Multipliers of locally compact quantum groups and Hilbert C*-modules

1. Locally compact groups, duality, and multiplier algebras

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June 2010
Let \( G \) be a locally compact group, and consider \( C_0(G) \), \( C^b(G) \) and \( L^\infty(G) \). These are two \( C^* \)-algebras and a von Neumann algebra: they depend only on the topological and measure space properties of \( G \).

We turn \( L^1(G) \) into a Banach algebra for the convolution product:

\[
(f * g)(s) = \int_G f(t)g(t^{-1}s) \, dt.
\]

This does remember the structure of \( G \), in the following sense: if \( L^1(G) \) and \( L^1(H) \) are isometrically isomorphic as Banach algebras, then \( G \) is, as a topological group, isomorphic to \( H \).
At the Operator algebra level

Define a map \( \Delta : L^\infty(G) \to L^\infty(G \times G) \) by

\[
\Delta(F)(s, t) = F(st) \quad (F \in L^\infty(G), s, t \in G).
\]

This is a unital, injective, \(*\)-homomorphism which is normal (weak*-continuous). The pre-adjoint is a map \( L^1(G \times G) \to L^1(G) \). As \( L^1(G) \otimes L^1(G) \) embeds into \( L^1(G \times G) \), we get a bilinear map on \( L^1(G) \). This is actually the convolution product, as

\[
\langle F, \Delta^*(f \otimes g) \rangle = \langle \Delta(F), f \otimes g \rangle = \int_{G \times G} F(st)f(s)g(t) \, ds \, dt
\]

\[
= \int_G F(t) \int_G f(s)g(s^{-1}t) \, ds \, dt = \langle F, f \ast g \rangle.
\]
For the C*-algebra

Notice that we can also interpret $\Delta$ as a $*$-homomorphism $C_0(G) \to C^b(G \times G)$,

$$\Delta(F)(s, t) = F(st) \quad (F \in C_0(G), s, t \in G).$$

but not as a map into $C_0(G \times G)$.

The map $\Delta : C_0(G) \to C^b(G \times G)$ “almost” maps into $C_0(G \times G)$. Indeed, for $f, g \in C_0(G)$,

$$((f \otimes 1)\Delta(g))(s, t) = f(s)g(st) \to 0 \text{ as } (s, t) \to \infty.$$ 

So $(f \otimes 1)\Delta(g) \in C_0(G \times G)$, and similarly $(1 \otimes f)\Delta(g) \in C_0(G \times G)$.

Notice also that the linear span of elements of the form $(f \otimes 1)\Delta(g)$ is dense in $C_0(G \times G)$. 

Multiplier algebras

For a C*-algebra $A$, we can regard $A$ as being a self-adjoint closed subalgebra of $B(H)$; or as $A$ being a subalgebra of its bidual $A^{**}$. If $A$ acts non-degenerately on $H$ (so $\text{lin}\{a(\xi) : a \in A, \xi \in H\}$ is dense in $H$) then the multiplier algebra of $A$ is

$$M(A) = \{ T \in B(H) : Ta, aT \in A \ (a \in A) \} = \{ x \in A^{**} : xa, ax \in A \ (a \in A) \}.$$ 

These are seen to be closed self-adjoint algebras containing $A$ as an ideal.

An abstract way to think of $M(A)$ is as the pairs of maps $(L, R)$ from $A$ to $A$ with $aL(b) = R(a)b$. A little closed graph argument shows that $L$ and $R$ are bounded, and that

$$L(ab) = L(a)b, \quad R(ab) = aR(b) \quad (a, b \in A).$$

The involution in this picture is $(L, R)^* = (R^*, L^*)$ where $R^*(a) = R(a^*)^*$, $L^*(a) = L(a^*)^*$. You can move between these pictures by a bounded approximate identity argument.

$M(A)$ is the largest C*-algebra containing $A$ as an essential ideal: if $x \in M(A)$ and $AXB = 0$ for all $a, b \in A$, then $x = 0$. 
We have that \( M(C_0(G)) = C^b(G) = \mathcal{C}(\beta G) \) (homework!)

Notice then that \( \Delta : C_0(G) \to M(C_0(G \times G)) \) (actually stronger, as \( (f \otimes 1)\Delta(g) \in C_0(G \times G) \).)

A useful topology to put on \( M(A) \) is the \textit{strict} topology:

\[
x_i \to x \iff x_i a \to xa, x_i^* a \to x^* a \quad (a \in A).
\]

**Theorem**

Let \( \theta : A \to M(B) \) be a \( \ast \)-homomorphism. Then the following are equivalent:

1. \( \theta \) is non-degenerate: \( \text{lin}\{\theta(a)b : a \in A, b \in B\} \) is dense in \( B \);
2. \( \theta(e_i) \to 1 \) strictly for some (or all) bai’s \( (e_i) \) in \( A \);
3. there is an extension \( \tilde{\theta} : M(A) \to M(B) \) which is unital, and strictly continuous on bounded sets.

