Introduction to Mean field games and applications

Wei Yang

Department of Mathematics and Statistics
University of Strathclyde Glasgow
Outline

1. Introduction

2. Our results and applications
mean field game (MFG) theory

- a new branch of game theory
- developed independently by Lions, Lasry, Guéant (2006, 2007...)
  Caines, Huang, Malhamé (2005, 2006...)
- community meeting: June 2015 Paris
mean field game (MFG) theory

- a new branch of game theory
- developed independently by Lions, Lasry, Guéant (2006, 2007...)
  Caines, Huang, Malhamé (2005, 2006...)
- community meeting: June 2015 Paris

maintaining a day to day interaction between mathematical research and real world applications

- The co-founders include P.-L. Lions and J.-M. Lasry
- Customers include banks, energy councils, twitter (big data)...
"Large numbers are much simpler than small ones?" - Maybe!

mean field game (MFG) theory
- to study large (stochastic) dynamic games
- inspired by ideas from statistical particle physics (particles are replaced by agents with strategic interactions)
  ... to use the concept of mean field
Mean field game Methodology

- consider an \( N \)-player stochastic dynamic game
- study a mean field game (a limit for \( N \to \infty \)) which can be expressed by a system of coupled equations:
  - Fokker-Planck equation
  - Hamilton-Jacobi-Bellman equation
- any solution to the mean field game is an \( \epsilon \)-equilibrium to the \( N \)-player game

.... an efficient decision-making process without paying too much attention to fine details of the system.
An $N$-player stochastic dynamic game

To sense dynamics and costs...
An $N$-player stochastic dynamic game

- $T > 0$: a finite time horizon.
- $\mathcal{U} \subset \mathbb{R}$: a compact control set.

For $i \in \{1, \ldots, N\}$, state dynamics $\{Z_i(t), t \geq 0\}$ is described by

$$dZ_i(t) = \frac{1}{N} \sum_{k=1}^{N} F(t, Z_i(t), u_i(t), Z_k(t)) dt + \sigma dW_i(t)$$

and the cost function is given as

$$J_i(u_i) =: \mathbb{E} \int_{0}^{T} \left[ \frac{1}{N} \sum_{k=1}^{N} L(t, Z_i(t), u_i(t), Z_k(t)) \right] dt$$

where $F, L : [0, T] \times \mathbb{R} \times \mathcal{U} \times \mathbb{R} \rightarrow \mathbb{R}$. $u_i = \{u_i(t) \in \mathcal{U}, t \geq 0\}$.

Players are coupled through dynamics and costs.
Example — a typical structure of interaction:

\[ dZ_i(t) = \frac{1}{N} \sum_{k=1}^{N} \left( f(t, Z_i(t), u_i(t)) + g(t, Z_k(t)) \right) dt + \sigma dW_i(t) \]

and

\[ J_i(u_i) =: \mathbb{E} \int_0^T \left[ \frac{1}{N} \sum_{k=1}^{N} \left( l(t, Z_i(t), u_i(t)) + h(t, Z_k(t)) \right) \right] dt \]

where \( f, l : [0, T] \times \mathbb{R} \times \mathcal{U} \to \mathbb{R} \) and \( g, h : [0, T] \times \mathbb{R} \to \mathbb{R} \).
Example — a typical structure of interaction:

\[
dZ_i(t) = \left( f(t, Z_i(t), u_i(t)) + \frac{1}{N} \sum_{j=1}^{N} g(t, Z_j(t)) \right) dt + \sigma dW_i(t)
\]

and

\[
J_i(u_i) =: \mathbb{E} \int_0^T \left[ l(t, Z_i(t), u_i(t)) + \frac{1}{N} \sum_{j=1}^{N} h(t, Z_j(t)) \right] dt
\]

where \( f, l : [0, T] \times \mathbb{R} \times \mathcal{U} \to \mathbb{R} \), \( g, h : [0, T] \times \mathbb{R} \to \mathbb{R} \).

- Dynamic and cost are closely related to its own states and control, while receiving an impact of the population.
- To analyse Nash strategies \( \{\hat{u}_1, \ldots, \hat{u}_N\} \), full information is needed!
The smaller the number of variables is, the more explainary the model is.

