Inverse Portfolio Problem and Restoring Risk Preferences

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Motivation: portfolio optimization

General deviation measures

Inverse portfolio problem formulation

Existence of solution

Exact and approximate solution in a simple form

Most conservative solution

Summary
Motivation: Portfolio Optimization

$X_i$ are random returns of $n$ risky instruments

$\mathcal{F} = \{X : X = \sum_{i=1}^{n} w_i X_i, \sum_{i=1}^{n} w_i = 1\}$ is a feasible set

$$\max_{X \in \mathcal{F}} U(X)$$

$U(.)$ - numerical representation of agent’s preferences:

$$X \succeq Y \iff U(X) \geq U(Y)$$

Question: What is your $U(.)$?
Theory of choice: preference relation $\succeq$

- Completeness
- Transitivity
- Continuity
- Independence

\[ \Rightarrow \exists u(.): X \succeq Y \iff Eu(X) \geq Eu(Y) \]

\[ U(X) = Eu(X), \quad u: \mathbb{R} \rightarrow \mathbb{R} \]

Question: What is your $u(.)$?
Empirical Realism

Set of questions for volunteers:

Q1: How much would you pay for an $x\%$ chance of getting $y$?
Q2: ...
Q3: ...
...

Answers to Q1, Q2, Q3 ... ⇒ $u(.)$

Problems:

- **Inaccuracy**: hard to answer $Q_i$ reliably
- **Cost**: costly exercise with real money, of limited use otherwise
- **Inappropriateness**: Returns $u(.)$ of volunteers, not investors.
Alternative Approach: Markowitz

$$\min_{X \in \mathcal{F}} \sigma(X)$$

s.t. $EX \geq \pi_0$

Classical Markowitz portfolio optimization problem

- Sets up a desired expected return $\pi_0$
- Minimizes deviation from it
- Uses standard deviation: $\sigma(X) = \sqrt{E[X - EX]^2}$

Properties

- Equal penalties of ups and downs
- No flexibility in risk preferences
- After all, why standard deviation?
General Deviation Measures

\[
\min_{X \in \mathcal{F}} \sigma(X) \quad \text{s.t. } EX \geq \pi_0 \rightarrow \min_{X \in \mathcal{F}} D(X) \quad \text{s.t. } EX \geq \pi_0
\]

General deviation measures

- Axiomatic approach
- Customization in risk preferences
- Nonsymmetric measures (non-smooth)
- Broad range of applications

(D1) insensitivity to constant shift
\[ \mathcal{D}(X + C) = \mathcal{D}(X) \text{ for all } X \text{ and constants } C \]

(D2) positive homogeneity
\[ \mathcal{D}(\lambda X) = \lambda \mathcal{D}(X) \text{ for all } X \text{ and all } \lambda > 0 \]

(D3) subadditivity
\[ \mathcal{D}(X + X') \leq \mathcal{D}(X) + \mathcal{D}(X') \text{ for all } X \text{ and } X' \]

(D4) nonnegativity
\[ \mathcal{D}(X) \geq 0 \text{ (equality for constant } X) \]

(D5) law-invariance
\[ X \sim Y \Rightarrow \mathcal{D}(X) = \mathcal{D}(Y) \]
Examples of Deviation Measures

- **Standard deviation**
  \[ \sigma(X) = \sqrt{E[(X - EX)^2]} \]

- **Standard lower and upper semideviations**
  \[ \sigma_+(X) = \sqrt{E[X - EX]^2_+} \]
  \[ \sigma_-(X) = \sqrt{E[X - EX]^2_-} \]
  where \( Y_+ = \max\{0, Y\} \) and \( Y_- = \max\{0, -Y\} \)

- **Mean absolute deviation (MAD)**
  \[ \text{MAD}(X) = E|X - EX| \]

- **Conditional Value-at-Risk (CVaR) deviation** for \( \alpha \in [0, 1] \)
  \[ \text{CVaR}_\alpha(X) = EX - \frac{1}{\alpha} \int_0^\alpha q_X(p) dp \]
The following are equivalent

\[ \succeq \text{satisfies axioms (list)} \]

\[ \exists \mathcal{D} \text{ and } V(., .) \text{ such that} \]

\[ X \succeq Y \iff V(EX, \mathcal{D}(X)) \geq V(EY, \mathcal{D}(Y)) \]

where \( \mathcal{D}(X) \) is a deviation measure which is

- continuous and law-invariant
- \[ \sup_{X \neq C} \frac{\mathcal{D}(X)}{EX - \inf X} < \infty \]

and \( V(m, d) \) satisfies (some natural properties)

Grechuk, B., Molyboha, A., Zabarankin, M.,
“Mean-Deviation Analysis in the Theory of Choice”
Risk Analysis, 2012
Coherent risk measures

\[ X \succeq Y \iff V(EX, D(X)) \geq V(EY, D(Y)) \]

