REPRESENTATIONS FOR THE DECAY PARAMETER OF A BIRTH-DEATH PROCESS

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1. birth-death processes

2. Karlin-McGregor representation

3. intermezzo: orthogonal polynomials

4. decay parameter: representations

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birth-death process

definition: birth-death process is Markov process \( \{X(t), \ t \geq 0\} \) on \( \{(-1, 0, 1, \ldots)\} \) with absorbing state -1, birth rates \( \lambda_n > 0 \) \( (n \geq 0) \) and death rates \( \mu_n > 0 \) \( (n > 0) \) and \( \mu_0 \geq 0 \)

assumption: rates determine process uniquely

notation: \( p_{ij}(t) := \Pr\{X(t) = j \mid X(0) = i\} \)

\[
p_j := \lim_{t \to \infty} p_{ij}(t)
\]

\( \pi_0 := 1, \pi_j := \frac{\lambda_0 \cdots \lambda_{j-1}}{\mu_1 \cdots \mu_j} \quad j > 0 \)

result \( (j \geq 0)\):

\[
p_j := \begin{cases} 
\frac{\pi_j}{\sum_n \pi_n} & \text{if } \mu_0 = 0 \text{ and } \sum_n \pi_n < \infty \\
0 & \text{otherwise}
\end{cases}
\]
birth-death process

\( i, j \geq 0: \)

\[ p_{ij}(t) := \Pr\{X(t) = j \mid X(0) = i\}, \quad p_j := \lim_{t \to \infty} p_{ij}(t) \]

notation:

\( \alpha_{ij} := \sup\{a > 0 : |p_{ij}(t) - p_j| = O(e^{-at}) \text{ as } t \to \infty\} \)

results (Kingman (1963)):

\[ \alpha_{ij} = -\lim_{t \to \infty} \frac{1}{t} \log |p_{ij}(t) - p_j| \]

and

\[ \alpha := \alpha_{00} \leq \alpha_{ij} \]

note: equality if \( \mu_0 > 0 \)

inequality for at most one value of \( i \) or \( j \) if \( \mu_0 = 0 \)

interest: decay parameter \( \alpha \)
representation \textit{(Karlin-McGregor (1957))}: \[
p_{ij}(t) = \pi_j \int_0^\infty e^{-xt} Q_i(x) Q_j(x) \psi(dx), \quad i, j \geq 0
\]
with \[
\pi_0 = 1, \quad \pi_j = \frac{\lambda_0 \ldots \lambda_{j-1}}{\mu_1 \ldots \mu_j} \quad j > 0
\]
and \[
\lambda_n Q_{n+1}(x) = (\lambda_n + \mu_n - x) Q_n(x) - \mu_n Q_{n-1}(x)
\]
\[
Q_0(x) = 1, \quad \lambda_0 Q_1(x) = \lambda_0 + \mu_0 - x
\]
definition decay parameter $\alpha$: \[
\alpha = \sup\{a > 0 : |p_{00}(t) - p_0| = O(e^{-at}) \text{ as } t \to \infty\}
\]
birth-death process: Karlin-McGregor representation

\[ p_{00}(t) = \pi_0 \int_0^\infty e^{-xt} Q_0(x) Q_0(x) \psi(dx) = \int_0^\infty e^{-xt} \psi(dx) \]

hence

\[ p_0 = \lim_{t \to \infty} p_{00}(t) = \psi(\{0\}) \]

so that

\[ p_{00}(t) - p_0 = \int_0^\infty e^{-xt} \psi(dx) \]

hence representation decay parameter

\[ \alpha = \begin{cases} \xi_1 & \text{if } \xi_1 > 0 \\ \xi_2 & \text{if } \xi_2 > \xi_1 = 0 \\ 0 & \text{if } \xi_1 = \xi_2 = 0 \end{cases} \]

where

\[ \xi_1 = \inf \text{supp}(\psi), \quad \xi_2 = \inf \{\text{supp}(\psi) \setminus \{\xi_1\}\} \]
birth-death process: Karlin-McGregor representation

representation:

\[ p_{ij}(t) = \pi_j \int_0^\infty e^{-xt}Q_i(x)Q_j(x)\psi(dx) \]

\[ t = 0: \quad \pi_j \int_0^\infty Q_i(x)Q_j(x)\psi(dx) = \delta_{ij} \]

\( \{Q_n(x)\} \) orthogonal polynomial sequence with respect to measure \( \psi \) on \([0, \infty)\)!

