Amenability of operator algebras on Banach spaces, II

Volker Runde

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The finite-dimensional case

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Amenability of $\mathcal{K}(E)$

Amenability of $B(E)$

A positive example $B(\ell^p \oplus \ell^q)$ with $p \neq q$

$B(\ell^p)$
The finite-dimensional case

Example

Let $E$ be a Banach space with $n := \text{dim}(E) < \infty$ so that $B(E) = K(E) \cong M_n$. Let $G$ be a finite subgroup of invertible elements of $M_n$ such that $\text{span}(G) = M_n$. Set $d := 1 |G| \sum_{g \in G} g \otimes g - 1$. Then $a \cdot d = d \cdot a$ ($a \in M_n$) and $\Delta d = I_n$. Hence, $K(E) = B(E)$ is amenable.
The finite-dimensional case

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Some more results

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A positive example
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Theorem (B. E. Johnson, 1972)
$\mathcal{K}(E)$ is amenable if $E = \ell^p$ with $1 < p < \infty$ or $E = C(T)$.

Amenable Banach algebras must have bounded approximate identities.

Theorem (N. Grønbæk & G. A. Willis, 1994)
Suppose that $E$ has the approximation property.
Then $\mathcal{K}(E)$ has a bounded approximate identity if and only if $E^*$ has the bounded approximation property.

Example
Let $E = \ell^2 \hat{\otimes} \ell^2$.
Then $E$ has the approximation property, but $E^* = \mathcal{B}(\ell^2)$ doesn’t.
Hence, $\mathcal{K}(E)$ does not have a bounded approximate identity and is thus not amenable.
Some more results

Theorem (B. E. Johnson, 1972)

For any Banach space $E$, the following conditions are equivalent:

1. $K(E)$ is amenable.
2. $E$ is isomorphic to an $l^p$-space with $1 < p < \infty$ or $E \cong C(T)$.

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Example

Let $E = l^2 \hat{\otimes} l^2$. Then $E$ has the approximation property, but $E^* = B(l^2)$ doesn't. Hence, $K(E)$ does not have a bounded approximate identity and is thus not amenable.
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Example: Let \( E = \ell^2 \mathbb{Z} \otimes \ell^2 \mathbb{Z} \). Then \( E \) has the approximation property, but \( E^* = \mathcal{B}(\ell^2) \) doesn’t. Hence, \( \mathcal{K}(E) \) does not have a bounded approximate identity and is thus not amenable.
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Definition

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A finite, biorthogonal system is a set \( \{ (x_j, \phi_k) : j, k = 1, \ldots, n \} \subset E \times E^* \) such that 
\[ \langle x_j, \phi_k \rangle = \delta_{j,k} \] 
for \( j, k = 1, \ldots, n \).

Remark

If \( \{ (x_j, \phi_k) : j, k = 1, \ldots, n \} \) is a finite, biorthogonal system, then \( \theta : M_n \to F(E), [\alpha_{j,k}] \mapsto \sum_{j,k=1}^n \alpha_{j,k} x_j \otimes \phi_k \) is an algebra homomorphism.
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Property (A)
Amenability of operator algebras on Banach spaces, II

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1. $\theta_{\lambda}(I_{n_{\lambda}}) \rightarrow \text{id}_E$ uniformly on compacts;
2. $\theta_{\lambda}(I_{n_{\lambda}}^*) \rightarrow \text{id}_{E^*}$ uniformly on compacts;
3. for each $\lambda$, there is a finite group $G_{\lambda}$ of invertible elements of $M_{n_{\lambda}}$ spanning $M_{n_{\lambda}}$ such that $\sup_{\lambda} \max_{g \in G_{\lambda}} \| \theta_{\lambda}(g) \| < \infty$. 
Property (\(\mathcal{A}\))

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Property (A) and the amenability of $\mathcal{K}(E)$
### Property \((A)\) and the amenability of \(\mathcal{K}(E)\)

#### The idea behind \((A)\)

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**Property \((A)\)**

Use the diagonals of the \(M_n\)'s to construct an approximate diagonal for \(\mathcal{K}(E)\).

**Theorem (N. Grønbæk, BEJ, & G. A. Willis, 1994)**

Suppose that \(E\) has property \((A)\).

**Then** \(\mathcal{K}(E)\) **is amenable.**

**Examples**

1. \(L^p(\mu)\) has property \((A)\) for all \(1 \leq p < \infty\) and all \(\mu\).
2. \(C(K)\) has property \((A)\) for each compact \(K\), therefore \(L^\infty(\mu)\) for each \(\mu\).
Property (A) and the amenability of $\mathcal{K}(E)$

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Theorem (N. Grønbæk, BEJ, & G. A. Willis, 1994)

Suppose that $E$ has property (A). Then $\mathcal{K}(E)$ is amenable.

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1. $L^p(\mu)$ has property (A) for all $1 \leq p < \infty$ and all $\mu$.
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Use the diagonals of the $M_{n\lambda}$’s to construct an approximate diagonal for $\mathcal{K}(E)$.

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Suppose that $E$ has property (A). Then $\mathcal{K}(E)$ is amenable.

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1. $L^p(\mu)$ has property (A) for all $1 \leq p < \infty$ and all $\mu$.
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1. $L^p(\mu)$ has property (A) for all $1 \leq p < \infty$ and all $\mu$.
2. $C(K)$ has property (A) for each compact $K$, as does therefore $L^\infty(\mu)$ for each $\mu$. 


The “scalar plus compact” problem
The “scalar plus compact” problem

Question
The “scalar plus compact” problem

**Question**

Is there an infinite-dimensional Banach space $E$
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There is a Banach space $E$ such that $\mathcal{B}(E) = \mathcal{K}(E) + \mathbb{C}\text{id}_E$ and $E^* = \ell_1$.
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**Corollary**
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There is an infinite-dimensional Banach space $E$.
The “scalar plus compact” problem

**Question**

Is there an infinite-dimensional Banach space $E$ such that $\mathcal{B}(E) = \mathcal{K}(E) + \mathbb{C}\, \text{id}_E$?

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**Theorem (N. Grønbæk, BEJ, & G. A. Willis, 1994)**

*Suppose that $E^*$ has property (A). Then so has $E$.***

**Corollary**

*There is an infinite-dimensional Banach space $E$ such that $\mathcal{B}(E)$ is amenable.*
Non-amenability of $\mathcal{B}(\ell^p \oplus \ell^q)$ for $p \neq q$, I
Non-amenability of $B(\ell^p \oplus \ell^q)$ for $p \neq q$, I

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Let $p, q \in (1, \infty)$ be such that $p \neq q$. Then $\mathcal{B}(\ell^p \oplus \ell^q)$ is not amenable.
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Ingredients
Non-amenability of $B(\ell^p \oplus \ell^q)$ for $p \neq q$, I

Theorem (G. A. Willis, unpublished)

Let $p, q \in (1, \infty)$ be such that $p \neq q$. Then $B(\ell^p \oplus \ell^q)$ is not amenable.

Ingredients

1. A quotient of an amenable Banach algebra is again amenable.
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Ingredients

1. A quotient of an amenable Banach algebra is again amenable.
2. Every complemented closed ideal of an amenable Banach algebra is amenable.
Non-amenability of $\mathcal{B}(\ell^p \oplus \ell^q)$ for $p \neq q$, I

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Let $p, q \in (1, \infty)$ be such that $p \neq q$. Then $\mathcal{B}(\ell^p \oplus \ell^q)$ is not amenable.

Ingredients

1. A quotient of an amenable Banach algebra is again amenable.
2. Every complemented closed ideal of an amenable Banach algebra is amenable.
3. Every amenable Banach algebra has a bounded approximate identity.
Non-amenability of $\mathcal{B}(\ell^p \oplus \ell^q)$ for $p \neq q$, I

**Theorem (G. A. Willis, unpublished)**

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**Ingredients**

1. A quotient of an amenable Banach algebra is again amenable.
2. Every complemented closed ideal of an amenable Banach algebra is amenable.
3. Every amenable Banach algebra has a bounded approximate identity.
4. **Pitt’s Theorem.**
Non-amenability of $\mathcal{B}(\ell^p \oplus \ell^q)$ for $p \neq q$.

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Let $p, q \in (1, \infty)$ be such that $p \neq q$. Then $\mathcal{B}(\ell^p \oplus \ell^q)$ is not amenable.

Ingredients

1. A quotient of an amenable Banach algebra is again amenable.
2. Every complemented closed ideal of an amenable Banach algebra is amenable.
3. Every amenable Banach algebra has a bounded approximate identity.
4. Pitt’s Theorem. If $p > q$, then $\mathcal{B}(\ell^p, \ell^q) = \mathcal{K}(\ell^p, \ell^q)$. 

Non-amenability of $\mathcal{B}(\ell^p \oplus \ell^q)$ for $p \neq q$, II

Proof. Suppose that $p > q$. Note that $\mathcal{B}(\ell^p \oplus \ell^q) = [\mathcal{B}(\ell^p), \mathcal{B}(\ell^q), \mathcal{B}(\ell^q), \mathcal{B}(\ell^q)]$ and $K(\ell^p \oplus \ell^q) = [K(\ell^p), K(\ell^q), K(\ell^q), K(\ell^q)]$, so that $C(\ell^p \oplus \ell^q) = [C(\ell^p), 0, C(\ell^q)]$. Then $I := [0, 0, 0]$ is a complemented ideal of $C(\ell^p \oplus \ell^q)$, thus is amenable, and thus has a BAI. But $I^2 = \{0\}$. . .
Non-amenability of $\mathcal{B}(\ell^p \oplus \ell^q)$ for $p \neq q$, II

Proof.

