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Consensus & Polarization in Constrained Opinion Dynamics

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Outline

Talk based on the preprint arXiv:1104.5147

Emergence of diversity is a complex problem in life and behavioural sciences: what are the main features to be incorporated in basic mathematical models? How to treat those models?

- Basic and unifying principles of evolutionary dynamics
- The 2-state voter model
- The 3-species constrained voter model
  - The model and solution method
  - The fixation probabilities
  - The mean fixation times
- Conclusion

Basic principles of evolutionary dynamics

Basic and unifying principles for modelling evolutionary dynamics:

- Dynamics proceeds by imitation: traits/strategies/individuals die and reproduce (are copied) at each time step.
- Selection: successful traits/strategies/individuals spread at the expense of the less fit
- Population is finite and there are demographic fluctuations ("genetic/random" drift)
- Mutations: traits/strategies/individuals can spontaneously switch with some rate
- Migration: traits/strategies/individuals can move and spread in space ("islands", "patches",...)

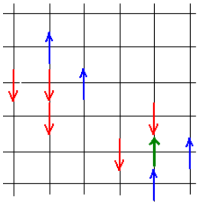
These principles transcend biology (genetics, ecology) and behavioural science (evolutionary game theory and opinion dynamics).

Key questions: how is diversity maintained?
In opinion dynamics: when is there consensus / cultural diversity?

Paradigmatic opinion dynamics model: **the voter model** (closely related to the Moran model and other model of imitation dynamics))

2-state Voter Model

Voter Model (Liggett 1985): Basic/paradigmatic two-state model where individuals are either in $+1$ ($\uparrow$) or -1 ($\downarrow$) opinion state
Dynamics: at each time step an individual adopt the opinion state of a randomly picked neighbour



N voters on a complete graph, initial fraction of $+1$ is x:

- Consensus is always reached
- Probability to reach $+1$ and -1 consensus is x and $1-x$, resp.
- Mean time to attain consensus is $T \sim N$

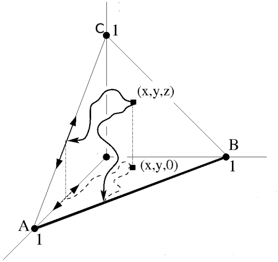
2-state VM cannot explain the emergence of cultural diversity
Axelrod (1997) & bounded comprise models (Deffuant et al., 2000): **consensus** limited by **incompatibility** $\Rightarrow$ *cultural diversity (?)*

The 3-species constrained voter model

N individuals of 3 species, A, B and C. A's (leftists) and B's (rightists) are *incompatible* (don't interact), but interact with centrists C's

$$AB \to AB; \quad AC \to AA \quad \text{rate}: \frac{1+q}{2}, \quad AC \to CC \quad \text{rate}: \frac{1-q}{2}$$
$$BC \to BB \quad \text{rate}: \frac{1+q}{2}, \quad BC \to CC \quad \text{rate}: \frac{1-q}{2}$$

4 possible outcomes: consensus with A, B or C, or frozen mixture of A and B (polarization) $\Rightarrow$ **What is the (fixation) probability and mean time for each of these events starting with densities x, y, $z = 1 - x - y$ of A, B and C? Here, $0 < |q| \leq 1$.**



- $q > 0$: bias towards *polarization* (extremisms), with absorbing line $x + y = 1$
- $q < 0$: bias towards *centrism* (appeasement)
- $q = 0$: driven by fluctuations [*Vazquez & Redner* in J.Phys.A **37**, 8479 (2004)]

Multi-faceted mathematical approach

The analysis of complex systems requires the complementary use of various mathematical tools

- Nonlinear dynamics and theory of ODEs & PDEs: mean field, separation of variables
- Individual-based modelling: stochastic processes, intrinsic noise
- Mapping onto related models across disciplines (here, population genetics, evolutionary game theory, quantum mechanics)
- Stochastic simulations (here, Gillespie algorithm)

Other related approaches (not in this work):

- Theory of network dynamics
- Spatially-extended interacting particle systems
- Theory of pattern formation

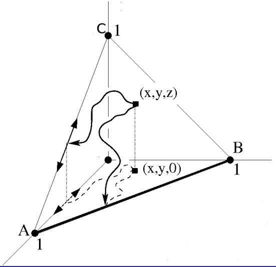
3-species constrained voter model: Mean-field