To reduce the complexity by the concept of empirical measure.

**Empirical measure on \( \mathbb{R} \)**

For \( z^{(N)} = (z_1, z_2, \ldots, z_N) \in \mathbb{R}^N \) define the empirical measure

\[
\eta_{z^{(N)}}(B) = \frac{1}{N} \sum_{i=1}^{N} \delta_{z_i}(B), \quad \forall B \in \mathcal{B} (\mathbb{R})
\]

where \( \delta_a \) is the Dirac measure at \( a \in \mathbb{R} \).
Denote \( Z^{(N)}(t) = (Z_1(t), \ldots, Z_N(t)) \), for any \( t \geq 0 \).

\[
dZ_i(t) = \frac{1}{N} \sum_{k=1}^{N} F(t, Z_i(t), u_i(t), Z_k(t)) dt + \sigma dW_i(t)
\]

\[
= \left( \int_{\mathbb{R}} F(t, Z_i(t), u_i(t), y) \eta_{Z^{(N)}(t)}^N(dy) \right) dt + \sigma dW_i(t)
\]
Denote $Z^{(N)}(t) = (Z_1(t), \ldots, Z_N(t))$, for any $t \geq 0$.

\[
    dZ_i(t) = \frac{1}{N} \sum_{k=1}^{N} F(t, Z_i(t), u_i(t), Z_k(t)) dt + \sigma dW_i(t)
\]

\[
    = \left( \int_{\mathbb{R}} F(t, Z_i(t), u_i(t), y) \eta^N_{Z^{(N)}(t)} (dy) \right) dt + \sigma dW_i(t)
\]

Let $N \to \infty$ and $Z(t) = (Z_1(t), \ldots, Z_N(t), \ldots)$.

Assume $\eta_Z(t) := \lim_{N \to \infty} \eta^N_{Z^{(N)}(t)}$ exits in weak sense, then

\[
    dZ_i(t) = \left( \int_{\mathbb{R}} F(t, Z_i(t), u_i(t), y) \eta_Z(t) (dy) \right) dt + \sigma dW_i(t).
\]

\[
    J_i(u_i) = \mathbb{E} \int_0^T \left( \int_{\mathbb{R}} L(t, Z_i(t), u_i(t), y) \eta_Z(t) (dy) \right) dt.
\]
Denote $\mathbf{Z}^{(N)}(t) = (Z_1(t), \ldots, Z_N(t))$, for any $t \geq 0$.

$$dZ_i(t) = \frac{1}{N} \sum_{k=1}^{N} F(t, Z_i(t), u_i(t), Z_k(t)) dt + \sigma dW_i(t)$$

$$= \left( \int_{\mathbb{R}} F(t, Z_i(t), u_i(t), y) \eta_{Z^{(N)}}^N (dy) \right) dt + \sigma dW_i(t)$$

Let $N \to \infty$ and $\mathbf{Z}(t) = (Z_1(t), \ldots, Z_N(t), \ldots)$.

Assume $\eta_{\mathbf{Z}(t)} := \lim_{N\to\infty} \eta_{Z^{(N)}}^N$ exits in weak sense, then

$$dZ_i(t) = \left( \int_{\mathbb{R}} F(t, Z_i(t), u_i(t), y) \eta_{\mathbf{Z}(t)} (dy) \right) dt + \sigma dW_i(t).$$

$$J_i(u_i) = \mathbb{E} \int_0^T \left( \int_{\mathbb{R}} L(t, Z_i(t), u_i(t), y) \eta_{\mathbf{Z}(t)} (dy) \right) dt.$$ 

**Underlying intuition: as $N$ increases...**

- the overall population’s behaviour (i.e. mean field) becomes relevant to a given agent’s dynamics.
Denote $Z^{(N)}(t) = (Z_1(t), \ldots, Z_N(t))$, for any $t \geq 0$.

