Consensus and Polarization in a Three-State Constrained Voter Model

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Big questions

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List of 25 big questions facing science and among them:

What Don't We Know?

Special Section

At *Science*, we tend to get excited about new discoveries that lift the veil a little on how things work, from cells to the universe. That puts our focus firmly on what has been added to our stock of knowledge. For this anniversary issue, we decided to shift our frame of reference, to look instead at what we *don't* know: the scientific puzzles that are driving basic scientific research.

WHAT DON'T WE KNOW?

What Determines Species Diversity

How Did Cooperative Behavior Evolve

Outline

Talk based on the paper: EPL (Europhysics Letters) **95**, 50002 (2011) [arXiv:1104.5147]

Emergence of diversity is a complex problem in life and behavioural sciences: main features to be incorporated in basic models?

- Basic and unifying principles of evolutionary dynamics
- The 2-state voter model
- The 3-species constrained voter model
  - The model and solution method
  - The exit/fixation probabilities
  - The mean exit/fixation times
- Conclusion

Basic principles of evolutionary dynamics

Key questions: how is diversity maintained?
Opinion dynamics context: when is there consensus / cultural diversity?

Basic and unifying principles for modelling evolutionary dynamics:

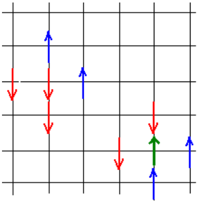
- Dynamics proceeds by imitation
- Selection: successful “opinions” spread at the expense of the others
- Population is finite and there are demographic fluctuations
- Mutations: spontaneously switch from an opinion to another
- Migration: opinions/individuals spread in space (“islands”, “patches”,...)

These principles transcend biology and behavioural science (opinion dynamics, evolutionary games, genetics, ecology, ...).

Paradigmatic opinion dynamics model: **the voter model** (related to the Moran and other evolutionary models)

2-state Voter Model

Voter Model (Liggett 1985, Galam 1990): Basic/paradigmatic two-state model where individuals are either in $+1$ (↑) or -1 (↓) opinion state. Dynamics: at each time step an individual adopts the opinion state of a random neighbour



Main properties for N voters on a complete graph, with an initial fraction x of $+1$:

- Consensus is always reached
- Probability to reach $+1$ and -1 consensus is x and $1-x$, resp.
- Mean time to attain consensus is $T \sim N$

2-state VM cannot explain the emergence of cultural diversity

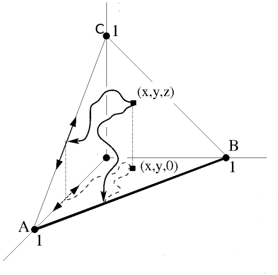
Axelrod'97, Deffuant'00, Weisbuch'02: competition between *consensus* **&** *incompatibility* $\Rightarrow$ **possible route to cultural diversity**

The 3-species constrained voter model

N individuals of 3 species. A's (leftists) and B's (rightists) are *incompatible* (don't interact), but interact with centrists C's ($|q| \leq 1$)

$$AB \to AB; \quad AC \to AA \quad \text{rate}: \frac{1+q}{2}, \quad AC \to CC \quad \text{rate}: \frac{1-q}{2}$$
$$BC \to BB \quad \text{rate}: \frac{1+q}{2}, \quad BC \to CC \quad \text{rate}: \frac{1-q}{2}$$

4 possible outcomes: consensus with A, B or C, or frozen mixture of A and B (polarization) $\Rightarrow$ **Probability and mean time for each of these events starting with densities x, y, $z = 1 - x - y$ of A, B, C?**



- $q > 0$: bias towards *polarization* (extremisms), with absorbing line $x + y = 1$
- $q < 0$: bias towards *centrism* (appeasement)
- $q = 0$: driven by fluctuations [*Vazquez & Redner* in J.Phys.A **37**, 8479 (2004)]

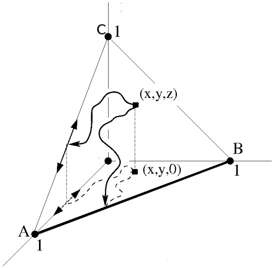
3-species constrained voter model: Mean-field

Mean field picture: all fluctuations are neglected (assume that $N = \infty$)
Deterministic dynamics in terms of rate equations:

$$\frac{d}{dt}a = qa(1-a-b), \quad \frac{d}{dt}b = qb(1-a-b)$$

$$a(t) = \frac{xe^{qt}}{1-(x+y)(1-e^{qt})}, \quad b(t) = \frac{ye^{qt}}{1-(x+y)(1-e^{qt})}$$

Ratio $a/b = x/y$ is conserved. 3 absorbing fixed points,
$(a,b,c) \in \{\mathscr{A} = (1,0,0), \mathscr{B} = (0,1,0), \mathscr{C} = (0,0,1)\}$+ *(polarization)*
line of fixed points $\mathscr{A}\mathscr{B} = (a, 1-a, 0)$, with $0 < a < 1$



