Supplemental Material to:

Evolution of a Fluctuating Population in Randomly Switching Environment

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In this Supplemental Material, we provide some technical details and supplementary information in support of the results discussed in the main text, as well as additional results concerning the population’s mean fixation time and its long-time distribution. We also comment the content of electronically available Videos [1] that illustrate the population dynamics in the pure resource competition and public good scenarios. In what follows, unless otherwise stated, the notation is the same as in the main text and the equations and figures refer to those therein. (As in the main, unless stated otherwise, below we tacitly assume \( x_0 = 1/2 \).)

1 Derivation of Equations (1,2) and (4)

As explained in the main text, the population dynamics is governed by multivariate birth-death process in which at each time increment an individual of species \( \alpha \in \{S,F\} \) is picked for reproduction, \( N_\alpha \to N_\alpha + 1 \), with transition rate \( \mathcal{T}_\alpha^+ = \mathcal{T}_\alpha^+ N_\alpha = g(x) f_\alpha N_\alpha / \bar{f} \) or death, \( N_\alpha \to N_\alpha - 1 \), with transition rate \( \mathcal{T}_\alpha^- = \mathcal{T}_\alpha^- N_\alpha = (N/K)N_\alpha \), or the carrying capacity is switched, \( K_+ \leftrightarrow K_- \), with rate \( \nu \). When internal noise is neglected, \( N \) and \( x \) evolve according to the mean-field rate equations

\[
\dot{N} = \sum_{\alpha=S,F} (\mathcal{T}_{\alpha}^+ - \mathcal{T}_{\alpha}^-) = N \left( g(x) - \frac{N}{K} \right),
\]

\[
\dot{x} = \frac{\mathcal{T}_S^+ - \mathcal{T}_S^-}{N} - x \frac{\dot{N}}{N} = -sxg(x) \frac{x(1 - x)}{1 - sx},
\]

where we have used \( f_F = 1 \), \( f_S = 1 - s \) and \( \bar{f} = 1 - sx \). These equations coincide with (1) and (2) and, when the carrying capacity \( K \) is constant, they provide a suitable description of the ecological and evolutionary (composition).

The deterministic description of the population dynamics in terms of (1) and (2) is valid only in the absence of internal and external noise.

When the carrying capacity randomly switches according to \( K(t) = \frac{1}{2} \left( [K_+ + K_-] + \xi(t) [K_+ - K_-] \right) \), where \( \xi \in \{-1, +1\} \) is the dichotomous noise defined in the main text, the equation for \( N \) becomes the following stochastic differential equation obtained by substituting \( K(t) \) into (1) and using \( \xi^2 = 1 \):

\[
\dot{N} = N \left( g(x) - \frac{2N}{K_+ + K_- + \xi(t) [K_+ - K_-]} \right) = N \left( g(x) - \frac{2N}{2K_+K_-} [K_+ + K_- - \xi(t) [K_+ - K_-]] \right)
\]

\[
= N \left( g(x) - \frac{N}{K} + \xi N \left( \frac{K_+ + K_-}{2K_+K_-} - \frac{2K_-}{2K_+K_-} \right) \right) = N \left( g(x) - \frac{N}{K} + \xi N \left( \frac{K_+ - K_-}{K_+K_-} \right) \right)
\]

where \( K = 2K_+K_- / (K_+ + K_-) \). This stochastic differential equation coincides with (4) and, together with (2), defines a piecewise deterministic Markov process (PDMP) [2, 3, 4] describing the population dynamics when the sole form of randomness is the random switching of the carrying capacity (internal noise is neglected).