Mean field picture: all fluctuations are neglected (assume that $N = \infty$)
Rate equations for density a of A's and b of B's (with $c = 1 - a - b$):

$$\frac{d}{dt}a = \frac{(1+q)}{2}ac - \frac{(1-q)}{2}ac = qa(1-a-b), \quad \frac{d}{dt}b = qb(1-a-b)$$

$$a(t) = \frac{xe^{qt}}{1-(x+y)(1-e^{qt})}, \quad b(t) = \frac{ye^{qt}}{1-(x+y)(1-e^{qt})}$$

Ratio $a/b = x/y$ is conserved. 3 absorbing fixed points:
$(a,b,c) \in \{\mathscr{A} = (1,0,0), \mathscr{B} = (0,1,0), \mathscr{C} = (0,0,1)\}$+ *(polarization)*
line of fixed points $\mathscr{A}\mathscr{B} = (a, 1-a, 0)$, with $0 < a < 1$



- When $q > 0$:
  $a \to \frac{x}{x+y}$, $b \to \frac{y}{x+y}$, $c \to 0$
  (*polarization*)
- When $q < 0$:
  $a \to 0$, $b \to 0$, $c \to 1$
  (*centrism*)

3-species constrained VM: Individual-based approach

When population is finite, $N < \infty$, fluctuations alter the mean field predictions: polarization is likely when $q > 0$, but still finite probability to reach a consensus; the opposite happens when $q < 0$

Finite and well-mixed ("complete graph") population ⇒ **Stochastic formulation**:
Probability $P^{\mathscr{A}\mathscr{B}}$ that the final state is a frozen mixture of extremists obeys the backward master equation (ME):

$$
\begin{aligned}
& (T_x^+ + T_x^- + T_y^+ + T_y^-) P^{\mathscr{A}\mathscr{B}}(x,y) = \\
& T_x^- P^{\mathscr{A}\mathscr{B}}(x-\delta, y) + T_x^+ P^{\mathscr{A}\mathscr{B}}(x+\delta, y) \\
+ \quad & T_y^- P^{\mathscr{A}\mathscr{B}}(x, y-\delta) + T_y^+ P^{\mathscr{A}\mathscr{B}}(x, y+\delta)
\end{aligned} \tag{1}
$$

+ boundary conditions. With $T_\xi^\pm \equiv (1 \pm q)\xi(1-x-y)/2$, $\xi \in (x,y)$ and $\delta = N^{-1}$
Analytical progress: expand the ME to 2nd order in $\delta \to$ Fokker-Planck equation. Analogy with models of population genetics and evolutionary game theory

3-species voter model: Mathematical treatment (II)

By Taylor expansion of the ME (1):

$$\left\{2s[x\partial_x + y\partial_y] + x\partial_x^2 + y\partial_y^2\right\} P^{\mathscr{A}\mathscr{B}}(x,y) = 0, \qquad (2)$$

with $P^{\mathscr{A}\mathscr{B}}(x,0) = P^{\mathscr{A}\mathscr{B}}(0,y) = 0$ and $P^{\mathscr{A}\mathscr{B}}(x,1-x) = 1$ and $\mathbf{s} \equiv \mathbf{Nq}$

Equation is **separable**. With polar coordinates $\sqrt{x} = \rho\cos\theta, \sqrt{y} = \rho\sin\theta$: $P^{\mathscr{A}\mathscr{B}} = \sum_n c_n R_n(\rho) u_n(\theta) \sin(2\theta)$, yielding

$$\rho^2 \frac{d^2 R_n}{d\rho^2} + \rho \frac{dR_n}{d\rho}[4s\rho^2 - 1] - \lambda_n R_n = 0$$

$$\frac{d^2 u_n}{d\theta^2} - \frac{3}{4}\left(\frac{1}{\sin^2\theta} + \frac{1}{\cos^2\theta}\right) u_n + (1+\lambda_n) u_n = 0$$

BCs: $P^{\mathscr{A}\mathscr{B}}(\rho = 0, \theta) = 0$ and $P^{\mathscr{A}\mathscr{B}}(\rho = 1, \theta) = 1$

Mapping onto a stationary Schrödinger equation with Pöschl-Teller potential $\Rightarrow \lambda_n = 4(n+1)(n+2)$

$$P^{\mathscr{A}\mathscr{B}}(x,y) = 2\sqrt{\frac{xy}{x+y}}\, e^{s(1-x-y)} \sum_{n\,\mathrm{odd}} \frac{2n+1}{n(n+1)} P_n^1\left(\frac{x-y}{x+y}\right) \frac{I_{n+1/2}(s(x+y))}{I_{n+1/2}(s)}$$

I_n's and P_n^1's: Modified Bessel functions & associated Legendre Polynomials

3-species voter model: Mathematical treatment (III)

