Similarity for C*-algebras
an introduction by a
non-expert.

Thanks to Gilles Pisier,
Erik Christensen, Stuart White
and Roger Smith for discussions

The definition of length and all results
are due to Gilles Pisier.

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Groups

Dixmier and Day [1950 independently] showed that a bounded representation of an amenable group on a Hilbert space can be unitized.

A representation

\[ \pi : G \to (\text{invertibles on } H) \]

is “strongly unitizable” if there is an invertible \( T \in (\pi(G), \pi(G)^*)'' \) such that \( g \mapsto T\pi(g)T^{-1} \) is a unitary representation.

**Theorem**  [Pisier, Simultaneous similarity, bounded generation and length, Archive 2005]

Every bounded representation of a discrete group \( G \to \text{Invertibles on } H \) is strongly unitizable if, and only if, \( G \) is amenable.
Kadison similarity conjecture [1955]

Let \( \mathcal{A} \) be a unital C*-algebra and let \( \theta \) be unital bounded homomorphism from \( \mathcal{A} \) into \( B(H) \). Show that there is an invertible \( T \in B(H) \) such that \( x \mapsto T\theta(x)T^{-1} \) is a \(*\)-homomorphism.

There are results due to Christensen, Haagerup and others on C*-algebras and Paulsen [1984] on operator algebras and complete boundedness.

A unital operator algebra \( \mathcal{A} \) has the similarity property if, and only if, each bounded homomorphism \( \pi : \mathcal{A} \to B(H) \) is completely bounded.
Theorem  [Pisier, St Petersburg M J '99] A unital operator algebra $\mathcal{A}$ has the similarity property if, and only if, it has finite length. The similarity degree and length are equal.

**Gilles intuition on similarity and length:**
We call this [generation by diagonals and similarity] the “dual” view point because it is reminiscent of the fact that the closed convex hull $C$ of a subset $B \subset E$ of a Banach space $E$ is characterized by the implication

$$\sup_{b \in B} f(b) \leq 1 \implies \sup_{s \in C} f(s) \leq 1$$

for all continuous real linear forms $f$. Although this is a wild analogy, we feel that our results on length are a kind of “nonlinear” analog of the very classical duality principle of convex hulls.
All integer values of length are attained for general operator algebras [Pisier] but the only current known values for C*-algebras are 1, 2 and 3.

*Allan’s intuition on length:* Every matrix over $\mathcal{A}$ can be factorized in a good metric way with the length of the factors tending to infinity by the Blecher-Paulsen Theorem or in a good algebraic way with length one; in general when the metric version is good, the algebraic one is poor and vice versa. Finite length encapsulates the opposing tensions of these two properties, metric/algebra, which lie at the core of operator algebras.
Idea

Scalar matrices and diagonal matrices over $\mathcal{A}$ are good.

Notation

$\mathcal{A}$ is subsequently a unital C*-algebra
$\mathbb{M}_{n,N} = n \times N$ matrices over $\mathbb{C}$
$\mathbb{M}_n = n \times n$ matrices over $\mathbb{C}$
$\mathbb{M}_n(\mathcal{A}) = n \times n$ matrices over $\mathcal{A}$
$\mathbb{D}_n(\mathcal{A}) = n \times n$ diagonal matrices over $\mathcal{A}$
If \((x_{ij}) \in \mathbb{M}_n(\mathcal{A})\), then \((x_{ij}) = V DW\), where

\[
V = \text{row}_n(1) \otimes I_n \\
= \begin{pmatrix}
1 & 1 & \cdots & 1 & 0 & \cdots & 0 & \cdots & 0 \\
0 & 0 & \cdots & 0 & 1 & \cdots & 1 & \cdots & 0 \\
\vdots & \vdots & \ddots & 0 & \ddots & 0 & \ddots & \ddots & 0 \\
0 & 0 & \cdots & 0 & 0 & \cdots & 0 & 1 & \cdots & 1 \\
\end{pmatrix}
\in \mathbb{M}_{n,n^2},
\]

\[
W = (I_n \ I_n \ \cdots \ I_n)^T \\
\in \mathbb{M}_{n^2,n} \quad \text{and}
\]

\[
D \\
= \text{diag}_{n^2}(x_{11}, x_{12}, \cdots, x_{1n}, x_{21}, x_{22}, \cdots, x_{nn}) \\
\in \mathbb{D}_{n^2}(\mathcal{A}).
\]

