Groups in stable and simple theories

Dugald Macpherson, School of Mathematics, University of Leeds, Leeds LS2 9JT,UK

April 7, 2010

These are sketch notes for my lecture on the MALOA Introductory Day of the meeting 'Geometric Model theory', March 25–28 2010, Oxford. They were written after the lecture, but have not been carefully polished. The bibliography is not carefully put together. For an old but wonderful introduction to stable group theory, see [13]. For an update, often with results stated in great generality, see [15]. On groups of finite Morley rank, [2] gives the state of play prior to serious intrusion of methods from the classification of finite simple groups, and [1] gives close-to-current information.

1 Motivation for stable group theory

Consider an algebraically closed field K, and a linear algebraic group $G(K) \leq \mathrm{GL}_n(K)$. This is a subgroup of $\mathrm{GL}_n(K)$ defined by the vanishing of some polynomials over K. Recall that the case when $G(K)$ is simple is classified – that is, the simple algebraic groups over K are classified, via their root systems and Dynkin diagrams (of types A_n, B_n, C_n, D_n, F_4, E_6, E_7, E_8, G_2). There is a more general notion of algebraic group $G(K)$– not necessarily linear – over K. This is obtained by glueing together affine varieties over K, with locally rational group operation. However, algebraic groups which are simple as groups have to be linear.

Any algebraically closed field is $\aleph_1$-categorical, and in fact is strongly minimal. Since any algebraic group $G(K)$ over an algebraically closed field K is definable in K, it follows that the group $G(K)$ itself has finite Morley rank.

These remarks motivate the following beautiful foundational conjecture, which emerged from considerations of Zilber in the 1970s.

Conjecture 1.1 (Cherlin-Zilber Conjecture/ Algebraicity conjecture) *Any infinite simple group of finite Morley rank is an algebraic group over an algebraically closed field.*

Remarks.

1. When tackling this conjecture, or more generally working with stable groups (or generalisations) we consider groups *possibly with extra structure*. Generally, the term *stable group* just means *group definable in a stable structure*, so there may be additional structure induced from the ambient structure.

2. By a result of Borovik and Poizat, the 2-Sylows in a group of finite Morley rank are conjugate. Furthermore, the connected component of a 2-Sylow has the form $B \times T$ where B is definable, connected, and nilpotent of bounded exponent, and T is divisible abelian. This suggests a case-division for the Algebraicity Conjecture into even type (where $T = 1$) mixed type (where B and T are non-trivial), odd type (where $B = 1$) and degenerate type (where $B = T = 1$). Even type should arise from an algebraic group over an algebraically closed field in even characteristic, odd type from odd or zero characteristic, and the other cases not exist. So far, as the culmination of very considerable work following ideas from the classification of finite simple groups, the even type case has been completed, and the mixed case eliminated [1]. There is considerable progress in odd type, but certain 'small' configurations are apparently hard to eliminate. The degenerate case appears inaccessible – there is no version of the Feit-Thompson Theorem in finite group theory. There appears now to be no great confidence in the truth of the overall conjecture, but the structure theory so far obtained is enlightening.

3. Any infinite simple group G of finite Morley rank has $\aleph_1$-categorical theory. This follows from the Zilber Indecomposability Theorem (see below).

5. The Algebraicity Conjecture would have followed from the Zilber Trichotomy Conjecture (in its strongest form) if the latter had been true.

6. Peterzil and Starchenko [10] proved an o-minimal version of the Zilber Trichotomy Conjecture is true. From this, Peterzil, Pillay and Starchenko [11, 12] proved the truth of an o-minimal version of the Algebraicity Conjecture, in the following form.

Theorem 1.2 (Peterzil, Pillay, Starchenko) *Let G be an infinite definably simple group definable in an o-minimal structure M. Then there is a definable real closed field k, and one of the following holds:*

(i) G is definably isomorphic to the $k(\sqrt{-1})$-rational points of a linear algebraic group defined over $k(\sqrt{-1})$;

(ii) G is definably isomorphic to the semialgebraic connected component of a group $\hat{G}(k)$, where $\hat{G}$ is an algebraic group defined over k.

7. Any group definable in an algebraically closed field can be definably given the structure of an algebraic group. This was proved by van den Dries in characteristic 0, and by Hrushovski in characteristic p, and is based on Weil's construction of an algebraic group from generic algebraic data.

8. By work of Zilber, any $\aleph_1$-categorical structure is built by a sequence of covers. Certain definable groups of permutations, which are either abelian of simple, play a key role (and have finite Morley rank, by definability).