2 Rescaling of the switching rate & fixation probability

We have studied the fixation probability \( \phi \) that, starting with a fraction \( x_0 \) of individuals of the slow type \( S \), the entire population eventually consists of \( N(t) \) individuals of species \( S \). The fixation of species \( F \) occurs with the complementary probability \( \tilde{\phi} = 1 - \phi \). We have investigated the joint effect of external (dichotomous) and internal (demographic) noise on these fixation probabilities by means of stochastic simulations (based on the Gillespie algorithm [5]) and,

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Figure S1: (Color online). (a) $\phi$ as function of $\nu$ with $(K_+, K_-) = (450, 50)$, $s = 0.02$ (o, blue/dark gray) and $s = 0.07$ (o, orange/light gray) in the pure competition case, $b = 0$. The solid lines are predictions from (6), they only match simulation results in the limits $\nu \rightarrow \infty$ and $\nu = 0$. Rescaling the switching rate to $\nu/s$ (inset), leaves the two limits untouched, but also provides accurate predictions in the intermediate regimes, see text. (b) $\phi$ as function of $b$ for $\nu = (0.002, 0.02, 0.2, 2)$ (top to bottom) and $(s, K_+, K_-) = (0.025, 450, 50)$ in log scale. Straight lines show $0.3e^{-0.35b}$ and $0.175e^{-0.75b}$ as eyeguides.

when $K_\rightarrow \gg 1$ and $\nu > 0$, with help of Eqs. (6) and (7) (see Figs. 1 and 3 in the main text). These formulas stem from a combination of the properties of a fitness-dependent Moran process [6, 7, 8] and the stationary probability density (PDF) (5) of the PDMP (2),(4) [2, 3, 4], with a rescaled switching rate. Here, we focus first on pure competition case ($b = 0$) and elaborate on the considerations underpinning the Ansatz (6):

- First of all, $0 < s \ll 1$ separates the timescale $O(1/s)$ on which $x(t)$ evolves, from that of $N$, which is $O(1)$, see Videos 1-2. This suggests to consider $N$ at quasi-stationarity by the time $S$ or $F$ fixates. When $K_\rightarrow \gg 1$, the stationary PDF $p^*_{\nu}(N)$ (5) describes well the behavior of the population size (see Figs. 2 and 3).

- In the limit of very fast switching rate, $\nu \rightarrow \infty$, the dichotomous noise of the environment self-averages, with $\xi \rightarrow \langle \xi \rangle = 0$ in Eq. (4) [4]. In this regime, the population quickly attains an effective size $N \approx \bar{K} \gg 1$, with $p^*_{\nu}(N)$ having a sharp peak at $N = \bar{K}$. The population’s internal evolution thus mirrors that of a population of constant size $\bar{K}$ obeying a fitness-dependent Moran process [6, 7, 8]. Therefore, the fixation probability of $S$, starting with an initial fraction $x_0$ is given by $\phi \rightarrow \infty = \phi_{\bar{K}} = (e^{-\bar{K}S(1-x_0)} - e^{-\bar{K}S})/(1 - e^{-\bar{K}S})$ [7], see main text. In the limit of very slow switching rate, $\nu \rightarrow 0$, the population is equally likely to be locked in either of the environmental state $\xi = -1$ or $\xi = +1$. The PDF $p^*_{\nu}(N)$ reflects this behavior by having singularities at $N = K_{\pm}$. As a consequence, fixation is as likely to occur in a population of fixed size $K_{\pm}$ or $K_{\pm}$. Hence, from the fixation properties of the Moran model [7], we infer $\phi \rightarrow 0 = (\phi_{K_-} + \phi_{K_+})/2$. We thus expect that averaging $\phi|_N$ over $p^*_{\nu}$ yields accurate predictions in the limits $\nu \rightarrow 0$ and $\nu \rightarrow \infty$. This is confirmed by the comparison of $\int_{K^-}^{K^+} \phi|_N p^*_{\nu}(N) \, dN$ with simulation results for $\phi$, see Fig. S1. However, this figure also shows that $\int_{K^-}^{K^+} \phi|_N p^*_{\nu}(N) \, dN$ underestimate or overestimate $\phi$ over a broad range of intermediate switching rate values (corresponding to a finite correlation time of the dichotomous noise).