Linear: \[ V(EX, D(X)) = EX - \lambda D(X) = -R(X) \]

(R1) **constant translation**
\[ R(X + C) = R(X) - C \text{ for all } X \text{ and constants } C \]

(R2) **positive homogeneity**
\[ R(0) = 0, \text{ and } R(\lambda X) = \lambda R(X) \text{ for all } X \text{ and all } \lambda > 0 \]

(R3) **subadditivity**
\[ R(X + Y) \leq R(X) + R(Y) \text{ for all } X \text{ and } Y \]

(R4) **monotonicity**
\[ R(X) \leq R(Y) \text{ when } X \succeq Y \text{ (almost surely)} \]

**Artzner, Delbaen, Eber, and Heath**
Inverse Portfolio Problem

\[
\min_{X \in \mathcal{F}} \mathcal{D}(X) \\
\text{s.t. } EX \geq \pi_0
\]  

Same Problem: What is your \( \mathcal{D} \)?

The idea:

- Observe agent’s current portfolio \( X^* \)
- Assume he/she is happy with it
- **Inverse problem**: find \( \mathcal{D}^* \) such that \( X^* \) is optimal in (1)
- Use \( \mathcal{D}^* \) to solve (1) with different \( \mathcal{F} \)
Empirical Realism vs Inverse Portfolio

Empirical Realism:

- **Hard** to answer reliably “How much would you pay for ...?”
- **Costly** exercise with real money, of limited use otherwise
- Returns utility of volunteers, not investors.

Inverse Portfolio:

- **Easy** to answer “What is your current portfolio?”
- **No cost**
- Returns $D$ of this particular investor.
Step 1: Choose your favourite axioms/theory:

- Expected utility theory: $U(X) = Eu(X)$, or
- Deviation measure theory: $U(X) = V(EX, D(X))$, or
- Coherent risk measure theory: $U(X) = -\mathcal{R}(X) = EX - \lambda D(X)$, or...

Step 2: Choose your favourite portfolio $X^*$

Step 3: Find $\mathcal{D} (\mathcal{R}, u, \text{etc.})$ such that $X^*$ is optimal.
Inverse problem: Find $D^*$ such that given $X^*$ is an optimal portfolio

Definition: $X \succ_S Y$ ($X$ super-dominates $Y$) if $D(X) < D(Y)$ for all $D$.

**Theorem**

$$X \succ_S Y \iff \text{CVaR}_{\alpha}^\Delta(X) < \text{CVaR}_{\alpha}^\Delta(Y),$$

where

$$\text{CVaR}_{\alpha}^\Delta(X) = EX - \frac{1}{\alpha} \int_0^\alpha q_X(p) dp$$

Definition: $X$ is $\succ_S$-undominated in $\mathcal{F}$ if no $Y \in \mathcal{F}$ super-dominates $X$

**Theorem**

The inverse problem has a solution if and only if $X^*$ is $\succ_S$-undominated in $\mathcal{F}$. 
Inverse problem: Find $D^*$ such that given $X^*$ is an optimal portfolio.

Solution exists $\iff X^*$ is $\succ_S$-undominated in $\mathcal{F}$

**Theorem**

Let $X^*$ be $\succ_S$-undominated in $\mathcal{F}$. Then a solution to the inverse problem can be chosen in the form

$$D^*(X) = \int_0^1 \text{CVaR}_{\lambda}^\Delta(X) \, d\lambda(\alpha)$$

with some $\lambda(\alpha) \geq 0$ such that $\int_0^1 d\lambda(\alpha) = 1$.

Even simpler solution? In general, no.

**Theorem**

For every $\lambda(\alpha)$ there exists $\mathcal{F}$ such that $D^*(X)$ is the only solution in the form (2).
Inverse problem: Find $D^*$ such that given $X^*$ is an optimal portfolio

**Theorem**

Assume that all $X \in \mathcal{F}$ are discrete r.v.s. If $X^*$ is $\succ_S$-undominated in $\mathcal{F}$, a solution to the inverse problem can be chosen in the form

$$D(X) = \sum_{i=1}^{n+1} \lambda_i \text{CVaR}^\Delta_{a_i}(X),$$  \hspace{1cm} (3)

for some $a_1, \ldots, a_{n+1}$, $a_i \in (0, 1)$, and non-negative weights $\lambda_1, \ldots, \lambda_{n+1}$, $\sum_{i=1}^{n+1} \lambda_i = 1$, where $n$ is the number of assets.