This last remark is based on the notion of a *binding group*, introduced by Zilber and developed in other contexts by Hrushovski. For example, we have the following, taken from [13].

Theorem 1.3 *Let T be ω-stable, $M \models T$ be prime over $\emptyset$, and let P, Q be $\emptyset$-definable sets, with P Q-internal (that is, $P \subset \mathrm{dcl}(Q \cup F)$ for some finite F). Then the group of automorphisms of M which fix Q pointwise induces a definable group of automorphisms of P, called the* binding group *of P over Q.*

We also mention the following theorem of Hrushovski, proved in greater generality by Hrushovski, and proved earlier in the ω-categorical case by Zilber. It is another indication that the study of definable (so stable) groups is essential to the development of stability theory.

Theorem 1.4 *Let M be a locally modular non-trivial strongly minimal set. then there is an infinite group definable in M.*

In fact, Theorem 1.4 carries much additional information. There is an infinitely definable group G (which is an intersection of definable groups). This group has the structure of a vector space over the division ring of quasi-endomorphisms, and model theoretic independence in M is governed by linear independence in G.

Several important methods of group construction are based on the following theorem of Hrushovski, an abstract model-theoretic generalisation of Weil's result mentioned above.

Theorem 1.5 *In a structure with stable theory, let p be a complete stationary type, and $m(x, y)$ a partial definable function, and suppose*

(i) if $a, b \models p$ are independent, then $m(a, b) \models p$ and is independent from both a and b,

(ii) if $a, b, c \models p$ are independent, then $m(a, m(b, c)) = m(m(a, b), c)$.

Then there is a connected ∞-definable group G and a bijection mapping the set of realisations of p to the generic of G, taking m to group multiplication.

Often, this is applied in conjunction with Hrushovski's 'Group Configuration' arguments. We remark that, in a stable group (in fact, in some more general situations), any ∞-definable group is an intersection of definable groups. This is due to Hrushovski.

2 One-based groups

First, recall the following theorem of Baur and Monk. A *positive primitive* (p.p.) formula $\phi(\bar{x})$ is one of the form $\exists \bar{y} \psi(\bar{x}, \bar{y})$, where ψ is a conjunction of atomic formulas.

Theorem 2.1 *Let R be a ring, and let T be any complete theory of right R-modules (in the usual language of right R-modules, with a unary function symbol f_r for each $r \in R$, for scalar multiplication by r). Then every formula $\phi(\bar{x})$ is equivalent to a boolean combination of positive primitive formulas. In particular, if $M \models T$ then every definable subset of M is a boolean combination of cosets of p.p. definable subgroups of M.*

Since every abelian group may be viewed as a $\mathbb{Z}$-modules, we obtain

Corollary 2.2 *Every abelian group (as a structure in the language of groups) has stable theory.*

Recall that a stable theory T is *1-based* if any sets A and B are independent over $\mathrm{acl}^{eq}(A) \cap \mathrm{acl}^{eq}(B)$.

For example, any $\aleph_1$-categorical structure with a locally modular strongly minimal set has 1-based theory. Also, any (pure) abelian group has 1-based theory. In the other direction, we have

Theorem 2.3 (Hrushovski, Pillay [7]) *Any group definable in a stable 1-based theory is abelian-by-finite.*

Remark 2.4 We may talk also of a definable set in a theory being 1-based. If G is a definable group in a stable structure, then G is 1-based if and only if, for all n, every definable subset of G^n is a boolean combination of ($\emptyset$-definable subgroups of G^n.

3 Groups with tame theory

Though the Algebraicity Conjecture is stated under the very strong finite Morley rank assumption, I am not aware of any known example of a simple non-abelian group with 'tame' theory, other than ones closely related to the rational points of an algebraic group over a 'tame field'. I do not define 'tame' here. It could be taken to mean that the underlying theory is stable, or simple, or NIP, or rosy, or..., or possibly just that there is a quantifier elimination in a reasonable language.

More generally, I am aware of the following classes of groups with tame theories.

– Abelian-by-finite groups (by the Baur-Monk p.p. elimination)

– Mekler Constructions. Let Γ be a graph with vertex set $V\Gamma$. If p is an odd prime, we consider the group $G(\Gamma)$ which is free on the generating set $V\Gamma$ subject to being of nilpotency class 2 and exponent p, and subject to the condition that for any $u, v \in V\Gamma$, $[u, v] = 1$ if and only if u and v are adjacent in Γ. By results of Mekler [9], much of the model theory of Γ (e.g. stability, but not Morley rank or $\aleph_0$-categoricity) transfers to $G(\Gamma)$. See [6, Appendix A.3] for a further exposition.