- Since $0 < s \ll 1$ and $x$ relaxes on a timescale $O(1/s)$ while $N$ evolves on a much faster timescale $O(1)$, see Videos 1-2 [1], the ratio $\nu/s$ is the average number of switches experienced by $K(t)$ while $x$ relaxes (prior to fixation). When $\nu = O(1), \nu/s \gg 1$ and we expect a large number of switches prior to fixation and thus $\phi$ can be approximated by assuming the self-averaging of $\xi$, yielding $\phi \approx \phi^\infty$, as confirmed by Fig. S1(a). Furthermore, when $\nu \ll s$ no switches are expected prior to fixation and thus $\phi \approx \phi^{(0)}$ when $\nu \ll s$, as shown in Fig. S1(a). This suggests to carry out the average of $\phi|_N$ over the PDF (5) with a rescaled switching rate $\nu \rightarrow \nu/s$. This allows us to “synchronize” the environmental noise onto $x$’s relaxation timescale on which fixation occurs (see below). This yields $\int_{K^-}^{K^+} \phi|_N p^*_{\nu/s}(N) \, dN$ which is in excellent agreement with $\phi$ over a
very broad range of \( \nu \), see Figs. 2(a) and S1(a).

- In other words, to “synchronize” the environmental noise onto \( x \)'s relaxation timescale we rescale \( \dot{t} \to t = s \dot{t} \) and \( \xi(\dot{t}) \to \xi(t) \), where \( \dot{t} \) is the time measured in \( N \)'s faster timescale. Hence, the average of the noise stays \( \langle \xi(\dot{t}) \rangle \to \langle \xi(t) \rangle = 0 \), but its autocorrelation becomes \( \langle \xi(t) \xi(t') \rangle = e^{-2(\nu/s)|t-t'|} \to \langle \xi(t) \xi(t') \rangle = e^{-2\nu|t-t'|} \). This shows that on \( x \)'s relaxation timescale, the rate \( \nu \) is rescaled into \( \nu/s \). This is the switching rate to be considered to compute the fixation properties with Eq. (6) of the main text. As Figs. 2(a) and S1(a) show, with this rescaling, the predictions of Eq. (6) agree excellently with the simulation results.

The same lines of reasoning hold also for the case \( b > 0 \). However, since Eqs. (2) and (4) for \( N \) and \( x \) are coupled in this case, we use an effective parameter \( q \geq 0 \) in our analysis that is introduced by considering the auxiliary stochastic differential equation obtained by substituting \( g = 1 + q \) in Eq. (4), see Sec. 1, which yields

\[
\frac{\dot{N}}{N} = 1 + q - \frac{N}{K} = 1 + q - \frac{N}{K} + \xi \frac{N(K_+ - K)}{K K_+}. \tag{S1}
\]

This equation is decoupled from the rate equation (2) for \( x \) and corresponds to a PDMP [2, 4], describing how the size of an effective population evolves under the sole effect of the environmental noise. This PDMP is characterized by a probability \( p_{\nu,q}^\pm(N, t) = p_{\nu,q}(N, \xi = \pm 1, t) \) to be in state \( \{N, \xi\} \) at time \( t \) for \( q \) given, where

\[
\frac{\partial}{\partial t} p_{\nu,q}^\pm(N, t) = -\frac{\partial}{\partial N} \left[ N \left( 1 + q - \frac{N}{K} \right) p_{\nu,q}^\pm(N, t) \right] - \nu [p_{\nu,q}^\pm(N, t) - p_{\nu,q}^\mp(N, t)].
\]

By assuming that the probability current is zero at \( N = (1 + q)K_\pm \) (natural boundary conditions [4]) and \( \nu > 0 \), the stationary marginal probability density function \( p_{\nu,q}^\pm(N) = \lim_{t \to \infty} p_{\nu,q}^\pm(N, t) + p_{\nu,q}^\mp(N, t) \) of (S1) is given by [2, 4, 9]

\[
p_{\nu,q}^\pm(N) = \frac{Z_{\nu,q}}{N^2} \frac{\{(1 + q)K_+ - N\} \{N - (1 + q)K_-\}}{N^2} \tilde{\tau}_\nu^{-1}, \tag{S2}
\]

where \( Z_{\nu,q} \) is the normalization constant, \( (1 + q)K_\pm \) are the effective carrying capacities, whose harmonic mean is \( (1 + q)K \), and \( [(1 + q)K_-, (1 + q)K_+] \) is the support of \( p_{\nu,q}^\pm \).

