An Introduction to Evolutionary Game Theory: Lecture 1

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Evolutionary Game Theory: What is it about?

- Modelling of the animal world
- Description of behavioural science and population dynamics (e.g. in ecology, economics, ...)
- Dynamical version of *classic (rational) game theory*
- Mathematical description of complex phenomena: interacting agents, spatial patterns, noise, non-linearity...

Some of the founders & pioneers:

- John von Neumann & Oskar Morgenstern (1944), "Theory of games and economic behavior"
- John Nash (1994 Nobel prize in Economics) → Nash equilibrium
- John Maynard Smith, "Evolution and the Theory of Games" (1972) → Evolutionary stability

Some books:

- J. Hofbauer & K. Sigmund, "Evolutionary Games and Population Dynamics" (1998)
- M. Nowak, "Evolutionary Dynamics" (2006)
- J. Maynard Smith, "Evolution and the Theory of Games" (1972)

Outline

The goal of this lecture is to give some insight into the following topics:

- Basics of Classic (Rational) Game Theory
- Notion of Nash Equilibrium
- Concept of Evolutionary Stability
- Examples of Popular Games
- Concept of Fitness and Evolutionary Dynamics
- The (deterministic) Replicator Dynamics

Classic (Rational) Game Theory in a Nutshell

Assumptions: complete information and perfect rationality Normal form of classic game is given by the triple $(\{N\},\{\mathscr{E}\},\{A\})$:

- $\{N\} = \{1,2,...,N\}$: set of players
- $\{\mathscr{E}\} = \{\mathbf{e}_1,\mathbf{e}_2,...,\mathbf{e}_Q\}$: same set of Q *pure strategies* for each player
- $\{\mathbf{A}\} = \{\mathbf{A}^{(1)},\mathbf{A}^{(2)},...,\mathbf{A}^{(N)}\}$: set of payoff (utility) functions for each player

Other ingredients:

- Mixed strategies (ME): allow to play each pure strategy $\mathbf{e}_j$ with probability $p_j \Rightarrow$ strategy profile is defined by the simplex $\{S\} = \{\mathbf{p} = (p_1,...,p_Q) : p_j \geq 0 \text{ and } \sum_{j=1}^{Q} p_j = 1\}$
- Assume pairwise contests and symmetry (identical players) $\Rightarrow$ one $(Q \times Q)$ payoff matrix $\mathbf{A} = (A_{ij})$ with $i,j = 1,...,Q$

Player 1 plays $\mathbf{p} \in \{S\}$ against player 2 playing $\mathbf{q} \in \{S\}$:
Payoff of player 1 is $\mathscr{P}_1$ and payoff player 2 is $\mathscr{P}_2$

$$\mathscr{P}_1 = \mathbf{p}.A\mathbf{q}, \quad \mathscr{P}_2 = \mathbf{q}.A^T\mathbf{p}$$

Example 1: Hawks & Doves

- Homogeneous population with individuals competing for their reproductive succcess (food, territory or mates)
- During each contest: individuals compete for resources and can win / lose a fight (possible injury) or run away
- 2 strategies: either Hawk (agressive, escalate) or Dove (avoid fights)

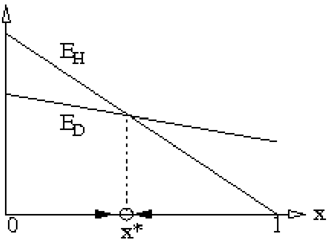
Strategy 1	Strategy 2	$\mathscr{P}_1$	$\mathscr{P}_2$	Because ...
Hawk	Dove	G	0	Hawk wins & Dove runs
Hawk	Hawk	$\frac{G-C}{2}$	$\frac{G-C}{2}$	50% chance of win/injury
Dove	Hawk	0	G	Dove runs & Hawk wins
Dove	Dove	$\frac{G}{2}$	$\frac{G}{2}$	Doves share resources