– Baudisch groups. There are various versions. For example, Baudisch built a connected group of finite Morley rank and nilpotency class 2 which does not interpret an algebraically closed field, by a 'Hrushovski construction'.

– Rational points of an algebraic group over a tame field (possibly taking a semialgebraic connected component, the fixed points of some automorphism, the rational points over a valuation ring, or some such tweaking).

– Free groups, and more generally hyperbolic groups (by work of Sela).

4 The Baldwin-Saxl Theorem, and chain conditions

We say that a group G has the *descending chain condition* on subgroups with property P if there are no infinite strictly descending chains of subgroups having the property P.

Proposition 4.1 *Let G be an ω-stable group. Then G has the dcc on definable subgroups.*

The idea of the proof is that pairs (Morley rank, Morley degree) are lexicographically well-ordered. If $H_1 < H_2$ are definable subgroups of G, then if $|H_2 : H_1|$ is finite then they have the same Morley rank but H_1 has smaller Morley degree, and if the index is infinite, H_1 has smaller Morley rank.

The following generalisation is originally due to Baldwin and Saxl [3]. There is a full development of the ideas in [15].

Theorem 4.2 *Let G be a group definable in a stable theory T. Then G has icc, the uniform chain condition on intersections of uniformly definable subgroups: for any formula $\phi(x, \bar{y})$, there is $n_\phi \in \mathbb{N}$ such that any descending chain of intersections of ϕ-definable subgroups has length at most n_ϕ.*

Sketch Proof(a) Since G is stable, it is NIP (i.e. does not have the independence property). It follows that there is n such that every intersection of finitely many ϕ-definable subgroups is an intersection of at most n ϕ-definable subgroups.

Indeed, otherwise, for all m we can find $\{H_i = \phi(G, \bar{a}_i) : 1 \leq i \leq m\}$ such that the intersection is not a proper subintersection. Given such a family, for each i, choose $b_i \in (\bigcap_{j \neq i} H_j) \setminus H_i$. Now for each $I \subset \{1, \ldots, m\}$, $b_I := \Pi_{i \in I} b_i \in (\bigcap_{j \notin I} H_j) \setminus \bigcup_{j \in I} H_j$.

Thus, $\phi(b_I, a_i)$ holds if and only if $i \notin I$, contradicting NIP.

(b) The collection of intersections of n many ϕ-definable subgroups is a uniformly definable family, so chains of such intersections have bounded length (for example because the theory T does not have the strict order property).

Remark 4.3 It follows that if G is stable and $A \subset G$ then there is finite $A_0 \subseteq A$ such that $C_G(A) = C_G(A_0)$ – to see this, apply the theorem to the formula $\phi(x, y)$ which says $xy = yx$. In particular, we have dcc for centralisers, and centralisers are definable. Since the formula $xy = yx$ is quantifier-free, this is inherited by subgroups of stable groups. In particular, as is well-known, linear groups have dcc on centralisers, as they are subgroups of $\mathrm{GL}_n(K)$ for some algebraically closed field K and some n.

With a little work (induction on derived length or nilpotency class), one obtains

Corollary 4.4 *Let G be a group definable in a stable theory. Then*

(i) if $H < G$ is soluble of derived length d, then H lies in a soluble definable subgroup of G of derived length d.

(ii) if $H < G$ is nilpotent of class c then H is contained in a definable nilpotent subgroup of G of class c.

In fact, it can be shown [15] that if G is stable, then the collection of all nilpotent normal subgroups of G generates a nilpotent normal subgroup of G (the Fitting subgroup). This will be definable. It is not known if every stable group has a largest soluble normal subgroup.

The Baldwin-Saxl Theorem gives a good notion of 'connected component' in stable groups. Indeed, let G be a stable group. For each formula $\phi(x, \bar{y})$ there is a smallest subgroup of G of finite index defined by intersections of instances of ϕ. The intersection of these, as ϕ varies, will be an ∞-definable subgroup of G of index at most $2^{|T|}$, and is denoted G^o and called the *connected component* of G. In particular, G is *connected* if it has no proper definable subgroups of finite index. If G is saturated, then G/G^o is profinite. If G is ω-stable, then by Proposition 4.1, G^o is definable and $|G : G^o|$ is finite. In groups of finite Morley rank, this is the generalisation, first identified by Zilber, of the connected component of an algebraic group over an algebraically closed field.