Suppose that $p > q$. Note that

$$\mathcal{B}(\ell^p \oplus \ell^q) = \left[ \mathcal{B}(\ell^p) \mathcal{B}(\ell^q, \ell^p) \mathcal{B}(\ell^q) \right]$$

and

$$\mathcal{K}(\ell^p \oplus \ell^q) = \left[ \mathcal{K}(\ell^p) \mathcal{K}(\ell^q, \ell^p) \mathcal{K}(\ell^p, \ell^q) \mathcal{K}(\ell^q) \right],$$

so that

$$\mathcal{C}(\ell^p \oplus \ell^q) = \left[ \mathcal{C}(\ell^p) \ast 0 \mathcal{C}(\ell^q) \right].$$

Then

$$I := \left[ 0 \ast 0 0 \right] \neq \{0\}$$

is a complemented ideal of $\mathcal{C}(\ell^p \oplus \ell^q)$, thus is amenable, and thus has a BAI. But

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Non-amenability of $\mathcal{B}(\ell^p \oplus \ell^q)$ for $p \neq q$, II

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Non-amenability of $B(\ell^p \oplus \ell^q)$ for $p \neq q$, II

Proof.

Suppose that $p > q$. Note that

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Proof.

Suppose that $p > q$. Note that

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so that

$$C(\ell^p \oplus \ell^q) = \begin{bmatrix} C(\ell^p) & * \\ 0 & C(\ell^q) \end{bmatrix}.$$

Then $I := \begin{bmatrix} 0 & * \\ 0 & 0 \end{bmatrix} \neq \{0\}$ is a complemented ideal of $C(\ell^p \oplus \ell^q)$,
Non-amenability of $\mathcal{B}(\ell^p \oplus \ell^q)$ for $p \neq q$, II

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Suppose that $p > q$. Note that

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Non-amenability of $B(\ell^p \oplus \ell^q)$ for $p \neq q$, II

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Suppose that $p > q$. Note that

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Non-amenability of $\mathcal{B}(\ell^p \oplus \ell^q)$ for $p \neq q$, II

Proof.

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Non-amenability of \( \mathcal{B}(\ell^p) \) for \( p = 1, 2, \infty \)
Non-amenability of $B(\ell^p)$ for $p = 1, 2, \infty$

Theorem (C. J. Read, <2006)
Non-amenability of $\mathcal{B}(\ell^p)$ for $p = 1, 2, \infty$

**Theorem (C. J. Read, <2006)**

$\mathcal{B}(\ell^1)$ is not amenable.
Non-amenability of $\mathcal{B}(\ell^p)$ for $p = 1, 2, \infty$

Theorem (C. J. Read, <2006)

$\mathcal{B}(\ell^1)$ is not amenable.

Progress since

2. Simultaneous proof for the non-amenability of $\mathcal{B}(\ell^p)$ for $p = 1, 2, \infty$ by N. Ozawa, 2006.

Question

Is $\mathcal{B}(\ell^p)$ amenable for any $p \in (1, \infty) \setminus \{2\}$?
Non-amenability of $\mathcal{B}(\ell^p)$ for $p = 1, 2, \infty$

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Is $\mathcal{B}(\ell^p)$ amenable for any $p \in (1, \infty) \setminus \{2\}$?
What if $B(\ell^p)$ were amenable?
What if $\mathcal{B}(\ell^p)$ were amenable?

**Theorem (M. Daws & VR, 2008)**

The following are equivalent for a Banach space $E$ and $p \in [1, \infty)$:  
1. $\mathcal{B}(\ell^p(E))$ is amenable;  
2. $\ell^\infty(\mathcal{B}(\ell^p(E)))$ is amenable.

Idea  
$\ell^p(\ell^p(E)) \sim \ell^p(E)$  
$\ell^\infty(\mathcal{B}(\ell^p(E))) \sim \text{block diagonal matrices in } \mathcal{B}(\ell^p(E))$  

**Corollary**

Suppose that $\mathcal{B}(\ell^p)$ is amenable for some $p \in [1, \infty)$. 

Then so are the Banach algebras $\ell^\infty(\mathcal{B}(\ell^p))$ and $\ell^\infty(K(\ell^p))$. 
What if $\mathcal{B}(\ell^p)$ were amenable?

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Amenability of operator algebras on Banach spaces, II

Volker Runde

Amenability of $K(E)$

Amenability of $B(E)$

A positive example $B(\ell^p \oplus \ell^q)$ with $p \neq p$

$B(\ell^p)$

What if $B(\ell^p)$ were amenable?

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The following are equivalent for a Banach space $E$ and $p \in [1, \infty)$:

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Idea

$\ell^p(E) \cong \ell^p(\ell^p(E)) \cong \ell^p(\ell^q)$

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Suppose that $B(\ell^p)$ is amenable for some $p \in [1, \infty)$. Then so are the Banach algebras $\ell^\infty(B(\ell^p))$ and $\ell^\infty(K(\ell^p))$. 
What if $\mathcal{B}(\ell^p)$ were amenable?

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**Idea**

- $\ell^p(\ell^p(E)) \cong \ell^p(E)$
- $\ell^\infty(\mathcal{B}(\ell^p(E))) \cong$ block diagonal matrices in $\mathcal{B}(\ell^p(\ell^p(E)))$
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What if $\mathcal{B}(\ell^p)$ were amenable?

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The following are equivalent for a Banach space $E$ and $p \in [1, \infty)$:

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**Idea**

- $\ell^p(\ell^p(E)) \cong \ell^p(E)$
- $\ell^\infty(\mathcal{B}(\ell^p(E))) \cong$ block diagonal matrices in $\mathcal{B}(\ell^p(\ell^p(E)))$

**Corollary**

Suppose that $\mathcal{B}(\ell^p)$ is amenable for some $p \in [1, \infty)$. 
What if $\mathcal{B}(\ell^p)$ were amenable?

### Theorem (M. Daws & VR, 2008)

The following are equivalent for a Banach space $E$ and $p \in [1, \infty)$:

1. $\mathcal{B}(\ell^p(E))$ is amenable;
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Suppose that $\mathcal{B}(\ell^p)$ is amenable for some $p \in [1, \infty)$. Then so are the Banach algebras $\ell^\infty(\mathcal{B}(\ell^p))$. 

---

**Amenability of operator algebras on Banach spaces, II**

Volker Runde

**Amenability of $\mathcal{K}(E)$**

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A positive example $\mathcal{B}(\ell^p \oplus \ell^q)$ with $p \neq p$
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Suppose that $\mathcal{B}(\ell^p)$ is amenable for some $p \in [1, \infty)$. Then so are the Banach algebras $\ell^\infty(\mathcal{B}(\ell^p))$ and $\ell^\infty(\mathcal{K}(\ell^p))$. 
Amenability of operator algebras on Banach spaces, II

Volker Runde

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Amenability of $B(E)$

A positive example

$B(\ell^p \oplus \ell^q)$ with $p \neq q$

$B(\ell^p)$
\( \mathcal{L}^p \)-spaces, I

**Definition (J. Lindenstrauss & A. Pełczyński, 1968)**

Let \( p \in [1, \infty] \) and let \( \lambda \geq 1 \).

A Banach space \( E \) is called a \( \mathcal{L}^p_\lambda \)-space if, for every finite-dimensional subspace \( X \) of \( E \), there is a finite-dimensional subspace \( Y \supset X \) of \( E \) with

\[
\| Y, \ell^p \| \dim Y \leq \lambda.
\]

We call \( E \) an \( \mathcal{L}^p \)-space if it is an \( \mathcal{L}^p_\lambda \)-space for some \( \lambda \geq 1 \).

**Examples**

1. All Banach spaces isomorphic to an \( \mathcal{L}^p \)-space are \( \mathcal{L}^p \)-spaces.
2. Let \( p \in (1, \infty) \setminus \{2\} \).

Then \( \ell^p(\ell^2) \) and \( \ell^2 \oplus \ell^p \) are \( \mathcal{L}^p \)-spaces, but not isomorphic to \( \mathcal{L}^p \)-spaces.
\textbf{$\mathcal{L}^p$-spaces, I}

\textbf{Definition (J. Lindenstrauss & A. Pełczyński, 1968)}

Let $p \in [1, \infty]$ and let $\lambda \geq 1$. 

\begin{enumerate}
\item All Banach spaces isomorphic to an $\mathcal{L}^p$-space are $\mathcal{L}^p$-spaces.
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Amenability of operator algebras on Banach spaces, II

Volker Runde

Amenability of $\mathcal{K}(E)$

Amenability of $\mathcal{B}(E)$

A positive example $\mathcal{B}(\ell^p \oplus \ell^q)$ with $p \neq p$

$\mathcal{B}(\ell^p)$

$L^p$-spaces, I

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Examples
$L^p$-spaces, 1

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**Examples**

1. All Banach spaces isomorphic to an $L^p$-space are $L^p$-spaces.
**Definition (J. Lindenstrauss & A. Pełczyński, 1968)**

Let $p \in [1, \infty]$ and let $\lambda \geq 1$. A Banach space $E$ is called a $\mathcal{L}_p^\lambda$-space if, for every finite-dimensional subspace $X$ of $E$, there is a finite-dimensional subspace $Y \supset X$ of $E$ with $d(Y, \ell_p^{\dim Y}) \leq \lambda$. We call $E$ an $\mathcal{L}_p$-space if it is an $\mathcal{L}_p^\lambda$-space for some $\lambda \geq 1$.