The parameter \( 0 \leq q \leq b \) is determined by considering the limit \( \nu \to \infty \) where the environmental noise self-averages and (S1) is characterized by the effective population size \( N = (1 + q)K \) yielding a probability \( \phi_{\nu(1+q)K} \) that, when \( \nu \to \infty \), species \( S \) fixates in the auxiliary model defined by (S1). For given parameters \( (K_\pm, s, b, x_0) \), the effective value of \( q \) is thus obtained by matching \( \phi_{\nu(1+q)K} \) with the fixation probability of \( S \) obtained in our simulations with \( \nu \gg 1 \) [10]. In the realm of this effective theory, once \( q \) is determined, the fixation probability is obtained by averaging \( \phi_{\nu,N} \) over (S2) with a rescaled switching rate \( \nu \to \nu/s \), as above. This yields Eq. (7) for \( \phi_q \) whose expression has been used in Figs. 4(b)-4(d) in lieu of \( \phi \), see also Sec. 4 below.

By setting \( q = b > 0 \) in (S2), we obtain the (marginal) PDF \( p_{\nu,b}(N) \) conditioned to the fixation of species \( S \) (but unconditioned of whether \( \xi = \pm 1 \)) in the public good scenario. Similarly, by setting \( q = 0 \) in (S2), we obtain \( p_{\nu,0}(N) = p_{\nu}^N \) which coincides with (5) and is the marginal PDF conditioned to the fixation of \( F \) (but unconditioned of \( \xi = \pm 1 \)) in the public good scenario and the marginal PDF in the pure resource scenario. In the latter case, \( p_{\nu}^N \) is used to obtain the expression (6) for the fixation probability \( \phi \).

### 2.1 Appraisal of formula (6) and (7)

It is worth noting that formula (6) and (7) explicitly reflect the coupling between internal and external noise.

The formula (6) provides an excellent approximation of the fixation probability of \( S \) for all the values of \( \nu > 0 \), when \( K_- \gg 1 \). Moreover, it captures the fact that external and internal noise can jointly significantly enhance the fixation probability of the slow type with the respect to its counterpart in a population of constant size \( \langle K \rangle \gg 1 \) subject to non-random environment, where this probability is exponentially small \( (\phi(\langle K \rangle) \approx e^{-\langle K \rangle}s/2 \) when \( x_0 = 1/2 \) and \( \langle K \rangle s \gg 1 \). This is true also in the limit \( \nu \to 0 \) where the population is as likely to be subject to a low or a large carrying capacity which in general greatly increases the probability that \( S \) fixates, even there may be no switching prior to fixation. For instance, in Figs. 2(a) and S1(a) we find that \( \phi \approx 0.20 - 0.30 \) when \( s = 0.02 \) while \( \phi(\langle K \rangle) = 0.08 \).
and for $s = 0.07$ we have obtained $\phi \approx 0.05 - 0.07$ while $\phi|_{(K) = 250} \approx 0.002$.

The expression (7) of $\phi_q$ is very close to $\phi$ when $\nu/s \gg 1$ (high switching rate) and $K_- \gg 1$, but slightly deviates from it when $\nu/s \ll 1$. This stems from the fact that the effective theory underpinning (7) builds on the value of $q$ inferred at high switching rate.