In simple theories, all this breaks down. An example to bear in mind is an infinite extraspecial group of exponent p (where p is an odd prime). Here $P' = Z(P) \cong C_p$, and $P/Z(P)$ is elementary abelian. This group is ω-categorical and supersimple of SU-rank 1, but not stable (it has the independence property, witnessed by the formula $[x, y] = 1$). The centre $Z(P)$ is an intersection of centralisers of finite index in P.

5 Generic types

Let $\mathcal{U}$ be a saturated model of a stable theory T, and G be a group definable (over the parameter set A) in $\mathcal{U}$. We shall denote by $S(G)$ the set of all types over $\mathcal{U}$ which contain the formula $x \in G$. A formula $\phi(x, \bar{c})$ (which implies $x \in G$) is *left generic* (over A) if for all $g \in G$, $\phi(gx, \bar{c})$ does not fork over A. The formula is *right generic* if for all $g \in G$, $\phi(xg, \bar{c})$ does not fork over A.

Theorem 5.1 *In the above setting, let $X = \phi(G, \bar{c})$. Then $\phi(x, \bar{c})$ is left generic if and only if $\phi(x, \bar{c})$ is right generic if and only if there are $g_1, \ldots, g_k$ such that $G = g_1X \cup \ldots \cup g_kX$.*

Definition 5.2 *We say that the formula $\phi(x, \bar{c})$ is* generic *if any/all of the conclusions of Theorem 5.1 hold.*

A type $p \in S(G)$ is generic *if all formulas in p are generic.*

The above notions are relevant also to groups definable in simple or NIP theories, but some of the equivalences break down.

The group G acts on the elements of $S(G)$: $\phi(x, \bar{c}) \in gp$ if and only if $\phi(g^{-1}x, \bar{c}) \in p$. Let $p \in S(G)$ be a type of elements of G. Then we define $\mathrm{Stab}(p) := \{a \in G : ap = p\}$. By stability, the type p is definable; that is, for each formula $\phi(x, \bar{y})$ there is a formula $d\phi(\bar{y})$ which holds of $\bar{c}$ if and only if $\phi(x, \bar{c}) \in p$. The stabiliser of the ϕ-part of p is thus definable, and it follows that $\mathrm{Stab}(p)$ is an intersection of definable groups.

It can be shown that $p \in S(G)$ is generic if and only if $\mathrm{Stab}(p) = G^o$. Also, the set of all generic types of G is in 1-1 correspondence with G/G^o. In an ω-stable group G, a type $p \in S(G)$ is generic if and only if $RM(p) = RM(G)$.

In supersimple theories, some aspects of this theory go through. However, one defines the notion of stabiliser differently. Given $p \in S(G)$, define

$$\mathrm{St}(p) := \{a \in G : \exists x \models p \text{ such that } a \downarrow x \text{ and } ax \models p\}$$

and $\mathrm{Stab}(p) = \mathrm{St}(p).\mathrm{St}(p)$. Then both $\mathrm{St}(p)$ and $\mathrm{Stab}(p)$ are ∞-definable, and $\mathrm{Stab}(p)$ is a group.

6 Zilber Indecomposability Theorem

The following crucial theorem, due to Zilber, generalises a well-known result for algebraic groups. It has generalisations for superstable (even supersimple) groups. First, a definable subset A of a definable group G is *indecomposable* if for every definable $H < G$, either A lies in a single coset of H, or A intersects infinitely many cosets of H non-trivially.

Theorem 6.1 *Let G be a group of finite Morley rank. Let $\{A_i : i \in I\}$ be indecomposable subsets of G, each containing the identity. Then $H := \langle A_i : i \in I\rangle$ is definable and connected and equals a finite product $A_{i_1} \dots A_{i_m}$.*

Sketch proof. Find $B = A_{i_1} \dots A_{i_m}$ such that $\mathrm{RM}(B) = \mathrm{RM}(A_i B)$ for each i. Let p be a global type of maximal rank of elements of B. Put $H := \mathrm{Stab}(p)$. Note
(i) $A_i \subset H$ for all i, and
(ii) p is the unique generic type of H (so H is connected).

This theorem has many consequences, some of which follow formulaically, i.e. tend to carry through to other contexts with a version of the Indecomposability Theorem. For example:

– if G is a group of finite Morley rank which is definably simple (i.e has no proper non-trivial definable normal subgroup) then G is simple.