**Examples**

1. All Banach spaces isomorphic to an $L^p$-space are $\mathcal{L}^p$-spaces.
2. Let $p \in (1, \infty) \setminus \{2\}$. 
**Definition (J. Lindenstrauss & A. Pełczyński, 1968)**

Let \( p \in [1, \infty] \) and let \( \lambda \geq 1 \). A Banach space \( E \) is called a \( \mathcal{L}_\lambda^p \)-space if, for every finite-dimensional subspace \( X \) of \( E \), there is a finite-dimensional subspace \( Y \supset X \) of \( E \) with \( d(Y, \ell^p_{\dim Y}) \leq \lambda \). We call \( E \) an \( \mathcal{L}^p \)-space if it is an \( \mathcal{L}_\lambda^p \)-space for some \( \lambda \geq 1 \).

**Examples**

1. All Banach spaces isomorphic to an \( L^p \)-space are \( \mathcal{L}^p \)-spaces.
2. Let \( p \in (1, \infty) \setminus \{2\} \). Then \( \ell^p(\ell^2) \) and \( \ell^2 \oplus \ell^p \) are \( \mathcal{L}^p \)-spaces,
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Let $p \in [1, \infty]$ and let $\lambda \geq 1$. A Banach space $E$ is called a $\mathcal{L}_\lambda^p$-space if, for every finite-dimensional subspace $X$ of $E$, there is a finite-dimensional subspace $Y \supset X$ of $E$ with $d(Y, \ell^p_{\dim Y}) \leq \lambda$. We call $E$ an $\mathcal{L}^p$-space if it is an $\mathcal{L}_\lambda^p$-space for some $\lambda \geq 1$.

Examples

1. All Banach spaces isomorphic to an $L^p$-space are $\mathcal{L}^p$-spaces.

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Amenability of operator algebras on Banach spaces, II

Volker Runde

Amenability of $\mathcal{K}(E)$

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A positive example

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$\mathcal{B}(\ell^p)$
\( \mathcal{L}^p \)-spaces, II

Theorem (M. Daws & VR, 2008)

\[ \text{Theorem (M. Daws & VR, 2008)} \]

Let \( p \in [1, \infty] \). Then one of the following is true:

1. \( \ell_\infty(K(E)) \) is amenable for every \( L^p \)-space \( E \) with \( \dim E = \infty \);
2. \( \ell_\infty(K(E)) \) is not amenable for any \( L^p \)-space \( E \) with \( \dim E = \infty \).

Corollary

Suppose that \( \mathcal{B}(\ell^p) \) is amenable for some \( p \in [1, \infty) \).

Then \( \ell_\infty(K(\ell^p)) \) is amenable.

Question

Is \( \ell_\infty(K(\ell^p \oplus \ell^q)) \) amenable for any \( p \in (1, \infty) \setminus \{2\} \)?
Theorem (M. Daws & VR, 2008)

Let $p \in [1, \infty]$. Then one of the following is true:

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Corollary

Suppose that $B(\ell^p)$ is amenable for some $p \in [1, \infty)$.

Then $\ell_\infty(K(\ell^p \oplus \ell^q))$ is amenable for every $L^p$-space $E$ with $\dim E = \infty$.

Question

Is $\ell_\infty(K(\ell^p \oplus \ell^q))$ amenable for any $p \in (1, \infty) \setminus \{2\}$?
\[ \mathcal{L}^p \text{-spaces, II} \]

**Theorem (M. Daws & VR, 2008)**

Let \( p \in [1, \infty] \). Then one of the following is true:

1. \( \ell^\infty(\mathcal{K}(E)) \) is **amenable** for every \( \mathcal{L}^p \)-space \( E \) with \( \dim E = \infty \);
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Corollary

Suppose that $\mathcal{B}(\ell^p \oplus \ell^q)$ is amenable for some $p \in [1, \infty)$. Then $\ell^\infty(\mathcal{K}(\ell^2 \oplus \ell^p))$ is amenable for every $\mathcal{L}^p$-space $E$ with $\dim E = \infty$. 

Question

Is $\ell^\infty(\mathcal{K}(\ell^2 \oplus \ell^p))$ amenable for any $p \in (1, \infty) \setminus \{2\}$?
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Let $p \in [1, \infty]$. Then one of the following is true:

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Suppose that $\mathcal{B}(\ell^p)$ is amenable for some $p \in [1, \infty)$. 

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Suppose that \( \mathcal{B}(\ell^p) \) is amenable for some \( p \in [1, \infty) \). Then \( \ell^\infty(\mathcal{K}(E)) \) is amenable.
Amenability of operator algebras on Banach spaces, II

Volker Runde

Amenability of $\mathcal{K}(E)$
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Suppose that $\mathcal{B}(\ell^p)$ is amenable for some $p \in [1, \infty)$. Then $\ell^\infty(K(E))$ is amenable for every $\mathcal{L}^p$-space $E$ with $\dim E = \infty$.

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Is $\ell^\infty(K(\ell^p \oplus \ell^q))$ amenable for any $p \in (1, \infty) \setminus \{2\}$?
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Corollary

Suppose that \( \mathcal{B}(\ell^p) \) is amenable for some \( p \in [1, \infty) \). Then \( \ell^\infty(\mathcal{K}(E)) \) is amenable for every \( \mathcal{L}^p \)-space \( E \) with \( \dim E = \infty \).

Question

Is \( \ell^\infty(\mathcal{K}(\ell^2 \oplus \ell^p)) \) amenable for any \( p \in (1, \infty) \setminus \{2\} \)?
Ozawa’s proof revisited, I
Definition
Ozawa’s proof revisited, I

**Definition**

A locally compact group $G$ has **Kazhdan’s property** $(T)$. 

Examples:
1. All compact groups have property $(T)$, as does $SL(3, \mathbb{Z})$. 
2. Amenable groups have property $(T)$ if and only if they are compact. 
3. $F_2$ and $SL(2, \mathbb{R})$ are not amenable, but lack property $(T)$. 

Definition

A locally compact group $G$ has Kazhdan’s property ($T$) if there are $\epsilon > 0$
Ozawa’s proof revisited, I

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\|\pi(k)\xi - \xi\| > \epsilon.
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Ozawa’s proof revisited, 1

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Examples

1. All compact groups have property ($T$), as does $\text{SL}(3, \mathbb{Z})$.
2. Amenable groups have property ($T$) if and only if they are compact.
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Ozawa’s proof revisited, I

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1. All compact groups have property ($T$), as does $\text{SL}(3, \mathbb{Z})$.
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Examples

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Ozawa’s proof revisited, II

Amenability of operator algebras on Banach spaces, II
Volker Runde

Amenability of $\mathcal{K}(E)$
Amenability of $\mathcal{B}(E)$
A positive example
$\mathcal{B}(\ell^p \oplus \ell^q)$
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Ozawa’s proof revisited, II

The setup
Ozawa’s proof revisited, II

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Since $\text{SL}(3, \mathbb{Z})$ has property ($T$),

...
Ozawa’s proof revisited, II

The setup

Since $\text{SL}(3, \mathbb{Z})$ has property $(T)$, it is finitely generated
Ozawa’s proof revisited, II

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Since $\text{SL}(3, \mathbb{Z})$ has property ($T$), it is finitely generated by $g_1, \ldots, g_m$, say.
The setup

Since $\text{SL}(3,\mathbb{Z})$ has property $(T)$, it is finitely generated by $g_1, \ldots, g_m$, say.
Write $\mathbb{P}$ for the set of prime numbers.
Ozawa’s proof revisited, II

The setup

Since $\text{SL}(3, \mathbb{Z})$ has property ($T$), it is finitely generated by $g_1, \ldots, g_m$, say.
Write $\mathbb{P}$ for the set of prime numbers.
Let $p \in \mathbb{P},$
Ozawa’s proof revisited, II

The setup

Since \( \text{SL}(3, \mathbb{Z}) \) has property \((T)\), it is finitely generated by \( g_1, \ldots, g_m \), say.

Write \( \mathbb{P} \) for the set of prime numbers.

Let \( p \in \mathbb{P} \), and let \( \Lambda_p \) be the projective plane over \( \mathbb{Z}/p\mathbb{Z} \).
Ozawa’s proof revisited, II

The setup

Since $\text{SL}(3, \mathbb{Z})$ has property $(T)$, it is finitely generated by $g_1, \ldots, g_m$, say.
Write $\mathbb{P}$ for the set of prime numbers.
Let $p \in \mathbb{P}$, and let $\Lambda_p$ be the projective plane over $\mathbb{Z}/p\mathbb{Z}$.
Then $\text{SL}(3, \mathbb{Z})$ acts on $\Lambda_p$ through matrix multiplication.
Ozawa’s proof revisited, II

The setup

Since $\text{SL}(3, \mathbb{Z})$ has property ($T$), it is finitely generated by $g_1, \ldots, g_m$, say.
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Then $\text{SL}(3, \mathbb{Z})$ acts on $\Lambda_p$ through matrix multiplication.
This group action induces a unitary representation
$\pi_p : \text{SL}(3, \mathbb{Z}) \to \mathcal{B}(\ell^2(\Lambda_p))$. 