This factorization is algebraically good, analytically poor as

\[
\|V\|\|D\|\|W\| \leq n\|X\|.
\]
If $d, n \in \mathbb{N}$, define $\| \cdot \|_{(d)}$ on $\mathbb{M}(\mathcal{A})$ by

$$\|X\|_{(d)} = \inf \left\{ \prod_{j=0}^{d} \|V_j\| \prod_{j=1}^{d} \|D_j\| : X = V_0D_1V_1\cdots D_dV_d \text{ with} \\
V_0, V_d^* \in \mathbb{M}_{n,N} \\
V_j \in \mathbb{M}_N (1 \leq j \leq d-1) \text{ and} \\
D_j \in \mathbb{D}_N(\mathcal{A}) (1 \leq j \leq d) \right\}$$

**Lemma**

(1) $\| \cdot \|_{(d)}$ is an operator space norm,

(2) $\|X\| \leq \|X\|_{(d+1)} \leq \|X\|_{(1)} \leq n\|X\|$, 

(3) $\|XY\|_{(d+r)} \leq \|X\|_{(d)}\|Y\|_{(r)}$

(4) $\| \cdot \|_{(1)} = \| \cdot \|_{\text{MAX}}$

is the maximal operator space norm.
Theorem
[Blecher + Paulsen,PAMS, 1991]

If $\mathcal{A}$ is a unital operator algebra, then

$$\lim_{d \to \infty} \|X\|_d = \|X\|
$$

for all $X \in M_n(\mathcal{A})$ and all $n \in \mathbb{N}$.

Good analytically, poor algebraically.

Gilles Pisier’s definition of length asks for efficiency both algebraically and analytically
Definition of length  [Pisier, 1999]

The algebra $\mathcal{A}$ has length $\leq d$ if, and only if, there is a constant $K$ such that $\|X\|_{(d)} \leq K\|X\|$ for all $X \in M_n(\mathcal{A})$ and all $n \in \mathbb{N}$. The length $l(\mathcal{A})$ is the minimum of $d$ such that $\mathcal{A}$ has length $\leq d$.

Length can be calculated via similarity and direct calculation of length.

*Generally*

Similarity calculation of degree (= length) $\leq$ Direct calculation of length
**Definition**

If $d, n \in \mathbb{N}$, let

$$K_{(d)}(n) = K_{(d)}(n, \mathcal{A}) = \sup\{\|X\|_{(d)} : X \in \mathbb{M}_n(\mathcal{A}), \|X\| \leq 1\}.$$ 

If $K \geq 1$, let

$$N_{(d)}(n, K) = \min\{N_0 : X \in \mathbb{M}_n(\mathcal{A}), \|X\|_{(d)} < K\|X\| \text{ with } N \leq N_0 \text{ in factorization.}\}.$$ 

Then $1 \leq K_{(d)}(n) \leq n$. 
Lemma (Pisier) If $\mathcal{A}$ is a unital C*-algebra and $p_1, p_2 \cdots p_n$ are projections in $\mathcal{A}$ with $\sum_1^n p_j = 1$, then
\[
\|(p_1, \cdots, p_n)\|_1 = 1 = \|(p_1, \cdots, p_n)\|.
\]

Proof Here row.row* = 1 gives the second equality. Let $W = (w_{ij})$ be a unitary matrix in $\mathbb{M}_n$ with $|w_{ij}| = n^{-1/2}$ for $1 \leq i, j \leq n$. Let
\[
V = (1, 1, \cdots, 1) \in \mathbb{M}_{1,n} \quad \text{and}
\]
\[
D = \text{diag}(\sum_{j=1}^n w_{ji} p_j) \in \mathbb{D}(\mathcal{A}).
\]
Then
\[
(p_1, \cdots, p_n) = VDW \quad \text{and}
\]
\[
\|V\| = n^{1/2}, \quad \|D\| = n^{-1/2}, \quad \|W\| = 1.
\]
Examples

1. $\mathcal{A} = \mathbb{C}^k = l^k_\infty$ has

$$\frac{(k/2)^{1/2}}{2} \leq K(1) \leq (k - 1)^{1/2}$$

using duality, Clifford algebras and $C^*_r(F_{k-1})$. Here $K(2) = 1$.