defining

\[ P_n(x) := (-1)^n\lambda_0\lambda_1 \ldots \lambda_{n-1}Q_n(x) \]

we have

\[ P_n(x) = (x - \lambda_{n-1} - \mu_{n-1})P_{n-1}(x) - \lambda_{n-2}\mu_{n-1}P_{n-2}(x) \]

\[ P_0(x) = 1, \quad P_1(x) = x - \lambda_0 - \mu_0 \]
Associate Editor Stochastic Processes and Their Applications:

... Van Doorn’s paper is difficult to read because hardly anybody knows orthogonal polynomials anymore.
intermezzo: orthogonal polynomials
intermezzo: orthogonal polynomials

**definition:** \( \{P_n(x), \ n = 0, 1, \ldots\} \) (monic, \( \deg(P_n) = n \)) is orthogonal polynomial sequence (OPS) if there exists (Borel) measure \( \psi \) (of total mass 1) such that

\[
\int_{-\infty}^{\infty} P_n(x)P_m(x)\psi(dx) = k_n \delta_{nm}
\]

with \( k_n > 0 \)

**Favard's theorem:**

\( \{P_n(x), \ n = 0, 1, \ldots\} \) is OPS \( \iff \) there exist \( c_n \in \mathbb{R}, \ d_n > 0 \) such that

\[
P_n(x) = (x - c_n)P_{n-1}(x) - d_nP_{n-2}(x)
\]

\[
P_0(x) = 1, \quad P_1(x) = x - c_1
\]

**remark:** \( \psi \) not necessarily unique (Hamburger moment problem)
intermezzo: orthogonal polynomials

\{P_n(x), \ n = 0, 1, \ldots\} is OPS with zeros \(x_{ni}\)

zeros of \(P_n(x)\) real and distinct:

\[x_{n1} < x_{n2} < \ldots < x_{nn}\]

interlacing property:

\[x_{n+1,i} < x_{ni} < x_{n+1,i+1}\]

hence

\[\xi_i := \lim_{n \to \infty} x_{ni}\]

exist, and

\[-\infty \leq \xi_i \leq \xi_{i+1} < \infty\]
intermezzo: orthogonal polynomials

\{P_n(x), \ n = 0, 1, \ldots\} is OPS satisfying

\[ P_n(x) = (x - c_n)P_{n-1}(x) - d_nP_{n-2}(x) \]
\[ P_0(x) = 1, \quad P_1(x) = x - c_1 \]

**Theorem:** if \( \xi_1 > -\infty \) there exists a (unique) orthogonalizing measure \( \psi \) for \( \{P_n(x)\} \), that is,

\[ \int_{-\infty}^{\infty} P_m(x)P_n(x)\psi(dx) = k_n\delta_{mn} \]

such that

\[ \xi_1 = \lim_{n \to \infty} x_{n1} = \inf \text{supp}(\psi) \]
{\( P_n(x) \), \( n = 0, 1, \ldots \)} is OPS satisfying

\[
P_n(x) = (x - c_n)P_{n-1}(x) - d_nP_{n-2}(x)
\]

\[
P_0(x) = 1, \quad P_1(x) = x - c_1
\]

**theorem:**

(i) \( \xi_1 \geq 0 \iff \{P_n\} \text{ OPS w.r.t measure } \psi \text{ on } [0, \infty) \)

\( \iff \)

there exist numbers \( \lambda_n > 0, \mu_{n+1} > 0 \) \( (n \geq 0) \) and \( \mu_0 \geq 0 \) such that \( c_1 = \lambda_0 + \mu_0 \), and, for \( n > 1 \),

\[
c_{n+1} = \lambda_n + \mu_n
\]

\[
d_{n+1} = \lambda_{n-1}\mu_n
\]

(ii) if there exist \ldots and \( \mu_0 > 0 \) then \( \psi(\{0\}) = 0 \)
intermezzo: orthogonal polynomials on \([0, \infty)\)

**summary:**

\(\{P_n(x), \ n = 0, 1, \ldots\}\) satisfies

\[
P_n(x) = (x - \lambda_n - \mu_n)P_{n-1}(x) - \lambda_{n-1}\mu_nP_{n-2}(x)
\]

\[
P_0(x) = 1, \quad P_1(x) = x - \lambda_0 - \mu_0
\]

with \(\lambda_n > 0, \mu_{n+1} > 0\) and \(\mu_0 \geq 0\)

\[
\Rightarrow
\]