Even simpler solution with bounded number of $\text{CVaR}^\Delta_{a_i}$?
Inverse Portfolio: Approximate Solution

\[ d(\mathcal{D}_1, \mathcal{D}_2) = \sup_{X: |X| \leq 1} |\mathcal{D}_1(X) - \mathcal{D}_2(X)|. \]  

(4)

Problem: Find “simple” \( \mathcal{D} \) which minimizes \( d(\mathcal{D}, \mathcal{D}^*) \), where \( \mathcal{D}^* \) is (complicated) exact solution.

\[ \mathcal{D}(X) = \int_0^1 \text{CVaR}_\alpha(X) d\lambda(\alpha) = \int_0^1 g(\alpha) d(q_X(\alpha)), \]  

(5)

for some positive concave function \( g : (0, 1) \to \mathbb{R} \).

**Theorem**

Let \( \mathcal{D}_1 \) and \( \mathcal{D}_2 \) be in the form (5) with \( g_1 \) and \( g_2 \), respectively. Then

\[ d(\mathcal{D}_1, \mathcal{D}_2) = 2 \sup_{\alpha \in (0,1)} |g_1(\alpha) - g_2(\alpha)|. \]  

(6)

The problem reduces to piecewise linear approximation of \( g \)!
Inverse problem: Find $D^*$ such that given $X^*$ is an optimal portfolio

$X^*$ is $\succ^s$-dominated in $\mathcal{F}$ $\Rightarrow$ no solution

$X^*$ is $\succ^s$-undominated in $\mathcal{F}$ $\Rightarrow$ many solutions

Option 1: Choose simple

Option 2: Choose most conservative:

$$D^*(X) = \sup_{D \in \mathcal{W}} D(X)$$  \hspace{1cm} (7)

where $\mathcal{W}$ is the set of all solutions $D$ with $D(X^*) = 1$.

Theorem

$D^*$ in (7) can also be written as:

$$D^*(X) = \sup_{\alpha \in [0,1]} \frac{\text{CVaR}^\Delta_\alpha(X)}{\text{CVaR}^\Delta_\alpha(X^*)} \quad \text{for all } X .$$  \hspace{1cm} (8)
$X_i$ are random returns of $n$ risky instruments

$$\mathcal{F} = \{X : X = \sum_{i=1}^{n} w_i X_i, \sum_{i=1}^{n} w_i = 1\}$$ is a feasible set

$$\max_{X \in \mathcal{F}} U(X)$$

**Question:** What is your $U(.)$?

For example, if $U(X) = Eu(X)$, what is your $u(.)$?

$$\min_{X \in \mathcal{F}} D(X)$$

s.t. $EX \geq \pi_0$

**Same Problem:** What is your $D$?

**Answer:** No idea what is my $D$, but my current portfolio is $X^*$.
Answer: No idea what is my $\mathcal{D}$, but my current portfolio is $X^*$. 

Hence: My $\mathcal{D}$ is probably such that $X^*$ is optimal.

$X^*$ is $\succ_s$-dominated in $\mathcal{F}$ $\Rightarrow$ no solution, the model does not work

$X^*$ is $\succ_s$-undominated in $\mathcal{F}$ $\Rightarrow$ many solutions

Option 1: Choose simple

Option 2: Choose most conservative:

$$
\mathcal{D}^*(X) = \sup_{\alpha \in [0,1]} \frac{\text{CVaR}_\alpha^\Delta (X)}{\text{CVaR}_\alpha^\Delta (X^*)} \quad \text{for all } X.
$$

I can then use $\mathcal{D}^*$ for portfolio optimization in the future.
Let $\mathcal{R}$ be a coherent risk measure.

$$\min_{X \in \mathcal{F}} \mathcal{R}(X) \quad (s.t. \ EX \geq \pi_0)$$

(9)

**Same Problem: What is your $\mathcal{R}$?**

**Inverse problem:** Given $X^*$, find $\mathcal{R}^*$ such that $X^*$ is optimal in (9)

Because $\mathcal{R}(X) = \mathcal{D}(X) - EX$, solution is similar.
Existence of Law-Invariant Solution

**Deviation:** (Law-invariant) solution exists ⇔ There is no $Y \in \mathcal{F}$ such that

$$\text{CVaR}^\Delta_{\alpha}(X^*) < \text{CVaR}^\Delta_{\alpha}(Y) \quad \forall \alpha$$

**Coherent risk:** (Law-invariant) solution exists ⇔

**SSD-efficiency:** there is no $Y \in \mathcal{F}$ such that

$$\text{CVaR}_\alpha(X^*) < \text{CVaR}_\alpha(Y) \quad \forall \alpha$$

Equivalently: there is no $\epsilon > 0$ and $Y \in \mathcal{X}$ such that $Y \succeq_{\text{SSD}} X^* + \epsilon$
Problem: Standard market indices are *not SSD-efficient!*

\(X^*\) is *not SSD-efficient \iff \) Law-invariant \(R^*\) does *not exist.*

**Theorem**

*A general solution exists \iff market is arbitrage-free.*

\[
R(X) = \sup_{P \in \mathcal{P}} E_P[-X] = \sup_{Q \in \mathcal{Q}} E[Q(-X)]
\]