Ozawa’s proof revisited, II

The setup

Since SL(3, \mathbb{Z}) has property (\mathcal{T}), it is finitely generated by \(g_1, \ldots, g_m\), say.
Write \(\mathbb{P}\) for the set of prime numbers.
Let \(p \in \mathbb{P}\), and let \(\Lambda_p\) be the projective plane over \(\mathbb{Z}/p\mathbb{Z}\).
Then SL(3, \mathbb{Z}) acts on \(\Lambda_p\) through matrix multiplication.
This group action induces a unitary representation \(\pi_p : SL(3, \mathbb{Z}) \to B(\ell^2(\Lambda_p))\).
Choose \(S_p \subset \Lambda_p\) with \(|S_p| = \frac{|\Lambda_p| - 1}{2}\).
Ozawa’s proof revisited, II

The setup

Since SL(3, \mathbb{Z}) has property (T), it is finitely generated by \( g_1, \ldots, g_m \), say.

Write \( \mathbb{P} \) for the set of prime numbers.

Let \( p \in \mathbb{P} \), and let \( \Lambda_p \) be the projective plane over \( \mathbb{Z}/p\mathbb{Z} \).

Then SL(3, \mathbb{Z}) acts on \( \Lambda_p \) through matrix multiplication.

This group action induces a unitary representation \( \pi_p : SL(3, \mathbb{Z}) \to \mathcal{B}(\ell^2(\Lambda_p)) \).

Choose \( S_p \subset \Lambda_p \) with \( |S_p| = \frac{\Lambda_p| - 1}{2} \) and define a unitary \( \pi_p(g_{m+1}) \in \mathcal{B}(\ell^2(\Lambda_p)) \).
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This group action induces a unitary representation $\pi_p : \text{SL}(3, \mathbb{Z}) \to \mathcal{B}(\ell^2(\Lambda_p))$.
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$$\pi_p(g_{m+1}) e_\lambda = \begin{cases} e_\lambda, & \lambda \in S_p, \end{cases}$$
Ozawa’s proof revisited, II

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$$\pi_p(g_{m+1})e_\lambda = \begin{cases} 
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\end{cases}$$
Ozawa’s proof revisited, III

Ozawa’s proof revisited, III

It is impossible to find, for each $\epsilon > 0$, a number $r \in \mathbb{N}$ with the following property: for each $p \in P$ there are $\xi_1, \xi_2, \ldots, \xi_r, \eta_1, \eta_2, \ldots, \eta_r \in \ell_2(\Lambda_p)$ such that

$$\sum_{k=1}^r \xi_k \otimes \eta_k \neq 0$$

and

$$\left\| \sum_{k=1}^r \xi_k \otimes \eta_k - \left( \pi_p(g_j) \otimes \pi_p(g_j) \right) \right\|_{\ell_2(\Lambda_p) \otimes \ell_2(\Lambda_p)} \leq \epsilon \left\| \sum_{k=1}^r \xi_k \otimes \eta_k \right\|_{\ell_2(\Lambda_p) \otimes \ell_2(\Lambda_p)}$$

for $j = 1, \ldots, m + 1$. 
Ozawa’s Lemma

*It is impossible to find, for each $\epsilon > 0$, a number $r \in \mathbb{N}$ with the following property: for each $p \in \mathbb{P}$ there are $\xi_{1,p}, \eta_{1,p}, \ldots, \xi_{r,p}, \eta_{r,p} \in \ell^2(\Lambda_p)$ such that $\sum_{k=1}^{r} \xi_{k,p} \otimes \eta_{k,p} \neq 0$ and

\[
\left\| \sum_{k=1}^{r} \xi_{j,p} \otimes \eta_{k,p} - (\pi_p(g_j) \otimes \pi_p(g_j))(\xi_{k,p} \otimes \eta_{k,p}) \right\|_{\ell^2(\Lambda_p) \hat{\otimes} \ell^2(\Lambda_p)} \\
\leq \epsilon \left\| \sum_{k=1}^{r} \xi_{k,p} \otimes \eta_{k,p} \right\|_{\ell^2(\Lambda_p) \hat{\otimes} \ell^2(\Lambda_p)} (j = 1, \ldots, m + 1).
\]
Ozawa’s proof revisited, IV

Ingredients

1. $\text{SL}(3, \mathbb{Z})$ has Kazhdan’s property (T).
2. The non-commutative Mazur map is uniformly continuous.
3. A key inequality.

For $p = 1, 2, \infty$, $N \in \mathbb{N}$, $S \in \mathcal{B}(\ell^p, \ell^p_N)$, and $T \in \mathcal{B}(\ell^p', \ell^p_N)$:

$$\sum_{n=1}^{\infty} \| S e_n \|_{\ell^2_N} \| T e_n^* \|_{\ell^2_N} \leq N \| S \| \| T \|.$$

(This estimate is no longer true for $p \in (1, \infty) \setminus \{2\}$.)'
Amenability of operator algebras on Banach spaces, II

Volker Runde

Amenability of \( \mathcal{K}(E) \)

Amenability of \( B(E) \)

A positive example \( B(\ell^p \oplus \ell^q) \) with \( p \neq p \)

\( B(\ell^p) \)

Ozawa’s proof revisited, IV

Ingredients
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Ozawa’s proof revisited, IV

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Ozawa’s proof revisited, IV

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\[
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Ozawa’s proof revisited, IV

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1. $\text{SL}(3, \mathbb{Z})$ has Kazhdan’s property (T).
2. The non-commutative Mazur map is uniformly continuous.
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Ozawa’s proof revisited, IV

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1. **SL(3, \mathbb{Z})** has Kazhdan’s property \((T)\).
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Ozawa’s proof revisited, IV

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(This estimate is no longer true for \( p \in (1, \infty) \setminus \{2\} \).)
A non-amenability result for $\ell_\infty(\mathcal{K}(\ell^2 \oplus E))$, I

Let $E$ be a Banach space with a basis $(x_n)_{n=1}^{\infty}$ such that there is $C > 0$ with

$$\sum_{n=1}^{\infty} \|Sx_n\| \|Tx_n^*\| \leq CN \|S\| \|T\|$$

($N \in \mathbb{N}, S \in B(E, \ell_2^N), T \in B(E^*, \ell_2^N)$).

Then $\ell_\infty(\mathcal{K}(\ell^2 \oplus E))$ is not amenable.

Example

It is easy to see that the following spaces satisfy the hypotheses of the theorem: $c_0$, $\ell_1$, and $\ell_2$. 
A non-amenability result for $\ell^\infty(\mathcal{K}(\ell^2 \oplus E))$, I

Theorem (VR, 2009)

Let $E$ be a Banach space with a basis $(x_n)_{n=1}^\infty$ such that there is $C > 0$ with

$$\sum_{n=1}^\infty \|S x_n\| \|T^*_n\| \leq C N \|S\| \|T\|$$

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A non-amenability result for $\ell^\infty(K(\ell^2 \oplus E))$, I

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A non-amenability result for $\ell^\infty(\mathcal{K}(\ell^2 \oplus E))$, I

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Let $E$ be a Banach space with a basis $(x_n)_{n=1}^\infty$ such that there is $C > 0$ with

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A non-amenability result for $\ell^\infty(K(\ell^2 \oplus E))$, I

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Amenability of operator algebras on Banach spaces, II
Volker Runde

Amenability of $\mathcal{K}(E)$
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A non-amenability result for $\ell^\infty(\mathcal{K}(\ell^2 \oplus E))$, II
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Lemma

Let $\mathcal{A}$ be an amenable Banach algebra, and let $e \in \mathcal{A}$ be an idempotent. Then,
A non-amenability result for $\ell^\infty(\mathcal{K}(\ell^2 \oplus E))$, II

Lemma

Let $\mathcal{A}$ be an amenable Banach algebra, and let $e \in \mathcal{A}$ be an idempotent. Then, for any $\epsilon > 0$
A non-amenability result for $\ell^\infty(K(\ell^2 \oplus E))$, II

**Lemma**

Let $\mathcal{A}$ be an amenable Banach algebra, and let $e \in \mathcal{A}$ be an idempotent. Then, for any $\epsilon > 0$ and any finite subset $F$ of $e\mathcal{A}e$,
A non-amenability result for \( \ell^\infty(\mathcal{K}(\ell^2 \oplus E)), \) II

**Lemma**

Let \( \mathcal{A} \) be an amenable Banach algebra, and let \( e \in \mathcal{A} \) be an idempotent. Then, for any \( \epsilon > 0 \) and any finite subset \( F \) of \( e\mathcal{A}e \), there are \( a_1, b_1, \ldots, a_r, b_r \in \mathcal{A} \)
A non-amenability result for $\ell^\infty(\mathcal{K}(\ell^2 \oplus E))$, II

**Lemma**

Let $\mathfrak{A}$ be an amenable Banach algebra, and let $e \in \mathfrak{A}$ be an idempotent. Then, for any $\epsilon > 0$ and any finite subset $F$ of $e\mathfrak{A}e$, there are $a_1, b_1, \ldots, a_r, b_r \in \mathfrak{A}$ such that

$$\sum_{k=1}^{r} a_k b_k = e$$

and

$$\|\sum_{k=1}^{r} x_a a_k \otimes b_k - x_a a_k \otimes b_k x\| < \epsilon \quad (x \in F).$$
A non-amenability result for $\ell^\infty(\mathcal{K}(\ell^2 \oplus E))$, II