2. $\mathcal{A} = \mathbb{M}_k$ has

$$K(1)(n) = \min\{n, k^{3/2}\}, \quad K(2) \leq k$$

$$K(3) \leq k^{1/2} \quad \text{and}$$

$$K(4) = 1 \quad \text{with} \quad N(4)(n, 1) \leq k^2 n.$$

3. $\mathcal{A} = \mathcal{M}$ is a $II_1$ factor with property $\Gamma$, then

$$3 \leq l(\mathcal{M}) \leq 5 \quad \text{with} \quad K(5) = 1 \quad [\text{Pisier}]$$

$$l(\mathcal{M}) = 3 \quad [\text{Christensen}].$$

4. $\mathcal{A} = \mathcal{N}$ is a properly infinite von Neumann algebra, then $l(\mathcal{N}) = 3$,

$$K(3) = 1 \quad \text{and} \quad N(3)(n, 1) = n.$$
Corollary of [Pisier] and [Christensen, Smith, S] using Popa’s constructive methods
Let $\mathcal{M}$ be separable $II_1$ factor with property $\Gamma$. There is a hyperfinite subfactor $R$ in $\mathcal{M}$ such that each continuous $R$-bimodule map $\phi$ from $\mathcal{M}$ is completely bounded with $\|\phi\|_{cb} = \|\phi\|$.
**Proposition** [Pisier] A unital C*-algebra has length 1 if, and only if, it is finite dimensional.

**Theorem** [Pisier] A unital C*-algebra has length 2 if, and only if, \( \mathcal{A} \) is amenable.

**Theorem** [Pisier] For each \( d \in \mathbb{N} \) there is an operator algebra \( \mathcal{A} \) with length \( d \).

**Theorem** [Pisier] Every C*-algebra has finite length if, and only if, there are \( d, K \in \mathbb{N} \) such that \( K_d(n, \mathcal{A}) \leq K \) for all \( n \in \mathbb{N} \) and all (unital) C*-algebras \( \mathcal{A} \).

**Pisier’s conjecture**

\[ l^\infty \left( l^\infty \left( \mathbb{M}_k : k \in \mathbb{N} \right) \right) \]

has infinite length.
Table of lengths of various algebras calculated by similarity or by length arguments.

? = currently calculable
?? = unknown

\[ \mathcal{S} = \text{Estimate by similarity.} \]
\[ \mathcal{L} = \text{Estimate by length.} \]

<table>
<thead>
<tr>
<th>Algebra</th>
<th>Length</th>
<th>( S )</th>
<th>( S )</th>
<th>( \mathcal{L} )</th>
<th>( \mathcal{L} )</th>
<th>( \mathcal{L} )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \mathcal{A} )</td>
<td>( l(\mathcal{A}) )</td>
<td>( d )</td>
<td>( K(d) )</td>
<td>( d )</td>
<td>( K(d) )</td>
<td>( N(d)(n, K) )</td>
</tr>
<tr>
<td>Abelian ( \mathbb{C}^k )</td>
<td>1</td>
<td>?</td>
<td>?</td>
<td>1</td>
<td>?</td>
<td>?</td>
</tr>
<tr>
<td>Matrix ( \mathbb{M}_k )</td>
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<td>?</td>
<td>?</td>
<td>1</td>
<td>( K(4) = 1 )</td>
<td>( nk^2 )</td>
</tr>
<tr>
<td>Amenable ( \mathcal{A} )</td>
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<td>2</td>
<td>1</td>
<td>?</td>
<td>??</td>
<td>??</td>
</tr>
<tr>
<td>( I_\infty,II_\infty,III )</td>
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<td>3</td>
<td>1</td>
<td>?</td>
<td>3</td>
<td>1</td>
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<tr>
<td>( II_1 R )</td>
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<td>3</td>
<td>1</td>
<td>?</td>
<td>4</td>
<td>1</td>
</tr>
<tr>
<td>( \Gamma )-factor ( \mathcal{M} )</td>
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<td>3</td>
<td>?</td>
<td>5</td>
<td>1</td>
<td>( n^2 )</td>
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