$$dZ_i(t) = \frac{1}{N} \sum_{k=1}^{N} F(t, Z_i(t), u_i(t), Z_k(t)) dt + \sigma dW_i(t)$$

$$= \left( \int_{\mathbb{R}} F(t, Z_i(t), u_i(t), y) \eta_{Z^{(N)}(t)}^N(dy) \right) dt + \sigma dW_i(t)$$

Let $N \to \infty$ and $Z(t) = (Z_1(t), \ldots, Z_N(t), \ldots)$. Assume $\eta_Z(t) := \lim_{N \to \infty} \eta_{Z^{(N)}(t)}^N$ exits in weak sense, then

$$dZ_i(t) = \left( \int_{\mathbb{R}} F(t, Z_i(t), u_i(t), y) \eta_Z(t)(dy) \right) dt + \sigma dW_i(t).$$

$$J_i(u_i) = \mathbb{E} \int_0^T \left( \int_{\mathbb{R}} L(t, Z_i(t), u_i(t), y) \eta_Z(t)(dy) \right) dt.$$ 

$\eta_Z(t)$ (mean field) contains only statistical property of the mass $Z(t)$. 

Wei Yang

Introduction to Mean field games and applications
To sense a mean field parameter....
\( \eta_Z(t) \) contains statistical property of \( Z(t) = (Z_1(t), \ldots, Z_N(t), \ldots) \)

Let \( \mathcal{P}_Z(t) \) be the probability distribution of \( Z_i(t) \). In the continuum limit \( N \rightarrow \infty \),

\[
dZ_i(t) = \left( \int_{\mathbb{R}} F(t, Z_i(t), u_i(t), y) \mathcal{P}_Z(t)(dy) \right) dt + \sigma dW_i(t)
\]

\[
J_i(u_i) = \mathbb{E} \int_0^T \left( \int_{\mathbb{R}} L(t, Z_i(t), u_i(t), y) \mathcal{P}_Z(t)(dy) \right) dt.
\]

**Underlying intuition: as \( N \) increases...**

- at any time \( t \geq 0 \), an individual’s distribution \( \mathcal{P}_Z(t) \) can effectively represent the empirical distribution \( \eta_Z(t) \).
Model assumptions

- players become infinitesimal and indistinguishable
  
  .......the dynamics of the mass is the result of what a single player does

- players respond to a mean-field
  
  .......restating game theory as an interaction between individual and the mass
Mean field game (limit for $N \to \infty$)

For a representative agent,

**The controlled dynamics $(X(t), t \geq 0)$**

For a control policy $u. = (u(t) \in \mathcal{U}, t \geq 0)$,

$$dX(t) = f(X(t), u(t), \mu(t))dt + \sigma dW(t)$$  \hspace{1cm} (1)

where $f : \mathbb{R} \times \mathcal{U} \times P(\mathbb{R}) \to \mathbb{R}$.

The solution is $\{(X(t), \mu(t)), t \geq 0\}$ such that

- $\{X(t), t \geq 0\}$ is a solution to Eq. (1)
- $\mu(t)$ is the probability distribution of $X(t)$ for any $t \geq 0$. 

Wei Yang

Introduction to Mean field games and applications
Mean field game (limit for $N \to \infty$)

For a representative agent,

The controlled dynamics $(X(t), t \geq 0)$

For a control policy $u_\ast = (u(t) \in \mathcal{U}, t \geq 0)$,

$$dX(t) = f(X(t), u(t), \mu(t))dt + \sigma dW(t) \quad \text{McKean-Vlasov} \quad (1)$$

where $f : \mathbb{R} \times \mathcal{U} \times P(\mathbb{R}) \to \mathbb{R}$. 
**Mean field game (limit for $N \rightarrow \infty$)**

For a representative agent,

**The controlled dynamics $(X(t), t \geq 0)$**

For a control policy $u. = (u(t) \in U, t \geq 0)$,

$$dX(t) = f(X(t), u(t), \mu(t))dt + \sigma dW(t) \quad \text{McKean-Vlasov} \quad (1)$$

where $f : \mathbb{R} \times U \times \mathcal{P}(\mathbb{R}) \rightarrow \mathbb{R}$.