\(\{P_n(x), \ n = 0, 1, \ldots\}\) is OPS with respect to measure \(\psi\) with support in \([0, \infty)\) (with \(\psi(\{0\}) = 0\) if \(\mu_0 > 0\)), \(x_{n1}\) decreasing in \(n\), and

\[
\xi_1 = \lim_{n \to \infty} x_{n1} = \inf \text{supp}(\psi)
\]
birth-death processes: decay parameter

representation:

\[ p_{ij}(t) = \pi_j \int_0^\infty e^{-xt} Q_i(x)Q_j(x)\psi(dx) \]

\[ p_{00}(t) - p_0 = \int_{0+}^\infty e^{-xt}\psi(dx) \]

interest: decay parameter

\[ \alpha = \inf\{\text{supp}(\psi)\backslash\{0\}\} = \begin{cases} \xi_1 & \text{if } \xi_1 > 0 \\ \xi_2 & \text{if } \xi_2 > \xi_1 = 0 \\ 0 & \text{if } \xi_1 = \xi_2 = 0 \end{cases} \]

where

\[ \xi_1 = \lim_{n \to \infty} x_{n1} = \inf\supp(\psi) \]

\[ \xi_2 = \lim_{n \to \infty} x_{n2} = \inf\{\text{supp}(\psi)\backslash\{\xi_1\}\} \]
birth-death processes: decay parameter

\[ p_{00}(t) - p_0 = \int_{0+}^{\infty} e^{-xt} \psi(dx) \]

decay parameter

\[ \alpha = \inf\{\text{supp}(\psi) \setminus \{0\}\} = \begin{cases} 
\xi_1 & \text{if } \xi_1 > 0 \\
\xi_2 & \text{if } \xi_2 > \xi_1 = 0 \\ 0 & \text{if } \xi_1 = \xi_2 = 0 
\end{cases} \]

recall: \( \mu_0 > 0 \Rightarrow p_0 = 0 \Rightarrow \psi(\{0\}) = 0 \)

hence

\[ \alpha = \begin{cases} 
\xi_2 & \text{if } \mu_0 = 0 \text{ and } \psi(\{0\}) > 0 \\
\xi_1 & \text{otherwise} 
\end{cases} \]
birth-death processes: decay parameter

**problem**: determine decay parameter

\[ \alpha = \begin{cases} 
\xi_2 & \text{if } \mu_0 = 0 \text{ and } \psi(\{0\}) > 0 \\
\xi_1 & \text{otherwise}
\end{cases} \]

**approach** if \( \mu_0 = 0 \): dual process

**definition**: given \( \lambda_n, \mu_{n+1} \ (n \geq 0) \) and \( \mu_0 = 0 \) the dual process has rates

\[ \lambda_n^* := \mu_{n+1}, \ \mu_{n+1}^* := \lambda_{n+1}, \ \mu_0^* := \lambda_0 \ (> 0) \]

then \( \{P_n^*(x)\} \) OPS w.r.t \( \psi^* \) and

\[ \text{supp}(\psi^*) = \text{supp}(\psi) \setminus \{0\} \]

so that

\[ \alpha = \inf\{\text{supp}(\psi) \setminus \{0\}\} = \xi_1^* \]
assumption: $\mu_0 > 0$

hence

$$\alpha = \xi_1 = \inf \text{supp}(\psi) = \lim_{n \to \infty} x_{n1}$$

with

$$x_{n1} < x_{n2} < \ldots < x_{nn}$$

zeros of $P_n(x)$:

$$P_n(x) = (x - \lambda_{n-1} - \mu_{n-1})P_{n-1}(x) - \lambda_{n-2}\mu_{n-1}P_{n-2}(x)$$

$$P_0(x) = 1, \quad P_1(x) = x - \lambda_0 - \mu_0$$

1st approach: orthogonal polynomial systems
\begin{align*}
\xi_1 &= \inf \text{supp}(\psi) = \lim_{n \to \infty} x_{n1} \\
\textbf{1st approach: orthogonal polynomial systems} \\
\textbf{example 1:} & \quad \lambda_n = \lambda, \quad \mu_n = \mu \\
\text{then} & \quad \xi_1 = (\sqrt{\lambda} - \sqrt{\mu})^2 \\
\textbf{example 2:} & \quad \lambda_n = \lambda, \quad \mu_n = \frac{\mu}{n + 1} \\
\text{then} & \quad \xi_1 = \lambda - \frac{1}{2} \left( \sqrt{\mu^2 + 4\lambda\mu} - \mu \right)
\end{align*}
\[ \xi_1 = \inf \text{supp}(\psi) = \lim_{n \to \infty} x_{n1} \]