- For $G = 4$, $C = 10$, payoff matrix:

$$\mathbf{A} = \begin{array}{c} \\ \textit{Hawk} \\ \textit{Dove} \end{array} \begin{array}{c} \begin{array}{cc} \text{Hawk} & \text{Dove} \end{array} \\ \left(\begin{array}{cc} -3,-3 & 4,0 \\ 0,4 & 2,2 \end{array} \right) \end{array}$$

Example 1: Hawks & Doves (continued)

- Strategies *H* and *D* played with frequencies x and $1-x$, resp.
- Expected payoff H-player is $E_H = -3x + 4(1-x) = 4 - 7x$
- Expected payoff D-player is $E_D = 0x + 2(1-x) = 2 - 2x$
- $E_H = E_D$ for $x = x^* = 2/5$
- $x^* = 2/5$ is a *mixed strategy*
- For $x > x^*$: reproductive success of H is *lower* than for the D's. Therefore, the frequency of D's increases and moves towards x^*
- For $x < x^*$: reproductive success of D is *lower* than for the H's. Therefore, the frequency of H's increases and moves towards x^*
- Hence, we call x^* an **evolutionary stable strategy (ESS)**



Nash Equilibrium

What strategy to choose and how to make such a choice?

$$\mathbf{p}.\mathbf{A}\bar{\mathbf{q}} \leq \bar{\mathbf{q}}.\mathbf{A}\bar{\mathbf{q}}, \ \forall \mathbf{p} \neq \bar{\mathbf{q}}$$

$\bar{\mathbf{q}}$ is a Nash equilibrium (NE), or a strategy which is the best reply to itself.
A strict Nash equilibrium (sNE) $\bar{\mathbf{q}}$ is the unique best reply to itself:

$$\mathbf{p}.\mathbf{A}\bar{\mathbf{q}} < \bar{\mathbf{q}}.\mathbf{A}\bar{\mathbf{q}}, \ \forall \mathbf{p} \neq \bar{\mathbf{q}}$$

Every normal form game admits at least one NE (how many of them?)

Problems: Dynamics? How to discriminate between NEs? Rationality seems to restrictive $\rightarrow$ no cooperation
Nonstrict NE are *not* proof against invasion: *invaders may use a strategy doing as well as* $\bar{\mathbf{q}}$ *and may spread (if reproductive advantage),* **unless evolutionary stability strategy (ESS)**

Evolutionary Stability

A strategy is *evolutionary stable* (ESS) if, whenever all members of the population adopt it, no dissident behaviour could invade the population under natural selection

Consider a population in which the majority of the players (fraction $1-\varepsilon$) plays strategy $\mathbf{p}^*$ and a minority, ε plays mutant strategy $\mathbf{p}$. $\mathbf{p}^*$ is an ESS iff it performs strictly better than the mutant strategy $\mathbf{p}$ against the composed population, i.e.

$$\mathbf{p}^*.\mathbf{A}[(1-\varepsilon)\mathbf{p}^* + \varepsilon\mathbf{p}] > \mathbf{p}.\mathbf{A}[(1-\varepsilon)\mathbf{p}^* + \varepsilon\mathbf{p}]$$

This can be rewritten as

$$(1-\varepsilon)(\mathbf{p}^*.\mathbf{A}\mathbf{p}^* - \mathbf{p}.\mathbf{A}\mathbf{p}^*) + \varepsilon(\mathbf{p}^*.\mathbf{A}\mathbf{p} - \mathbf{p}.\mathbf{A}\mathbf{p}) > 0$$

Thus, 2 conditions:

- NE condition:
  $\mathbf{p}^*.\mathbf{A}\mathbf{p}^* \geq \mathbf{p}.\mathbf{A}\mathbf{p}^*, \forall \mathbf{p} \in S$
- Stability condition:
  if $\mathbf{p}^* \neq \mathbf{p}$ and $\mathbf{p}^*.\mathbf{A}\mathbf{p}^* = \mathbf{p}.\mathbf{A}\mathbf{p}^*$, then $\mathbf{p}^*.\mathbf{A}\mathbf{p} > \mathbf{p}.\mathbf{A}\mathbf{p}, \forall \mathbf{p} \in S$

Evolutionary Stability (continued)