**Lemma**

Let $\mathcal{A}$ be an amenable Banach algebra, and let $e \in \mathcal{A}$ be an idempotent. Then, for any $\epsilon > 0$ and any finite subset $F$ of $e\mathcal{A}e$, there are $a_1, b_1, \ldots, a_r, b_r \in \mathcal{A}$ such that

$$\sum_{k=1}^{r} a_k b_k = e$$

and

$$\left\| \sum_{k=1}^{r} x a_k \otimes b_k - a_k \otimes b_k x \right\|_{\mathcal{A} \hat{\otimes} \mathcal{A}} < \epsilon \quad (x \in F).$$
A non-amenability result for $\ell^\infty(\mathcal{K}(\ell^2 \oplus E))$, III
A non-amenability result for $\ell^\infty(K(\ell^2 \oplus E))$, III
A non-amenability result for $\ell^\infty(\mathcal{K}(\ell^2 \oplus E))$, III

Sketched proof of the Theorem

Embed

$$\ell^\infty - \bigoplus_{p \in \mathbb{P}} B(\ell^2(\Lambda_p))$$
A non-amenability result for $\ell^\infty(\mathcal{K}(\ell^2 \oplus E))$, III

Sketched proof of the Theorem

Embed

$$\ell^\infty - \bigoplus_{p \in \mathbb{P}} B(\ell^2(\Lambda_p)) \subset \ell^\infty - \bigoplus_{p \in \mathbb{P}} \mathcal{K}(\ell^2 \oplus E)$$
A non-amenability result for $\ell^\infty(K(\ell^2 \oplus E))$, III

Sketched proof of the Theorem

Embed

$$\ell^\infty \bigoplus_{p \in \mathbb{P}} B(\ell^2(\Lambda_p)) \subset \ell^\infty \bigoplus_{p \in \mathbb{P}} K(\ell^2 \oplus E) =: \mathfrak{A}$$
A non-amenability result for $\ell^\infty(K(\ell^2 \oplus E))$, III

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Embed

$$\ell^\infty - \bigoplus_{p \in \mathbb{P}} B(\ell^2(\Lambda_p)) \subset \ell^\infty - \bigoplus_{p \in \mathbb{P}} K(\ell^2 \oplus E) =: \mathcal{A}$$

as “upper left corners”.

Amenability of operator algebras on Banach spaces, II
Volker Runde

Amenability of $K(E)$

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A positive example $B(\ell^p \oplus \ell^q)$ with $p \neq q$ in $B(\ell^p)$
A non-amenability result for $\ell^\infty(\mathcal{K}(\ell^2 \oplus E))$, III

Sketched proof of the Theorem

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$$\ell^\infty - \bigoplus_{p \in \mathbb{P}} B(\ell^2(\Lambda_p)) \subset \ell^\infty - \bigoplus_{p \in \mathbb{P}} \mathcal{K}(\ell^2 \oplus E) =: \mathcal{A}$$

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$$\ell^2(\mathbb{P}, \ell^2 \oplus E)$$
A non-amenability result for $\ell^\infty(K(\ell^2 \oplus E))$, III

Sketched proof of the Theorem

Embed

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$$\ell^2(P, \ell^2 \oplus E) \cong \ell^2(P, \ell^2) \oplus \ell^2(P, E).$$
A non-amenability result for $\ell^\infty(K(\ell^2 \oplus E))$, IV
A non-amenability result for $\ell^\infty(K(\ell^2 \oplus E))$, IV

Sketched proof of the Theorem (continued)

For $p \in \mathbb{P}$,
Sketched proof of the Theorem (continued)

For $p \in \mathbb{P}$, let $P_p \in B(\ell^2)$ be the canonical projection
Sketched proof of the Theorem (continued)

For $p \in \mathbb{P}$, let $P_p \in \mathcal{B}(\ell^2)$ be the canonical projection onto the first $|\Lambda_p|$ coordinates.
Sketched proof of the Theorem (continued)

For $p \in \mathbb{P}$, let $P_p \in \mathcal{B}(\ell^2)$ be the canonical projection onto the first $|\Lambda_p|$ coordinates of the $p^{\text{th}}$ $\ell^2$-summand.
A non-amenability result for $\ell^\infty(K(\ell^2 \oplus E))$, IV

Sketched proof of the Theorem (continued)

For $p \in \mathbb{P}$, let $P_p \in B(\ell^2)$ be the canonical projection onto the first $|\Lambda_p|$ coordinates of the $p^{th}$ $\ell^2$-summand of

$$\ell^2(\mathbb{P}, \ell^2) \oplus \ell^2(\mathbb{P}, E).$$
A non-amenability result for $\ell^\infty(K(\ell^2 \oplus E))$, IV

Sketched proof of the Theorem (continued)

For $p \in \mathbb{P}$, let $P_p \in B(\ell^2)$ be the canonical projection onto the first $|\Lambda_p|$ coordinates of the $p^{th}$ $\ell^2$-summand of

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Set $e = (P_p)_{p\in\mathbb{P}}$. 

A non-amenability result for $\ell^\infty(\mathcal{K}(\ell^2 \oplus E))$, IV

Sketched proof of the Theorem (continued)

For $p \in \mathbb{P}$, let $P_p \in \mathcal{B}(\ell^2)$ be the canonical projection onto the first $|\Lambda_p|$ coordinates of the $p^{th}$ $\ell^2$-summand of

$$\ell^2(\mathbb{P}, \ell^2) \oplus \ell^2(\mathbb{P}, E).$$

Set $e = (P_p)_{p \in \mathbb{P}}$. Then $e$ is an idempotent in $\mathcal{A}$.
A non-amenability result for $\ell^\infty(K(\ell^2 \oplus E))$, IV

Sketched proof of the Theorem (continued)

For $p \in \mathbb{P}$, let $P_p \in \mathcal{B}(\ell^2)$ be the canonical projection onto the first $|\Lambda_p|$ coordinates of the $p^{th}$ $\ell^2$-summand of

$$
\ell^2(\mathbb{P}, \ell^2) \oplus \ell^2(\mathbb{P}, E).
$$

Set $e = (P_p)_{p \in \mathbb{P}}$. Then $e$ is an idempotent in $\mathcal{A}$ with

$$
e \mathcal{A} e = \ell^\infty - \bigoplus_{p \in \mathbb{P}} \mathcal{B}(\ell^2(\Lambda_p)).$$
A non-amenability result for $\ell^\infty(\mathcal{K}(\ell^2 \oplus E))$, V
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Sketched proof of the Theorem (continued)
A non-amenability result for $\ell^\infty(\mathcal{K}(\ell^2 \oplus E))$, V

Sketched proof of the Theorem (continued)

Assume towards a contradiction that $\ell^\infty(\mathbb{P}, \mathcal{K}(\ell^2 \oplus E))$ is amenable.
A non-amenability result for $\ell^\infty(\mathcal{K}(\ell^2 \oplus E))$, V

Sketched proof of the Theorem (continued)

Assume towards a contradiction that $\ell^\infty(\mathcal{P}, \mathcal{K}(\ell^2 \oplus E))$ is amenable.
Let $\epsilon > 0$ be arbitrary.
Sketched proof of the Theorem (continued)

Assume towards a contradiction that \( \ell^\infty(\mathcal{P}, \mathcal{K}(\ell^2 \oplus E)) \) is amenable.

Let \( \epsilon > 0 \) be arbitrary. By the previous Lemma
A non-amenability result for $\ell^\infty(K(\ell^2 \oplus E))$, V

**Sketched proof of the Theorem (continued)**

Assume towards a contradiction that $\ell^\infty(P, K(\ell^2 \oplus E))$ is amenable.

Let $\epsilon > 0$ be arbitrary. By the previous Lemma there are thus $a_1, b_1, \ldots, a_r, b_r \in \mathcal{A}$
A non-amenability result for $\ell^\infty (\mathcal{K}(\ell^2 \oplus E))$, V

Sketched proof of the Theorem (continued)

Assume towards a contradiction that $\ell^\infty (\mathbb{P}, \mathcal{K}(\ell^2 \oplus E))$ is amenable.

Let $\epsilon > 0$ be arbitrary. By the previous Lemma there are thus $a_1, b_1, \ldots, a_r, b_r \in \mathcal{A}$ such that $\sum_{k=1}^{r} a_k b_k = e$
A non-amenability result for $\ell^\infty(K(\ell^2 \oplus E))$, V

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Assume towards a contradiction that $\ell^\infty(\mathbb{P}, K(\ell^2 \oplus E))$ is amenable. Let $\epsilon > 0$ be arbitrary. By the previous Lemma there are thus $a_1, b_1, \ldots, a_r, b_r \in \mathcal{A}$ such that $\sum_{k=1}^r a_k b_k = e$ and

$$\left\| \sum_{k=1}^r x a_k \otimes b_k - a_k \otimes b_k x \right\| < \frac{\epsilon}{(C + 1)(m + 1)} \quad (x \in F),$$
A non-amenability result for $\ell^\infty(\mathcal{K}(\ell^2 \oplus E))$, V

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Assume towards a contradiction that $\ell^\infty(\mathbb{P}, \mathcal{K}(\ell^2 \oplus E))$ is amenable.