let

\[
R_{n+1} = \begin{pmatrix}
-\lambda_0 - \mu_0 & \lambda_0 & 0 & \cdots & \cdots \\
\mu_1 & -\lambda_1 - \mu_1 & \lambda_1 & \cdots & \cdots \\
\vdots & \vdots & \ddots & \ddots & \vdots \\
\cdots & \cdots & \cdots & -\lambda_{n-1} - \mu_{n-1} & \lambda_{n-1} \\
\cdots & \cdots & \cdots & 0 & \mu_n & -\lambda_n - \mu_n
\end{pmatrix}
\]

truncated q-matrix of birth-death process, then

\[ P_n(x) = \det(xI_n + R_n), \quad n > 0 \]

so zeros of \( P_n(x) \) are eigenvalues of \(-R_n\)
decay parameter: representations

\[ \xi_1 = \inf \text{supp}(\psi) = \lim_{n \to \infty} x_{n1} \]

more generally, let \( a_j > 0 \) and

\[
T_{n+1} := \begin{pmatrix}
\lambda_0 + \mu_0 & \lambda_0 \mu_1/a_1 & 0 & \cdots & \cdots \\
a_1 & \lambda_1 + \mu_1 & \lambda_1 \mu_2/a_2 & \cdots & \cdots \\
\vdots & \vdots & \ddots & \ddots & \vdots \\
\cdots & \cdots & \cdots & \lambda_{n-1} + \mu_{n-1} & \lambda_{n-1} \mu_n/a_n \\
\cdots & \cdots & \cdots & 0 & a_n \\
\end{pmatrix}
\]

then

\[ P_n(x) = \det(xI_n - T_n) \]

2nd approach: eigenvalue techniques
Theorem

\[ x_{n1} = \max_{a > 0} \min_{0 \leq i < n} \left\{ \lambda_i + \mu_i - (1 - \delta_{i,n-1})a_i + \frac{\lambda_i - 1 \mu_i}{a_i} \right\} \]

\[ x_{n1} = \min_{a > 0} \max_{0 \leq i < n} \left\{ \lambda_i + \mu_i - (1 - \delta_{i,n-1})a_i + \frac{\lambda_i - 1 \mu_i}{a_i} \right\} \]

so that

\[ \xi_1 = \max_{a > 0} \inf_{i \geq 0} \left\{ \lambda_i + \mu_i - a_{i+1} - \frac{\lambda_i - 1 \mu_i}{a_i} \right\} \]

\[ \xi_1 = \lim_{n \to \infty} \min_{a > 0} \max_{0 \leq i \leq n} \left\{ \lambda_i + \mu_i - (1 - \delta_{i,n})a_i + \frac{\lambda_i - 1 \mu_i}{a_i} \right\} \]

where \( \lambda_{-1} = 0, \ a_0 = 1, \ a = (a_1, a_2, \ldots) \)

Proof: Geršgorin discs, Collatz-Wielandt formula, Perron-Frobenius theory
decay parameter: representations

**Theorem**

\[ x_{n1} = \max_h \min_{1 \leq i < n} \frac{1}{2} \left\{ c_i + c_{i+1} - \sqrt{(c_{i+1} - c_i)^2 + \frac{4d_{i+1}}{(1 - h_i)h_{i+1}}} \right\} \]

where

\[ c_{i+1} = \lambda_i + \mu_i, \quad d_{i+1} = \lambda_{i-1}\mu_i \]

and

\[ h = (h_1, \ldots, h_n), \quad h_1 = 0, \quad h_n = 1, \quad 0 < h_i < 1 \quad (1 < i < n) \]

so that

\[ \xi_1 = \max_b \inf_{i \geq 1} \frac{1}{2} \left\{ c_i + c_{i+1} - \sqrt{(c_{i+1} - c_i)^2 + 4d_{i+1}/b_i} \right\} \]