Remarkably, both (6) and (7) are able to capture the nontrivial dependence of $\phi$ on the switching rate $\nu$, see Figs. 2(a), S1(a) and 4(a): $\phi$ increases with $\nu$ when $\phi(\infty) > \phi(0)$ and decreases when $\phi(\infty) < \phi(0)$. The former situation arises under sufficiently low selection pressure, whereas the latter scenario occurs above a certain selection intensity. The intuitive explanation for this is that $\nu \approx 0$ corresponds to a high-volatility-high-reward setting, in which $S$ is equally likely to end up in an environment with relatively high demographic noise ($K_-$), where its fixation probability is high, or in one ($K = K_+$) with low noise and lower fixation probability. When $\nu \gg 1$, on the other hand, the species $S$ is in a low-volatility-low-reward setting: it faces an almost constant population size ($N \approx K_+$). When the selection intensity $s$ is increased, it becomes increasingly less favorable for $S$ to be in the low-volatility-low-reward setting, and thus $\phi(\infty) < \phi(0)$ and thus $\phi$ decreases with $\nu$. In the case of Figs. 2(a) and S1(a), we can explicitly determine the critical selection pressure $s_c$ below which $\phi(\infty) > \phi(0)$. When $K_+ \gg K_- \gg 1$, we have $K = 2K_- (1 + \mathcal{O}(K_-/K_+))$ and therefore $\phi(\infty) \approx (e^{-K_- s} - e^{-2K_- s})/(1 - e^{-2K_- s})$ while $\phi(0) \approx (e^{-K_- s/2} - e^{-K_- s})/[2(1 - e^{-K_- s})]$. Hence, the condition $\phi(\infty) > \phi(0)$ for $\phi$ to increase with $\nu$ leads to $2y^2/(1 + y^2) > y/(1 + y)$, where $y = e^{-K_- s/2}$. Therefore, $\phi$ is an increasing function of $\nu/s$ when $y^2 + 2y - 1 > 0$, i.e. if $y = e^{-K_- s/2} > \sqrt{2} - 1$, while $\phi$ decreases with $\nu$ if $e^{-K_- s/2} < \sqrt{2} - 1$. The critical selection pressure is thus defined by $e^{-K_- s_c/2} = \sqrt{2} - 1$. For $(K_+, K_-, s) = (450, 50, 0.02)$, we find $s_c \approx 0.035$. Hence, $s = 0.02 < s_c$ and $s = 0.07 > s_c$. Therefore, $\phi$ increases with $\nu$ when $s = 0.02$, and it decreases with $\nu$ when $s = 0.07$, as reported in Figs. 2(a) and S1(a).

Finally, we note that while (7) is useful to obtain an approximation of $\phi$ and its dependence on $\nu$ and $s$, it is unable to capture its dependence on the public good parameter $b > 0$. However, we know that the typical population size increases with $b$ when $x \approx 1$ and $S$ is close to fixation, and therefore the intensity of the demographic fluctuations is reduced by increasing $b$. Based on the properties of the Moran process, we thus expect $\phi$ to decay exponentially with $b$ [11], which is confirmed by Fig. S1(b).

3 Mean fixation time

We have also investigated the mean fixation time $T(x_0)$, which is the unconditional mean time until the fixation of either species $S$ or $F$ starting from a initial fraction $x_0$ of individuals of type $S$ in the population.

![Figure S2: (Color online). (a) $T(x_0) = T$ vs. $\nu/s$ in the pure competition case $b = 0$ with $s = 0.02$ (⊙, blue/black) and $s = 0.07$ (○, orange/gray). Symbols are simulation results for $T$, solid lines are from (S3), dashed and dotted lines show $T|_K$ and $(T|_{K_+} + T|_{K_-})/2$, respectively. (b) $T$ vs. $\nu/s$ in the public good scenario with $b = 0.2$ (⊙, blue/gray for $s = 0.01$; ○, red/black for $s = 0.05$) and $b = 2$, $s = 0.05$ (▽, green/dark gray). $(K_+, K_-, x_0) = (450, 50, 1/2)$ in both panels.](image-url)
3.1 Mean fixation time when $b = 0$

In the case $b = 0$, $N$ evolves independently of which species has fixated the population, see Videos 1-2 [1]. This allows us to proceed as we did with (6) for the fixation probability, and estimate the mean fixation time by $T_{\nu/s}$. This quantity is obtained by averaging the unconditional mean fixation time $T(x_0)|_N$ [7, 8] obtained in a Moran process for a population of constant size $N$ over $p^*_\nu/N (N)$ given by (5) with a rescaled switching rate $\nu \rightarrow \nu/s$. This yields

$$T_{\nu/s} = \int_{K_+}^{K_-} T(x_0)|_N p^*_\nu/N (N) \, dN,$$

(S3)