Notice that \( \Delta \) is non-degenerate.
Group C*-algebras

We let $G$ act on $L^2(G)$ by the left-regular representation:

$$(\lambda(s)f)(t) = f(s^{-1}t) \quad (f \in L^2(G), s, t \in G).$$

The $s^{-1}$ arises to make $G \mapsto B(H); s \mapsto \lambda(s)$ a group homomorphism.

We can integrate this to get a contractive homomorphism

$\tilde{\lambda} : L^1(G) \to B(L^2(G))$. The action of $L^1(G)$ on $L^2(G)$ is just convolution.

Let the norm closure of $L^1(G)$ in $B(L^2(G))$ be $C^*_r(G)$, the (reduced) group C*-algebra. The weak-operator closure is $VN(G)$, the group von Neumann algebra. Equivalently, $VN(G)$ is $\{\lambda(s) : s \in G\}''$.

We claim that there is a normal, unital injective $\ast$-homomorphism

$\Delta : VN(G) \to VN(G \times G)$ satisfying

$$\Delta(\lambda(s)) = \lambda(s) \otimes \lambda(s) = \lambda(s, s).$$

Here we identify $VN(G) \overline{\otimes} VN(G)$ with $VN(G \times G)$. If $\Delta$ exists, then it's uniquely defined by this property.
Construction of $\Delta$

Define $\hat{\mathcal{W}} : L^2(G \times G) \to L^2(G \times G)$ by

$\hat{\mathcal{W}}\xi(s, t) = \xi(ts, t)$ \quad ($\xi \in L^2(G \times G), \xi, \eta \in G$).

Then $\hat{\mathcal{W}}$ is unitary, and

$$(\hat{\mathcal{W}}^*(1 \otimes \lambda(r))\hat{\mathcal{W}}\xi)(s, t) = ((1 \otimes \lambda(r))\hat{\mathcal{W}}\xi)(t^{-1}s, t)$$
$$= (\hat{\mathcal{W}}\xi)(t^{-1}s, r^{-1}t) = \xi(r^{-1}s, r^{-1}t)$$
$$= (\lambda(r) \otimes \lambda(r))\xi(s, t).$$

So we could define $\Delta$ by

$$\Delta(x) = \hat{\mathcal{W}}^*(1 \otimes x)\hat{\mathcal{W}} \quad (x \in \text{VN}(G)).$$

Then obviously $\Delta$ is an injective, unital, normal $\ast$-homomorphism, and

$\Delta(\lambda(s)) = \lambda(s) \otimes \lambda(s)$, so by normality, $\Delta$ must map into $\text{VN}(G \times G)$. 

At the $C^*$-algebra level

We expect that $\Delta$ should restrict to give a non-degenerate map

$$C^*_r(G) \to M(C^*_r(G \times G)).$$

This is indeed so:

- Notice that $\lambda(s)f \ast \xi = (s \cdot f) \ast \xi$ for $s \in G, f \in L^1(G), \xi \in L^2(G)$, where $(s \cdot f)(t) = f(s^{-1}t)$. So $\lambda(s)\bar{\lambda}(L^1(G)) \subseteq \bar{\lambda}(L^1(G))$.

- By density, $\lambda(s) \in M(C^*_r(G))$ for all $s$.

- So also $\lambda(s, s) \in M(C^*_r(G \times G))$, and we can integrate the map

  $$G \to M(C^*_r(G \times G)); s \mapsto \lambda(s, s)$$

  to get a homomorphism

  $$L^1(G) \to M(C^*_r(G \times G)).$$

  This “is” the map $\Delta$, so by density, we’re done.

Checking that $\Delta$ is non-degenerate is a touch more work:

- We can find “nice” bai’s in $L^1(G)$, and then $\Delta$ takes these to a strict bai in $M(C^*_r(G \times G))$;

- It’s not too hard to check that for $f \in L^1(G)$ and $h \in L^1(G \times G)$, we have

  $$\Delta(\bar{\lambda}_G(f))\bar{\lambda}_{G \times G}(h) = \bar{\lambda}_{G \times G}(g)$$

  for some $g \in L^1(G \times G)$. Then $\Delta$ is non-degenerate by density.
Back to $C_0(G)$

There is a unitary $W$ associated to $C_0(G)$ and $L^\infty(G)$, given by $W\xi(s, t) = \xi(s, s^{-1}t)$, and which satisfies

$$\Delta(x) = W^*(1 \otimes x)W \quad (x \in L^\infty(G) \text{ or } x \in C_0(G)).$$

The map $\tilde{\lambda} : L^1(G) \to C_r^*(G)$ is actually

$$\tilde{\lambda}(f) = (f \otimes \iota)W \quad (f \in L^1(G)).$$

This needs some explanation!