$\mu. = \{\mu(t) \in \mathcal{P}(\mathbb{R}), t \geq 0\}$ represents the mean field.

**The cost function**

For a mean field $\mu. = \{\mu(t) \in \mathcal{P}(\mathbb{R}), t \geq 0\}$,

$$J(t, x, u., \mu.) = \mathbb{E}_x \int_t^T L(X(s), u(s), \mu(s))ds \quad (2)$$

where $L : \mathbb{R} \times U \times \mathcal{P}(\mathbb{R}) \rightarrow \mathbb{R}$.

To find a $\hat{u}.$ which is an optimal response to $\mu.$ and produces $\mu.$
Mean field game (limit for $N \to \infty$)

The controlled dynamics $(X(t), t \geq 0)$
For a control policy $u. = (u(t) \in U, t \geq 0)$,

$$dX(t) = f(X(t), u(t), \mu(t))dt + \sigma dW(t)$$

$\Rightarrow$ Fokker-Planck (FP) equation (forward Kolmogorov equation)

The cost function
For a mean field $\mu. = \{\mu(t) \in P(\mathbb{R}), t \geq 0\}$,

$$J(t, x, u., \mu.) = \mathbb{E}_x \int_t^T L(x(s), u(s), \mu(s))ds$$

$\Rightarrow$ Hamilton-Jacobi-Bellman (HJB) equation
Mean field game (limit for $N \to \infty$)

**FP equation**

For a control policy $u. = (u(t) \in \mathcal{U}, t \geq 0)$,

$$\frac{d}{dt} m(t, x) = - \frac{d}{dx} \left( f(x, u(t), m(t, x)) m(t, x) \right) + \frac{\sigma^2}{2} \frac{d^2}{dx^2} m(t, x)$$

$m(0, x) = m_0(x)$

...describe the aggregation of the action of all players.

**HJB equation**

For a mean field $\mu. = \{\mu(t) \in \mathcal{P}(\mathbb{R}), t \geq 0\}$,

$$- \frac{\partial V}{\partial t} = \inf_{u(t) \in \mathcal{U}} \left\{ f(x, u(t), \mu(t)) \frac{\partial V}{\partial x} + L(x, u(t), \mu(t)) \right\} + \frac{\sigma^2}{2} \frac{\partial^2 V}{\partial x^2}$$

$V(T, x) = 0$

...the reaction of players to the mass
## Mean field game (limit for $N \to \infty$)

### FP equation - forward equation

For a control policy $u. = (u(t) \in \mathcal{U}, t \geq 0)$,

$$
\frac{d}{dt} m(t, x) = - \frac{d}{dx} \left( f(x, u(t), m(t, x)) m(t, x) \right) + \frac{\sigma^2}{2} \frac{d^2}{dx^2} m(t, x)
$$

$$
m(0, x) = m_0(x)
$$

...describe the aggregation of the action of all players.

...where the population actually end up, based on the initial dist.

### HJB equation

For a mean field $\mu. = \{\mu(t) \in P(\mathbb{R}), t \geq 0\}$,

$$
- \frac{\partial V}{\partial t} = \inf_{u(t) \in \mathcal{U}} \left\{ f(x, u(t), \mu(t)) \frac{\partial V}{\partial x} + L(x, u(t), \mu(t)) \right\} + \frac{\sigma^2}{2} \frac{\partial^2 V}{\partial x^2}
$$

$$
V(T, x) = 0
$$

...the reaction of players to the mass
Mean field game (limit for \( N \to \infty \))

**FP equation - forward equation**

For a control policy \( u. = (u(t) \in \mathcal{U}, t \geq 0) \),

\[
\frac{d}{dt} m(t, x) = -\frac{d}{dx} \left( f(x, u(t), m(t, x)) m(t, x) \right) + \frac{\sigma^2}{2} \frac{d^2}{dx^2} m(t, x)
\]

\[
m(0, x) = m_0(x)
\]

...describe the aggregation of the action of all players.
...where the population actually end up, based on the initial dist.