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$$\left\| \sum_{k=1}^r x a_k \otimes b_k - a_k \otimes b_k x \right\| < \frac{\epsilon}{(C + 1)(m + 1)} \quad (x \in F),$$

where

$$F := \{ (\pi_p(g_j))_{p \in \mathbb{P}} : j = 1, \ldots, m + 1 \}.$$
A non-amenability result for $\ell^\infty(K(\ell^2 \oplus E))$, VI

Amenability of operator algebras on Banach spaces, II

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Amenability of $B(E)$

A positive example $B(\ell^p \oplus \ell^q)$ with $p \neq q$

$B(\ell^p)$
A non-amenability result for $\ell^\infty(K(\ell^2 \oplus E))$, VI

Sketched proof of the Theorem (continued)
A non-amenability result for $\ell^\infty(\mathcal{K}(\ell^2 \oplus E))$, VI

Sketched proof of the Theorem (continued)

For $p, q \in \mathbb{P}$ and $n \in \mathbb{N}$, define
A non-amenability result for $\ell^\infty(K(\ell^2 \oplus E))$, VI

**Sketched proof of the Theorem (continued)**

For $p, q \in \mathbb{P}$ and $n \in \mathbb{N}$, define

$$T_p(q, n) := \sum_{k=1}^{r} P_p a_k(e_q \otimes e_n) \otimes P^*_p b_k^*(e_q^* \otimes e_n^*)$$

$$+ P_p a_k(e_q \otimes x_n) \otimes P^*_p b_k^*(e_q^* \otimes x_n^*)$$

Note that $T_p(q, n) \in \ell^2(\Lambda_p \hat{\otimes} \ell^2(\Lambda_p))$. 
A non-amenability result for $\ell^\infty(\mathcal{K}(\ell^2 \oplus E))$, VI

Sketched proof of the Theorem (continued)

For $p, q \in \mathbb{P}$ and $n \in \mathbb{N}$, define

$$T_p(q, n) := \sum_{k=1}^{r} P_p a_k (e_q \otimes e_n) \otimes P_p^* b_k^* (e_q^* \otimes e_n^*)$$

$$+ P_p a_k (e_q \otimes x_n) \otimes P_p^* b_k^* (e_q^* \otimes x_n^*)$$

Note that

$$T_p(q, n) \in \ell^2(\Lambda_p) \hat{\otimes} \ell^2(\Lambda_p).$$
A non-amenability result for $\ell^\infty(\mathcal{K}(\ell^2 \oplus E))$, VII
A sketched proof of the Theorem (continued)

It follows that

\[ \sum_{q \in P} \sum_{n=1}^{\infty} \| T_{p}(q, n) - (\pi_{p}(g_{j}) \otimes \pi_{p}(g_{j})) T_{p}(q, n) \| \leq \epsilon m + 1 |\Lambda_{p}| \]

for \( j = 1, \ldots, m + 1 \) and \( p \in P \) and thus

\[ \sum_{q \in P} \sum_{n=1}^{m+1} \sum_{j=1}^{m+1} \| T_{p}(q, n) - (\pi_{p}(g_{j}) \otimes \pi_{p}(g_{j})) T_{p}(q, n) \| \leq \epsilon |\Lambda_{p}| . \]
A non-amenability result for $\ell^\infty(K(\ell^2 \oplus E))$, VII

Sketched proof of the Theorem (continued)

It follows that
A non-amenability result for $\ell^\infty(\mathcal{K}(\ell^2 \oplus E))$, VII

**Sketched proof of the Theorem (continued)**

It follows that
A non-amenability result for $\ell^\infty(K(\ell^2 \oplus E))$, VII

**Sketched proof of the Theorem (continued)**

It follows that

$$\sum_{q \in \mathbb{P}} \sum_{n=1}^\infty \| T_p(q, n) - ((\pi_p(g_j) \otimes \pi_p(g_j)) T_p(q, n) \| \leq \frac{\epsilon}{m+1} |\Lambda_p|$$
A non-amenability result for $\ell^\infty(\mathcal{K}(\ell^2 \oplus E))$, VII

Sketched proof of the Theorem (continued)

It follows that

$$\sum_{q \in \mathcal{P}} \sum_{n=1}^{\infty} \left\| T_p(q, n) - \left( (\pi_p(g_j) \otimes \pi_p(g_j)) T_p(q, n) \right) \right\| \leq \frac{\epsilon}{m + 1} |\Lambda_p|$$

for $j = 1, \ldots, m + 1$
A non-amenability result for $\ell^\infty(K(\ell^2 \oplus E))$, VII

Sketched proof of the Theorem (continued)

It follows that

$$\sum_{q \in \mathbb{P}} \sum_{n=1}^{\infty} \| T_p(q, n) - ((\pi_p(g_j) \otimes \pi_p(g_j)) T_p(q, n) \| \leq \frac{\epsilon}{m + 1} |\Lambda_p|$$

for $j = 1, \ldots, m + 1$ and $p \in \mathbb{P}$
A non-amenability result for $\ell^\infty(K(\ell^2 \oplus E))$, VII

Sketched proof of the Theorem (continued)

It follows that

$$\sum_{q \in \mathbb{P}} \sum_{n=1}^{\infty} \left\| T_p(q, n) - ((\pi_p(g_j) \otimes \pi_p(g_j)) T_p(q, n) \right\| \leq \frac{\epsilon}{m+1} |\Lambda_p|$$

for $j = 1, \ldots, m+1$ and $p \in \mathbb{P}$ and thus

$$\sum_{q \in \mathbb{P}} \sum_{n=1}^{\infty} \sum_{j=1}^{m+1} \left\| T_p(q, n) - ((\pi_p(g_j) \otimes \pi_p(g_j)) T_p(q, n) \right\| \leq \epsilon |\Lambda_p|.$$
A non-amenability result for $\ell^\infty(\mathcal{K}(\ell^2 \oplus E))$, VIII
A non-amenability result for $\ell^\infty(K(\ell^2 \oplus E))$, VIII

Sketched proof of the Theorem (continued)
A non-amenability result for $\ell^\infty(\mathcal{K}(\ell^2 \oplus E))$, VIII

Sketched proof of the Theorem (continued)

On the other hand:

$$\sum_{q \in \mathcal{P}} \sum_{n=1}^{\infty} \| T_p(q, n) \| \geq \sum_{n=1}^{\infty} \left| \sum_{k=1}^{r} \langle P_p a_k, p e_n, P_p^* b_k, p e_n^* \rangle + \sum_{k=1}^{r} \langle P_p a_k, p x_n, P_p^* b_k, p x_n^* \rangle \right|$$
Sketched proof of the Theorem (continued)

On the other hand:

\[
\sum_{q \in \mathbb{P}} \sum_{n=1}^{\infty} \| T_p(q, n) \| \\
\geq \sum_{n=1}^{\infty} \left| \sum_{k=1}^{r} \langle P_p a_k, p e_n, P_p b_k^* p e_n^* \rangle + \sum_{k=1}^{r} \langle P_p a_k, p x_n, P_p b_k^* p x_n^* \rangle \right| \\
= \text{Tr} \sum_{k=1}^{r} b_k p P_p a_k p
\]
Sketched proof of the Theorem (continued)

On the other hand:

\[
\sum_{q \in \mathbb{P}} \sum_{n=1}^{\infty} \| T_p(q, n) \| \geq \sum_{n=1}^{\infty} \left| \sum_{k=1}^{r} \langle P_p a_k, p e_n, P_p^* b_k, p e_n^* \rangle + \sum_{k=1}^{r} \langle P_p a_k, p x_n, P_p^* b_k, p x_n^* \rangle \right| \\
= \text{Tr} \sum_{k=1}^{r} b_k, p P_p a_k, p \\
= \text{Tr} \sum_{k=1}^{r} P_p a_k, p b_k, p
\]
A non-amenability result for $\ell^\infty(\mathcal{K}(\ell^2 \oplus E))$, VIII

Sketched proof of the Theorem (continued)

On the other hand:

$$\sum_{q \in \mathcal{P}} \sum_{n=1}^{\infty} \| T_p(q, n) \| \geq \sum_{n=1}^{\infty} \left| \sum_{k=1}^{r} \langle P_p a_k, p e_n, P_p^* b_k^*, p e_n^* \rangle + \sum_{k=1}^{r} \langle P_p a_k, p x_n, P_p^* b_k^*, p x_n^* \rangle \right|$$

$$= \text{Tr} \sum_{k=1}^{r} b_{k,p} P_p a_{k,p}$$

$$= \text{Tr} \sum_{k=1}^{r} P_p a_{k,p} b_{k,p}$$

$$= \text{Tr} P_p$$
A non-amenability result for $\ell^\infty(\mathcal{K}(\ell^2 \oplus E))$, VIII

Sketched proof of the Theorem (continued)

On the other hand:

$$\sum_{q \in \mathbb{P}} \sum_{n=1}^\infty \| T_p(q, n) \| \geq \sum_{n=1}^\infty \left| \sum_{k=1}^r \langle P_p a_k, p e_n, P_p^* b_k, p e_n^* \rangle + \sum_{k=1}^r \langle P_p a_k, p x_n, P_p^* b_k, p x_n^* \rangle \right|$$

$$= \text{Tr} \sum_{k=1}^r b_k, p P_p a_k, p$$

$$= \text{Tr} \sum_{k=1}^r P_p a_k, p b_k, p$$

$$= \text{Tr} P_p = |\Lambda_p|.$$
A non-amenability result for $\ell^\infty(K(\ell^2 \oplus E))$, IX
A non-amenability result for $\ell^\infty(\mathcal{K}(\ell^2 \oplus E))$, IX

