Lecture 4: Local Properties of Group C*-Algebras

Zhong-Jin Ruan

at Leeds, Thursday, 20 May, 2010
Local Property of Banach Spaces

It is known from the Hahn-Banach theorem that given any Banach space $V$, there exists an index set $I$ such that we have the isometric inclusion

$$V \hookrightarrow \ell_\infty(I).$$

Usually $I$ is an infinite index (even $V$ is finite dimensional).

**Question:** If $V$ is finite dimensional, can we

“approximately embed” $V$ into a finite dimensional $\ell_\infty(n)$

for some positive integer $n \in \mathbb{N}$?
Finite Representability in $\{\ell_\infty(n)\}$

**Theorem:** Let $E$ be a f.d. Banach space. For any $\varepsilon > 0$, there exist $n(\varepsilon) \in \mathbb{N}$ and $F \subseteq \ell_\infty(n(\varepsilon))$ such that

$$ E \overset{1+\varepsilon}{\cong} F, $$

i.e., there exists a linear isomorphism $T : E \to F$ such that

$$ \|T\| \|T^{-1}\| < 1 + \varepsilon. $$

Therefore, we say that

- every f.d. Banach space $E$ is **representable** in $\{\ell_\infty(n)\}$;
- every Banach space $V$ is **finitely representable** in $\{\ell_\infty(n)\}$. 
Proof: Since $E^*$ is finite dim, the closed unit ball $E_1^*$ is totally bounded. For arbitrary $1 > \varepsilon > 0$, there exists finitely many functionals $f_1, \cdots, f_n \in E_1^*$ such that for every $f \in E_1^*$, there exists some $f_j$ such that

$$\|f - f_j\| < \frac{\varepsilon}{1 + \varepsilon}.$$ 

Then we obtain a linear contraction

$$T : x \in E \rightarrow (f_1(x), \cdots, f_n(x_n)) \in \ell_1^n.$$ 

For any $f \in E_1^*$, we let $f_j$ such that $\|f - f_j\| < \frac{\varepsilon}{1 + \varepsilon}$. Then we get

$$\|T(x)\| \geq |f_j(x)| \geq |f(x)| - |f(x) - f_j(x)| \geq |f(x)| - \frac{\varepsilon\|x\|}{1 + \varepsilon}.$$ 

This shows that

$$\|T(x)\| \geq \|x\| - \frac{\varepsilon\|x\|}{1 + \varepsilon} = \frac{\|x\|}{1 + \varepsilon}.$$ 

Therefore, $\|T^{-1}\| < 1 + \varepsilon$. 
Finite Representatibility of Operator Spaces in \( \{M_n\} \)

An operator space \( V \) is called finitely representable in \( \{M_n\} \) if for every f.d. subspace \( E \) and \( \varepsilon > 0 \), there exist \( n(\varepsilon) \in \mathbb{N} \) and \( F \subseteq M_n(\varepsilon) \) such that

\[
E \overset{1+\varepsilon}{\cong} \text{cb } F,
\]
i.e., there exists a linear isomorphism \( T : E \rightarrow F \) such that

\[
\|T\|_{\text{cb}} \|T^{-1}\|_{\text{cb}} < 1 + \varepsilon.
\]

It is natural to ask whether every finite dim operator space is representable in \( \{M_n\} \), or whether every operator space is finitely representable in \( \{M_n\} \)?

**Theorem:** Let \( A \subseteq B(H) \) be a C*-algebra. Then \( A \) is finitely representable in \( \{M_n\} \) if and only if there exists two nets of completely contractive maps

\[
S_\alpha : A \rightarrow M_n(\alpha) \text{ and } T_\alpha : M_n(\alpha) \rightarrow B(H)
\]
such that \( \|T_\alpha \circ S_\alpha(x) - x\| \rightarrow 0 \) for all \( x \in A \).
Exact C*-algebras

We recall from Kirchberg that a C*-algebra $A$ is an exact C*-algebra if we have the short exact sequence

$$0 \to K(\ell_2) \otimes A \hookrightarrow B(\ell_2) \otimes A \to Q(\ell_2) \otimes A \to 0,$$

where $Q(H) = B(\ell_2)/K(\ell_2)$.

**Theorem [Kirchberg (Pisier) 1995]:** A C*-algebra $A$ is exact if and only if there exists two nets of completely positive and contractive maps (complete contractions)

$$S_\alpha : A \to M_{n(\alpha)}$$

and

$$T_\alpha : M_{n(\alpha)} \to B(H)$$

such that $\|T_\alpha \circ S_\alpha (x) - x\| \to 0$ for all $x \in A$.

Therefore, $A$ is finitely representable in $\{M_n\}$ iff $A$ is exact.
Theorem (Pisier 1995): Let \( \ell_1(n) \) be the operator dual of \( \ell_\infty(n) \). If

\[
T : \ell_1(n) \to F \subseteq M_k
\]

is a linear isomorphism, then for \( n \geq 3 \)

\[
\|T\|_{cb}\|T^{-1}\|_{cb} \geq \frac{n}{2\sqrt{n-1}}.
\]

Hence for \( n \geq 3 \),

\[
\ell_1(n) \hookrightarrow C^*(\mathbb{F}_{n-1}) \subseteq B(H_\pi)
\]

are not finitely representable in \( \{M_n\} \).