**HJB equation - backward equation**

For a mean field \( \mu. = \{\mu(t) \in \mathcal{P}(\mathbb{R}), t \geq 0\} \),

\[
-\frac{\partial V}{\partial t} = \inf_{u(t) \in \mathcal{U}} \left\{ f(x, u(t), \mu(t)) \frac{\partial V}{\partial x} + L(x, u(t), \mu(t)) \right\} + \frac{\sigma^2}{2} \frac{\partial^2 V}{\partial x^2}
\]

\[
V(T, x) = 0
\]

...the reaction of players to the mass
...decisions based on where you want to be in the future
Mean field game (limit for $N \to \infty$)

Mean field equations - coupled system of two equations

**FP equation - forward equation**

For a control policy $u. = (u(t) \in \mathcal{U}, t \geq 0)$,
\[
\frac{d}{dt} m(t, x) = -\frac{d}{dx} \left( f(x, u(t), m(t, x)) m(t, x) \right) + \frac{\sigma^2}{2} \frac{d^2}{dx^2} m(t, x)
\]
\[m(0, x) = m_0(x)\]

\[\hat{u}(t) \uparrow \quad \downarrow \mu(t)(dx) = m(t, x)dx\]

**HJB equation - backward equation**

For a mean field $\mu. = \{\mu(t) \in P(\mathbb{R}), t \geq 0\}$,
\[
-\frac{\partial V}{\partial t} = \inf_{u(t) \in \mathcal{U}} \left\{ f(x, u(t), \mu(t)) \frac{\partial V}{\partial x} + L(x, u(t), \mu(t)) \right\} + \frac{\sigma^2}{2} \frac{\partial^2 V}{\partial x^2}
\]
\[V(T, x) = 0.\]

Let $\hat{u}(t) = \hat{u}(t, x, \mu.)$ be the best response to the mean field.
Outline

1. Introduction

2. Our results and applications
Joint work with Vassili Kolokoltsov at Warwick University

We developed a unified framework where a larger class of Markov processes is considered.

The dynamic of the $N$ players $\left( Z^N(t) \in \mathbb{R}^N : t \in [0, T] \right)$ is associated to a family of linear and bounded operators $\left\{ A[t, \mu, u] \in \mathcal{L}(C^2, C) : t \in [0, T], \mu \in \mathcal{P}(\mathbb{R}^d), u \in \mathcal{U} \right\}$. 

\[
\left\{ A[t, \mu, u] \in \mathcal{L}(C^2, C) : t \in [0, T], \mu \in \mathcal{P}(\mathbb{R}^d), u \in \mathcal{U} \right\}.
\]
For each \((t, \mu, u) \in [0, T] \times \mathcal{P}(\mathbb{R}^d) \times \mathcal{U}\), \(A[t, \mu, u] : C^2 \rightarrow \mathbb{C}\) is assumed to generate a Feller process with values in \(\mathbb{R}^d\) and to be of the form

\[
A[t, \mu, u]f(z) = (h(t, z, \mu, u), \nabla f(z)) + R[t, \mu]f(z)
\]

- \(h : [0, T] \times \mathbb{R}^d \times \mathcal{P}(\mathbb{R}^d) \times \mathcal{U} \rightarrow \mathbb{R}^d\)
- \(R[t, \mu] \in \mathcal{L}(C^2, \mathbb{C})\) is of the form:

\[
R[t, \mu]f(z) = \frac{1}{2} (G(t, z, \mu)\nabla, \nabla)f(z) + (b(t, z, \mu), \nabla f(z))
\]

\[
+ \int_{\mathbb{R}^d} (f(z + y) - f(z) - (\nabla f(z), y)1_{B_1}(y)) \nu(t, z, \mu, dy).
\]
For each \((t, \mu, u) \in [0, T] \times \mathcal{P}(\mathbb{R}^d) \times \mathcal{U}\), \(A[t, \mu, u] : C^2 \mapsto C\) is assumed to generate a Feller process with values in \(\mathbb{R}^d\) and to be of the form

\[
A[t, \mu, u]f(z) = (h(t, z, \mu, u), \nabla f(z)) + R[t, \mu]f(z)
\]

**Example**

If \( R[t, \mu] = \frac{1}{2} \sigma^2 \Delta \) with a constant \( \sigma \), i.e,

\[
A[t, \mu, u]f(z) = (h(t, z, \mu, u), \nabla f(z)) + \frac{1}{2} \sigma^2 \Delta f(z),
\]

\((Z^N(t) : t \in [0, T])\) can also be described by the SDE

\[
dZ^N(t) = h(t, Z^N(t), \mu_t, u_t) \, dt + \sigma \, dW_t.
\]
Optimal control problem for each player...