– if G is a group of finite Morley rank then its derived subgroup G' is definable. To see this, we may suppose G is connected. Observe that $G' = \langle (x^{-1})^G x : x \in X\rangle$, and that the set $(x^{-1})^G x$ are indecomposable and contain 1.

7 Groups of low Morley rank.

We briefly mention some early results due to Cherlin [4]. This was perhaps the first intrusion of ideas from finite group theory to groups of finite Morley rank. The rank one case is due to Reineke.

Theorem 7.1 *Let G be a connected group of finite Morley rank.*

(i) If $\mathrm{RM}(G) = 1$, then G is abelian, and is elementary abelian or torsion-free divisible.

(ii) If $\mathrm{RM}(G) = 2$, then G is soluble of derived length at most two.

(iii) If $\mathrm{RM}(G) = 3$ and G is simple and has a definable rank 2 subgroup then G is isomorphic to $\mathrm{PSL}_2(K)$ for some definable algebraically closed field K.

Related to this, there is the following important result of Hrushovski. It is crucial to understanding the interaction between two non-orthogonal minimal types, through the action of the binding group.

Theorem 7.2 *In a stable theory T, let G be a definable group acting definably and transitively on a strongly minimal set X. Then G is connected, and one of the following holds.*

(i) $\mathrm{RM}(G) = 1$ *and G acts regularly on X.*

(ii) $\mathrm{RM}(G) = 2$*, and* $G = \mathrm{AGL}_1(K)$ *for some definable algebraically closed field K, acting on the affine line (by maps $x \mapsto ax + b$).*

(iii) $\mathrm{RM}(G) = 3$*, and* $G = \mathrm{PSL}_2(K)$ *(K a definable algebraically closed field) acting on the projective line* $\mathrm{PG}_1(K)$*.*

These results have various generalisations. Both generalise to groups definable in o-minimal theories [8], and to pseudofinite supersimple groups of finite SU-rank [5], and Theorem 7.1 generalises to groups of low 'Cantor rank' (an analogue of Morley rank appropriate for working in a particular model, not necessarily saturated) [14].

References

[1] T. Altinel, A. Borovik, G. Cherlin, *Simple groups of finite Morley rank*, Mathematical Surveys and Monographs 146, Amer. Math. Soc., Providence, RI. 2007.

[2] A. Borovik, A. Nesin, *Groups of finite Morley rank*, Oxford Logic Guides 26, Oxford University Press, 1994.

[3] J. Baldwin, J. Saxl, 'Logical stability in group theory', J. Austral. Math. Soc. 21 (1976), 267–276.

[4] G. Cherlin, 'Groups of small Morley rank, Ann. Math. Logic 17 (1979), 1–28.

[5] R. Elwes, E. Jaligot, H.D. Macpherson, M.J. Ryten, 'Groups in supersimple and pseudofinite theories', preprint.

[6] W. Hodges, *Model theory*, Cambridge University Press, Cambridge, 1993.

[7] E. Hrushovski, A. Pillay, ‘Weakly normal groups’, in *Logic colloquium ’85* (Eds. C. Berline et al.), North-Holland, Amsterdam, 1987, pps. 233-244.

[8] H.D. Macpherson, A.J. Mosley, K. Tent, ‘Permutation groups in o-minimal structures’, J. London Math. Soc., (2) 62 (2000), 650–670.

[9] A. Mekler, ‘Stability of nilpotent groups of class 2 and prime exponent’, J. Symb. Logic 46 (1981), 781–788.

[10] Y. Peterzil, S. Starchenko, ‘A trichotomy theorem for o-minimal structures’, Proc. London Math. Soc. (3) 77 (1998), 481–523.

[11] Y. Peterzil, A. Pillay, S. Starchenko, ‘Definably simple groups in o-minimal structures’, Trans. Amer. Math. Soc. 352 (2000), 4397–4419.

[12] Y. Peterzil, A. Pillay, S. Starchenko, ‘Simple algebraic and semialgebraic groups over real closed fields’, Trans. Amer. Math. Soc. 352 (000), 4421–4450.

[13] B. Poizat, *Stable groups* (translated from the 1987 volume *Groupes stables*, Math. Surveys and Monographs 87, Amer. Math. Soc. Providence, RI, 2001.

[14] B. Poizat, ‘Groups of small Cantor rank’, J. Symb. Logic 75 (2010), 346–354.

[15] F. Wagner, *Stable Groups*, London Math. Soc. Lecture Notes 240, Cambridge University Press, Cambridge, 1997.