where $T(x_0)|_N \sim O(1/s)$ when $e^{\nu/s} \gg K_-$. As Figure S2(a) shows, this expression agrees well with the leading contribution $T(x_0) \simeq T_{\nu/s} \sim O(1/s)$ when $x_0$ is well separated from the absorbing boundaries. The scaling of the mean fixation time in the presence of EN is therefore the same as $T(x_0)|_N = T|_N$ [7, 8]. The main effect of the EN is to affect the subleading prefactor of $T$ [11]: as shown in Fig. S2 (a) and captured by (S3), the mean fixation time decreases when $\nu$ increases. This stems from the fact that $\langle N \rangle^*$ decreases with $\nu$, see Fig. 2(b). In the case of pure resource competition, our theory is therefore able to correctly predict that the mean fixation time always scales as $1/s$ but is shortened when the switching rate is increased.

3.2 Mean fixation time when $b > 0$

In the public good scenario ($b > 0$), the mean fixation time still scales as $T(x_0) \sim O(1/s)$ and decreases with the environmental switching rate $\nu$, as shown in S2(b). This is because the average population size also decreases with $\nu$ (see Fig. 3). In this case, however the fixation of the $S$ type happens in larger populations (and, hence, after longer times) than the fixation of $F$, see Videos 5-7 [1]. As a result, to accurately compute $T(x_0)$, it is necessary to determine the two conditional mean fixation times (which are equal only when $b = 0$) [11]. Clearly, this cannot be achieved by assuming a timescale separation between $N$ and $x$, and is beyond the reach of our effective theory.

4 Population size quasi-stationary distribution: additional discussion and results

In this section, we provide additional discussion and results about the population size distribution after the occurrence of fixation. An important common feature of the $b = 0$ and $b > 0$ scenarios is that long-time population size distribution is well described by $p^*_\nu/N (5)$ when $b = 0$, and by combining the conditional PDFs $p^*_\nu$ and $p^*_{\nu,b}$ (S2) with $\phi$ when $b > 0$, as explained in the main text.

4.1 Noise-induced transitions

The quasi-stationary population size distributions are thus characterized by different regimes in which they are unimodal, bimodal, or even multimodal, see Figs. 3, 4 and S3. The transitions between these various regimes are called “noise-induced transitions” because they are solely caused by the environmental noise [4, 9]. In fact, if the carrying capacity in (S1) was oscillating periodically (deterministically), the corresponding PDF would always be bimodal: the transition to the unimodal regime is only possible for randomly fluctuating $K$ [9].

4.2 Simulation and prediction of the population size steady state distribution

To assess the theoretical predictions for the long-time population size distribution inferred from (5) and (S2), we have generated $10^5$ replicas that we let run until 99% of them reached fixation. The outcome has then been binned to generate the histograms shown as solid lines in Figs. 3, 4(c,d) and in Fig.S3. In the pure competition case ($b = 0$), see Fig. 3, these simulation results are compared with $p^*_\nu/N (5)$ multiplied by the number of replicas. (In this case, $N$ evolves independently of $x$, therefore it is not necessary to wait until 99% of fixation has occurred, see Videos 3-4 [1]. We have proceeded in this way for consistency with the case $b > 0$.

In the public good scenario ($b > 0$), as Figs. 4(c,d) and S3, we have waited until fixation had occurred in almost all replicas (99% of them) to collect the data to build the histograms that correctly reflect the quasi-stationary state distributions of the population size (now depending on $x$), see Videos 7-9 [1]. Via our effective theory, we have computed the fixation probability of the strain $S$ and $F$. Multiplying these values by $10^5$ (number of samples), we have obtained the expected number of replicas to fixate to $S$ and to $F$. By multiplying these numbers by $p^*_\nu (5)$ and $p^*_{\nu,b}$
we obtain the histograms associated with the conditional probability distributions (unconditioned of $\xi = \pm 1$). These are shown by dotted lines in Fig. S3 and their sum gives the histogram of the marginal distribution (orange dashed lines in Fig S3), which can be directly compared with the histogram from the simulations.

