A new proof of the stable arithmetic regularity lemma

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Background on Regularity Lemmas

Szemerédi's Regularity Lemma (1976):

Given $\epsilon > 0$, any finite graph has an equitable partition of size $n \leq \exp^{O(1/\epsilon^5)}(1)$, in which at least $(1-\epsilon)n^2$ pairs are ϵ-regular.

Malliaris-Shelah (2013): "Szemerédi regularity for stable graphs"

Given $k \geq 1$ and $\epsilon > 0$, any sufficiently large finite "k-stable" graph has an equitable partition of size $(1/\epsilon)^{O_k(1)}$, in which all pairs are ϵ-regular with edge densities within ϵ of 0 or 1.

Green (2005): "Arithmetic regularity for vector spaces over $\mathbb{F}_2$"

Suppose G is a finite abelian group of exponent 2 and $A \subseteq G$. Then for any $\epsilon > 0$, there is a subgroup H of index $n \leq \exp^{\epsilon^{-O(1)}}(1)$ such that A is "Fourier-uniform" in all but ϵn cosets of H.

Terry & Wolf: Stable arithmetic regularity

A subset A of a group G is k**-stable** if there do not exist $a_1, \ldots, a_k, b_1, \ldots, b_k \in G$ such that $a_i b_j \in A$ if and only if $i \leq j$.

Theorem (Terry-Wolf 2017/2018)
Suppose G is a finite abelian group and $A \subseteq G$ is k-stable. Then, for any $\epsilon > 0$, there is a subgroup $H \leq G$ of index $\exp((1/\epsilon)^{O_k(1)})$ such that for any $x \in G$,

$$|xH \cap A| < \epsilon|H| \text{ or } |xH \backslash A| < \epsilon|H|.$$

So if $D = \bigcup \left\{ xH : |xH \cap A| \geq \epsilon|H| \right\}$ then $|A \triangle D| < \epsilon|G|$.

The proof is modeled after Malliaris-Shelah and, in particular, the correspondence between the order property and $R(-, \phi, 2)$-rank. They also use tools from discrete Fourier analysis.

Background: Stable arithmetic regularity

Theorem (C.-Pillay-Terry 2017)
Suppose G is a finite group and $A \subseteq G$ is k-stable. Then, for any $\epsilon > 0$, there is a normal subgroup $H \leq G$ of index $O_{k,\epsilon}(1)$, and a set D which is a union of cosets of H, such that $|A \triangle D| < \epsilon|H|$.

Note: for any $x \in G$, either $xH \cap A \subseteq A \triangle D$ or $xH \backslash A \subseteq A \triangle D$.

Compare & Contrast

Summary

- TW: A k-stable set in a finite abelian group G is $\epsilon|G|$-approximated by a union of cosets of a subgroup H of index $\exp((1/\epsilon)^{O_k(1)})$.
- CPT: A k-stable set in a finite group G is $\epsilon|H|$-approximated by a union of cosets of a normal subgroup H of index $O_{k,\epsilon}(1)$.

Questions

(1) Can the bound in TW be improved to $(1/\epsilon)^{O_k(1)}$?

(2) Can $\epsilon|G|$ in TW be improved to $\epsilon|H|$ with comparable bounds?

(3) What is an explicit bound for $O_{k,\epsilon}(1)$ in CPT?

Goal: A new proof of CPT, which answers these three questions.

Main Result

(1) Can the bound in TW be improved to $(1/\epsilon)^{O_k(1)}$?

(2) Can $\epsilon|G|$ in TW be improved to $\epsilon|H|$ with comparable bounds?

(3) What is an explicit bound for $O_{k,\epsilon}(1)$ in CPT?

Theorem (C. 2020)
Suppose G is a finite group and $A \subseteq G$ is k-stable. Then, for any $\epsilon > 0$, there is a subgroup $H \leq G$ of index $(1/\epsilon)^{O_k(1)}$, and a set D which is a union of left cosets of H, such that $|A \triangle D| < \epsilon|H|$.

- This gives positive answers to (1) and (2).
- For (3), a variation of the argument can be use to obtain a normal subgroup of index $\exp^{O_k(1)}(1/\epsilon)$.

Pseudofinite Setting

Let G be an ultraproduct of finite groups, and let μ be the normalized pseudofinite counting measure on internal subsets of G.

Fix an internal set $A \subseteq G$, which is k-stable for some $k \geq 1$.

Theorem (Pseudofinite stable arithmetic regularity, CPT)
There is an internal finite-index subgroup $H \leq G$ such that, for any $x \in G$, either $\mu(xH \cap A) = 0$ or $\mu(xH \backslash A) = 0$.

Remarks

- The previous theorem (and Łoś) yields the stable arithmetic regularity lemma, but with no explicit bounds.
- The proof in CPT uses local stability theory (Hrushovski-Pillay), including definability of types, symmetry of forking, finite equivalence relation theorem, dynamics of generic types.

Proof Sketch

Given $\epsilon \geq 0$, let $S_\epsilon(A) = \{x \in G : \mu(Ax \triangle A) \leq \epsilon\}$ ("ϵ-stabilizer" of A).

$S_\epsilon(A)^{-1} = S_\epsilon(A)$ and $S_\epsilon(A)^2 \subseteq S_{2\epsilon}(A)$. So $S_0(A)$ is a subgroup of G.