So \( C^*(\mathbb{F}_{n-1}) \) and \( B(H_\pi) \) are examples of non-exact C*-algebras.
Examples of Exact $C^*$-algebras

• For $C^*$-algebras, we have

$$\text{Nuclearity} \Rightarrow \text{CBAP} \Rightarrow \text{Strong OAP} \Rightarrow \text{Exactness}$$

• For any discrete group $G$, we have

$$\text{Nuclearity} \Rightarrow \text{Weakly Amenable} \Rightarrow \text{AP} \Rightarrow \text{Exact, i.e. } C^*_\lambda(G') \text{ is exact}$$

• Groups like $G = \mathbb{F}_n, \mathbb{Z}^2 \rtimes SL(2, \mathbb{Z}), G = SL(3, \mathbb{Z})$ are exact.
Some Interesting Theorems

It is easy to see that if $A$ is an exact C*-algebra, then any C*-subalgebra or subspace of $A$ is also exact. Therefore, every C*-subalgebra of nuclear C*-algebra is exact.

Theorem [Kirchberg and Phillips 2000]: If $A$ is a separable exact C*-algebra, then $A$ is *-isomorphic to a C*-subalgebra of $O_2$.

How about group C*-algebras?
Now let $G$ be a discrete group. Then

$$\mathcal{UC}(G) = \overline{\text{span}\{f\lambda_s : f \in \ell_\infty(G), s \in G\}} \subseteq B(\ell_2(G))$$

is a unital C*-algebra, which is called uniform algebra, or uniform Roe algebra. In fact, $\mathcal{UC}(G) = \ell_\infty(G) \rtimes G$.

The following theorem was first observed by Guentner and Kaminker, but was finally proved by Ozawa.

**Theorem [Ozawa]:** Let $G$ be a discrete group. Then TFAE:

1. $G$ is exact;

2. for any finite subset $E \subseteq G$ and $\varepsilon > 0$, there exists a subset $F \subseteq G$ and a positive definite kernel $u : G \times G \to \mathbb{C}$ such that

   $$|u(s, t) - 1| < \varepsilon \ \text{if} \ \ st^{-1} \in E \ \text{and} \ u(s, t) = 0 \ \text{if} \ \ st^{-1} \notin F.$$ 

3. $\mathcal{UC}(G) = \ell_\infty(G) \rtimes C^*_\lambda(G)$ is nuclear.
Finite Representability in $\{\ell_1^n\}$

In Banach space theory it is known that a Banach space $V$ is finitely representable in $\{\ell_1^n\}$ if and only

$$V \hookrightarrow L_1(\mu)$$

is isometric to a closed subspace of some $L_1(\mu)$ space.
Finite Representability in \( \{T_n\} \)

An operator space \( V \) is finitely representable in \( \{T_n\} \) if for any f.d. subspace \( E \) and \( \varepsilon > 0 \), there exist \( n(\varepsilon) \in \mathbb{N} \) and \( F \subseteq T_n(\varepsilon) \) such that

\[
E \overset{1+\varepsilon}{\simeq}_{cb} F,
\]

i.e., there exists a linear isomorphism \( T : E \to F \) such that

\[
\|T\|_{cb} \|T^{-1}\|_{cb} < 1 + \varepsilon.
\]

- If \( A \) is a nuclear C*-algebra, then \( A^* \) and \( A^{***} \) are finitely representable in \( \{T_n\} \). For example

\[
C(X)^*, \quad T(\ell_2), \quad B(\ell_2)^*.
\]

- \( C_\lambda^*(\mathbb{F}_2)^* \) is finitely representable in \( \{T_n\} \).
Question: Is the predual $M_*$ of a von Neumann algebra is finitely representable in $\{T_n\}$?

Theorem [E-J-R 2000]: Let $M$ be a von Neumann algebra. Then $M_*$ is finitely representable in $\{T_n\}$ if and only if $M$ has the QWEP, i.e. $M$ is a quotient of a C*-algebra with Lance’s weak expectation property.

A C*-algebra has the WEP if for the universal representation $\pi : A \to B(H)$, there exists a completely positive and contraction $P : B(H) \to A^{**}$ such that $P \circ \pi = id_A$.

A. Connes’ conjecture 1976: Every finite von Neumann algebra with separable predual is *-isomorphic to a von Neumann subalgebra of the ultrapower of the hyperfinite $II_1$ factor

$$M \hookrightarrow \prod_{U} R_0.$$ 

E. Kirchberg’s conjecture 1993: Every $C^*$-algebra has QWEP.
Residually Finite Groups

Let $G$ be a discrete group. We say that $G$ is residually finite if for any finitely many distinct elements $s_1, \ldots, s_n$ in $G$ there exists a group homomorphism $\theta$ from $G$ into a finite group $H$ such that $\theta(s_1), \ldots, \theta(s_n)$ are distinct in $H$.

**Theorem [Kirchberg 1993, Wassermann 1994]:** If a discrete group $G$ is residually finite, then $G$ has property (F) and thus $L(G)$ has the QWEP.
More Examples

For the following groups $G$, $C^*_\lambda(G)$ are exact C*-algebras, i.e. finitely representable in $\{M_n\}$, and $A(G) = L(G)_*$ and $B_\lambda(G) = (C^*_\lambda(G))^*$ are finitely representable in $\{T_n\}$.

- For $n \geq 2$, $G = SL(n, \mathbb{Z})$ is residually finite since for any distinct $s_1, \cdots s_n$ in $G$, we can find a sufficiently large prime numbers $p$ such that the homomorphism

$$\theta_p : SL(n, \mathbb{Z}) \to SL(n, \mathbb{Z}_p)$$

with distinct image $\theta_p(s_1), \cdots, \theta_p(s_n)$ in finite group $SL(n, \mathbb{Z}_p)$.

- $G = \mathbb{Z}^2 \rtimes SL(2, \mathbb{Z})$ is residually. We can consider

$$\theta_p : \mathbb{Z}^2 \rtimes SL(2, \mathbb{Z}) \to \mathbb{Z}_p^2 \rtimes SL(2, \mathbb{Z}_p).$$