Given \( \{\eta^N(t) : t \in [0, T]\}\), the optimal payoff of \(i\)-th player, \(i \in [1, N]\), starting at \(x\) and \(t\) is

\[
V_{i,N}(t, x) = \inf_{u.} \mathbb{E}_x \left[ \int_t^T L(s, Z_{i,s}^N, \eta_s^N, u_{i,s}) \, ds + V_T(Z_{i,T}^N) \right]
\]

- \(L : [0, T] \times \mathbb{R}^d \times \mathcal{P}(\mathbb{R}^d) \times \mathcal{U} \to \mathbb{R}\)
- \(V^T : \mathbb{R}^d \to \mathbb{R}\)
Formally, if $\eta^N_t \to \mu_t$ and $V_{i,N} \to V$ as $N \to \infty$ such that

**Coupled forward-backward equations**

**forward kinetic equation in weak form:**

$$
\begin{cases}
\frac{d}{dt}(g, \mu_t) = (A[t, \mu_t, \Gamma(t, .., \{\mu_s : t \leq s \leq T\})]g, \mu_t) \\
\mu_0 = \mu
\end{cases}
$$

**backward HJB equation:**

$$
\begin{cases}
\frac{\partial V(t,x)}{\partial t} + H_t(x, \nabla V(t, x), \mu_t) + R[t, \mu_t]V(t, x) = 0 \\
H_t(x, p, \mu) = \inf_{u \in U} (h(t, x, \mu, u)p + L(t, x, \mu, u)) \\
V|_{T} = V^T
\end{cases}
$$

$$
V(t, x) \quad \Gamma(t, x, \{\nu_t : t \in [0, T]\}) \quad (X_t : t \in [0, T])
$$

$$
\uparrow \quad \uparrow
\begin{cases}
\{\nu_t : t \in [0, T]\} \\
\{\mu_t : t \in [0, T]\}
\end{cases}
$$
Formally, if $\eta_t^N \to \mu_t$ and $V_{i,N} \to V$ as $N \to \infty$ such that

**Coupled forward-backward equations**

**forward kinetic equation in weak form:**

$$\begin{cases} \frac{d}{dt}(g, \mu_t) = (A[t, \mu_t, \Gamma(t, .., \{\mu_s : t \leq s \leq T\}))g, \mu_t) & (FE) \\ \mu_0 = \mu \end{cases}$$

**backward HJB equation:**

$$\begin{cases} \frac{\partial V(t,x)}{\partial t} + H_t(x, \nabla V(t,x), \mu_t) + R[t, \mu_t]V(t,x) = 0 & (BE) \\ H_t(x, p, \mu) = \inf_{u \in U}(h(t, x, \mu, u)p + L(t, x, \mu, u)) \\ V|_T = V^T \end{cases}$$

$$V(t, x) \implies \Gamma(t, x, \{\nu_t : t \in [0, T]\}) \implies (X_t : t \in [0, T])$$

$$\uparrow \quad \{\nu_t : t \in [0, T]\} = \{\mu_t : t \in [0, T]\} \quad \downarrow$$

fixed point, consistency
Theorem 1 (KY2013): mean field limit

For arbitrary $T > 0$, there exists a solution to the Cauchy problem for (FE)

\[
\frac{d}{dt}(g, \mu_t) = (A[t, \mu_t, \Gamma(t, .., \{\mu_s : t \leq s \leq T\})]g, \mu_t)
\]

\[
\mu_0 = \mu.
\]

- For $T$ small enough, the solution is unique.
- Requirement: $\Gamma$ has feedback regularity property.

