Banach Algebras with an algebraic structure of Kakutani-Kodaira flavour

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May 24, 2010
Let $A$ be a Banach algebra with a faithful multiplication.

- $\mu : A \rightarrow A$ is a left multiplier if $\mu(ab) = \mu(a)b$, a right multiplier if $\mu(ab) = a\mu(b)$.

- For $a \in A$, $\ell_a : x \mapsto ax$ is a left multiplier, $r_a : x \mapsto xa$ is a right multiplier.

- $LM(A) :=$ the left multiplier algebra of $A$ ($\subseteq B(A)$), $RM(A) :=$ the right multiplier algebra of $A$ ($\subseteq B(A)^{op}$).

Then $LM(A)$ and $RM(A)$ are Banach algebras.

The multiplier algebra $M(A)$ of $A$ is also defined.

- $a \mapsto \ell_a$ and $a \mapsto r_a$ are injective and contractive.
If $A$ has a bounded approximate identity (BAI), then

$$\| \cdot \|_{LM(A)} \sim \| \cdot \|_A \sim \| \cdot \|_{RM(A)}$$
on $A$.

In this case, $A$ is identified with a left closed ideal in $LM(A)$, and a right closed ideal in $RM(A)$.

- For $\mu \in LM(A)$, we write $\mu \in A$ if $\mu = \ell_a$.
- For $\mu \in RM(A)$, we write $\mu \in A$ if $\mu = r_a$.

**Question**: How can $A$ be characterized inside $LM(A)$, $RM(A)$?
Motivation – a range space problem

For $\mathbb{G} = L_\infty(G)$, $VN(G)$, in the representation

$$\Theta^r : M^r_{cb}(L_1(\mathbb{G})) \cong CB^\sigma_{L_\infty(\mathbb{G})}(B(L_2(\mathbb{G})))$$

by Neufang-Ruan-Spronk (08),

$$\Theta^r(L_1(\mathbb{G})) = CB^\sigma_{L_\infty(\mathbb{G})}(B(L_2(\mathbb{G})))$$

One may ask the same question for the representation of general LCQGs by Junge-Neufang-Ruan (09).

- Using a measure theoretic proof, N-R-S showed that
  $$\Theta^r(L_1(G)) = CB^\sigma_{VN(G)}(B(L_2(G))).$$

- The question was open for $\mathbb{G} = VN(G)$ (N.-R.-S.).

- We will consider a **Banach algebraic** approach to this range space problem.
Any Banach algebra $A$ is a right $RM(A)$-module via

$$A \times RM(A) \longrightarrow A, \ (a, \mu) \longmapsto \mu(a).$$

Then $A^*$, $A^{**}$ are naturally left, right $RM(A)$-modules, resp.

$$\langle AA^* \rangle = \text{span}\{a \cdot f : a \in A, f \in A^*\} \|\cdot\| \subseteq \langle RM(A) \cdot A^* \rangle \subseteq A^*.$$ 

Recall: a Banach space $X$ is weakly sequentially complete (WSC) if every $w$-Cauchy sequence in $X$ is $w$-convergent.

The predual of a von Neumann algebra is WSC.
Proposition. (H.-N.-R.) Let $A$ be a WSC Banach algebra with a sequential BAI. Then for $\mu \in RM(A)$, T.F.A.E.

(i) $\mu \in A$.

(ii) $\mu \cdot A^* \subseteq \langle AA^* \rangle$.

(iii) $\exists m \in A^{**}$ such that $n \cdot \mu = n \diamond m$ ($n \in A^{**}$).

The left version also holds.

- The following inequality is crucial in the proof:
  \[
  \text{card(BAI)} \leq \text{a cardinal level of weak completeness of } A.
  \]

- However, $A$ can have a BAI but without any sequential BAI.
Banach algebraic approach: More general situation

- $L_1(G)$ has a sequential BAI $\iff$ $G$ is metrizable.
- $A(G)$ has a sequential BAI $\iff$ $G$ is amenable $\sigma$-compact.