- When $q > 0$:
  $a \to \frac{x}{x+y}, b \to \frac{y}{x+y}, c \to 0$
  (*polarization*)
- When $q < 0$:
  $a \to 0, b \to 0, c \to 1$
  (*centrism*)

3-species constrained VM: Individual-based approach

Finite population ($N < \infty$): fluctuations alter the mean field predictions. Polarization is likely when $q > 0$, but still probability to reach a consensus. (The opposite when $q < 0$)

Stochastic formulation of a finite and well-mixed ("complete graph") population
Starting from (x, y), probability $P^{\mathscr{A}\mathscr{B}}(x, y)$ that final state is a frozen mixture of extremists obeys the backward master equation (ME)

$$\begin{aligned} &(T_x^+ + T_x^- + T_y^+ + T_y^-)P^{\mathscr{A}\mathscr{B}}(x, y) = \\ &T_x^- P^{\mathscr{A}\mathscr{B}}(x-\delta, y) + T_x^+ P^{\mathscr{A}\mathscr{B}}(x+\delta, y) \\ + \quad &T_y^- P^{\mathscr{A}\mathscr{B}}(x, y-\delta) + T_y^+ P^{\mathscr{A}\mathscr{B}}(x, y+\delta) \end{aligned} \tag{1}$$

+ boundary conditions. With $T_\xi^\pm \equiv (1 \pm q)\xi(1 - x - y)/2$, $\xi \in (x, y)$ and $\delta = N^{-1}$
Analytical progress: expand the ME to 2nd order in $\delta \rightarrow$ Fokker-Planck equation. *Analogy with models of population genetics and evolutionary games*

3-species voter model: Mathematical treatment (I)

By Taylor expansion of the ME (1):

$$\mathscr{L}_{\rm bFP}(x,y)P^{\mathscr{A}\mathscr{B}}(x,y) = \left\{2s[x\partial_x + y\partial_y] + x\partial_x^2 + y\partial_y^2\right\} P^{\mathscr{A}\mathscr{B}}(x,y) = 0, \quad (2)$$

with $\mathbf{s} \equiv \mathbf{Nq}$ and $P^{\mathscr{A}\mathscr{B}}(x,0) = P^{\mathscr{A}\mathscr{B}}(0,y) = 0$ and $P^{\mathscr{A}\mathscr{B}}(x,1-x) = 1$.
Equation is **separable** ($\sqrt{x} = \rho\cos\theta, \sqrt{y} = \rho\sin\theta$):
$P^{\mathscr{A}\mathscr{B}} = \sum_n c_n R_n(\rho) u_n(\theta) \sin(2\theta)$, yielding

$$\rho^2 \frac{d^2 R_n}{d\rho^2} + \rho \frac{dR_n}{d\rho}[4s\rho^2 - 1] - \lambda_n R_n = 0$$

$$\frac{d^2 u_n}{d\theta^2} - \frac{3}{4}\left(\frac{1}{\sin^2\theta} + \frac{1}{\cos^2\theta}\right) u_n + (1+\lambda_n) u_n = 0$$

BC:$P^{\mathscr{A}\mathscr{B}}(\rho = 0, \theta) = 0$ and $P^{\mathscr{A}\mathscr{B}}(\rho = 1, \theta) = 1 \Rightarrow \lambda_n = 4(n+1)(n+2)$

$$P^{\mathscr{A}\mathscr{B}}(x,y) = 2\sqrt{\frac{xy}{x+y}} e^{s(1-x-y)} \sum_{n\,\mathrm{odd}}^{\infty} \frac{2n+1}{n(n+1)} \left(\frac{I_{n+1/2}(s(x+y))}{I_{n+1/2}(s)}\right) P_n^1\left(\frac{x-y}{x+y}\right)$$

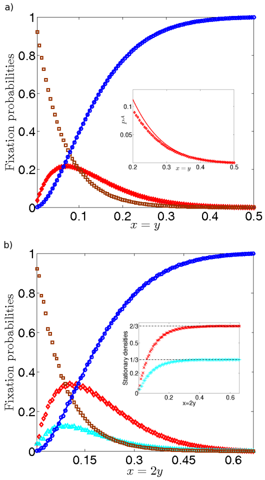
I_n's and P_n^1's: Modified Bessel functions & associated Legendre Polynomials.

Exit probabilities when $s = Nq > 0$

- When $s = N|q| \gg 1$: drift dominates over diffusion $\Rightarrow$ mean field
- When $s \ll 1$: drift negligible $\Rightarrow$ like in JPA **37**, 8479 (2004))
- **Effective (interesting) competition arises when** $s = Nq = \mathscr{O}(1)$

Top: Exit probabilities for $s > 0$ as functions of x (for $N = 200, s = 4$, i.e. $q = 0.02$): $P^{\mathscr{A}}$ ($\diamond$), $P^{\mathscr{B}}$ ($\triangle$), $P^{\mathscr{C}}$ ($\square$); $P^{\mathscr{A}\mathscr{B}}$ ($\circ$)
Solid line: analytical sol. of (2)

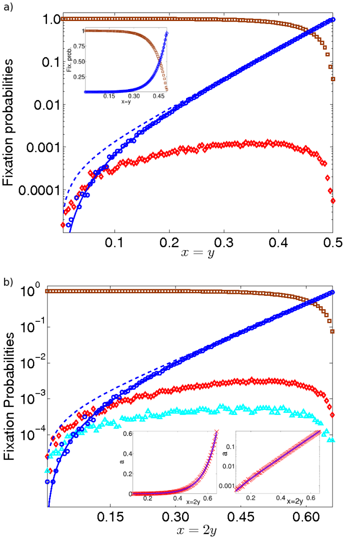
Bottom: same as above, but with $x = 2y$.
Inset: final densities of species A ($\times$) and B ($+$) as functions of $x = 2y$.