VC-theory (Haussler; Komlos-Pach-Woeginger)

(*a*) For $\epsilon > 0$, G can be covered by $(30/\epsilon)^{k-1}$ right translates of $S_\epsilon(A)$.

(*b*) If $X \subseteq G$ is internal and ℓ-stable, and $\mu(X) > 0$, then G can be covered by at most $8(\ell - 1)\mu(X)^{-2}$ right translates of X.

Suppose we know: $H := S_0(A) = S_\epsilon(A)$ for some $\epsilon > 0$.

By VC(*a*), H has finite index (at most $(30/\epsilon)^{k-1}$).

Exercise: H is internal.

Regularity

Assumption: $H := S_0(A) = S_\epsilon(A)$ for some $\epsilon > 0$.

Proposition

For any $g \in G$, either $\mu(gH \cap A) = 0$ or $\mu(gH \backslash A) = 0$.

Proof.

Suppose we have $g \in G$ such that $\mu(gH \cap A) > 0$ and $\mu(gH \backslash A) > 0$.

Set $B = H \cap g^{-1}A$ and $C = H \backslash g^{-1}A$.

Then B is k-stable and $\mu(B) > 0$. So G is covered by finitely many right translates of B by VC(b).

Since $\mu(C) > 0$, there is some $x \in G$ such that $\mu(Bx \cap C) > 0$.

Rewrite: $Bx \cap C = Hx \cap H \cap g^{-1}(Ax \backslash A)$.

So $x \in H$ and $\mu(Ax \backslash A) > 0$. This contradicts $H = S_0(A)$. □

The Stabilizer

Theorem
$S_0(A) = S_\epsilon(A)$ for some $\epsilon > 0$.

Proof. Suppose not.

Let $\phi(x; y)$ be the formula $x \in Ay_1 \triangle Ay_2$, where $y = (y_1, y_2)$.

Then $\phi(x; y)$ is k_*-stable, where $k_* = 2^{4^k - 1}$.

Given $b \in G \times G$, let $X(b) = \phi(G; b)$.

For all $\epsilon > 0$, there is some b such that $0 < \mu(X(b)) \leq \epsilon$.

Pick b_1 such that $\mu(X(b_1)) > 0$.

Pick c such that $0 < \mu(X(c)) < \mu(X(b_1))$.

There is some $g \in G$ such that $\mu\big(X(b_1) \cap X(c)g\big) > 0$.

Note that $X(c)g = X(b_2)$ for some $b_2 \in G \times G$.

We have $\mu\big(X(b_1) \cap X(b_2)\big) > 0$ and $\mu\big(X(b_1) \cap \neg X(b_2)\big) > 0$.

Stabilizer Proof (continued)

We have $\mu\big(X(b_1) \cap X(b_2)\big) > 0$ and $\mu\big(X(b_1) \cap \neg X(b_2)\big) > 0$.

Pick c' such that

$$0 < \mu(X(c')) < \min\left\{\mu\big(X(b_1) \cap X(b_2)\big), \mu\big(X(b_1) \cap \neg X(b_2)\big)\right\}$$

Find b_3 such that the following sets have positive measure:

$$\begin{aligned} a_3 &\in X(b_1) \cap X(b_2) \cap X(b_3) \\ a_2 &\in X(b_1) \cap X(b_2) \cap \neg X(b_3) \\ a_1 &\in X(b_1) \cap \neg X(b_2) \cap \neg X(b_3) \end{aligned}$$

Then $a_i \in X(b_j)$ (i.e., $\phi(a_i, b_j)$ holds) if and only if $i \geq j$.

Construct $b_1, \ldots, b_{k^*}$, violating k^*-stability of $\phi(x; y)$. $\square$

Finitization

Theorem (pseudofinite)

If G is a pseudofinite group and $A \subseteq G$ is internal and stable, then $S_0(A) = S_\epsilon(A)$ for some $\epsilon > 0$.

Theorem (finite)

Suppose G is a finite group and $A \subseteq G$ is k-stable. For any function $f\colon (0,1) \to (0,1)$ and $\epsilon > 0$, there is $\delta = \delta(\epsilon, f, k) < \epsilon$ and $\eta \in (\delta, \epsilon)$ such that $S_\eta(A) \subseteq S_{f(\eta)}(A)$.

Finitization

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If $f(\eta) \leq \frac{\eta}{2}$, then $H = S_\eta(A)$ is a subgroup of index $\leq (30/\eta)^{k-1}$.

Let $f(x) = x^{4k}$. For any $g \in G$, $|gH \cap A| < \frac{\eta}{m}|H|$ or $|gH \backslash A| < \frac{\eta}{m}|H|$.

Direct proof: $\delta = h^{k^*}(\epsilon)$, where $h(x) = xf(\frac{1}{2}x)^2/8k^*$.

So $\delta \geq \epsilon^{O_k(1)}$ and $m \leq (30/\eta)^{k-1} \leq (30/\delta)^{k-1} \leq (1/\epsilon)^{O_k(1)}$.

Choosing $f(x) = \exp(-x^{-k})$, one can replace H by $\bigcap_{g \in G} gHg^{-1}$. But this pushes the bound on the index to $\exp^{O_k(1)}(1/\epsilon)$.