Mathematics, meta-mathematics and conceptual change

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Overview

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Setting up the problem

Change in mathematics

Change in mathematics

Most general accounts of conceptual change either do not include mathematics because of its significant differences to natural sciences or they deny that conceptual change occurs in mathematics altogether.

Because these models of conceptual change have been largely drawn from Kuhn's framework of scientific revolutions, the argument is: Since scientific revolutions do not occur in mathematics, conceptual change (as we know it from the natural sciences) does not occur either.

Change in mathematics – negative

Michael Crowe in his “Ten ‘laws’ concerning patterns of change in the history of mathematics” (1975, 165) states:

> *Tenth law: Revolutions never occur in mathematics.*

However, the scope of the law hinges on the meaning of “in”. Crowe (1975, 166) warns the reader that

> *stress in law 10 on the preposition “in” is crucial, for [...] revolutions may occur in mathematical nomenclature, symbolism, metamathematics (e.g. the metaphysics of mathematics), methodology (e.g. standards of rigor), and perhaps even in the historiography of mathematics.*

Let’s call this the Argument-from-stressing-“in”.

Change in mathematics – negative

The Argument-from-stressing-“in” is used by Crowe e.g. for the case of the rise of non-Euclidean geometry: it is a revolutionary change in the philosophy of mathematics, not *in* mathematics.

The Argument-from-stressing-“in” has been used by other authors, such as Dunmore (1992), too: Revolutions may occur only in meta-mathematics, not *in* mathematics.

Against the argument-from-stressing-“in”

Pourciau (2000) offers a direct rebuttal to the Argument-from-stressing-“in” by arguing that revolutions in mathematics are conceptually and historically possible:
Example: Brouwer’s intuitionist mathematics as a failed revolution.

Also François and Van Bendegem (2010, 113) sensibly interject: “What is left *in* mathematics when we strip all those aspects of mathematics [nomenclature, symbolism, metamathematics, rigor, etc.] away?”

(A similar argument can be found in Mehrtens (1976).)

Change in mathematics – positive

Other historians accept the "revolution"-talk in mathematics perhaps on a more inclusive definition of "in mathematics":

Cohen (1985) discusses revolutions that occurs when the method of solving mathematical problems are radically changed on a large scale (altering the nature of the problems posed as well as the scope and content of mathematics). Example: Descartes' *Geometry*.

Dauben (1992) makes the case for the existence of 'critical' or 'discontinuous' transitions or shifts in mathematical development. Example: Cantor's introduction of transfinite numbers.

Taking stock: The question if conceptual change occurs in mathematics seems to hinge on (amongst others) the differentiation between mathematics and meta-mathematics.

The distinction between mathematics and meta-mathematics

Mathematics and (mathematical) logic

Consider the following quote from the literature on mathematical explanation: "To avoid the corrupting influence of philosophical intuitions, I have [...] tried to use examples from workaday mathematics rather than from logic, set theory, and other parts of mathematics that have important philosophical connections." (Lange 2014) *and* (Lange 2016)

In *Mathematical knowledge and the Interplay of Practices*, Ferreiros (2016) regards set-theoretic practice as mathematical up to the point of (roughly) the 1960's whereas later set-theoretic work is classified as meta-mathematical.

Also Feferman argues for set theory to be meta-mathematics (and equals this with logic), stating that this leads to loss of interest from mathematicians in questions such as the Continuum Hypothesis.

Arguments for the classification of logic as meta-mathematics

Arguments vary, but they mostly seem to revolve around three topics:

- The argument via objects,
- the argument via methods, and
- the argument via epistemic interest.

("Mathematical logic" here refers to the fields of set theory, model theory, proof theory and computability theory; however examples will mostly be from set theory.)

Argument via objects

Claim: Mathematical logic is not mathematics, because it does not deal with mathematical objects (or concepts, entities etc.). Instead it deals with meta-mathematical/logical objects.

For set theory this supports the thesis of a shift from mathematics to meta-mathematics (taken from Feferman):

Objects from “early” set theory: set, ordinal, cardinal etc.

Objects from later, more modern, set theory: constructible set, elementary embedding, superstrong cardinal etc.

Borderline cases: weakly and strongly compact cardinals etc.

Argument via methods

Claim: Mathematical logic is not mathematics, because it does not use (at its core) mathematical methods. Instead it uses meta-mathematical methods.

Again Feferman on set theory: "[One] also sees that [the *Handbook of set theory*] is almost entirely concerned with set theory as an axiomatic subject for which the methods to be applied are those of mathematical logic. It is true that various of the concepts involved may be understood in ordinary mathematical terms, but the essence of their use lies in their logical properties."