More general, it can be shown that

- \[
    \min \{ \text{card}(J) : (e_j)_{j \in J} \text{ is a BAI of } L_1(G) \} = \text{the local weight } \chi(G) \text{ of } G.
    \]

- \[
    \min \{ \text{card}(J) : (e_j)_{j \in J} \text{ is a BAI of } A(G) \} = \text{the compact covering number } \kappa(G) \text{ of } G.
    \]

**Our approach:** Consider Banach algebras $A$ with a “Large” family of “Small” subalgebras. More precisely, $A$ has a family $\{A_i\}$ of subalgebras such that each $A_i$ is WSC with a sequential BAI, and $\{A_i\}$ is large so that each $\mu \in RM(A)$ is determined by its behavior on these subalgebras.
**Definition.** (H.-N.-R.) Let $A$ be a Banach algebra with a BAI. Suppose that for every $\mu \in RM(A)$, there is a closed subalgebra $B$ of $A$ with a BAI satisfying the following conditions.

1. $\mu|_B \in RM(B)$.
2. $f|_B \in BB^*$ for all $f \in AA^*$.
3. There is a family $\{B_j\}$ of closed right ideals in $B$ such that
   - (i) each $B_j$ is WSC with a sequential BAI;
   - (ii) for all $j$, there is a left $B_j$-module projection from $B$ onto $B_j$;
   - (iii) $\mu \in A$ if $\mu|_{B_j} \in B_j$ for all $j$.

Then $A$ is said to be of type $(RM)$.

Similarly, Banach algebras of type $(LM)$ are defined.

- $A$ is of type $(M)$ if $A$ is both of type $(LM)$ and of type $(RM)$. 

The classical Kakutani-Kodaira theorem.

Let $G$ be a $\sigma$-compact locally compact group. Then
\( \forall \) sequence \((U_n)\) of neighborhoods of \(e\),
\( \exists \) a compact normal subgroup \(N\) of \(G\) such that

\[ N \subseteq \bigcap U_n \quad \text{and} \quad G/N \text{ is metrizable.} \]
Theorem. (H. 05) Let $G$ be a locally compact group. Then
\[ \forall \ \text{family} \ (U_j)_{j \in J} \ \text{of neighborhoods of } e \ \text{with } \text{card}(J) \leq \kappa(G) \aleph_0 , \]
\[ \exists \ \text{a compact normal subgroup } N \ \text{of } G \ \text{such that} \]
\[ N \subseteq \bigcap U_j \ \text{and} \ \chi(G/N) \leq \kappa(G) \aleph_0 . \]

- In fact, this generalized K-K theorem was motivated by its dual version (H. 02), which was used to study the ENAR of $A(G)$ in the sense of Granirer (96). We give below a unified K-K theorem in the setting of Kac algebras.

- Recently, this generalized K-K theorem is used by Filali-Neufang-SanganiMonfared (09) and Losert-Neufang-Pachl-Steprāns in their study of topological centres of $A(G)$ and $M(G)$, resp.
Recall: Kac algebras and reduced Kac algebras

Let $\mathbb{K} = (\mathcal{M}, \Gamma, \kappa, \varphi)$ be a **Kac algebra**, where $\kappa : \mathcal{M} \to \mathcal{M}$ is an involutive anti-automorphism satisfying

\[(\kappa \otimes \kappa) \circ \Gamma = \Sigma \circ \Gamma \circ \kappa.\]

Let $p \in \mathcal{M}$ be a central projection such that $\Gamma(p) \geq p \otimes p$ and $\kappa(p) = p$.

Let $r : \mathcal{M} \to \mathcal{M}_p$ be the canonical surjection $x \mapsto xp$. Then $\mathbb{K}_p = (\mathcal{M}_p, \Gamma_p, \kappa_p, \varphi_p)$ is a **reduced Kac algebra** of $\mathbb{K}$, where

$\Gamma_p(r(x)) = (r \otimes r)\Gamma(x)$ and $\kappa_p(r(x)) = r(\kappa(x))$, 

and $\varphi_p$ is obtained by reduction from $\varphi$. 
Recall: Kac algebras and reduced Kac algebras

- $L_\infty(G)$ and $VN(G)$ are Kac algebras, and $\widehat{L_\infty(G)} = VN(G)$.