- Given $\xi, \eta \in L^2(G)$, we define $(\omega_{\xi, \eta} \otimes \iota)W \in B(L^2(G))$ by
  $$((\omega_{\xi, \eta} \otimes \iota)W \gamma | \delta) = (W(\xi \otimes \gamma) | \eta \otimes \delta) \quad (\gamma, \delta \in L^2(G)).$$

- We let $f = \xi \overline{\eta} \in L^1(G)$ (pointwise product). Then part of the claim is that $(\omega_{\xi, \eta} \otimes \iota)W$ depends only on $f$. 
Do the calculation!

\[(\omega_{\xi,\eta} \otimes \iota) W \gamma|\delta) = (W(\xi \otimes \gamma)|\eta \otimes \delta) = \int_{G \times G} \xi(s)\gamma(s^{-1}t)\overline{\eta(s)\delta(t)} \, ds \, dt = (f \ast \gamma|\delta),\]

Thus indeed \((\omega_{\xi,\eta} \otimes \iota) W = \tilde{\lambda}(f)\) where \(f = \xi\overline{\eta} \).

Similarly, we calculate \((\iota \otimes \omega_{\xi,\eta}) W:\)

\[((\iota \otimes \omega_{\xi,\eta}) W \gamma|\delta) = (W(\gamma \otimes \xi)|\delta \otimes \eta) = \int_{G \times G} \gamma(s)\xi(s^{-1}t)\overline{\delta(s)\eta(t)} \, ds \, dt = (f\gamma|\delta).\]

Here \(f \in C_0(G)\) is the map \(f(s) = \int_G \xi(s^{-1}t)\overline{\eta(t)} \, dt\). Such \(f\) are linearly dense in \(C_0(G)\).

So \(W\) determines \(\Delta\), the algebra \(C_0(G)\), and the map \(\tilde{\lambda}\). In this sense, \(W\) completely determines \(G\).
What happens for $VN(G)$?

Using the coproduct $\Delta$, we can turn the predual of $VN(G)$ into a Banach algebra. This is the Fourier algebra $A(G)$: for the moment, we just view this abstract as the predual of $VN(G)$.

Given $\xi, \eta \in L^2(G)$, let $\omega_{\xi,\eta} \in A(G)$ be the normal functional

$$VN(G) \to \mathbb{C}; \quad x \mapsto (x(\xi)|\eta).$$

As $VN(G)$ is in standard position (big von Neumann algebra machinery) on $L^2(G)$, it follows that actually every member of $A(G)$ takes this form.

Let’s try to define $\hat{\lambda}: A(G) \to C_0(G)$ by

$$\omega_{\xi,\eta} \mapsto (\omega_{\xi,\eta} \otimes \iota)\hat{W}.$$
The Fourier algebra $A(G)$

\[
((\omega_\xi,\eta \otimes \iota) \hat{W} \gamma | \delta) = \int_{G \times G} \xi(ts) \gamma(t) \overline{\eta(s) \delta(t)} \, ds \, dt = (f \gamma | \delta),
\]

where $f \in C_0(G)$ is the map $f(t) = \int_G \xi(ts) \overline{\eta(s)} \, ds = (\lambda(t^{-1}) \xi | \eta)$. As $\lambda$ is weak-operator continuous, it follows immediately that $f$ is continuous, and it's easy to see that actually $f \in C_0(G)$. So

\[
\hat{\lambda}(\omega_\xi,\eta) = (\omega_\xi,\eta \otimes \iota) \hat{W} = f, \quad f(t) = \langle \lambda(t^{-1}), \omega_\xi,\eta \rangle.
\]

As $\{\lambda(t^{-1}) : t \in G\}$ generates $VN(G)$, we see that $\hat{\lambda}$ is injective.

