$G$, locally compact group

representation of $G$ by unitary operators on a Hilbert space $H_{\pi}$ strongly continuous

(1) inner product of $H_{\pi}$

For $f, g \in H_{\pi}$

$\langle f, g \rangle = (\pi(x)f | g) \quad \text{coefficient function of } \pi$

$B(G) = \{ \langle f, g \rangle : \pi \text{ as above, } f, g \in H_{\pi} \}$

Fourier-Stieltjes algebra of $G$

(Kraus, 1940)

$\|u\|_{B} = \inf \| \langle f, g \rangle \| : u = \langle f, g \rangle$

$B(G) \subseteq C_{b}(G)$ subalgebra

contains the constant functions

Banach algebra w.r.t. $\| \|_{B}$

pointwise product

$u_{f} \cdot u_{g} = u_{f \circ g}$

coeff. of $\pi \otimes \pi$, (repr. on $H \otimes H$)

tensor product of repr.
\( \mathcal{H} = L^2(G) \) (w.r.t. left Haar measure)

\((\mathcal{H} \cdot f) (y) = f(x^{-1} y) \quad (f \in \mathcal{H}) \)

left regular representation

\( A(G) = \{ u_{f,g} : f,g \in L^2(G) \} \) (coeff. of \( \lambda \))

Fourier algebra of \( G \) (Eymard 1965)

\( \| u \|_A = \inf \{ \| f \|_1 : u = u_{f,g} , f,g \in L^2(G) \} \)

\( A(G) \) closed subalgebra (ideal) of \( B(G) \)
$\text{VN}(G) = \text{von Neumann algebra on } \ell^2(G) \text{ generated by } \chi_x, x \in G \subset B(H)$

$\text{strong closure in } B(H) \text{ of linear subspace generated by this set}$

$= \{ T \in B(H) : T \cdot \rho(x) = \rho(x) \cdot T \} \forall x \in G \subset B(H)$

$\rho(x)$ right translation

$(\rho(x) f)(y) = f(y+x) \Delta(x^{-1})$

\text{Duality:}

$(T, \chi_g) = (T \chi_g, 1)$

Precisely for $T \in \text{VN}(G)$ this does not depend on the representation of $\chi = \chi_g$.

It makes $\text{VN}(G)$ the dual of the Banach space $A(G)$

$(A(G) = \text{predual of } \text{VN}(G))$

\text{Similarly:}

$B(G)$ is the dual of $C^*(G)$ (= enveloping $C^*$-algebra of $L^1(G)$)
Convolution:
\[
(f * g)(y) = \int f(x)g(x-y)\,dx
\]
\[
t, g \in L^1(\mathbb{R}) \implies f * g \in L^1(\mathbb{R})
\]
**Bounded algebra**

extension:
\[
\mathbb{R} + \mathbb{R} \ni f, g \in M(\mathbb{R}) \implies M(\mathbb{R}) \text{ is a bounded algebra}
\]

(for finite Radon measures on \(\mathbb{R}\))

\[
x + f \in L^2(\mathbb{R}) \text{ for } x \in M(\mathbb{R}), f \in L^2(\mathbb{R})
\]
defines \(J_x\) convolution operator
\[
J_x f = \int f(y)\,dy
\]

(in particular \(J_x f \in \mathcal{H}(\mathbb{R})\) \(\forall \) \(f \in L^2(\mathbb{R})\))

\(\mathcal{H}(\mathbb{R})\) is isomorphic to \(\mathcal{H}(\mathbb{R})\)

\[
\left(J_x f, g\right)_{\mathcal{H}(\mathbb{R})} = \left(\int f(y)\,dy, g\right)_{L^2(\mathbb{R})}
\]

\[
= \left(\int \int f(x)g(x-y)\,dy\,dx\right)_{L^2(\mathbb{R})}
\]

\[
= \int \int f(x)g(x-y)\,dy\,dx
\]

\[
\overset{\text{by duality}}{=} \left(J_x f, g\right)_{\mathcal{H}(\mathbb{R})} - \left(\int f\right)_{\mathcal{H}(\mathbb{R})} \left(\int g\right)_{\mathcal{H}(\mathbb{R})}
\]

\Rightarrow duality between \(\mathcal{H}(\mathbb{R}), \mathcal{A}(\mathbb{R})\) extends the usual duality between measures and functions on \(\mathbb{R}\).

\(\delta_x\) point measure at \(x\)
\[
\delta_x + f = J_x f \quad \text{i.e. } J_x(\delta_x) = f
\]

generalization: \(\pi\) rep. of \(\mathbb{G}\) (as before)
\[
\pi(x)f = \int \pi(x) f \, dm(x) \quad \text{(Bochner integral) extends } \pi \text{ to a}
\]
\[
\pi \in \mathcal{H}(\mathbb{R}) \text{ on } \mathcal{H}(\mathbb{R})
\]
Examples:

\( G = \mathbb{Z} \):

\[ A(\mathbb{Z}) = \text{set of Fourier coefficients of all integrable functions on } T = \mathbb{R}/\mathbb{Z} \]  

(torus group)

\[ B(\mathbb{Z}) = \text{Fourier-Stieltjes coeff. of all measures on } T \]

more generally:

\( G \) abelian (ie. compact)

\( \hat{G} \) dual group (character, 1-dim. reps.)