2 conditions for evolutionary stability:

1. NE condition:
   $\mathbf{p}^*.\mathbf{A}\mathbf{p}^* \geq \mathbf{p}.\mathbf{A}\mathbf{p}^*, \forall \mathbf{p} \in S$
2. Stability condition:
   if $\mathbf{p}^* \neq \mathbf{p}$ and $\mathbf{p}^*.\mathbf{A}\mathbf{p}^* = \mathbf{p}.\mathbf{A}\mathbf{p}^*$, then $\mathbf{p}^*.\mathbf{A}\mathbf{p} > \mathbf{p}.\mathbf{A}\mathbf{p}$

(1) says that $\mathbf{p}^*$ is a NE which is **not enough for non-invadibility**: there might be another *alternative best reply* **p**. In such a case, (2) states that $\mathbf{p}^*$ fares better against **p** than **p** itself ($\Rightarrow$ **non-invadibility**)

- Strict-NE are ESS (symmetric games)
- All ESS are NEs (but not necessarily strict-NEs)
- A game with 2 pure strategies always has an ESS

How to find an ESS? Necessary condition (for an interior NE) given by the Bishop-Cannings Theorem: If $\mathbf{p}^* = \sum_{i=1}^{Q} p_i \mathbf{e}_i \in \mathrm{int}(S)$ with $p_i \geq 0$ is an interior ESS, then

$$\begin{aligned}\mathbf{e}_1.\mathbf{A}\mathbf{p}^* &= (\mathbf{A}\mathbf{p}^*)_1 = \ldots = (\mathbf{A}\mathbf{p}^*)_i = \ldots = (\mathbf{A}\mathbf{p}^*)_Q = \mathbf{p}^*.\mathbf{A}\mathbf{p}^* \\ p_1 &+ \ldots + p_Q = 1\end{aligned}$$

Hawks-Doves & game revisited (Example 1)

Hawks & Doves: gain from resources G and cost for injury $C > G$

Payoff matrix: Hawk Dove

$$\mathbf{A} = \begin{matrix} \textit{Hawk} \\ \textit{Dove} \end{matrix} \begin{pmatrix} (G-C)/2 & G \\ 0 & G/2 \end{pmatrix}$$

Mixed strategy $\mathbf{p}^* = (G/C)\mathbf{e_H} + (1 - G/C)\mathbf{e_D}$ is an NE

However, is it an ESS?

With $\mathbf{p} = p\mathbf{e_H} + (1-p)\mathbf{e_D} = (p, 1-p)^T$ we check the conditions for $\mathbf{p}^*$ to be an ESS:

- $\mathbf{p}.\mathbf{A}\mathbf{p}^* = \mathbf{p}^*.\mathbf{A}\mathbf{p}^* = G(1 - G/C)/2, \forall p \in \mathscr{S}$

  Thus, condition 1 (for NE) is satisfied for all $\mathbf{p}$'s with equality

  One has thus to check condition 2 (for stability):

- With $\mathbf{p}.\mathbf{A}\mathbf{p} = (G - Cp^2)/2$ and $\mathbf{p}^*.\mathbf{A}\mathbf{p} = (G^2/2C) + G(1-2p)/2$,

  $\mathbf{p}^*.\mathbf{A}\mathbf{p} - \mathbf{p}.\mathbf{A}\mathbf{p} = \frac{(G-Cp)^2}{2C} > 0$, $\forall \mathbf{p} \neq \mathbf{p}^* \Rightarrow$ stability guaranteed!