For $\omega_1, \omega_2 \in A(G)$, we have that

\[
\hat{\lambda}(\omega_1 \omega_2)(t) = \langle \lambda(t^{-1}), \omega_1 \omega_2 \rangle = \langle \Delta(\lambda(t^{-1})), \omega_1 \otimes \omega_2 \rangle
\]

\[
= \langle \lambda(t^{-1}) \otimes \lambda(t^{-1}), \omega_1 \otimes \omega_2 \rangle = \hat{\lambda}(\omega_1)(t) \hat{\lambda}(\omega_2)(t).
\]

Thus $\hat{\lambda}$ is a homomorphism. It is usual to identify $A(G)$ with it's image in $C_0(G)$; so $A(G)$ is a commutative Banach algebra, dense in $C_0(G)$ (and actually with spectrum $G$).
Finishing the duality picture

We perform a similar calculation:

\[
((\iota \otimes \omega_{\xi,\eta}) \hat{W} \gamma | \delta) = (\hat{W}(\gamma \otimes \xi) \mid \delta \otimes \eta) = \int_{G \times G} \gamma(ts)\xi(t)\overline{\delta(s)\eta(t)} \, ds \, dt
\]

\[
= \int_{G \times G} \gamma(t^{-1}s)\xi(t^{-1})\nabla(t^{-1})\overline{\delta(s)\eta(t^{-1})} \, ds \, dt
\]

\[
= \int_{G} \int_{G} \xi(t^{-1})\nabla(t^{-1})\overline{\eta(t^{-1})}\gamma(t^{-1}s) \, dt \, \overline{\delta(s)} \, ds
\]

\[
= (f \ast \gamma | \delta),
\]

where \( f \in L^1(G) \) is the function \( f(t) = \xi(t^{-1})\nabla(t^{-1})\overline{\eta(t^{-1})} \). Here \( \nabla \) is the group modular function.

So we have that operators of the form \((\iota \otimes \omega_{\xi,\eta}) \hat{W}\) are linearly dense in \( C^*_r(G) \).

Again, \( \hat{W} \) allows us to build the algebra \( C^*_r(G) \), the coproduct \( \Delta \) and the map \( \hat{\lambda} \).

In fact, \( \hat{W} = \sigma W^* \sigma \), where \( \sigma \in B(L^2(G \times G)) \) is the swap map \( \sigma \xi(s, t) = \xi(t, s) \).
Where does $W$ live?

We have been considering $W$ as a unitary in $B(L^2(G \times G))$; however, right slices of $W$ land in $C_0(G)$, and left slices in $C^*_r(G)$.

Set $H = L^2(G)$. The $C^*$-algebra $C_0(G) \otimes B_0(H)$ is the closure of the algebraic tensor product $C_0(G) \circ B_0(H)$ acting on $L^2(G \times G)$; here $B_0(H)$ is the compact operators on $H$. We can thus identify $M(C_0(G) \otimes B_0(H))$ with a subalgebra of $B(L^2(G \times G))$.

**Theorem**

*We can identify $C_0(G) \otimes B_0(H)$ with $C_0(G, B_0(H))$, the (norm) continuous functions $f : G \rightarrow B_0(H)$ which vanish at infinity. This identifies $a \otimes T$ with $f$ where $f(s) = a(s)T$.***

**Proof.**

Consider $L^2(G \times G)$ as the vector-valued $L^2(G, H)$. Then $C_0(G, B_0(H))$ acts on $L^2(G, H)$ pointwise: $(f_{\xi})(s) = f(s)\xi(s)$. It follows that $C_0(G, B_0(H))$ is isometrically represented on $L^2(G \times G)$, in a way compatible with the action of $C_0(G) \circ B_0(H)$. It remains to show that $C_0(G) \circ B_0(H)$ is dense in $C_0(G, B_0(H))$: this follows by a partition of unity argument.
Where does $W$ live? (continued)

Recall that we identified $M(C_0(G))$ with $C^b(G)$. The multiplier algebra of $B_0(H)$ is simply $B(H)$.

Similarly, we can identify $M(C_0(G, B_0(H)))$ with $C^b_{str}(G, B(H))$, the bounded functions $F : G \to B(H)$ which are strictly continuous. Such a function $F$ acts on $L^2(G, H)$ by $(F\xi)(s) = F(s)\xi(s)$.

Theorem

The operator $W$ is a member of $M(C_0(G) \otimes B_0(H)) = C^b_{str}(G, B(H))$.

Proof.

Under the identification $L^2(G \times G) = L^2(G, H)$, $W$ acts as $(W\xi)(s) = \lambda(s)\xi(s)$. Thus $W \in C^b_{str}(G, B(H))$ is the map $s \mapsto \lambda(s)$; we simply check that this is strictly continuous.
Where does $\hat{W}$ live?

**Theorem**

The operator $\hat{W}$ is a member of $M(C_r^*(G) \otimes B_0(H))$.

**Proof.**

Homework! (Actually, I know of no particularly nice proof).

**Theorem**

The operator $W$ is a member of $M(C_0(G) \otimes C_r^*(G))$.

**Proof.**

Again, $C_0(G) \otimes C_r^*(G) = C_0(G, C_r^*(G))$ and the multiplier algebra is $C^b_{str}(G, M(C_r^*(G)))$. Then check that $s \mapsto \lambda(s)$ does map into $M(C_r^*(G))$, and is strictly continuous.