- It is known (Takesaki-Tatsuuma 71) that
  - For $K = L_\infty(G)$, $K_P$ is a reduced Kac algebra of $L_\infty(G)$ iff $K_P = L_\infty(H)$ for some open subgroup $H$ of $G$.
  - For $K = VN(G)$, $K_P$ is a reduced Kac algebra of $VN(G)$ iff $K_P = VN(G/N)$ for some comp. normal subgroup $N$ of $G$. 
Recall: Decomposability number

For a von Neumann algebra $\mathcal{M}$, the **decomposability number** $\text{dec}(\mathcal{M})$ of $\mathcal{M}$ is the greatest carnality of a family of pairwise orthogonal non-zero projections in $\mathcal{M}$.

E.g., $\text{dec}(B(H)) = \dim(H)$ and $\text{dec}(B(H)^{**}) = 2^{2\dim(H)}$.

**Theorem.** (H.-N. 06) Let $G$ be an infinite LCG. Then

(i) $\text{dec}(L_\infty(G)) = \kappa(G)\aleph_0$.

(ii) $\text{dec}(VN(G)) = \chi(G)\aleph_0$. 
A Kac algebraic Kakutani-Kodaira theorem

Definition (H. 05) For a Kac algebra $K$ and a cardinal $\alpha$, the $\alpha^{th}$ Kakutani-Kodaira number $\delta_\alpha(K)$ of $K$ is the least cardinal $\kappa$ such that

\[
\forall \text{ family } (\mathcal{U}_i)_{i \in J} \text{ of } w^*-\text{nbhds of } id_K \text{ with } \text{card}(J) \leq \alpha,
\exists \text{ a reduced Kac algebra } K_p \text{ of } K \text{ such that}
\]
\[
p \in \bigcap \mathcal{U}_j \text{ and } \text{dec}(K_p) \leq \kappa.
\]

Then $\delta_\alpha(K) \leq \delta_\beta(K)$ if $\alpha \leq \beta$. We denote $\delta_1(K)$ by $\delta(K)$.

Theorem. (H. 05) If $K = L_\infty(G)$ or $VN(G)$, then

\[
\delta(K) \leq \text{dec}(\hat{K}),
\]

and the equality holds for many $K$ with uncountable $\text{dec}(\hat{K})$.

Equivalently, we have $\delta_{\text{dec}(\hat{K})}(K) \leq \text{dec}(\hat{K})$. 
More on dual relation between $K$ and $\hat{K}$ can be described quantitatively in terms of these Kac algebraic invariants. For example, we have the following.

**Theorem.** (H. 05) For $K = L_\infty(G)$ or $VN(G)$, there exists a one-to-one correspondence between the families

\[
\{\text{maximally decomposable sub Kac algebras of } K\}
\]

and

\[
\{\text{norm closed } \hat{K}\text{-invariant } \ast\text{-subalgebras } A \text{ of } L_1(\hat{K}) \text{ with dense}(A) = \text{dec}(K)\}. \]
Using the Kac algebraic Kakutani-Kodaira theorem, we showed that the class of Banach algebras of type \((M)\) includes:

- group algebras \(L_1(G)\);
- weighted convolution (Beurling) algebras \(L_1(G, \omega)\);
- Fourier algebras \(A(G)\) of amenable \(G\).

This class also includes:

- WSC Banach algebras \(A\) with a central BAI and \(A\) being an ideal in \(A^{**}\);
- WSC Banach algebras with a sequential BAI, in particular, quantum group algebras \(L_1(G)\) of co-amenable \(G\) with \(L_1(G)\) separable.
It turns out that Banach algebras $A$ of type $(M)$ behave well regarding multipliers and structures on $A^{**}$.

**Theorem.** (H.-N.-R.) Let $A$ be a Banach algebra of type $(RM)$. Then for $\mu \in RM(A)$, T.F.A.E.

(i) $\mu \in A$.

(ii) $\mu \cdot A^* \subseteq \langle AA^* \rangle$.