General solution:

\[
\mathcal{Q} = \{ Q \mid Q \geq 0, EQ = 1, E[QX^*] \geq 0 \} 
\]
General Solution with Extra Requirements

(a) \( \mathcal{R}(X_1) \leq 0, \ldots, \mathcal{R}(X_k) \leq 0 \iff \mathcal{Q} \subset \mathcal{Q}_M, \)
\[ \mathcal{Q}^M = \{ Q \in \mathcal{Q}^{MAX} \mid E[QX_1] \geq 0, \ldots, E[QX_k] \geq 0 \}; \]

(b) Order “reference” rates \( X_1, \ldots, X_k \) according to a given set \( S \) of orderings: \( (i,j) \in S \iff X_i \succeq X_j. \)
\[ \mathcal{Q}^M = \{ Q \in \mathcal{Q}^{MAX} \mid E[QX_i] \geq E[QX_j], (i,j) \in S \}; \]

(c) Exclude “too concentrated”
\[ \mathcal{Q}^M = \{ Q \in \mathcal{Q}^{MAX} \mid 1/2 \leq Q \leq 3/2 \}; \]

(d) and so on...

Inverse problem solution: \( \mathcal{Q} = \{ Q \in \mathcal{Q}^M \mid E[QX^*] \geq 0 \}. \)
General Solution in Multi-period

\[ t \in \mathcal{T} = \{0, 1, 2, 3, \ldots \} \]

\[ \min_{X_{t+1} \in \mathcal{F}_t} \mathcal{R}(X_{t+1}). \]

Let \( X^*(t) \) - portfolio return at \( t \in \mathcal{T} \).

**Strong inverse:** find \( \mathcal{R} \) such that \( X^*(t) \) is **optimal** for each \( t \in \mathcal{T} \).

**Weak inverse:** find \( \mathcal{R} \) such that \( X^*(t) \) is **acceptable** for each \( t \in \mathcal{T} \).

Solution: \( \mathcal{Q} = \{ Q \in \mathcal{Q}^M \mid E[QX^*(t)] \geq 0 \ \forall t \in \mathcal{T} \} \).
Back to Law-Invariant Case

\[
\min_{X \in \mathcal{F}} \mathcal{R}(X) \quad \quad \quad (10)
\]

(s.t. \( EX \geq \pi_0 \))

Inverse problem: Given \( X^* \), find \( \mathcal{R}^* \) such that \( X^* \) is optimal in (10)

\( X^* \) is not SSD-efficient \( \iff \) Law-invariant \( \mathcal{R}^* \) does not exist.

Approximate inverse problem:

Given \( \hat{X} \), find \( \mathcal{R}^* \) s.t. \( \hat{X} \) is almost optimal:

\[
\min_{\mathcal{R}} \left( \mathcal{R}(\hat{X}) - \mathcal{R}(X^*) \right) \quad \text{subject to} \quad X^* \in \arg\min_{X \in \mathcal{F}} \mathcal{R}(X).
\]
Approximate Inverse Problem

\[
\min_{\mathcal{R}} \left( \mathcal{R}(\hat{X}) - \mathcal{R}(X^*) \right) \quad \text{subject to} \quad X^* \in \arg\min_{X \in \mathcal{F}} \mathcal{R}(X).
\]

Let \( \mathcal{R}(X) = \sum_{i=1}^{m} \lambda_i \text{CVaR}_{\alpha_i}(X) \), \( \lambda_i \geq 0 \) with \( \sum_{i=1}^{m} \lambda_i = 1 \).

\[ \iff \min_{\mathcal{R}} \max_{X \in \mathcal{F}} \left( \mathcal{R}(\hat{X}) - \mathcal{R}(X) \right), \]

\[ \iff \max_{X \in \mathcal{F}} \min_{\mathcal{R}} \left( \mathcal{R}(\hat{X}) - \mathcal{R}(X) \right), \]

\[ \iff \max_{X \in \mathcal{F}} \min_{i \in \{1, \ldots, m\}} \left( \text{CVaR}_{\alpha_i}(\hat{X}) - \text{CVaR}_{\alpha_i}(X) \right), \]

\[ \iff \text{linear program for finding } X^* \]
\[
\min_{X \in \mathcal{F}} R(X)
\]

What is your \( R \)?

Answer: My \( R \) is such that given portfolio \( X^* \) is optimal.

\( X^* \) is SSD-undominated in \( \mathcal{F} \) \( \Rightarrow \) law-invariant solution

\( X^* \) is SSD-dominated in \( \mathcal{F} \):

Option 1: Choose general solution:

\[
Q = \{ Q \in Q^M | E[QX^*] \geq 0 \}
\]

Option 2: Choose approximate solution.
Thank you