Sketched proof of the Theorem (conclusion)
A non-amenability result for $\ell^\infty(\mathcal{K}(\ell^2 \oplus E))$, IX

Sketched proof of the Theorem (conclusion)

It follows that,
A non-amenability result for $\ell^\infty(\mathcal{K}(\ell^2 \oplus E))$, IX

Sketched proof of the Theorem (conclusion)

It follows that, for each $p \in \mathbb{P}$,
A non-amenability result for $\ell^\infty(\mathcal{K}(\ell^2 \oplus E))$, IX

**Sketched proof of the Theorem (conclusion)**

It follows that, for each $p \in \mathbb{P}$, there are $q \in \mathbb{P}$ and $n \in \mathbb{N}$
A non-amenability result for $\ell^\infty(\mathcal{K}(\ell^2 \oplus E))$, IX

Sketched proof of the Theorem (conclusion)

It follows that, for each $p \in \mathbb{P}$, there are $q \in \mathbb{P}$ and $n \in \mathbb{N}$ with $T_p(q, n) \neq 0$
Sketched proof of the Theorem (conclusion)

It follows that, for each $p \in \mathbb{P}$, there are $q \in \mathbb{P}$ and $n \in \mathbb{N}$ with $T_p(q, n) \neq 0$ and

$$\| T_p(q, n) - ((\pi_p(g_j) \otimes \pi_p(g_j)) T_p(q, n) \| \leq \epsilon \| T_p(q, n) \|$$

for $j = 1, \ldots, m + 1$. 
Sketched proof of the Theorem (conclusion)

It follows that, for each $p \in \mathbb{P}$, there are $q \in \mathbb{P}$ and $n \in \mathbb{N}$ with $T_p(q, n) \neq 0$ and

$$\|T_p(q, n) - ((\pi_p(g_j) \otimes \pi_p(g_j)) T_p(q, n))\| \leq \epsilon \|T_p(q, n)\|$$

for $j = 1, \ldots, m + 1$, which violates Ozawa’s Lemma.
A non-amenability result for $\ell^\infty(\mathcal{K}(\ell^2 \oplus E))$, IX

Sketched proof of the Theorem (conclusion)

It follows that, for each $p \in \mathbb{P}$, there are $q \in \mathbb{P}$ and $n \in \mathbb{N}$ with $T_p(q, n) \neq 0$ and

$$\|T_p(q, n) - ((\pi_p(g_j) \otimes \pi_p(g_j)) T_p(q, n))\| \leq \epsilon \|T_p(q, n)\|$$

for $j = 1, \ldots, m + 1$, which violates Ozawa’s Lemma.
$p$-summing operators
$p$-summing operators

Definition

Let $p \in [1, \infty)$, and $E$ and $F$ be Banach spaces. A linear map $T : E \to F$ is called $p$-summing if the amplification $id_{\ell^p} \otimes T : \ell^p \otimes E \to \ell^p \otimes F$ extends to a bounded map from $\ell^p \hat{\otimes} E$ to $\ell^p (F)$. The operator norm of $id_{\ell^p} \otimes T : \ell^p \hat{\otimes} E \to \ell^p (F)$ is called the $p$-summing norm of $T$ and denoted by $\pi_p(T)$.

Theorem (Y. Gordon, 1969)

$\pi_p(id_{\ell^2 N^n}) \sim N^{1/2}$ for all $p \in [1, \infty)$. 
\( p \)-summing operators

### Definition

Let \( p \in [1, \infty) \),
$p$-summing operators

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Let $p \in [1, \infty)$, and $E$ and $F$ be Banach spaces.
p-summing operators

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Let $p \in [1, \infty)$, and $E$ and $F$ be Banach spaces. A linear map $T : E \to F$ is called $p$-summing.
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Let $p \in [1, \infty)$, and $E$ and $F$ be Banach spaces. A linear map $T : E \to F$ is called $p$-summing if the amplification

\[ \text{id}_{\ell^p} \otimes T : \ell^p \otimes E \to \ell^p \otimes F \]

is $p$-summing.
Amenability of operator algebras on Banach spaces, II

Volker Runde

Amenability of $\mathcal{K}(E)$

Amenability of $B(E)$

A positive example $B(\ell^p \oplus \ell^q)$ with $p \neq q$

$B(\ell^p)$

$p$-summing operators

**Definition**

Let $p \in [1, \infty)$, and $E$ and $F$ be Banach spaces. A linear map $T : E \to F$ is called $p$-summing if the amplification $\text{id}_{\ell^p} \otimes T : \ell^p \otimes E \to \ell^p \otimes F$ extends to a bounded map from $\ell^p \hat{\otimes} E$ to $\ell^p(F)$. 

Theorem (Y. Gordon, 1969)

$\pi_p(\text{id}_{\ell^2}) \sim \frac{1}{2}$ for all $p \in [1, \infty)$. 

The operator norm of $\text{id}_{\ell^p} \otimes T : \ell^p \otimes E \to \ell^p \otimes F$ is called the $p$-summing norm of $T$ and denoted by $\pi_p(T)$. 

The operator $\text{id}_{\ell^p}$ is called the $p$-summing operator on $\ell^p$. 

A positive example $B(\ell^p \oplus \ell^q)$ with $p \neq q$. 

$B(\ell^p)$
**p-summing operators**

**Definition**

Let $p \in [1, \infty)$, and $E$ and $F$ be Banach spaces. A linear map $T : E \to F$ is called **$p$-summing** if the amplification $\text{id}_{\ell^p} \otimes T : \ell^p \otimes E \to \ell^p \otimes F$ extends to a bounded map from $\ell^p \hat{\otimes} E$ to $\ell^p(F)$. The operator norm of $\text{id}_{\ell^p \otimes T} : \ell^p \hat{\otimes} E \to \ell^p(F)$ is called the **$p$-summing norm** of $T$.
Definition

Let \( p \in [1, \infty) \), and \( E \) and \( F \) be Banach spaces. A linear map \( T : E \to F \) is called \( p \)-summing if the amplification \( \text{id}_{\ell^p} \otimes T : \ell^p \otimes E \to \ell^p \otimes F \) extends to a bounded map from \( \ell^p \hat{\otimes} E \) to \( \ell^p(F) \). The operator norm of \( \text{id}_{\ell^p \otimes T} : \ell^p \hat{\otimes} E \to \ell^p(F) \) is called the \( p \)-summing norm of \( T \) and denoted by \( \pi_p(T) \).
\begin{defn}
Let \( p \in [1, \infty) \), and \( E \) and \( F \) be Banach spaces. A linear map \( T : E \to F \) is called \textit{\( p \)-summing} if the amplification \( \text{id}_{\ell^p} \otimes T : \ell^p \otimes E \to \ell^p \otimes F \) extends to a bounded map from \( \ell^p \hat{\otimes} E \) to \( \ell^p(F) \). The operator norm of \( \text{id}_{\ell^p \otimes T} : \ell^p \hat{\otimes} E \to \ell^p(F) \) is called the \textit{\( p \)-summing norm} of \( T \) and denoted by \( \pi_p(T) \).
\end{defn}

\begin{thm}[Y. Gordon, 1969]
\end{thm}
$p$-summing operators

**Definition**

Let $p \in [1, \infty)$, and $E$ and $F$ be Banach spaces. A linear map $T : E \to F$ is called $p$-summing if the amplification $\text{id}_{\ell^p} \otimes T : \ell^p \otimes E \to \ell^p \otimes F$ extends to a bounded map from $\ell^p \hat{\otimes} E$ to $\ell^p(F)$. The operator norm of $\text{id}_{\ell^p \otimes T} : \ell^p \hat{\otimes} E \to \ell^p(F)$ is called the $p$-summing norm of $T$ and denoted by $\pi_p(T)$.

**Theorem (Y. Gordon, 1969)**

$$\pi_p(\text{id}_{\ell^2_N}) \sim N^{\frac{1}{2}}$$
\textit{p-summing operators}

\section*{Definition}
Let $p \in [1, \infty)$, and $E$ and $F$ be Banach spaces. A linear map $T : E \to F$ is called \textit{p-summing} if the amplification $\text{id}_{\ell^p} \otimes T : \ell^p \otimes E \to \ell^p \otimes F$ extends to a bounded map from $\ell^p \hat{\otimes} E$ to $\ell^p(F)$. The operator norm of $\text{id}_{\ell^p \otimes T} : \ell^p \hat{\otimes} E \to \ell^p(F)$ is called the \textit{p-summing norm} of $T$ and denoted by $\pi_p(T)$.

\begin{theorem}[Y. Gordon, 1969]
\[ \pi_p(\text{id}_{\ell^2_N}) \sim N^{\frac{1}{2}} \]
for all $p \in [1, \infty)$.
\end{theorem}
A Lemma

Amenability of \( \mathcal{K}(E) \)

Amenability of \( \mathcal{B}(E) \)

A positive example \( \mathcal{B}(\ell^p \oplus \ell^q) \) with \( p \neq q \)

\( \mathcal{B}(\ell^p) \)
A Lemma

Lemma

Let $p \in (1, \infty)$. Then there is $C > 0$ such that

$$\sum_{n=1}^{\infty} \| S e_n \|_{\ell^2} \| T e_n^* \|_{\ell^2} N \| S \| \| T \|_N \leq C N \| S \| \| T \|$$

for $S \in B(\ell^p, \ell^2 N)$, $T \in B(\ell^p', \ell^2 N)$. 
A Lemma

Let $p \in (1, \infty)$. 