Thus, $\mathbf{p}^* = (G/C)\mathbf{e_H} + (1 - G/C)\mathbf{e_D}$ is an ESS for the Hawk-Dove game with $C > G$

Example 2: Prisoner's Dilemma (PD)

Two suspects in a crime are put in separate cells and questioned, they can confess (i.e. cooperation, C) or defect (i.e. remain silent, D)

- If both confess (C C), each is sentenced 3 years (major crime)
- If none confess (D D), each is sentenced 1 year (minor crime because no strong evidences)
- One confess and the other remain silent: C-strategist is used as witness and goes free while the S-strategist is sentenced 5 years
- Payoff matrix for the PD:

$$\mathbf{A} = \begin{array}{c} \\ C \\ D \end{array} \begin{array}{c} \begin{array}{cc} C & D \end{array} \\ \left(\begin{array}{cc} 3,3 & 0,5 \\ 5,0 & 1,1 \end{array} \right) \end{array}$$

Defection is the only (strict) Nash equilibrium (i.e. an ESS): D D gives a payoff $(1,1)$. Cooperation gives a higher payoff $(3,3)$, but it is "irrational" because of free ride danger

For each (rational) player it is beneficial to always defect (sNE). However, if both defect they get a payoff 1, less than for mutual cooperation (which gives a payoff 3). *That's the dilemma !*

Example 3: Jean-Jacques Rousseau's stag-hunt game (SHG)

2 individuals can decide to hunt a stag (S) or a hare (H). A hare is worth less than a stag but can be get individually

- If both cooperate to hunt a stag (S S), the payoff is 4
- If one individual chooses to hunt a hare (H), his/her payoff is always 3
- An individual stag-hunter (S) has little chance of success and her/his payoff is 1
- Payoff matrix for the SHG:

$$\mathbf{A} = \begin{array}{c} \\ S \\ H \end{array} \begin{array}{c} \begin{array}{cc} S & \;\; H \end{array} \\ \left(\begin{array}{cc} 4,4 & 1,3 \\ 3,1 & 3,3 \end{array} \right) \end{array}$$

There are 2 strict NEs (i.e. ESS): the pure strategies S S, giving payoff $(4,4)$, and H H giving payoff $(3,3)$

S S is *payoff (Pareto) dominant* because $(5,5) > (3,3)$

H H is *risk dominant* ("safer") because it gives a payoff 3 for 2 of the 4 choices (while there is only 1 option out of 4 to get payoff 4)

In addition: the mixed strategy $\mathbf{p} = (2/3)S + (1/3)H = (2/3, 1/3)$ is a nonstrict NE

SH is a coordination game, prototype of the "social contract"

Classic games with 2 pure strategies: summary

Games with 2 pure strategies, $\mathbf{e}_1$ and $\mathbf{e}_2$, and payoff matrix:

$$\mathbf{A} = \begin{matrix} \mathbf{e_1} \\ \mathbf{e_2} \end{matrix} \overset{\begin{matrix} \mathbf{e_1} & \mathbf{e_2} \end{matrix}}{\begin{pmatrix} a & b \\ c & d \end{pmatrix}}$$

3 cases:

1. There is one single pure ESS *(e.g. prisoner's dilemma)*
2. Both pure strategies $\mathbf{e_1}$ and $\mathbf{e_2}$ are ESS *(e.g. stag-hunt game)*
3. A mixed strategy is an ESS *(e.g. hawk-dove with $C > G$)*

More precisely:

- If $a > c$, $\mathbf{e_1}$ is ESS (strict-NE)
- If $d > b$, $\mathbf{e_2}$ is ESS (strict-NE)
- If $c > a$, and $b > d$, mixed strategy $\mathbf{p} = p\mathbf{e_1} + (1-p)\mathbf{e_2}$ with $p = \frac{b-d}{b+c-a-d}$ is an ESS (strict-NE)

Evolutionary dynamics & the concept of fitness

Alternative representation in terms of population dynamics:
Population of pure strategists with proportion with fraction x_i of $\mathbf{e_i}$-players. In this case, $\mathbf{x} = (x_1, ..., x_i, ..., x_Q)$ is an ESS under the same conditions. In this language $\mathbf{e_i}$-strategists are regarded as individuals of species i

Nash equilibrium and Evolutionary stability are *static* concepts

Evolutionary dynamics relies on the concept of **fitness**:

- Evolutionary "forces": *selection, reproduction, mutation, imitation*
- $\mathbf{x}(t)$ state of population at time t, with $\sum_{i=1}^{Q} x_i = 1$
- Dynamics of the system: $\frac{d}{dt} x_i = F(\mathbf{x})$
- It seems natural to assume that $F(\mathbf{x})$ is a function of the poulation's *fitness*, where
- **Fitness of a species i**, denoted $f_i(\mathbf{x})$, measures the success of reproduction of that species. This quantity depends on the state of the whole population

Goals: Which kind of dynamics? What are the steady states?