Figure S3: (Color online). Long-time population size distributions for $\nu = 20$ (a), $\nu = 1.2$ (b), $\nu = 0.2$ (c), and $\nu = 0.02$ (d) with $(K_+, K_-, x_0, s, b) = (450, 50, 0.5, 0.02, 2)$ similar to Fig. 4(c,d) but now showing also the results obtained from the $S$-conditional (dotted, yellow/light gray) and $F$-conditional (dotted, blue/dark gray) PDFs. The histogram of the marginal PDF (dashed) is the sum of the $S/F$-conditional histograms weighted by $\phi_q (7)$, see text.

4.3 Long-time population size distribution in the public good scenario $(b > 0)$

To understand the properties of the quasi-stationary marginal population size distribution when $b > 0$, it is useful to notice that when $S$ fixates ($x = 1$), the relevant conditional PDF (unconditioned of $\xi = \pm 1$) is $p_{\nu,b}^*$ which is unimodal and peaked at $N = (1+b)K$ when $\nu > 1+b$, while it is bimodal with peaks at $N = (1+b)K_{\pm}$ if $\nu < 1+b$. Similarly, $p_{\nu}^*$ is the PDF conditioned to fixation of $F$ (but unconditioned of $\xi = \pm 1$): it is unimodal and peaked at $N \approx K$ if $\nu > 1$, whereas it is bimodal with peaks at $N \approx K_{\pm}$ when $\nu < 1$. The sum of the conditional PDFs weighted by $\phi_q$ yields the marginal PDF (unconditioned of $\xi = \pm 1$ and of whether $S$ or $F$ fixates) that, depending on $\nu$ and $b$, is either bimodal or multimodal. Therefore, as shown in Figs. 4(c,d) and S3 as well as in Videos 7-9 [1], the marginal quasi-stationary population size distribution is characterized by

- two peaks at about $N = K$ and $N = (1+b)K$ when $\nu > 1 + p$, see Video 7.
- three peaks located about $N = K$ and $N = (1+b)K_{\pm}$ when $1 < \nu < 1 + p$, see Video 8.
- four peaks located around $N = (1+b)K_{\pm}$ and $N = K_{\pm}$ when $\nu < 1$, see Video 9.

The peaks at $N = (1+b)K_{\pm}$ and $N = (1+b)K$ stem from the fixation of $S$ and thus are less marked than those at $N \approx K_{\pm}$ and $N \approx K$ which result from the more likely fixation of $F$.

4.4 Figure 4(c, d) revisited

In Fig. 4(c,d), we report the histograms of the stationary marginal population distribution at $\nu = 20$ and $\nu = 0.02$ with $b = 2$. For the sake of completeness, in Fig. S3 we also consider the intermediate switching rates $\nu = 1.2$ and $\nu = 0.2$. 


and show the conditional PDFs $p^\ast_{\nu}$ and $p^\ast_{\nu,b}$. The marginal PDF is obtained as the sum of $p^\ast_{\nu}$ and $p^\ast_{\nu,b}$ weighted by $\phi_q$ and $1 - \phi_q$ given by (7).

4.5 Deviations from the PDF predictions

In Figs. 3, 4 and S3, we notice that Eqs. (5) and (S2) are able to predict when the long-time population size distributions are unimodal, bimodal or multimodal and the location of the peaks. However, the quasi-stationary size distributions obtained from the simulations are not strictly confined with in the support of the PDFs (5) and (S2), especially at low $\nu$. These deviations appear because, due to demographic noise, the population in the simulations is allowed to fluctuate around the fixed points $N = K_+$ and $N = (1 + b)K_+$, see Video 9. The deviations are more pronounced (yielding broader peaks) when the the values of $K_+$ are increased resulting in a weaker attraction exerted by these fixed points [11].

5 Supporting videos

The dynamics of the models and our findings are illustrated by a series of videos available electronically [1].

5.1 Videos 1-4: $b = 0$

Videos 1-4 illustrate the population dynamics in the pure resource competition scenario for the parameters $(s, K_+, K_-, x_0) = (0.02, 450, 50, 0.5)$ and different switching rates.

- Video 1 shows the sample paths $N(t)$ (left) and $x(t)$ (right) of five replicas for $\nu = 20$. We clearly notice a timescale separation: the population size quickly starts to endlessly fluctuate about $N \approx K = 90$ while $x(t)$ evolves much more slowly, with fixation occurring in time $t \sim O(1/s)$.

