!]_c$ iff $\varphi \in T$.

Canonical logics

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Definition

An extension Λ of K (also called a **normal logic**) is **canonical** if its canonical model is based on a **Λ-frame**.

Example: K4 $:=$ K $+$ 4

The μ-calculus

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- $[\![\mu p.\varphi(p)]\!]$ is the **least fixed point** of $X \mapsto [\![\varphi(X)]\!]$.
- $\nu p.\varphi(p) := \neg\mu p.\neg\varphi(\neg p)$ is the **greatest fixed point** of $X \mapsto [\![\varphi(X)]\!]$.

Example: Transitive closure

Define $\Diamond^* \varphi := \mu p.(\varphi \vee \Diamond p)$.

Least fixed point of monotone operators

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Theorem
Every monotone operator has a least fixed point.

Lemma
If p appears positively in $\varphi(p)$, then $X \mapsto [\![\varphi(X)]\!]$ is a monotone operator.

(Topological) closure semantics of modal logic

If $\mathcal{X} = (X, \mathcal{T})$ is a topological space, we may also define

$$[\![\Diamond\varphi]\!] := \boldsymbol{c}\,[\![\varphi]\!]\,.$$

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Theorem

The logic

$$\mathsf{S4} := \mathsf{K} + 4 + T$$

is sound and complete for the class of **closure spaces** *(topological spaces equipped with the closure operator).*

Cantor derivative semantics

If X is a topological space and $A \subseteq X$, define the **Cantor derivative** or **set of limit points of A** by

$$dA = \{x \in X : x \in c(A \setminus \{x\})\}.$$

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Theorem

The logic

$$\mathrm{wK4} := \mathrm{K} + w4$$

is sound and complete for the class of topological spaces.

Soundness of $w4$

Kripke semantics of $wK4$

A relation $\sqsubset \subseteq W \times W$ is **weakly transitive** if $T \sqsubset S \sqsubset U$ implies that $T \sqsubseteq U$.

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A relation $\sqsubset \subseteq W \times W$ is **weakly transitive** if $T \sqsubset S \sqsubset U$ implies that $T \sqsubseteq U$.

Theorem

The logic wK4 *is sound and complete for the class of weakly transitive frames. Moreover,* wK4 *is canonical.*

Unifying Kripke and topological semantics

Definition

A **derivative space** is a pair (X, d) where X is a set and $d\colon 2^X \to 2^X$ satisfies

- $d\varnothing = \varnothing$
- $d(A \cup B) = dA \cup dB$
- $ddA \subseteq dA \cup A$

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Examples:

- If X is a topological space and d its Cantor derivative, (X, d) is a derivative space.
- If $(W, \sqsubset)$ is a wK4 frame, define $d_{\sqsubset}A := \sqsubset^{-1}(A)$. Then, $(W, d_{\sqsubset})$ is a derivative space.

The derivational μ-calculus

If $\mathcal{X} = (X, d)$ is a derivative space, a valuation $[\![\cdot]\!]$ on $\mathcal{X}$ is defined by setting $[\![\Diamond\varphi]\!] := d\,[\![\varphi]\!]$.

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Fact: If p is positive on $\varphi(p)$, then $A \mapsto [\![\varphi(A)]\!]$ is a monotone operator.

Hence the μ-calculus extends to derivative spaces by letting $[\![\mu p.\varphi(p)]\!]$ be the least fixed point of $A \mapsto [\![\varphi(A)]\!]$.

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Theorem (Dawar and Otto 2009)

Every formula of the μ-calculus is equivalent to a formula in $\mathcal{L}_{\Diamond\Diamond^\infty}$ over the class of T_D spaces: derivative spaces validating

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Theorem (Baltag, Bezhanishvili, F-D)

The language $\mathcal{L}_{\Diamond\Diamond^{\infty}}$ is not expressively complete over T_0 spaces.

Axiomatizing the μ-calculus

If Λ is a normal logic, define μ-Λ by adding

- $\varphi(p) \to \varphi(\mu p.\varphi(p))$
- $\dfrac{\varphi(\psi) \to \psi}{\mu p.\varphi(p) \to \psi}$

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Theorem (Walukiewicz, 2000)

μ-K *is sound and complete for the class of Kripke frames.*

Theorem (Goldblatt, Hodkinson 2018)

μ-S4 *is sound and complete for the class of finite closure spaces, and for any dense-in-itself metric space.*

The final submodel

Let $\mathcal{M}_c = (W_c, \sqsubset_c, [\![\cdot]\!]_c)$ be the canonical model for μ-wK4. This model is based on a wK4 frame, since wK4 is canonical.

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Final submodel: $\mathcal{M}_c^\Sigma = (W_c^\Sigma, \sqsubset_c^\Sigma, [\![\cdot]\!]_c^\Sigma)$ is the submodel of Σ-final theories.

Truth lemma for the final submodel

Lemma (Σ-Final Truth Lemma)

Let Σ be finite and closed under subformulas (and a few other operations, such as single negation). Let

$$\mathcal{M}_c^\Sigma = (W_c^\Sigma, \sqsubset_c^\Sigma, [\![\cdot]\!]_c^\Sigma)$$

be the canonical wK4 *model.*

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Theorem (Baltag, Bezhanishvili, F-D)

*The logic μ-*wK4 *is sound and complete for the class of* wK4 *frames.*

Shallow model property

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Fact: If Σ is finite, $\mathcal{M}_c^\Sigma$ is shallow.

Fact: Shallow frames are bisimilar to finite frames, so we further obtain the following:

Theorem (Baltag, Bezhanishvili, F-D)

The logic μ-wK4 *has the finite model property, hence is decidable.*

Cofinal subframe logics

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This includes μ-S4, μ-K4, and many other examples.

Topological completeness

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3. *The logic* μ-S4 *is sound and complete for the class of* T_D *spaces with topological closure.*
4. *The logic* μ-wK4T$_0$ *(which I won't define here) is sound and complete for the class of* T_D *spaces with topological closure.*

Proof of topological completeness

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Thank you!