(iii) $\exists m \in A^{**}$ such that $n \cdot \mu = n \diamond m$ $(n \in A^{**})$.

The left version holds for $A$ of type $(LM)$ and $\mu \in LM(A)$. 
Theorem. (H.-N.-R.) Let $\mathbb{G}$ be a LCQG and let

$$\Theta^r : M^r_{cb}(L_1(\mathbb{G})) \cong CB^{\sigma,L_\infty(\mathbb{G})}_{L_\infty(\hat{\mathbb{G}})}(B(L_2(\mathbb{G})))$$

be the completely isometric representation by J.-N.-R.

If $L_1(\mathbb{G})$ is of type $(M)$ (e.g., $\mathbb{G}$ is co-amenable and $L_1(\mathbb{G})$ is separable), then

$$\Theta^r(L_1(\mathbb{G})) = CB^{\sigma,(L_\infty(\mathbb{G}),RUC(\mathbb{G}))}_{L_\infty(\hat{\mathbb{G}})}(B(L_2(\mathbb{G}))),$$

where $RUC(\mathbb{G}) := \langle L_1(\mathbb{G}) \star L_\infty(\mathbb{G}) \rangle$. 
A characterization of amenability

In particular, for every amenable LCG $G$, we have

$$
\Theta(A(G)) = CB_{L_\infty(G)}^{\sigma, (VN(G), UC(\hat{G}))}(B(L_2(G))).
$$

where $UC(\hat{G}) = \langle VN(G) \cdot A(G) \rangle = \langle A(G) \cdot VN(G) \rangle$.

This answers the open question by N.-R.-S. (08).

The converse of the above is also true. That is, if we let

$$A_\Theta(G) := \{ \mu \in M_{cb}A(G) : \mu \cdot VN(G) \subseteq UC(\hat{G}) \},$$

then we have

**Corollary.** $G$ is amenable $\iff A_\Theta(G) = A(G)$.
Recall: Arens products

- The **left Arens product** \(\square\) on \(A^{**}\) is naturally defined when \(A\) is considered as a left \(A\)-module:
  for \(a, b \in A\), \(f \in A^*\), and \(m, n \in A^{**}\), we have
  \[
  \langle f \cdot a, b \rangle = \langle f, ab \rangle, \quad \langle n \square f, a \rangle = \langle n, f \cdot a \rangle, \quad \langle m \square n, f \rangle = \langle m, n \square f \rangle.
  \]

- The **right Arens product** \(\diamond\) on \(A^{**}\) is defined similarly.

  Equivalently,
  \[
  m \square n = \text{\(w^*\)-lim}_{\alpha} \text{\lim}_{\beta} a_{\alpha} b_{\beta}
  \quad \text{and} \quad
  m \diamond n = \text{\(w^*\)-lim}_{\beta} \text{\lim}_{\alpha} a_{\alpha} b_{\beta}
  \]
  whenever \((a_{\alpha}), (b_{\beta})\) are nets in \(A\) \(w^*\)-convergent to \(m, n\).

- Both \(\square\) and \(\diamond\) extend the multiplication on \(A\).
Recall: Arens regularity

- $A$ is said to be **Arens regular** if $\Box$ and $\Diamond$ coincide.

- Every operator algebra (in particular, every $C^*$-algebra) and every quotient algebra thereof are Arens regular.

It is known that

(i) $L_1(G)$ is Arens regular $\iff G$ is finite (Young 73).

(ii) for amenable $G$, $A(G)$ is Arens regular $\iff G$ is finite (Lau 81).

(i) and (ii) can be seen dual to each in the setting of LCQG, noticing that $L_\infty(G)$ is always co-amenable, and $VN(G)$ is co-amenable iff $G$ is amenable.

It is still open whether (ii) holds for all LCGs $G$. 
Recall: Topological centres

- $(A^{**}, \square)$ is a right topological semigroup under $w^*$-top:
  
  for any fixed $m \in A^{**}$, $n \mapsto n\square m$ is $w^*$-$w^*$ cont.