where \( b = (b_1, b_2, \ldots) \) is a chain sequence

**Proof:** ovals of Cassini
Decay parameter: representations

Theorem

\[ x_{n1} = \min_{h \geq 0} \left\{ \sum_{i=0}^{n} (h_i^2(\lambda_i + \mu_i) - 2h_{i-1}h_i\sqrt{\lambda_i\mu_{i+1}}) \right\} \]

where \( h = (h_0, \ldots, h_n) \), \( h_0 = 0 \), \( \sum_{i=1}^{n} h_i^2 = 1 \), whence

\[ \xi_1 = \inf_{h \geq 0} \left\{ \lim_{n \to \infty} \inf_{n \to \infty} \left\{ \sum_{i=1}^{n} (h_i^2(\lambda_i + \mu_i) - 2h_{i-1}h_i\sqrt{\lambda_i\mu_{i+1}}) \right\} \right\} \]

where \( h = (h_0, h_1, \ldots) \), \( h_0 = 0 \), \( \sum_{i=1}^{\infty} h_i^2 = 1 \)

Proof: symmetrize \( T_n \) by suitable choice of \( a_i \), then

\[ x_{n1} = \min_{y \neq 0} \frac{y^T T_n y}{y^T y} \]

Courant-Fischer theorem
decay parameter: representations

setting: $\mu_0 \geq 0$

$$p_{00}(t) - p_0 = \int_{0+}^{\infty} e^{-xt} \psi(dx)$$

$$\alpha = \inf\{\text{supp}(\psi) \setminus \{0\}\} = \begin{cases} 
\xi_2 & \text{if } \mu_0 = 0 \text{ and } \psi(\{0\}) > 0 \\
\xi_1 & \text{otherwise}
\end{cases}$$

notation:

$$\pi_0 = 1, \quad \pi_n = \frac{\lambda_0 \cdots \lambda_{n-1}}{\mu_1 \cdots \mu_n}, \quad n > 0$$

and

$$K_n = \sum_{i=0}^{n} \pi_i \quad K_\infty = \sum_{i=0}^{\infty} \pi_i$$

$$L_n = \sum_{i=1}^{n} \frac{1}{\mu_i \pi_i} \quad L_\infty = \sum_{i=1}^{\infty} \frac{1}{\mu_i \pi_i}$$
intermezzo: birth-death processes

setting: $\mu_0 \geq 0$

$$p_{00}(t) - p_0 = \int_{0+}^{\infty} e^{-xt} \psi(dx)$$

$$\alpha = \inf\{\text{supp}(\psi) \backslash \{0\}\} = \begin{cases} \xi_2 & \text{if } \mu_0 = 0 \text{ and } \psi(\{0\}) > 0 \\ \xi_1 & \text{otherwise} \end{cases}$$

results (Karlin and McGregor (1957)):

$$\mu_0 = 0 \quad \Rightarrow \quad \psi(\{0\}) = K_{\infty}^{-1} \geq 0$$

$$\mu_0 = 0 \quad \text{and} \quad K_{\infty} = \infty \quad \Rightarrow \quad \int_{(0,\infty)} \frac{\psi(dx)}{x} = L_{\infty} \leq \infty$$

$$\mu_0 > 0 \quad \Rightarrow \quad \psi(\{0\}) = 0 \quad \text{and} \quad \int_{(0,\infty)} \frac{\psi(dx)}{x} \leq \frac{1}{\mu_0}$$
intermezzo: birth-death processes

recall:

\[ K_\infty = \sum_{i=0}^{\infty} \pi_i \quad \text{and} \quad L_\infty = \sum_{i=1}^{\infty} \frac{1}{\mu_i \pi_i} \]

assumption \( \psi \) unique \( \Rightarrow \) \( K_\infty + L_\infty = \infty \)

results for \( \mu_0 = 0 \):

process is

\[
\begin{align*}
\text{positive recurrent} & \quad \text{if} \quad K_\infty < \infty \quad \text{and} \quad L_\infty = \infty \\
\text{null recurrent} & \quad \text{if} \quad K_\infty = \infty \quad \text{and} \quad L_\infty = \infty \\
\text{transient} & \quad \text{if} \quad L_\infty < \infty
\end{align*}
\]

for \( \mu_0 > 0 \):

absorption is

\[
\begin{align*}
\text{certain and } E(\text{time}) < \infty & \quad \text{if} \quad K_\infty < \infty \quad \text{and} \quad L_\infty = \infty \\
\text{certain and } E(\text{time}) = \infty & \quad \text{if} \quad K_\infty = \infty \quad \text{and} \quad L_\infty = \infty \\
\text{not certain} & \quad \text{if} \quad L_\infty < \infty
\end{align*}
\]
decay parameter: bounds

\[ K_\infty = \sum_{i=0}^{\infty} \pi_i, \quad L_\infty = \sum_{i=1}^{\infty} \frac{1}{\mu_i \pi_i} \]

assumption: \( K_\infty + L_\infty = \infty \)

fact: \( K_\infty = L_\infty = \infty \Rightarrow \xi_1 = \xi_2 = 0 \)