Theorem 2 (KY2013): optimal control

(a) For a given flow \( \{ \nu_t : t \in [0, T] \} \), the Cauchy problem

\[
\begin{aligned}
\frac{\partial V(t,x)}{\partial t} + H_t(x, \nabla V(x), \nu_t) + R[t, \nu_t] V(t, x) &= 0 \\
H_t(x, p, \mu) &= \inf_{u \in U} (h(t, x, \mu, u)p + L(t, x, \mu, u)) \\
V|_{T} &= V^T
\end{aligned}
\]

is wellposed;
**Theorem 2 (KY2013): optimal control - sensitivity**

(b) For any \( \{\nu_1^t : t \in [0, T]\} , \{\nu_2^t : t \in [0, T]\} \), there exists \( k > 0 \) such that

\[
\sup_{t \in [0, T]} \| V(t, \cdot ; \{\nu_1^t\}) - V(t, \cdot ; \{\nu_2^t\}) \|_{C^1} \leq k \sup_{t \in [0, T]} \| \nu_1^t - \nu_2^t \| (C^2)^* \]

and

\[
\sup_{t \in [0, T]} \| \nabla V(t, \cdot ; \{\nu_1^t\}) - \nabla V(t, \cdot ; \{\nu_2^t\}) \|_{C} \leq k \sup_{t \in [0, T]} \| \nu_1^t - \nu_2^t \| (C^2)^* ;
\]
The unique optimal control function $\Gamma(t, x; \{\nu_t : t \in [0, T]\})$ denote has the feedback regularity required by Theorem 1, i.e. for $\{\nu^1_t : t \in [0, T]\}, \{\nu^2_t : t \in [0, T]\}$,

$$\sup_{t, x} \left| \Gamma(t, x; \{\nu^1\}) - \Gamma(t, x; \{\nu^2\}) \right| \leq k_1 \sup_{s \in [0, T]} ||\nu^1_s - \nu^2_s|| (C^2)^*$$

with some constant $k_1 > 0$.

- prove, rather than assume, the feedback regularity property

Theorem 3 (KY2012): convergence

The $N$-player games converge to the limit system, i.e.

$$\mu_t^N(\mu_0^N) \rightarrow \mu_t(\mu_0) \quad \text{as} \quad N \rightarrow \infty, \quad \forall t \in [0, T]$$

with a convergence rate $O(1/N)$, where $\mu_t$ is a solution to (FE).

- Technical issue: smoothness of the solutions to equation (FE) with respect to initial data $\mu_0$.
- Improvement to the existing models: convergence rate $1/N$, instead of $1/\sqrt{N}$.

**Definition: \(\epsilon\) Nash equilibrium**

For \(\epsilon > 0\), a strategy profile \(\Gamma\) is a \(\epsilon\)-Nash equilibrium if

\[
J_i(\Gamma) \leq J_i(\Gamma_{-i}, u_i) + \epsilon, \quad \forall i = 1, \ldots, N
\]

where \((\Gamma_{-i}, u_i)\) denotes the profile obtained from \(\Gamma\) by substituting the strategy of player \(i\) with any eligible strategy \(u_i\).

**Theorem 4 (KY2012): \(\epsilon\) Nash equilibrium**

Any solution of the limit model

\[
\Gamma(t, x, \{\mu_s : t \leq s \leq T\})
\]

represents an \(\epsilon\)-equilibrium for an \(N\) players dynamic game, with \(\epsilon = \mathcal{O}(1/N)\).

- In a general setting, solutions are not in a closed-form.

A setting with a major player:

**Inspection games in mean field setting**

- consider one inspector with $N$ inspectees
- introduction of a deterministic major player
- Markov Chain on a finite state space $\{l_1, \ldots, l_d\}$ (crime levels)
- obtain a convergence result without a convergence rate

Kolokoltsov, Yang (2014). Inspection games in mean field setting, in preparation
MFG theory attracts much attention from Mathematical society.

Other developments of MFG
- cooperative population
- stability of the limit system over infinite horizon
- with a stochastic major player
- mean field type controls
- ...
References:


Thanks for your attention.