He gives the ultraproduct construction and forcing as the main examples.

Argument via epistemic interest

Claim: Mathematical logic pursues epistemic interests that are of no interest or different from those mathematicians pursue.

In mathematics one cares about if a proposition is true or false; in meta-mathematics it is true, false or undecidable with respect to some axiomatization or model.

Combining the discussions

Conceptual change in meta-mathematics

Combining the two above accounts seems to imply the following unified picture: Conceptual change is not possible in mathematics, but it is possible in meta-mathematics. Then it follows that

1. it is possible to have conceptual change in areas such as set theory, model theory, proof theory and recursion theory; and
2. if we do find examples of conceptual change in these areas, this does not provide us with examples of conceptual change in mathematics.

I think that the unified picture is not correct. In particular, 1. is right but for the wrong reasons and 2. is simply wrong by itself.

Problems

Problem 1: What is meant by meta-mathematics in the conceptual change literature is not the same as the meta-mathematics talked about in delineating mathematical logic from mathematics.

Problem 2: Neither of these two distinctions are appropriate for their purposes.

Considering problem 1

For Crowe (1975, 166), meta-mathematics is the metaphysics of mathematics; for Dunmore (1992) it is about the mathematicians' beliefs about their discipline.

This is however not the same as the "arguments via objects" or "arguments via epistemic interests". In the conceptual change debate the gap between mathematics and meta-mathematics is much wider; the proposed meta-level encompasses classical areas of philosophy.

Considering problem 1

The distinction for mathematical logic as meta-mathematics does not go so far; it is more reminiscent of the object- vs. meta-language distinction:

Mathematics is on the object-level that is investigated on the "meta-level" of meta-mathematics.

Example: Mathematicians study mathematical objects such as numbers, functions etc; set-theorists study these object on the "meta-level" of models and axiomatizations. Also, model-theorists study the models themselves, therefore operating on a meta-(meta?-)level.

Considering problem 1

Problem 1: What is meant by meta-mathematics in the conceptual change literature is not the same as the meta-mathematics talked about in delineating mathematical logic from mathematics.

Clarification: In the conceptual change literature the meta-level that allows change refers to genuinely philosophical issues about mathematics; here mathematical logic would count as mathematics.

Making the debates compatible

To make the debates compatible, we can either argue that mathematical logic, as meta-mathematics, should not count towards mathematics; therefore if a change occurs in logic it does still not signify a change in mathematics.

Or we can argue that mathematical logic should count towards mathematics, therefore making a change in logic a valid example of a change in mathematics.

Here, we will take the second route by refuting the arguments by objects, by methods and by epistemic interest.

Mathematics and meta-mathematics, again

Contra the argument via epistemic interest

Claim: Mathematical logic pursues epistemic interests that are of no interest or different from those mathematicians pursue.

In mathematics one cares about if a proposition is true or false; in meta-mathematics it is true, false or undecidable with respect to some axiomatization or model.

At least for set-theory, this is not accurate: In large parts of set theory concerned with independence, the work with different models and axiomatizations follows the same epistemic interest as what is ascribed to (normal) mathematicians. It only requires more nuanced methods.

Contra the argument via methods

Claim: Mathematical logic is not mathematics, because it does not use (at its core) mathematical methods. Instead it uses meta-mathematical methods.

Consider the example of forcing: Forcing is a logical method in theory; at its core stand relations between models. However from its practice it is mathematical in the objects and methods it employs: Using sets to build partial orders, proving properties about cardinals etc.

The logical part of forcing (proving axioms, getting access to the extension etc.) is used as a black box, the "real work" that is done is mathematical.

Contra the argument via objects

Claim: Mathematical logic is not mathematics, because it does not deal with mathematical objects (or concepts, entities etc.). Instead it deals with meta-mathematical/logical objects.

What makes an object mathematical/meta-mathematical?

The example of models: They first occurred in logic as a "study of formal languages and their interpretations" (Hodges 2020). However then they became objects of mathematical interest and research: they are studied with mathematical methodology, following mathematical interests (investigating its properties, proving theorems about them etc). (See also Baldwin 2018).

The reverse argument: from philosophy to mathematics

For set theory (Kanamori 2008) offers an argument in reverse to Ferreiros (2016) and Feferman:

> *With Cohen there was an infusion of mathematical thinking and of method and a proliferation of models, much as in other modern, sophisticated fields of mathematics. [...] This further drew out that in set theory as well as in mathematics generally, it is a matter of method, not ontology. [...]*
>
> *Forcing has thus come to play a crucial role in the transformation of set theory into a modern, sophisticated field of mathematics [...].*

This is actually also what Cohen thought about his development of forcing (i.e. that it was less philosophy and more mathematics).