- Similarly, $(A^{**}, \diamond)$ is a left topological semigroup.

- The topological centres of $(A^{**}, \square)$ and $(A^{**}, \diamond)$ are

  $$Z_t(A^{**}, \square) = \{m \in A^{**} : n \mapsto m\square n \text{ is } w^*-w^* \text{ cont.}\}$$

  $$Z_t(A^{**}, \diamond) = \{m \in A^{**} : n \mapsto n\diamond m \text{ is } w^*-w^* \text{ cont.}\}$$

  simply called the left and right topological centres of $A^{**}$. 

Recall: Topological centres

- $A \subseteq Z_t(A^{**}, \Box) \subseteq A^{**}$; $A \subseteq Z_t(A^{**}, \Diamond) \subseteq A^{**}$.

- $Z_t(A^{**}, \Box) = A^{**} \iff A$ is AR $\iff Z_t(A^{**}, \Diamond) = A^{**}$.

- $A$ is said to be **left strongly Arens irregular** (LSAI) if $Z_t(A^{**}, \Box) = A$ (**Dales-Lau 05**).

Similarly, RSAI and SAI are defined.

- Every group algebra $L_1(G)$ is SAI (**Lau-Losert 88**).
Theorem. (H.-N.-R.) Let $A$ be a Banach algebra of type ($M$). Then for $m \in A^{**}$, T.F.A.E.

(i) $m \in A$.

(ii) $m \in \mathcal{Z}_t(A^{**}, \Box)$ and $m \cdot A \subseteq A$.

(iii) $m \in \mathcal{Z}_t(A^{**}, \Diamond)$ and $A \cdot m \subseteq A$.

Corollary. Let $A$ be a Banach algebra of type ($M$). Then

(1) $A$ is LSAI $\iff \mathcal{Z}_t(A^{**}, \Box) \cdot A \subseteq A$;

(2) $A$ is RSAI $\iff A \cdot \mathcal{Z}_t(A^{**}, \Diamond) \subseteq A$. 
Recall: The left quotient algebra $\langle A^*A \rangle^*$ of $A^{**}$

- $\langle A^*A \rangle$ is an $A$-submodule of $A^*$ and is **left introverted** in $A^*$ (i.e., a left $(A^{**}, \Box)$-submodule of $A^*$).

- $\Box$ on $A^{**}$ induces a product on $\langle A^*A \rangle^*$ such that the canonical quotient map $A^{**} \longrightarrow \langle A^*A \rangle^*$ yields

  $$(\langle A^*A \rangle^*, \Box) \cong (A^{**}, \Box)/\langle A^*A \rangle_{\perp}.$$ 

- $\langle A^*A \rangle^*$ is also a right topological semigroup under the $w^*$-topology. Its topological centre is defined by

  $$\mathcal{Z}_t(\langle A^*A \rangle^*) = \{ m \in \langle A^*A \rangle^* : n \mapsto m\Box n \text{ is } w^*-w^* \text{ cont.} \}.$$ 

- For every LCG $G$, $\mathcal{Z}_t(LUC(G)^*) = M(G)$ (Lau 86).
Let \( q : A^{**} \longrightarrow \langle A^* A \rangle^* \) be the canonical quotient. Then
\[
q(3_t(A^{**}, \Box)) \subseteq 3_t(\langle A^* A \rangle^*).
\]

If \( A \) has a BRAI, then \( RM(A) \hookrightarrow 3_t(\langle A^* A \rangle^*) \subseteq \langle A^* A \rangle^* \).

**Proposition.** (H.-N.-R.) If \( A \) has a BRAI, then
\[
3_t(\langle A^* A \rangle^*) = RM(A) \iff A \cdot 3_t(A^{**}, \Box) \subseteq A.
\]

Recall from the above: If \( A \) is of type \( (M) \), then
\[
3_t(A^{**}, \Box) = A \iff 3_t(A^{**}, \Box) \cdot A \subseteq A.
\]

In next lecture, we shall explain this asymmetry and show what is missing here.
The answer to an open question by Lau-Ülger

For $m \in A^{**}$, let $m_R : A^* \rightarrow A^*$ be the map $f \mapsto f \Diamond m$.