Probability density along polarization line is $F_a^{\mathscr{A}\mathscr{B}}$, with $P^{\mathscr{A}\mathscr{B}}(x,y) = \int_0^1 da F_a^{\mathscr{A}\mathscr{B}}(x,y) \Rightarrow F_a^{\mathscr{A}\mathscr{B}}$ obeys (2) with BC $F_a^{\mathscr{A}\mathscr{B}}(x,1-x) = \delta(a-x)$

Fixation probabilities $P^{\mathscr{S}}(x,y)$ in states $\mathscr{S} \in \{\mathscr{A},\mathscr{B},\mathscr{C},\mathscr{A}\mathscr{B}\}$ satisfy

$$P^{\mathscr{A}}(x,y) + P^{\mathscr{B}}(x,y) + P^{\mathscr{A}\mathscr{B}}(x,y) = a+b = 1 - P^{\mathscr{C}}(x,y) = \frac{1-e^{-2s(x+y)}}{1-e^{-2s}},$$

with $P^{\mathscr{B}}(x,y) = P^{\mathscr{A}}(y,x)$. Futhermore, steady state densities are:

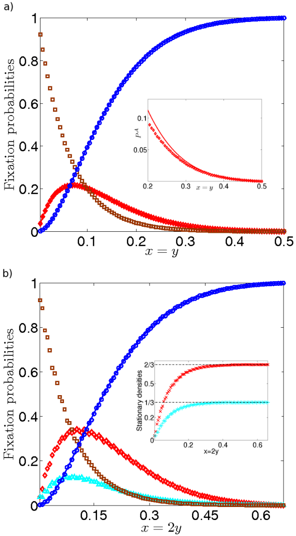
$$\begin{aligned} a &= P^{\mathscr{A}}(x,y) + \int_0^1 da'\, a' F_{a'}(x,y) \\ b &= P^{\mathscr{B}}(x,y) + P^{\mathscr{A}\mathscr{B}}(x,y) - \int_0^1 da'\, a' F_{a'}(x,y) \end{aligned}$$

- When $s = N|q| \gg 1$: drift dominates over diffusion terms $\Rightarrow$ like mean field
- When $s \ll 1$: drift negligible $\Rightarrow \approx$ like the unbiased case (JPA37, 8479 (2004))
- **Effective (interesting) competition arises when $s = Nq = \mathscr{O}(1)$**

Fixation probabilities when $s = Nq > 0$

Top: Fixation probabilities for $s > 0$ as functions of x (for $N = 200, s = 4$, i.e. $q = 0.02$): $P^{\mathscr{A}}$ ($\diamond$), $P^{\mathscr{B}}$ ($\triangle$), $P^{\mathscr{C}}$ ($\Box$); $P^{\mathscr{A}\mathscr{B}}$ ($\circ$)
Solid line: analytical sol. of (2)

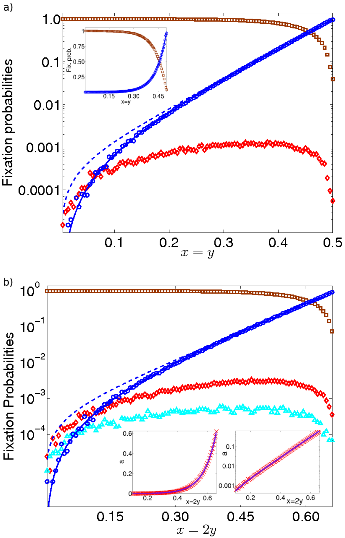
Bottom: same as above, but with $x = 2y$.
Inset: final densities of species A ($\times$) and B ($+$) as functions of $x = 2y$.



Fixation probabilities when $s < 0$

Top: as before, with $s = -4$
Comparison with analytics:
Solid: solution of (2)
Dashed:
$P^{\mathscr{A}\mathscr{B}} \approx 1 - P^{\mathscr{C}} = \frac{e^{2|s|(x+y)}-1}{e^{2|s|}-1}$

Bottom: as above, but with $x = 2y$.
Inset: stationary density of A ($\times$) as function of $x = 2y$ compared with analytics



3-species voter model: Mathematical treatment (IV)