Practices and uses

Conclusion: There is no clear demarcation line between mathematics and logic in the sense of mathematics vs. meta-mathematics.

(This could mean that logic is simply not a good example for meta-mathematics or that meta-mathematics is actually mathematics.)

In particular, practice in set theory is mathematical rather than meta-mathematical. This includes the methods and objects used in research and the epistemic interests that are pursued.

Of course, this does not exclude possible epistemic uses of these areas outside of mathematics, for example philosophy.

Outlook: Conceptual change in set theory

Taking stock

We have solved Problem 1 (making the debates compatible) by showing that set theory is mathematics, despite its meta-mathematical uses.

We have also addresses one part of Problem 2 by showing that the distinction between mathematics and meta-mathematics is not appropriate for the discussion about the practice, status and uses of set theory.

(The other part was already criticised by (François and Van Bendegem 2010), (Mehrtens 1976) and others.)

However we think that the discussion about the distinction between mathematics and meta-mathematics for set theory points towards something else, namely a conceptual change in set theory.

A shift from what to what?

Ferreiros (2016) and Feferman argue that set theory shifted from mathematics to meta-mathematics; Kanamori (2008) and Cohen (2002) argue that it went from philosophy to mathematics.

Both refer to forcing and the changes it introduced in the 1960's and 1970's.

We see this as evidence for a fundamental shift in set theory during this time, however not (primarily) a shift between disciplines in the above sense, but a conceptual change in set theory as a mathematical discipline.

This means that we agree with the observations of all of the above authors that something changed, but we disagree with what changed.

Tracing the change

Interestingly the change can be traced in the very elements used to argue for (or against) set theory being meta-mathematics:

A change in objects or concepts (e.g. models), a change in methods (e.g. forcing), a change in epistemic interests (e.g. independence), possible other areas of change. All in all this amounts to a change in the practice of set theory.

One possible way of interpreting this change is to say that set theory changed from a concrete to an algebraic mathematical theory (see Hamkins (2012)).

However, this still is a change in mathematics!

Thank you for your attention!

Time for questions, comments, critique, wider context, other vague ideas etc.

References i

John T. Baldwin. *Model Theory and the Philosophy of Mathematical Practice: Formalization without Foundationalism*. Cambridge University Press, 2018. doi: 10.1017/9781316987216.

Paul J. Cohen. The discovery of forcing. *Rocky Mountain Journal of Mathematics*, 32(4):1071–1100, 2002.

Michael J Crowe. Ten "laws" concerning patterns of change in the history of mathematics. *Historia Mathematica*, 2(2):161–166, 1975.

Caroline Dunmore. Meta-level revolutions in mathematics. In Donald Gillies, editor, *Revolutions in Mathematics*, pages 209–225. Oxford University Press, 1992.

Solomon Feferman. Is the continuum hypothesis a definite mathematical problem?

Jose Ferreiros. *Mathematical Knowledge and the Interplay of Practices*. Princeton, USA: Princeton University Press, 2016.

References ii

Karen François and Jean Paul Van Bendegem. Revolutions in mathematics. more than thirty years after crowe's "ten laws". a new interpretation. *PhiMSAMP. Philosophy of mathematics: Sociological aspects and mathematical practice*, 11:107–120, 2010.

Joel D. Hamkins. The set-theoretic multiverse. *The Review of Symbolic Logic*, 5:416–449, 2012.

Wilfrid Hodges. Model Theory. In Edward N. Zalta, editor, *The Stanford Encyclopedia of Philosophy*. Metaphysics Research Lab, Stanford University, winter 2020 edition, 2020.

Akihiro Kanamori. Cohen and set theory. *Bulletin of Symbolic Logic*, pages 351–378, 2008.

Marc Lange. Aspects of mathematical explanation: Symmetry, unity, and salience. *Philosophical Review*, 123(4):485–531, 2014. doi: 10.1215/00318108-2749730.

Marc Lange. Explanatory proofs and beautiful proofs. *Journal of Humanistic Mathematics*, 6:8–51, 01 2016. doi: 10.5642/jhummath.201601.04.

Herbert Mehrtens. Ts kuhn's theories and mathematics: A discussion paper on the "new historiography" of mathematics. *Historia Mathematica*, 3(3):297–320, 1976.

Bruce Pourciau. Intuitionism as a (failed) kuhnian revolution in mathematics. *Studies in History and Philosophy of Science Part A*, 31 (2):297–329, 2000.