**Question** (Lau-Ülger 96):

For a WSC Banach algebra $A$ with a BAI, if $m \in Z_t(A^{**}, \square)$, are $\ker(m_R)$ and $m_R(\text{ball}(A^*))$ $w^*$-closed in $A^*$?

**Answer**: It can be **negative** for $A$ of type $(M)$ with

**Property (X)** (Godefroy-Talagrand 81).

A special case for the answer is as follows.

**Proposition.** (H.-N.-R.) Let $\mathcal{M}$ be a von Neumann algebra with $A = \mathcal{M}_*$ separable with a BAI. Then, for any $m \in Z_t(A^{**}, \square) \setminus A$, either $\ker(m_R)$ or $m_R(\text{ball}(A^*))$ is not $w^*$-closed in $A^*$. 
An outline of the proof:

\[ A = M_\ast \text{ is separable} \implies A \text{ has the Mazur property} \]
(i.e., for \( m \in A^{**} \), we have \( m \in A \) if \( m \) is sequentially \( w^* \)-cont).

In this case,

\[
\ker(m_R) \text{ and } m_R(\text{ball}(A^*)) \text{ are both } w^*-\text{closed in } A^*
\]
\[ \iff m_R : A^* \rightarrow A^* \text{ is } w^*-w^* \text{ cont } \text{(Godefroy 89)}. \]

Let \( m \in Z_t(A^{**}, \square) \setminus A \). By our characterization of \( A \) inside \( Z_t(A^{**}, \square) \) given above, we have

\[
m \cdot A \notin A; \text{ i.e., } m^*_R(A) \notin A.
\]

Therefore, \( m_R : A^* \rightarrow A^* \) is not \( w^*-w^* \) cont.
Question: In the above Proposition, which of the sets $\ker(m_R)$ and $m_R(\text{ball}(A^*))$ is not $w^*$-closed in $A^*$?

Answer: Both are possible.

Example. Let $\triangleleft$ be the multiplication on $T(\ell_2(\mathbb{Z}))$ induced by the left fundamental unitary $W$ of $\ell_\infty(\mathbb{Z})$. Then $(T(\ell_2(\mathbb{Z})), \triangleleft)^{op}$ is just the convolution algebra $(T(\ell_2(\mathbb{Z})), \ast)$ introduced by Neufang (00). It is known from Auger-Neufang (07) the right topological centre of $(T(\ell_2(\mathbb{Z})), \ast)$ is $\ell_1(\mathbb{Z}) \oplus \ell_\infty(\mathbb{Z})^\perp$. Then

$$Z_t(T(\ell_2(\mathbb{Z}))^{**}, \square_\triangleleft) = \ell_1(\mathbb{Z}) \oplus \ell_\infty(\mathbb{Z})^\perp \nsubseteq T(\ell_2(\mathbb{Z})).$$

Let $A$ be the unitization of $(T(\ell_2(\mathbb{Z})), \triangleleft)$. Then

$$Z_t(A^{**}, \square) = \ell_1(\mathbb{Z}) \oplus \ell_\infty(\mathbb{Z})^\perp \oplus \mathbb{C} \nsubseteq A.$$
Proposition. (H.-N.-R.) Let $A$ be the same as above. Then $A$ is a unital Banach algebra with $A^*$ a von Neumann algebra.

Let $s \in \ell_\infty(\mathbb{Z}) \perp \ell_\infty(\mathbb{Z}) \perp$ and $m = (s, \alpha) \in \mathcal{Z}_t(A^{**}, \Box) \setminus A$.

(i) If $\alpha \neq 0$, then $\ker(m_R) = \{0\}$ is obviously $w^*$-closed in $A^*$, but $m_R(\text{ball}(A^*))$ is not $w^*$-closed in $A^*$.

(ii) If $\alpha = 0$, then $\ker(m_R)$ is not $w^*$-closed in $A^*$. In this case, $m_R(\text{ball}(A^*))$ is $w^*$-closed in $A^*$ iff $\|m\|$ is attainable.