- Video 2 shows similar paths for $\nu = 0.01$ (and a sped-up animation). We again see the timescale separation between $N(t)$ and $x(t)$. However, in the long run $N(t)$ endlessly jumps between $N \approx K_-$ and $N \approx K_+$. Moreover, the video shows how the behavior of the population size is unaffected by changes in $x$: $N$ relaxes at a faster timescale and maintains the same behavior also after fixation (of either species).

- Video 3 shows the histograms of the population size (left) and of the fraction of $S$ individuals (right) for fast switching ($\nu = 20$). The population size histogram rapidly becomes bell-shaped and centered about $N = K$. It reaches this form much before fixation typically occurs, and is independent of the distribution of $x$ (internal dynamics). On the other hand, the histogram of $x$ evolves slowly and is eventually characterized by asymmetric peaks at $x = 0$ and $x = 1$ corresponding to the fixation probability of $F$ and $S$, respectively.

- Video 4: as in Video 3, but for a slow-switching environment ($\nu = 0.2$). We notice that the population size distribution readily attains a right-tailed, bimodal shape with peaks about $N = K_+$, and is independent of the distribution of $x$ (internal dynamics). The histogram of $x$ has the same properties as in Video 3.

5.2 Movies 5-9: $b > 0$

Movies 5-7 illustrate the internal and ecological dynamics in the public good scenario, $b > 0$, for the parameters $(s, K_+, K_-, b, x_0) = (0.02, 450, 50, 2, 0.5)$ and different switching rates. In this scenario, the fast $N$ dynamics is enslaved to the slower evolution of $x$. The population size distribution is characterized by peaks that slowly emerge as occurrences of $S$ and $F$ fixation accumulate (right panels).

- Video 5 shows sample paths of $N$ and $x$ for five realizations with $\nu = 20$, as in Video 1. The population size and composition are correlated: the population size attains large values when $x$ dwells about 1, while $N$ is much smaller when $x \ll 1$ (for example, the green replica is almost always larger than the purple one). As the species fixate, the sample paths for $N$ separate into two distinct sets: those associated with the fixation of $S$ ($x = 1$) fluctuate about $N \approx K = 90$, while the paths associated to $x = 0$ (fixation of $F$) fluctuate around $N \approx (1 + b)K = 270$. 


• Video 6 shows similar sample paths for $\nu = 2$. In addition to showing the correlation between $N$ and $x$, the video illustrates how populations with a high fraction of $S$ ($x \approx 1$) experience random switching with an effectively reduced switching rate. For example, in the purple sample paths, which readily attains $x \approx 1$, $N$ evolves by large abrupt jumps, in agreement with the properties of the $S$-conditional PDF $p_{\nu,2}^*$, see (S2).

• Video 7 shows the histograms of $N$ and $x$ for fast switching ($\nu = 20$). The histogram of the population size (left) has first a right-tailed bell shape. As fixation occurrences build up, the distribution gradually splits into asymmetric peaks about $K = 90$ and $(1 + b)K = 270$. The histogram of $x$ is characterized by slowly-developing asymmetric peaks at $x = 0$ and $x = 1$.

• Video 8 shows the histograms of $N$ and $x$ for intermediate switching ($\nu = 1.2$). Similarly to Video 7, the histogram of $N$ changes from having first a right-tailed bell shape to its eventual quasi-stationary form. In this case, the quasi-stationary state is characterized by three asymmetric peaks, located at about $K = 90$ and $(1 + b)K = 150$, and about $(1 + b)K = 1350$, that slowly develop as fixation occurrences pile up (right panel).

• Video 9 shows the histograms of $N$ and $x$ for slow switching $\nu = 0.2$. Initially, the histogram of $N$ develops as in Videos 7 and 8, but now the quasi-stationary state is characterized by four slowly-developing asymmetric peaks, located at about $K = 50$, $(1 + b)K = 150$, and about $K = 450$ and $(1 + b)K = 1350$.

References


[10] For instance, for $(K_+, K_-, s, b, x_0) = (450, 50, 0.025, 2, 0.5)$ we have found $q = 1.1543$.