The unconditional mean fixation time (MFT) τ to reach *any* of the system's absorbing states obeys the backward Fokker-Planck equation with boundary conditions $\tau(1,0)=\tau(0,1)=\tau(0,0)=\tau(a,1-a)=0$. With $w\equiv x+y$:

$$\mathscr{L}_{\mathrm{bFP}} = \frac{w(1-w)}{N}\left[2s\frac{d}{dw}+\frac{d^2}{dw^2}\right]$$
$$\mathscr{L}_{\mathrm{bFP}}(w)\tau(w) = -1, \quad \text{with} \quad \tau(0)=\tau(1)=0 \Rightarrow \tag{3}$$

Useful mapping with a population genetics model (s is "selection strength"). Solution (for $w=1/2$):

$$\begin{aligned}\tau &= Nf_\tau(s,w=1/2) = \frac{N}{s(1+e^s)}\left[e^{2s}\mathrm{Ei}(-2s)+(e^s-1)\ln(2|s|)\right]\\ &+ \frac{N}{2se^s(1+e^s)}\left[e^s(e^s-1)\gamma_{\mathrm{E}}+\mathrm{Ei}(s)-\mathrm{Ei}(2s)\right]\\ &+ \frac{N}{2s(1+e^s)}\left[2\mathrm{Ei}(s)-2e^s(e^s+1)\mathrm{Ei}(-s)\right],\end{aligned}$$

where $\mathrm{Ei}(x)\equiv\int_{-\infty}^{x}\frac{e^t}{t}\,dt$ and $\gamma_{\mathrm{E}}=0.5772\ldots$

Unconditional & Conditional Mean Fixation Times

The unconditional MFT $\tau = \tau(x+y)$:

- is a function of initial density of extremists $(x+y)$
- scales linearly with N
- symmetry: invariant under $(s, x+y) \to (-s, 1-x-y) \Rightarrow$ $\tau(x+y) = Nf_\tau(s, x+y) = Nf_\tau(-s, 1-x-y)$
- f_τ has an inverted u-shape dependence on $w = x+y$

The *conditional* mean fixation times $\tau^{\mathscr{S}}$, to reach the specific absorbing state $\mathscr{S} \in (\mathscr{A}, \mathscr{B}, \mathscr{C}, \mathscr{A}\mathscr{B})$ obey:
$\mathscr{L}_{\mathrm{bFP}}(x,y)[P^{\mathscr{S}}(x,y)\tau^{\mathscr{S}}(x,y)] = -P^{\mathscr{S}}(x,y)$ *(+BC's)*,
where the $P^{\mathscr{S}}(x,y)$'s are the fixation probabilities

- All $\tau^{\mathscr{S}}$'s are found to scale linearly with N
- The $\tau^{\mathscr{S}}$'s do not depend on the sign of s *(more noisy when $s < 0$)*
- The extremists' MFTs, $\tau^{\mathscr{A}}$ and $\tau^{\mathscr{B}}$, are always the longest MFTS

Mean Fixation Time

Normalized unconditional MFT τ/N ($\times$) compared with sol. of (3) (solid)

Normalized conditional MFTs

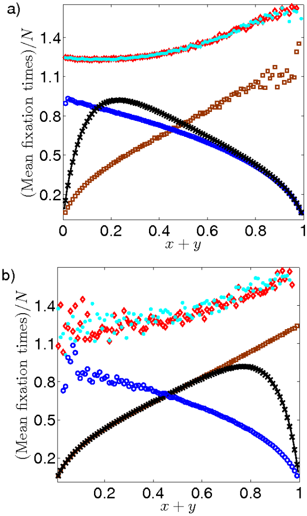
$\tau^{\mathscr{A}}/N$ ($\diamond$)

$\tau^{\mathscr{B}}/N$ ($\bullet$)

$\tau^{\mathscr{C}}/N$ ($\square$)

$\tau^{\mathscr{A}\mathscr{B}}/N$ ($\circ$).

$N = 200$, $x = y$, $s = 4$ (top) and $s = -4$ (bottom). Average is over 2×10^5 samples.



Outlook & Conclusion

Understanding the origin and maintenance of diversity in evolutionary dynamics

Seek of consensus & incompatibility: relevant ingredients for cultural diversity in opinion dynamics?

Study of a mathematically amenable model: the 3-state constrained voter model:

- Possible outcomes: consensus with extremists, consensus with centrists, polarization of extremism (“leftists” and “rightists” coexist)
- Bias ($\sim$ selection) $\rightarrow$ nonlinearity. Finite population $\rightarrow$ intrinsic noise
- Small bias ($q \sim N^{-1}$, “weak selection”): subtle competition between drift and fluctuations
- Combination of mathematical methods to tackle the problem

How relevant is all of this?

- Realistic ingredients: mutations, dispersal, network
- **Would it be possible to validate a variant of the model using real data?**