Fourier transform defines isomorphisms between \( A(G) \) and \( L^1(G) \)

\[ B(G) \quad M(G) \]
non-commutative case

\( \hat{G} \): equivalence classes of irreducible reps.

for compact \( G \) \( n \in \hat{G} \) finite dimensional.

\[
A(G) = \left\{ u \in C(G) : \sum_{\pi \in \hat{G}} \dim \pi \cdot \| \pi(f) \|_{T} < \infty \right\}
\]

more generally

\( G \) type I, unimodular, second countable

\( \mu \) Plancherel measure on \( \hat{G} \)

for \( u \in L^1_\mu L^2(G) \):
\[
\| u(x) \|_2^2 = \int \sum_{\pi \in \hat{G}} \dim \pi \cdot \| \pi(u) \|_2^2 \, d\mu(\pi)
\]

extends to isomorphisms between

\( L^2(G) \) and \( \int \oplus H_\pi \otimes \overline{H}_\pi \, d\mu(\pi) \) direct integral

\( VN(G) = \int \oplus B(H_\pi) \, d\mu(\pi) \)

\( A(G) = \int \oplus \mathcal{N}(H_\pi) \, d\mu(\pi) \) trace class operators

for \( u \in L^1 \cap A(G) \) non-commutative \( L^1 \)-space

\[
u(e) = \int \delta_{e}(\pi(u)) \, d\mu(\pi)
\]

inversion formula

Examples: \( G \) compact Lie group, either semisimple or nilpotent.

For \( G \) discrete: \( G \) type I \( \iff \) \( H \) abelian subgroup of finite index (Thoma)
Properties of $A(G)$:

$A(G) \subseteq C_c(G)$ subalgebra (proper for $G$ infinite)

Riemann-Lebesgue lemma

functions with compact support are dense in $A(G)$

translation invariant

separates pts. of $G$ $\Rightarrow$ dense in $C_c(G)$ w.r.t. $\mathcal{H}$

Gelfand spectrum of $A(G)$: $G$

$\mathcal{H}$ difficult to compute (in general)

special case: $v$ positive definite ($\Leftrightarrow v = v^*$

then $\mathcal{H} = v(1e)$

$A(G)$ is generated by pos. def. functions
Conditions on $G$: amenability

$G$ amenable $\iff$ left invariant mean on $L^2(G)$

Examples: amenable: abelian, compact...
non-amenable: $F_2$ (free groups, discrete), $SL(2, \mathbb{Z})$, semisimple Lie groups

$A(G)$ has unit $\implies G$ compact

TFAE:
(i) $G$ amenable
(ii) $A(G)$ has bounded approx. unit
(iii) $A(G)$ factorizes

(i.e. $A(G) = \{ u \in V : u \in A(G) \}$

Growth properties:

For $G$ amenable, non-compact decrease to 0 of $w(A(G))$ can be arbitrarily slow.

$w(G) \implies \exists x \in A(G): \forall G \geq 1, w(G) = x \in G$

For $G$ non-amenable the elements of $A(G)$ satisfy growth conditions:

$\exists \mu$ $\sigma$-finite, non-neg. measure on $G$ with $\mu(G) = \infty$

such that $A(G) \in L^1(G, \mu)$
for $G = \mathbb{F}_q^2$:

$$\sum_{x \in \mathbb{F}_q^2} \frac{|u(x)|}{3^{\frac{n}{2}} n} < \infty$$

(Kazhdan)

$n = k!$ word length

for semisimple Lie groups $G$ with finite centre:

Kunze-Stein phenomenon

(Coifling)

$A(G) \subseteq L^p(G)$

$\forall p > 2$

irred. reps. such that $u_g \in L^2(G):$ square-integrable reps.

this holds iff $\pi$ is a subrep. of $\mathbb{F}$

$(A(G) \subseteq L^2(G)$ \iff \text{ comp.})$

(Rickey)
Dress products

A Banach algebra, $A \subseteq A''$ (bidual space)

For $u \in A$, $u \mapsto u\otimes u$ bidual of $v \mapsto u\cdot v$

$A'' \to A''$ $A \to A$

(multiplication operator)

For $T \in A'$, $\langle u \cdot T, u \rangle = \langle u \otimes u, T \rangle$ defines $u \cdot T \in A'$

($A', A''$ - module)

For $\eta \in A''$, $\langle \eta \otimes u, T \rangle = \langle \eta, u \cdot T \rangle$ defines $\eta \otimes u : A' \to A''$

(first dress product)

$\eta = \lim u_{ij}$ (limits)

$\eta \otimes u_{ij} = \lim \lim (u_{ij})_{\cdot k}$

$\Rightarrow A''$ Banach algebra, $A$ subalgebra
For $A = A(G)$ commutative Banach algebra

$A(G)^{''}$ not commutative for "most" infinite $G$

if there exists an infinite $G$ such that

$A(G)^{''}$ commute. ("$A(G)$ is strongly regular")

then $G$ must be discrete

and must not contain any infinite amenable subgroup ($\Rightarrow G$ contains no

infinite abelian subgroup)

$A(G) \leq Z(A(G)^{''})$ (centre)

for many amenable groups $Z(A(G)^{''}) = A(G)$

("$A(G)$ strongly almost irregular")

for example: $G$ discrete amenable

or metrizable solvable

for many non-amenable groups $Z(A(G)^{''}) \neq A(G)$

for example: $G$ discrete, $G \cong F_2$

or connected group semisimple finite center
$A(c) = V(U(c))$ in Neumann algebras.

In particular, a C*-algebra can be realized as a representation of a C*-algebra.

In the realm of representations of $V(U(c))$ on $H$, consider $c$.

$\rho(a) = \lim_{\rho(a)}$ for $\rho(a)$.

In particular, $c$.

$A(c)$ is a limit of representations $V(U(c))$ on $H$.

$A(c)$ is a limit of representations $V(U(c))$ on $H$.