Exit probabilities when $s < 0$

Top: as before, with $s = -4$
Comparison with analytics:
Solid: solution of (2)
Dashed:
$P^{\mathscr{A}\mathscr{B}} \approx 1 - P^{\mathscr{C}} = \frac{e^{2|s|(x+y)}-1}{e^{2|s|}-1}$

Bottom: as above, but with $x = 2y$.
Inset: stationary density of A ($\times$) as function of $x = 2y$ compared with analytics



3-species voter model: Mathematical treatment (II)

The unconditional mean exit/fixation time (MET) τ to reach *any* of the system's absorbing states obeys the backward Fokker-Planck equation with BC $\tau(1,0)=\tau(0,1)=\tau(0,0)=\tau(a,1-a)=0$. With $w\equiv x+y$:

$$\mathscr{L}_{\rm bFP}(w) = \frac{w(1-w)}{N}\left[2s\frac{d}{dw}+\frac{d^2}{dw^2}\right]$$
$$\mathscr{L}_{\rm bFP}(w)\tau(w) = -1, \quad \text{with} \quad \tau(0)=\tau(1)=0 \Rightarrow \qquad (3)$$

Useful mapping with a population genetics model (s is "selection strength")

The unconditional MET $\tau=\tau(x+y)$:

- is a function of initial density of extremists $(x+y)$
- scales linearly with N
- symmetry: invariant under $(s,x+y)\to(-s,1-x-y)\Rightarrow$ $\tau(x+y)=Nf_\tau(s,x+y)=Nf_\tau(-s,1-x-y)$
- f_τ has an inverted u-shape dependence on $w=x+y$

Unconditional & Conditional Mean Exit Times

The *conditional* MET $\tau^{\mathscr{S}}$, to reach the specific absorbing state $\mathscr{S} \in (\mathscr{A}, \mathscr{B}, \mathscr{C}, \mathscr{A}\mathscr{B})$ obeys:

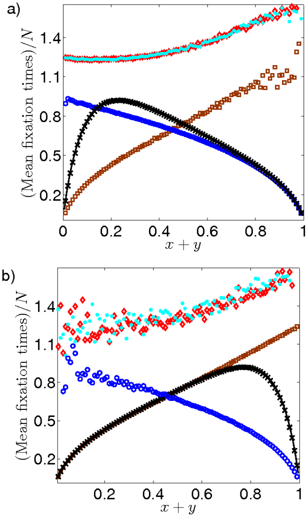
$$\mathscr{L}_{\rm bFP}(x,y)[P^{\mathscr{S}}(x,y)\tau^{\mathscr{S}}(x,y)] = -P^{\mathscr{S}}(x,y)$$

(+BC's),
where the $P^{\mathscr{S}}(x,y)$'s are exit/fixation probabilities

- All $\tau^{\mathscr{S}}$'s are found to scale linearly with N
- The $\tau^{\mathscr{S}}$'s do not depend on the sign of s *(more noisy when $s<0$)*
- The extremists' METs, $\tau^{\mathscr{A}}$ and $\tau^{\mathscr{B}}$, are always the longest METs

Mean Exit Times

Normalized unconditional MET τ/N **(**$\times$**)** compared with sol. of (3) (solid)
Normalized conditional METs
$\tau^{\mathscr{A}}/N$ ($\diamond$)
$\tau^{\mathscr{B}}/N$ ($\bullet$)
$\tau^{\mathscr{C}}/N$ ($\Box$)
$\tau^{\mathscr{A}\mathscr{B}}/N$ ($\circ$).
$N = 200$, $x = y$, $s = 4$ (top) and $s = -4$ (bottom). Average is over 2×10^5 samples.



Outlook & Conclusion

Understanding the origin and maintenance of diversity in evolutionary dynamics

Seek for consensus & incompatibility: relevant ingredients for cultural diversity in opinion dynamics(?)

The 3-state constrained voter model is mathematically amenable:

- Possible outcomes: consensus with extremists/centrists, or polarization of extremism ("leftists" and "rightists" coexist)
- Bias ($\sim$ selection) $\rightarrow$ nonlinearity. Finite population $\rightarrow$ noise
- Small bias ($q \sim N^{-1}$, "weak selection"): subtle competition between drift and fluctuations

How relevant all of this is?

- Realistic ingredients: mutations, dispersal, spatial structure
- **Validation a variant of the model using real data?**