Lemma
Lemma

Let $p \in (1, \infty)$. Then there is $C > 0$
Lemma

Let $p \in (1, \infty)$. Then there is $C > 0$ such that

$$\sum_{n=1}^{\infty} \left\| S e_n \right\|_{\ell^2_N} \left\| T e_n^* \right\|_{\ell^2_N} \leq C N \| S \| \| T \|$$

($N \in \mathbb{N}$, $S \in \mathcal{B}(\ell^p, \ell^2_N)$, $T \in \mathcal{B}(\ell^{p'}, \ell^2_N)$).
Proof of the Lemma

Proof.

Identify algebraically $\mathcal{B}(\ell^p, \ell^2 \mathbb{N}) = \ell^p \hat{\otimes} \ell^2 \mathbb{N} = \ell^p \hat{\otimes} \ell^2 \mathbb{N} = \ell^p(\ell^2 \mathbb{N})$, and $\mathcal{B}(\ell^p', \ell^2 \mathbb{N}) = \ell^p \hat{\otimes} \ell^2 \mathbb{N} = \ell^p \otimes \ell^2 \mathbb{N} = \ell^p(\ell^2 \mathbb{N})$.

Note that $\sum_{n=1}^{\infty} \|S e_n\|_{\ell^2 \mathbb{N}} \|T e_n^*\|_{\ell^2 \mathbb{N}} \leq \|S\|_{\ell^p'(\ell^2 \mathbb{N})} \|T\|_{\ell^p(\ell^2 \mathbb{N})}$, by H"older,

$\leq \pi_{\ell^p'}(\text{id}_{\ell^2 \mathbb{N}}) \pi_{\ell^p}((\text{id}_{\ell^2 \mathbb{N}}) \|S\| \|T\| \leq C_N \|S\| \|T\|$, by Gordon.
Proof of the Lemma

Proof.

Identify algebraically $B(\ell_p,\ell_2^N) = \ell_p^\prime \otimes \ell_2^N = \ell_p^\prime \otimes (\ell_2^N)$, and $B(\ell_p^\prime,\ell_2^N) = \ell_p \otimes \ell_2^N = \ell_p (\ell_2^N)$. Note that

$$\sum_{n=1}^{\infty} \|S e_n\|_{\ell_2^N} \|T e_n^*\|_{\ell_2^N} \leq \|S\|_{\ell_p^\prime (\ell_2^N)} \|T\|_{\ell_p (\ell_2^N)} \leq \pi_{p^\prime} (id_{\ell_2^N}) \pi_p (id_{\ell_2^N}) \|S\| \|T\| \leq C N \|S\| \|T\|,$$

by H"older, 

$$\leq \pi_{p^\prime} (id_{\ell_2^N}) \pi_p (id_{\ell_2^N}) \|S\| \|T\|,$$

by Gordon.
Proof of the Lemma

Proof.

Identify
Proof of the Lemma

Proof.

Identify

\[ \mathcal{B}(\ell^p, \ell^2_N) = \ell^p' \otimes \ell^2_N \]
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Proof of the Lemma

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Identify algebraically

\[ \mathcal{B}(\ell^p, \ell^2_N) = \ell^p \otimes \ell^2_N = \ell^p' \otimes \ell^2_N = \ell^p' (\ell^2_N), \]
Proof of the Lemma

**Proof.**

Identify **algebraically**

\[ \mathcal{B}(\ell^p, \ell^2_N) = \ell^p' \otimes \ell^2_N = \ell^p' \otimes \ell^2_N = \ell^p' (\ell^2_N), \quad \text{and} \]

\[ \sum_{n=1}^{\infty} \|S_e_n\|_{\ell^2_N} \|T_e^* n\|_{\ell^2_N} \leq \|S\| \|T\|_{\ell^p'(\ell^2_N)}, \quad \text{by H"older,} \]

\[ \leq \pi_{p'} \left( id_{\ell^2_N} \right) \pi_p \left( id_{\ell^2_N} \right) \|S\| \|T\|, \quad \text{by Gordon.} \]
Proof of the Lemma

Proof.

Identify algebraically

\[ \mathcal{B}(\ell^p, \ell^2_N) = \ell^p \hat{\otimes} \ell^2_N = \ell^p \otimes \ell^2_N = \ell^p(\ell^2_N), \quad \text{and} \]

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Proof of the Lemma

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Identify algebraically

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$$\mathcal{B}(\ell^{p'}, \ell^2_N) = \ell^{p'} \hat{\otimes} \ell^2_N = \ell^{p'} \otimes \ell^2_N = \ell^{p'} (\ell^2_N).$$

Note that
Proof of the Lemma

Proof.

Identify algebraically

\[ \mathcal{B}(\ell^p, \ell^2_N) = \ell^p' \otimes \ell^2_N = \ell^p' \otimes \ell^2_N = \ell^p' (\ell^2_N), \quad \text{and} \]
\[ \mathcal{B}(\ell^p', \ell^2_N) = \ell^p \otimes \ell^2_N = \ell^p \otimes \ell^2_N = \ell^p (\ell^2_N). \]

Note that

\[ \sum_{n=1}^{\infty} \| S e_n \|_{\ell^2_N} \| T e_n^* \|_{\ell^2_N} \leq \| S \|_{\ell^p' (\ell^2_N)} \| T \|_{\ell^p (\ell^2_N)}, \quad \text{by H"older}, \]
Proof of the Lemma

Identify algebraically

\[ \mathcal{B}(\ell^p, \ell^2_N) = \ell^p' \otimes \ell^2_N = \ell^p' \otimes \ell^2_N = \ell^p' (\ell^2_N), \quad \text{and} \]
\[ \mathcal{B}(\ell^p', \ell^2_N) = \ell^p \otimes \ell^2_N = \ell^p \otimes \ell^2_N = \ell^p (\ell^2_N). \]

Note that

\[
\sum_{n=1}^{\infty} \| Se_n \|_{\ell^2_N} \| Te_n^* \|_{\ell^2_N} \leq \| S \|_{\ell^p'(\ell^2_N)} \| T \|_{\ell^p(\ell^2_N)}, \quad \text{by Hölder,}
\]
\[
\leq \pi_{p'}(\text{id}_{\ell^2_N}) \pi_p(\text{id}_{\ell^2_N}) \| S \| \| T \|.
\]
Amenability of operator algebras on Banach spaces, II
Volker Runde

Amenability of $\mathcal{K}(E)$
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A positive example $\mathcal{B}(\ell^p \oplus \ell^q)$ with $p \neq p$
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Proof of the Lemma

Proof.

Identify algebraically

$$\mathcal{B}(\ell^p, \ell^2_N) = \ell^p' \otimes \ell^2_N = \ell^p' \otimes \ell^2_N = \ell^p' (\ell^2_N),$$

and

$$\mathcal{B}(\ell^p', \ell^2_N) = \ell^p \otimes \ell^2_N = \ell^p \otimes \ell^2_N = \ell^p (\ell^2_N).$$

Note that

$$\sum_{n=1}^{\infty} \|S e_n\|_{\ell^2_N} \|T e_n^*\|_{\ell^2_N} \leq \|S\|_{\ell^p' (\ell^2_N)} \|T\|_{\ell^p (\ell^2_N)},$$

by Hölder,

$$\leq \pi_{p'}(\text{id}_{\ell^2_N}) \pi_p(\text{id}_{\ell^2_N}) \|S\| \|T\|$$

$$\leq C \mathcal{N} \|S\| \|T\|$$
Proof of the Lemma

**Proof.**

Identify algebraically

$$\mathcal{B}(\ell^p, \ell^2_N) = \ell^p' \otimes \ell^2_N = \ell^p' \otimes \ell^2_N = \ell^p' (\ell^2_N), \quad \text{and}$$

$$\mathcal{B}(\ell^p', \ell^2_N) = \ell^p \otimes \ell^2_N = \ell^p \otimes \ell^2_N = \ell^p (\ell^2_N).$$

Note that

$$\sum_{n=1}^{\infty} \|S e_n\|_{\ell^2_N} \|T e_n^*\|_{\ell^2_N} \leq \|S\|_{\ell^p'(\ell^2_N)} \|T\|_{\ell^p(\ell^2_N)}, \quad \text{by Hölder},$$

$$\leq \pi_{p'}(\text{id}_{\ell^2_N}) \pi_{p}(\text{id}_{\ell^2_N}) \|S\| \|T\|\|, \quad \text{by Gordon.}$$
Non-amenability of $\mathcal{B}(\ell^p)$ for $p \in (1, \infty)$
Non-amenability of $\mathcal{B}(\ell^p)$ for $p \in (1, \infty)$

Corollary
Non-amenability of $\mathcal{B}(\ell^p)$ for $p \in (1, \infty)$

Corollary

Let $p \in (1, \infty)$ and let $E$ be an $L^p$-space with $\dim E = \infty$. 
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Let $p \in (1, \infty)$. Then $\mathcal{B}(\ell^p)$ and $\mathcal{B}(L^p[0, 1])$
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