4 scenarios:

1. \( K_\infty < \infty \) and \( \mu_0 = 0 \): \( \xi_2 > 0? \)
2. \( K_\infty < \infty \) and \( \mu_0 > 0 \): \( \xi_1 > 0? \)
3. \( L_\infty < \infty \) and \( \mu_0 = 0 \): \( \xi_1 > 0? \)
4. \( L_\infty < \infty \) and \( \mu_0 > 0 \): \( \xi_1 > 0? \)

facts: scenario 1: \( \xi_1 = 0 \) scenarios 2-4: \( \xi_1 = 0 \Rightarrow \xi_2 = 0 \)
results:

1: if $K_\infty < \infty$ and $\mu_0 = 0$, then $\xi_1 = 0$ and

\[ \xi_2 > 0 \iff \sup_n L_n(K_\infty - K_{n-1}) < \infty \]

Chen (2000)

2: if $K_\infty < \infty$ and $\mu_0 > 0$, then

\[ \xi_1 > 0 \iff \sup_n L_n(K_\infty - K_{n-1}) < \infty \]

Sirl et al. (2007)

3 and 4: if $L_\infty < \infty$, then

\[ \xi_1 > 0 \iff \sup_n K_n(L_\infty - L_n) < \infty \]

Chen (2010)
decay parameter: bounds

more detailed results:

1: if $K_\infty < \infty$ and $\mu_0 = 0$, then $\xi_1 = 0$ and

$$\frac{1}{4R} \leq \xi_2 \leq \frac{K_\infty}{R}, \quad R := \sup_n L_n(K_\infty - K_{n-1})$$

Chen (2000)

2: if $K_\infty < \infty$ and $\mu_0 > 0$, then

$$\frac{1}{4S} \leq \xi_1 \leq \frac{1}{S}, \quad S := \sup_n (L_n + \mu_0^{-1})(K_\infty - K_{n-1})$$

Sirl et al. (2007)

similar results in scenarios 3 and 4: Chen (2010)
decay parameter: bounds

setting: \( \mu_0 = 0 \)

recall dual process:

\[
\lambda_n^* = \mu_{n+1} \text{ and } \mu_n^* = \lambda_n \ (\mu_0^* > 0) \\
\text{supp}(\psi^*) = \text{supp}(\psi) \setminus \{0\}
\]

so that

\[
\alpha = \inf \{\text{supp}(\psi) \setminus \{0\}\} = \xi_1^*
\]

moreover

\[
L_\infty < \infty \iff K_\infty^* < \infty \text{ and } L_\infty^* < \infty \iff K_\infty < \infty
\]

hence solution scenario 3 \iff solution scenario 2
solution scenario 4 \iff solution scenario 1
decay parameter: bounds

recall: \( x_{n1} < x_{n2} < \cdots < x_{nn} \) zeros of \( P_n(x) \), \( \xi_i = \lim_{n \to \infty} x_{ni} \)

let

\[
T_{n+1} := \begin{pmatrix}
\lambda_0 + \mu_0 & \sqrt{\lambda_0 \mu_1} & \cdots & 0 & 0 \\
\sqrt{\lambda_0 \mu_1} & \lambda_1 + \mu_1 & \cdots & 0 & 0 \\
0 & \sqrt{\lambda_1 \mu_2} & \cdots & 0 & 0 \\
\vdots & \vdots & \ddots & \vdots & \vdots \\
0 & 0 & \cdots & \lambda_{n-1} + \mu_{n-1} & \sqrt{\lambda_{n-1} \mu_n} \\
0 & 0 & \cdots & \sqrt{\lambda_{n-1} \mu_n} & \lambda_n + \mu_n
\end{pmatrix}
\]

then

\[
\det(xI_n - T_n) = P_n(x), \quad n > 0
\]

hence zeros of \( P_n(x) \) are eigenvalues of symmetric matrix \( T_n \)
decay parameter: bounds