Replicator Dynamics

Population of Q different species: $\mathbf{e_1},...,\mathbf{e_Q}$, with frequencies $x_1,...,x_Q$

State of the system described by $\mathbf{x}=(x_1,...,x_Q)\in\mathscr{S}_Q$, where $\mathscr{S}_Q=\{\mathbf{x};x_i\geq 0,\sum_{i=1}^Q x_i=1\}$

To set up the dynamics, we need a functional expression for the fitness $f_i(\mathbf{x})$

Between various possibilities, a very popular choice is:

$$\dot{x}_i = x_i\left(f_i(\mathbf{x})-\bar{f}(\mathbf{x})\right),$$

where, one (out of many) possible choices, for the fitness is the *expected payoff*: $f_i(\mathbf{x})=\sum_{i=1}^Q A_{ij}x_j$

and $\bar{f}(\mathbf{x})$ is the *average fitness*: $\bar{f}(\mathbf{x})=\sum_i^Q x_i f_i(\mathbf{x})$

This choice corresponds to the so-called **replicator dynamics** on which most of evolutionary game theory is centered

Some Properties of the Replicator Dynamics (I)

Replicator equations (REs):

$$\dot{x}_i = x_i\left[(\mathbf{Ax})_i - \mathbf{x}.\mathbf{Ax}\right]$$

Set of coupled cubic equations (when $\mathbf{x}.\mathbf{Ax} \neq 0$)

Let $\mathbf{x}^* = (x_1^*, ..., x_Q^*)$ be a fixed point (steady state) of the REs

- $\mathbf{x}^*$ can be (Lyapunov-) stable, unstable, attractive (i.e. there is basin of attraction), asymptotically stable=attractor (=stable + attractive), globally stable (basin of attraction is $\mathscr{S}_Q$)
- Only possible interior fixed point satisfies (there is either 1 or 0):

$$(\mathbf{Ax}^*)_1 = (\mathbf{Ax}^*)_2 = ... = (\mathbf{Ax}^*)_Q = \mathbf{x}^*.\mathbf{Ax}^*$$
$$x_1 + ... + x_Q = 1$$

- Same dynamics if one adds a constant c_j to the payoff matrix $\mathbf{A} = (A_{ij})$: $\dot{x}_i = x_i\left[(\mathbf{Ax})_i - \mathbf{x}.\mathbf{Ax}\right] = x_i\left[(\widetilde{\mathbf{A}}\mathbf{x})_i - \mathbf{x}.\widetilde{\mathbf{A}}\mathbf{x}\right]$, where $\widetilde{\mathbf{A}} = (A_{ij} + c_j)$

Some Properties of the Replicator Dynamics (II)

Dynamic versus evolutionary stability: connection between dynamic stability (of REs) and NE/evolutionary stability?
Notions do not perfectly overlap $\Rightarrow$ *Folks Theorem* of EGT:

Let $\mathbf{x}^* = (x_1^*, ..., x_Q^*)$ be a fixed point (steady state) of the REs

- NEs are rest points (of the REs)
- Strict NEs are attractors
- A stable rest point (of the REs) is an NE
- Interior orbit converges to $\mathbf{x}^* \Rightarrow \mathbf{x}^*$ is an NE

- ESSs are attractors (asymptotically stable)
- Interior ESSs are global attractors

Converse statements generally do not hold!

- For 2×2 matrix games $\mathbf{x}^*$ is an ESS iff it is an attractor
- REs with Q strategies can be mapped onto Lotka-Volterra equations for $Q-1$ species: $\dot{y}_i = y_i \left(r_i + \sum_{j=1}^{Q-1} b_{ij} y_j \right)$
- Replicator dynamics is non-innovative: cannot generate new strategies