zeros $x_{ni}$ of $P_n(x)$ are eigenvalues of symmetric matrix $T_n$

\[ \xi_i = \lim_{n \to \infty} x_{ni} \]

recall Courant-Fischer Theorem:

\[ x_{n1} = \min_{y \neq 0} \frac{y^T T_n y}{y y^T}, \quad x_{n2} = \max_{\dim \mathcal{V} = n} \min_{y \in \mathcal{V}, y \neq 0} \frac{y^T T_n y}{y y^T}, \quad \ldots \]

trick:

\[ \xi_i = \tilde{\xi}_i = \lim_{n \to \infty} \tilde{x}_{ni} \]

or

\[ \xi_i = \tilde{\xi}_{i+1} = \lim_{n \to \infty} \tilde{x}_{n,i+1} \]

$\tilde{x}_{ni}$ zeros of suitable sequence of birth-death polynomials $\{\tilde{P}_n\}$
scenario 2: $K_\infty < \infty$ and $\mu_0 > 0$

recall (Sirl et al. (2007)):

$$\frac{1}{4R} \leq \xi_1 \leq \frac{1}{R}, \quad R = \sup_{n \geq 0} \left\{ \sum_{i=0}^{n} \frac{1}{\mu_i \pi_i} \sum_{i=n}^{\infty} \pi_i \right\}$$

alternative proof: $\xi_1 = \tilde{\xi}_1 = \lim_{n \to \infty} \tilde{x}_n$

$\tilde{x}_n$ smallest zero of

$$\tilde{P}_n(x) = P^*_n(x) + \lambda^*_n P^*_{n-1}(x)$$

where $P^*_n$ are polynomials corresponding to dual process

then $\{\tilde{P}_n\}$ is sequence of birth-death polynomials

whence $\tilde{x}_n$ is smallest eigenvalue of symmetric matrix $\tilde{T}_n$
scenario 2: \( K_\infty < \infty \) and \( \mu_0 > 0 \)

\( \tilde{x}_{n1} \) is smallest eigenvalue of symmetric matrix \( \tilde{T}_n \)

Courant-Fischer Theorem:

\[
\tilde{x}_{n1} = \min_{u \neq 0} \left\{ \frac{\sum_{i=0}^{n-1} \mu_i \pi_i u_i^2}{\left( \sum_{i=0}^{n-1} \pi_i \left( \sum_{j=0}^{i} u_j \right) \right)^2} \right\}
\]

\( u = (u_0, u_1, \ldots, u_{n-1}), u_i \in \mathbb{R}, \) so that

\[
\xi_1 = \tilde{\xi}_1 = \lim_{n \to \infty} \tilde{x}_{n1} = \inf_{u \in U} \left\{ \frac{\sum_{i=0}^{\infty} \mu_i \pi_i u_i^2}{\left( \sum_{i=0}^{\infty} \pi_i \left( \sum_{j=0}^{i} u_j \right) \right)^2} \right\}
\]
scenario 2: \( K_\infty < \infty \) and \( \mu_0 > 0 \)

\[
\xi_1 = \inf_{u \in U} \left\{ \frac{\sum_{i=0}^{\infty} \mu_i \pi_i u_i^2}{\sum_{i=0}^{\infty} \pi_i \left( \sum_{j=0}^{i} u_j \right)^2} \right\}
\]

hence

\[
\frac{1}{\xi_1} = \inf \left\{ A \leq \infty : \sum_{i=0}^{\infty} \pi_i \left( \sum_{j=0}^{i} u_j \right)^2 \leq A \sum_{i=0}^{\infty} \mu_i \pi_i u_i^2 \text{ for all } u \right\}
\]

weighted discrete Hardy’s inequalities (Miclo (1999)):

\[
\frac{1}{4R} \leq \xi_1 \leq \frac{1}{R}, \quad R = \sup_{n \geq 0} \left\{ \sum_{i=0}^{n} \frac{1}{\mu_i \pi_i} \sum_{i=n}^{\infty} \pi_i \right\}
\]
summary

interest: decay parameter of birth-death process

results:

1. representations in terms of support of orthogonalizing measure

2. representations as limits of eigenvalues of $n \times n$ sign-symmetric tridiagonal matrices

3. alternative proofs for findings of Chen and Sirl et al involving orthogonal-polynomial and eigenvalue techniques

reference: