# Model Theory of Valued fields

Dugald Macpherson (University of Leeds)

May 7, 2008

## 1 Notation, basics

These notes focus mainly on the model theory of *algebraically closed* valued fields (loosely referred to as ACVF). This subject begins with work by A. Robinson in the 1950s (see the proof of model completeness of algebraically closed valued fields in [41]). Thus, it predates the major work of Ax-Kochen and Ershov around 1963; and, unlike the latter (and much subsequent work on quantifier elimination for henselian fields), the model theory of algebraically closed valued fields is also well understood in residue characteristic $p$.

The viewpoint given is motivated by ideas from stability and o-minimality. My main purpose is to make more accessible the background to some recent work of Hrushovski and co-authors – see for example [19], [20], [21]. The bulk of the material for these notes is from [14], [15]. There is considerable overlap with some notes of David Lippel and myself, for a similar tutorial series for the workshop 'An Introduction to Recent Applications of Model Theory' at the Newton Institute, Cambridge, March 2005.

The context *algebraically closed valued fields* may seem rather specific. However, there is a geometric viewpoint whereby an algebraically closed valued field is a universal domain for an arbitrary valued field (of appropriate characteristics). Model-theoretic results on ACVF will yield information on quantifier-free definable sets in any valued field.

### 1.1 Basic Notation

Initially, $\Gamma$ will denote an ordered abelian group, written additively, and $K$ will denote a valued field with valuation $v : K \to \Gamma \cup \{\infty\}$ (usually assumed surjective, when we call $\Gamma$ the *value group of* $(K, v)$). The symbol $\infty$ is a notational convenience, to ensure $v(0)$ is defined. Later, we will be assuming that $K$ is algebraically closed, and we will move towards a viewpoint whereby $K$ actually denotes the field *sort* in a certain language.

The valuation axioms are as follows (for $x, y \in K$):
(i) $v(x) = \infty \Leftrightarrow x = 0$;
(ii) $v(xy) = v(x) + v(y)$
(iii) $v(x + y) \geq \text{Min}\{v(x), v(y)\}$.
These have the following easy consequences.

**Lemma 1.1.1** *(i)* $v(1) = 0$.
*(ii)* $v(x) = v(-x)$.
*(iii) if* $v(x) < v(y)$ *then* $v(x + y) = v(x)$.

*Note.* In the literature, terminology around valued fields in inconsistent. For example, in [40], there is a background assumption that $\Gamma$ is archimedean (i.e. has no proper convex subgroups), and the more general notion described above is called a *Krull valuation* (see Chapter 13 of [40]). Other sources define a notion of rank on $\Gamma$ based on the convex subgroups, so an infinite value group with no proper convex subgroups has rank one, and the valuation is called a *rank one valuation*. If $\Gamma$ is archimedean (i.e. rank one), then the valuation yields an *absolute value* $| - | : K \to \mathbb{R}^{\geq 0}$: indeed, as $\Gamma$ is archimedean we may assume $(\Gamma, <, +) \leq (\mathbb{R}, <, +)$, and for any fixed $\gamma \in \Gamma$ with $0 < \gamma < 1$ we obtain an absolute value (also sometimes called a valuation), putting $|x|_\gamma = \gamma^{v(x)}$. The axioms above then become:
$(i)'$ $|x| = 0 \Leftrightarrow x = 0$;
$(ii)'$ $|xy| = |x|.|y|$
$(iii)'$ $|x + y| \leq \text{Max}\{|x|, |y|\}$.

The convention of [14],[15] is to use the language of absolute values rather than valuations. To confuse matters further, the absolute value arising from an archimedean valuation is called a 'non-archimedean' absolute value, basically because of the 'ultrametric' form of the triangle inequality, which says that all triangles are isosceles.

Often, I will just refer to the 'valued field $(K, v, \Gamma)$', but this notation will mean that $v$ is surjective to $\Gamma$. Write $R := \{x \in K : v(x) \geq 0\}$. This is a local ring, with unique maximal ideal $\mathcal{M} := \{x \in K : v(x) > 0\}$. The *residue field* of $(K, v, \Gamma)$ is $k := R/\mathcal{M}$. Note that $(\text{char}(K), \text{char}(k)) \in \{(0, 0), (0, p), (p, p)\}$. If $U$ is the group of units of $R$, then $U$ is the kernel of the valuation map (regarded as a homomorphism $(K^*, .) \to (\Gamma, +)$, so we may identify $\Gamma$ with $K^*/U$. We have $xU(R) < yU(R)$ if there is $z \in R$ with $xzU(R) = yU(R)$.

Later on, we will be using these symbols $K, R, \Gamma, \mathcal{M}, k$ specifically when $K$ is algebraically closed.

## 1.2 Valued field extensions, henselian fields, completions

The following theorem is basically Chevalley's Place Extension Theorem.

**Theorem 1.2.1** *Let* $v : K \to \Gamma$ *be a valuation, and suppose* $L > K$ *is an extension of fields. Then there is a value group* $\Delta \geq \Gamma$ *and a valuation* $w : L \to \Delta$ *extending* $v$.

*Sketch Proof, based on [26].*

Write $R_K, \mathcal{M}_K$ for the valuation ring and maximal ideal of $K$. Put $k := R_K/\mathcal{M}_K$, and let $\tilde{k}$ denote its algebraic closure. Let $\phi_K$ be the residue map $R_K \to R_K/\mathcal{M}_K$. The key fact [26, Corollary 3.3 of IX] is that if $R_L$ is a maximal subring of $L$ such that $\phi_K$ extends to a homomorphism $\phi_L : R_L \to \tilde{k}$, then $R_L$ is a local ring such that if $x \in L \setminus \{0\}$ then $x \in R_L$ or $x^{-1} \in R_L$.

Now let $\mathcal{M}_L$ be the maximal ideal of $R_L$. Then $\mathcal{M}_L \cap R_K = \mathcal{M}_K$ (as $1 \notin \mathcal{M}_L \cap R_K \supset \mathcal{M}_K$). It follows that if $U_K$, $U_L$ denote the groups of units of $R_K, R_L$ respectively then $U_L \cap K = U_K$. Hence we have a canonical embedding $K^*/U_K \to L^*/U_L$. This gives a valuation on $L$ – the value group is just $L^*/U_L$.

**Definition 1.2.2** (i) A valued field $(K, v, \Gamma)$ is *henselian* if the valuation $v$ extends *uniquely* to $\widetilde{K}$.

(ii) A *henselisation* of a valued field $(K, v)$ is a valued field extension $(K^h, v^h)$ such that (a) $[K^h : K]$ is an algebraic field extension, (b) $(K^h, v^h)$ is henselian, and (c) for any other $(K', v')$ satisfying (a) and (b) there is a unique $K$-isomorphism $\rho : K^h \to K'$ such that $v' \circ \rho = v^h$.

Any two henselisations are isomorphic via a unique isomorphism over $K$. To obtain a henselisation, extend $(K, v)$ to $(K^s, v^s)$, where $K^s$ is the separable closure of $K$. Let $G := \{\sigma \in \mathrm{Aut}(K^s/K) : v^s \circ \sigma = v^s\}$. Then $\mathrm{Fix}(G)$, with the restriction of $v^s$, is a henselisation.

Observe that any algebraic extension of a henselian field is henselian.

There are many equivalent characterisations of henselian fields. We give some next. Below, if $a \in R$ then $\bar{a}$ denotes $\mathrm{res}(a)$, and we write $\bar{f}$ for the reduction of $f$ mod $\mathcal{M}$, where $f \in R[X]$.

**Theorem 1.2.3** *Let $(K, v, \Gamma)$ be a valued field. Then the following are equivalent.*

*(i) $(K, v, \Gamma)$ is henselian.*

*(ii) If $f \in R[X]$ is monic, and $\bar{f}$ has a simple root $\bar{b} \in k$, then $f$ has a root $a \in R$ with $\bar{a} = \bar{b}$.*

*(iii) If $f \in R[X]$ is monic, and $b \in R$ with $v(f(b)) > 0$ and $v(f'(b)) = 0$, then $f$ has a root $a \in R$ with $\bar{a} = \bar{b}$.*

If $(K, v) < (L, v)$ is an extension of valued fields then there are natural embeddings of the value group of $K$ into that of $L$, and likewise for the residue fields. We say that the extension is *immediate* if $K$ and $L$ have the same value groups and residue fields, i.e. if these embeddings are surjective.

There is a general notion of *complete* in the context of valued fields, which we shall not discuss. We mainly work with the notion of *maximally complete* valued field due to F.K. Schmidt and Krull [25]; that is, a valued field with no proper immediate extensions.

**Theorem 1.2.4** *Every valued field has an immediate maximally complete extension.*

An influential theory of maximally complete field extensions was developed by Kaplansky in [23, 24]. We mention that a valued field is maximally complete if and only if it is *spherically complete*; that is, if the intersection of any chain of non-empty balls, ordered by inclusion, is non-empty; see Definition 2.0.6 for a definition of *balls*. Any maximally complete field is henselian (indeed, any valued field with no immediate algebraic extensions is henselian). The power series fields $k((t^\Gamma))$ constructed in Example 1.3.2(2) below are maximally complete. For valued fields with value group $\mathbb{Z}$, 'complete' and 'maximally complete' coincide.

## 1.3 Algebraically closed valued fields

**Lemma 1.3.1** *Let $(K, v, \Gamma)$ be an algebraically closed valued field. Then*
*(i) $\Gamma$ is divisible,*
*(ii) $k$ is algebraically closed.*

*Proof.* (i) Let $\gamma \in \Gamma$, with $\gamma = v(x)$, say, and let $n \in \mathbb{N}^{>0}$. As $K$ is algebraically closed there is $y \in K$ with $y^n = x$. Then $nv(y) = \gamma$.
(ii) Exercise.

**Example 1.3.2 (Examples of algebraically closed valued fields)** (1) Start with $\mathbb{Q}_p$, and extend the valuation to the algebraic closure of $\mathbb{Q}_p$, or its completion, usually denoted $\mathbb{C}_p$. This completion $\mathbb{C}_p$ is also algebraically closed.

For ease of reference, we recall here the construction of $\mathbb{Q}_p$. Fix a prime $p$, and define a valuation $v_p : \mathbb{Q} \to \mathbb{Z} \cup \{\infty\}$: put $v_p(a/b) = s - t$, where $a = p^s m$ and $b = p^t n$, where $(p, m) = (p, n) = 1$. The residue field is the finite field $\mathbb{F}_p$. Convert this valuation to an absolute value $| - |_p : \mathbb{Q} \to \mathbb{R}^{\geq 0}$, putting $|x| = p^{-v_p(x)}$. Now $\mathbb{Q}_p$ is the completion of $\mathbb{Q}$ with respect to this absolute value – that is, the ring of Cauchy sequences modulo the maximal ideal of null sequences. The absolute value on $\mathbb{Q}$ extends to one on $\mathbb{Q}_p$, and this yields a valuation map on $\mathbb{Q}_p$, again with value group $\mathbb{Z}$, and residue field $\mathbb{F}_p$.

The valuation ring of $\mathbb{Q}_p$ is usually denoted $\mathbb{Z}_p$, the ring of *p-adic integers*. As an alternative construction, define $\mathbb{Z}_p$ to be an inverse limit of rings $\mathbb{Z}/p^n\mathbb{Z}$, and $\mathbb{Q}_p$ to be its fraction field.

It is also possible to construct $\mathbb{Q}_p$ as a set of power series of the form $\Sigma_{n \in \mathbb{Z}, n \geq n_0} a_n p^n$ where $n_0 \in \mathbb{Z}$, $a_n \in \mathbb{F}_p$ and we treat $p$ in calculations as a number, not an indeterminate. This is essentially the 'Witt vector' construction, over $\mathbb{F}_p$. The general construction is described in [40], and [26].

The algebraic closure $\tilde{\mathbb{Q}}_p$, and likewise its completion $\mathbb{C}_p$, will have value group $(\mathbb{Q}, <, +)$ and residue field $\tilde{\mathbb{F}}_p$.

(2) We sketch the 'Hahn product' construction. Fix any field $k$, and any non-trivial ordered abelian group $\Gamma$, and let $t$ be an indeterminate. If $f$ is an expression $\Sigma_{\gamma \in \Gamma} a_\gamma t^\gamma$ (where $a_\gamma \in k$), then $\mathrm{supp}(f) := \{\gamma \in \Gamma : a_\gamma \neq 0\}$. Now put

$$K = k((t^\Gamma)) := \{f = \Sigma_{\gamma \in \Gamma} a_\gamma t^\gamma : \mathrm{supp}(f) \text{ is well-ordered}\}.$$

Addition and multiplication are defined as for power series – the 'well-order' assumption ensures the definition of multiplication makes sense. Now $K$ is a field (Exercise: prove it; if $f = \Sigma a_\gamma t^\gamma \in K$ and $\gamma_0 := \mathrm{Min}(\mathrm{supp}(f))$, write $f = (a_{\gamma_0} t^{\gamma_0})(1 + h)$ where $\mathrm{supp}(h) \subset \{\gamma \in \Gamma : \gamma > 0\}$, and deduce that $f$ is invertible). The map $v : K \to \Gamma$ given by $v(f) = \mathrm{Min}(\mathrm{supp}(f))$ is a surjective valuation. The residue field if $k$. Clearly, $\mathrm{char}(K) = \mathrm{char}(k)$. As noted at the end of Section 1.2, power series fields $k((t^\Gamma))$ are maximally complete, so henselian.

It can be checked that if $\Gamma$ is divisible and $k$ is algebraically closed, then $K$ is also algebraically closed. (Exercise: prove this, using that $K$ is maximally complete.)

If $k = \mathbb{C}$, the field of *Puiseux series* is the subfield of $k((t^{\mathbb{Q}}))$ consisting of the union of fields $k((t^{\mathbb{Z}/n!}))$, where $\mathbb{Z}/n!$ denotes the group of rationals $a/n!$ where $a \in \mathbb{Z}$. This field of Puiseux series is an algebraic extension of $k((t))$ which is only algebraically closed in characteristic 0 (in characteristic $p$, consider for instance the series $\sum_{i=1}^{\infty} t^{-p^{-i}}$ in $k((t^{\mathbb{Q}}))$ which satisfies the equation $x^p - x - t^{-1} = 0$).

## 1.4 Quantifier elimination in valued fields

Model-theoretically, there are various languages appropriate for valued fields. If $(K, v, \Gamma)$ is a valued field, then the apparatus of definable sets can be expressed just using the ring language augmented by a predicate for the valuation ring $R$. For then, the group $U(R)$ of units of $R$ is definable, and $\Gamma$ may be identified with $K^*/U(R)$, with $v$ the natural homomorphism $K^* \to \Gamma$, and with $xU(R) < yU(R)$ if there is $z \in R$ with $xzU(R) = yU(R)$.

We shall denote by $L_v$ the language $(+, \times, -, 0, 1)$ of rings augmented by a binary relation symbol interpreted as $v(x) \leq v(y)$. From this we may define $R := \{x : v(1) \leq v(x)\}$, and hence everything else. The language $L_\Gamma$ is two-sorted, with the language of rings on $K$, the language $(<, +, -, 0)$ on $\Gamma$, and a function symbol $v : K \to \Gamma$. By $L_{k,\Gamma}$ we mean a 3-sorted language with sorts $K, k, \Gamma$. It includes the symbols of $L_\Gamma$, but also has the language of rings on $k$ and a function symbol $\mathrm{Res} : K^2 \to k$, with $\mathrm{Res}(x, y) = \mathrm{res}(xy^{-1})$ if $v(x) \geq v(y)$, and $\mathrm{res}(x, y) = 0$ otherwise.

The following is essentially due to Robinson [41].

**Theorem 1.4.1** *(i) In $L_v$, a complete theory is axiomatised by sentences which express (a) $K$ is algebraically closed, (b) there are $x, y \in K^*$ such that $v(x) < v(y)$, and (c) the values of* $\mathrm{char}(k)$, $\mathrm{char}(K)$.

*(ii) Algebraically closed valued fields have quantifier elimination in each of the languages $L_v$, $L_\Gamma$, $L_{k,\Gamma}$.*

A key point in the proof is that, given a valued field $F(, v)$, the extensions of $v$ to a finite normal extension are all conjugate under the Galois group.

We shall refer to the complete theory described in Theorem 1.4.1(i) (in any of the three languages, or sometimes in other languages) as ACVF.

We mention some more general results on quantifier elimination. First, if $(K, v, \Gamma)$ is a valued field, an *angular component map modulo $\mathcal{M}$* is a map $\overline{\mathrm{ac}}$ : $K \to k$ such that (a) $\overline{\mathrm{ac}}(0) = 0$, (b) $\overline{\mathrm{ac}}$ induces a multiplicative homomorphism $K^* \to k^*$, and (c) $\overline{\mathrm{ac}}$ restricted to the group $U$ of units of $R$ is the residue map.

Not all valued fields have an angular component, but it exists if the valued field has a cross section (a group homomorphism $\pi : \Gamma \to K^*$ such that $v \circ \pi = \mathrm{id}_\Gamma$), or if the residue field is $\aleph_1$-saturated [36]. For $\mathbb{Q}_p$ the natural angular component map is $\overline{\mathrm{ac}}(x) = \mathrm{res}(p - v(x).x)$. For a power series field $K = k((t^\Gamma)$, define $\overline{\mathrm{ac}}$ so that if $f = \Sigma_{\gamma > \gamma_0} a_\gamma t^\gamma$ with $a_{\gamma_0} \neq 0$, then $\overline{\mathrm{ac}}(f) = a_{\gamma_0}$.

In general, if there is an angular component it is not definable in the language $L_{k,\Gamma}$, though it is so definable in the $p$-adic case (see [36]).

The following theorem is due to Pas [35].

**Theorem 1.4.2 ([35])** *Let $T$ be the theory of henselian valued fields of residue characteristic 0 with angular component modulo $\mathcal{M}$, in the language $L_{k,\Gamma}$ expanded by the angular component map. Then $T$ has elimination of field quantifiers.*

There is also such a quantifier elimination, due to Delon (unpublished), in a conservative expansion of $L_{v,\Gamma}$.

Finally, we mention the sort $RV$, which binds together the value group and residue field into one structure. Define $RV^*(K) = K^*/(1 + \mathcal{M})$, and let $RV = RV^* \cup \{0\}$. There is an exact sequence

$$0 \to k^* \to RV^* \to \Gamma \to 0,$$

as $1 + \mathcal{M} < U(R) < K^*$ and $K^*/U(R) \cong \Gamma$. A major goal of [19] is, roughly, to set up a homomorphism, at the level of Grothendieck groups of definable sets, from $RV$ to $K$, and to identify the kernel.

A quantifier elimination result for henselian characteristic zero valued fields, using $RV$ and related sorts, is given in [8]. It is related to quantifier elimination results of [3] and [43]. In each case, *valued field* quantifiers are eliminated.

It appears that the RV sort was first introduced into the model theory literature by Basarab [2].

# 2 Consequences of quantifier elimination for ACVF

Throughout these notes, 'definable' will mean 'definable with parameters'.

**Definition 2.0.3** Let $T$ be a complete theory, $M \models T$, $D$ an $\emptyset$-definable set in $M^{\mathrm{eq}}$. We say $D$ is *stably embedded* if, for any $t$, any definable subset of $D^t$ is definable (uniformly in parameters) with parameters from $D$.

**Remark 2.0.4** Of course, if $M$ is sufficiently saturated, we may omit 'uniformly in parameters'. Various equivalents to this definition are mentioned in the Appendix of [6].

**Proposition 2.0.5** *In ACVF, we have the following.*

*(i) $\Gamma$ is stably embedded.*

*(ii) $k$ is stably embedded.*

*(iii) $k$ is a 'pure' algebraically closed field: any subset of $k^n$ which is $\emptyset$-definable is $\emptyset$-definable in $(k, +, \times)$.*

*(iv) $\Gamma$ is a pure ordered abelian group, possibly (in the case of $(0,p)$ characteristics), expanded by constants.*

*(v) $k$ is strongly minimal, and $\Gamma$ is o-minimal.*

*Proof.* The assertions about $\Gamma$ follow from QE in $L_\Gamma$, and those about the residue field follow from QE in $L_{k,\Gamma}$.

**Definition 2.0.6** In any valued field $(K, v, \Gamma)$, an *open ball* is a non-empty set $B_{>\gamma}(a) = \{x \in K : v(x-a) > \gamma\}$. A *closed ball* is a set $B_{\geq\gamma}(a) := \{x \in K : v(x-a) \geq \gamma\}$. Here, $a \in K$, and $\gamma \in \Gamma \cup \{\infty\}$ (so a singleton of $K$ is a closed ball).

Observe (Exercise) that if $B_1, B_2$ are balls, then either they are disjoint, or one contains the other. We shall sometimes view balls as subsets of $K$ (in which case, the meaning changes under taking elementary extensions) and sometimes as elements of $K^{\mathrm{eq}}$: indeed, note that the collection of all closed balls is identifiable with a sort of $K^{\mathrm{eq}}$, as is the collection of all open balls

We write $B_\gamma(a)$ if we do not wish to specify whether the ball is open or closed. Observe that $B_\gamma(0)$ is an $R$-submodule of $K$. Its cosets in $K$ all have form $B_\gamma(a)$.

There is a topology $\mathcal{T}$ on $K$, with $\mathcal{T}_{\geq} := \{B_{\geq\gamma}(a) : a \in K, \gamma \in \Gamma\}$ as a basis of open sets. In fact, $\mathcal{T}_{>} := \{B_{>\gamma}(a) : a \in K, \gamma \in \Gamma\}$ is also a basis of open sets of the same topology. If $\Gamma$ is discretely ordered, this is clear, since in this case an open ball $B_{>\gamma}(a)$ is also a closed ball. If $\Gamma$ is densely ordered, we have $B_{>\gamma}(a) = \bigcup_{\delta>\gamma} B_{\geq\delta}(a)$, so $\mathcal{T}_{>} \supseteq \mathcal{T}_{\geq}$, and $B_{\geq\gamma}(a) = \bigcup_{x \in B_{\geq\gamma}(a)} B_{>\gamma}(x)$, so $\mathcal{T}_{\geq} \supseteq \mathcal{T}_{>}$. In fact, both kinds of (non-singleton) balls are clopen in this topology: for example, $B := B_{\geq\gamma}(a) = K \setminus \bigcup_{\delta\in\Gamma, x \notin B} B_{>\delta}(x)$.

The field operations preserve this topology: that is, $K$ is a topological field. The fields $\mathbb{Q}_p$ and $\mathbb{F}_p((t))$ are *locally compact* in this topology: their valuation rings (respectively, $\mathbb{Z}_p$ and $\mathbb{F}_p[[t]]$) are compact. The corresponding topology on a model of ACVF is never locally compact.

Quantifier-elimination gives the following convenient description of *one-variable* definable sets.

**Proposition 2.0.7** *Let $(K, v, \Gamma) \models ACVF$. Then any definable subset of $K$ is a finite boolean combination of balls.*

*Proof.* It suffices to show that any atomic formula defines a finite boolean combination of balls. One must consider formulas $v(\Sigma(a_i x^i) = v(\Sigma b_i x^i)$ or $v(\Sigma(a_i x^i) = v(\Sigma b_i x^i)$ where $a_i, b_i \in K$. Since $K$ is algebraically closed, polynomials factorise completely, so these formulas reduce to the form $\Sigma v(x-\alpha_i)+\gamma = \Sigma v(x-\beta_i)$, or $\Sigma v(x-\alpha_i)+\gamma < \Sigma v(x-\beta_i)$, respectively. We leave it as an exercise to show that any such set defines a finite union of 'Swiss cheeses' (see Holly [17]), and hence that in fact any definable subsets of $K$ is a finite union of Swiss cheeses; a Swiss cheese is essentially a ball with finitely many sub-balls removed. In fact, Holly shows that there is a *canonical* such expression.

*Notation.* In these notes, $a$ will denote a tuple, possibly of length greater than one. We tend to write $AB$ for $A \cup B$, $Ab$ for $A \cup \{b_1, \ldots, b_n\}$ where $b = (b_1, \ldots, b_n)$. We write $\mathrm{dcl}_\Gamma(A)$ for $\mathrm{dcl}(A) \cap \Gamma$ (and likewise for acl). If $\mathrm{tp}(a/C) = \mathrm{tp}(a'/C)$, we write $a \equiv_C a'$.

## 2.1 Notions of Minimality

Recall that a complete theory $T$ is strongly minimal if every definable subset of every $M \models T$ is finite or cofinite; that is, it is quantifier-free definable in the language just with equality. And a theory $T$ of totally ordered structures is *o-minimal* if every definable subset of $M$ is a finite union of singletons and open intervals (with endpoints in $M \cup \{\pm\infty\}$), that is, every such set is definable quantifier-free just from the order relation. (In fact, because of very nice definable connectedness properties of o-minimal structures, one only needs to assume this for one model of $T$, but this is very special for o-minimality.)

This suggests a framework for other minimality definitions. Suppose $L \subset L^+$ are languages, and $T$ is a complete $L^+$-theory. We say $T$ is $L$-minimal if, for every $M \models T$, every definable subset of $M$ (i.e. a one variable definable set) is definable by a quantifier-free $L$-formula. Such definitions have potential provided (i) there is a reasonably rich supply of examples, and (ii) some theory can be developed, as in the o-minimal and strongly minimal cases.

Some initial observations on such notions are made in [32]. Cluckers and Loeser [8] have developed a notion of *b-minimality* intended as a similar and rather flexible framework. Cluckers will discuss this at this meeting.

Proposition 2.0.7 suggests such a notion of minimality appropriate for ACVF. The intuition is that, like disjoint subsets of a pure set, or disjoint intervals in a totally ordered set, at the combinatorial level there is very little interaction between disjoint balls in a valued field; indeed, given any two balls, either they are disjoint or one contains the other, and this suggests that valued fields have a treelike structure in which the balls determine the combinatorics. This leads to various notions of *$C$-minimal theory* appropriate for algebraically closed valued fields. The idea is that a complete theory of valued fields (possibly with extra structure) should be $C$-minimal if, for every $M \models T$, every 1-variable definable subset of the field sort of $M$ is a boolean combination of balls.

A version of $C$-minimality was introduced in [32] and developed in [12]. We give below the slightly less general definition, from [19].

**Definition 2.1.1 ([19])** Let $L$ be a language with two sorts $VF$ and $\Gamma_\infty$. Assume that the sort $\Gamma$ has a constant symbol $\infty$ and a binary relation symbol $<$, and there is a function symbol val : $VF^2 \to \Gamma_\infty$. Informally, write $\Gamma$ for $\Gamma_\infty \setminus \{\infty\}$. Let the theory $T_{um}$ state

(i) $\Gamma_\infty$ is a dense linear ordering with no least element, and with a greatest element $\infty$;

(ii) $\mathrm{val}(x, y) = \infty \Leftrightarrow x = y$.

(iii) for each $\alpha \in \Gamma_\infty$, the relation $\mathrm{val}(x, y) \geq \alpha$ is an equivalence relation, whose classes are called *closed $\alpha$-balls*, and, for $\alpha \in \Gamma$, $\mathrm{val}(x, y) > \alpha$ is also an equivalence relation, with classes called *open $\alpha$-balls*.

(iv) If $\alpha \in \Gamma$ then every closed $\alpha$-ball is a union of infinitely many open $\alpha$-balls.

A complete theory $T^+$ over a language $L^+ \supseteq L$ is called *$C$-minimal* if, for every $M \models T^+$, every definable subset of the $VF$ sort of $M$ is a finite boolean combination of balls.

The idea here is that $T_{um}$ is a theory of *ultrametric spaces*; see Poizat [37] for another presentation of such a theory. In the intended interpretation, $C$-minimal expansions of ACVF, $VF$ is the field sort and $val(x, y) = v(x - y)$.

Under the slightly broader definition from [32] and [12], the base language $L$ has a single ternary relation $C$. The idea is that $C$ should be the natural ternary relation on the set of maximal chains in a tree; here 'tree' is used in the sense of partial orders – any two elements have a common lower bound, and the set of lower bounds of an element is linearly ordered. So $C(x, y, z)$ holds if and only if $y \cap x \subset y \cap z$, where we view $x, y, z$ as sets of nodes of the tree. One can write down finitely many axioms about a structure $(M, C)$ which guarantee that there is an interpretable tree, and that the underlying structure consists of a dense set of maximal chains in this tree, with $C$ interpreted naturally. Under this definition, there is no ultrametric structure, as there is no notion of two nodes of the tree having the same *level*. But there is a reasonable theory of $C$-minimality in this sense: a local monotonicity theorem for definable functions, and a very rudimentary cell decomposition theorem. There are notions of $C$-group and $C$-field, where we assume compatibility of the operations with the $C$-relation. Simonetta (e.g. [45]) has developed a theory of $C$-minimal groups, and is is shown in [12] that every $C$-minimal field is an algebraically closed valued field. Fares Maalouf [30] has made considerable progress towards a $C$-minimal Zilber Trichotomy Theorem, under an assumption that algebraic closure has the exchange property.

We summarise below results on ACVF and $C$-minimality; they hold both under the broader notion from [12], and that of Hrushovski and Kazhdan [19].

**Theorem 2.1.2** *(i) [32] ACVF is $C$-minimal.*

*(ii) [12] Every $C$-minimal valued field is algebraically closed.*

*(iii) [28] The 'rigid analytic' expansions of ACVF introduced by Lipshitz [27] and investigated by Lipshitz and Robinson are $C$-minimal.*

We do not give details of the rigid analytic expansions. However, suppose that $K = \mathbb{C}_p$ (or another complete algebraically closed valued field with archimedean value group) and $f = \Sigma a_\nu T^\nu$ is a power series over the valuation ring $R$ of $\mathbb{C}_p$, where $\nu = (\nu_1, \ldots, \nu_n)$ is a multi-index, and assume $v(a_\nu) \to \infty$ as $|\nu_1 + \ldots + \nu_n| \to \infty$. Then $f$ defines a function $R^n \to R$, and we may view this as a function $K^n \to R$ taking value zero on $K^n \setminus R^n$. Thus, we may expand $K$ by function symbols for all such $f$ (over all $n$). The results of [28] ensure that such an expansion is $C$-minimal; in fact, these results apply to a slightly richer, but more complicated, language.

There are other similar notions of minimality. First, in [13] there is a definition of *$P$-minimal valued field*. It is shown in [11] that the expansion of $\mathbb{Q}_p$ by restricted analytic functions, in the sense of [9] is $P$-minimal.

Second, we mention the theory RCVF. Let $R$ be a non-archimedean real closed field (e.g. a non-principal ultrapower of $\mathbb{R}$), and let $V$ be the ring of finite elements, that is $V := \{x \in R : |x| \leq n \text{ for some } n \in \mathbb{N}\}$. Then $V$ is a local ring with as the unique maximal ideal the set of infinitesimals, namely

$\mathcal{M} := \{x \in V : |x| < \frac{1}{n}$ for all $n \in \mathbb{N}^{>0}\}$. Put $\Gamma := R^*/U$, where $R^*$ is the multiplicative group of $R$ and $U$ is the group of units of $V$. Then $\Gamma$ inherits a total order, and if $v : R \to \Gamma \cup \{\infty\}$ is the natural map, then $(R, v, \Gamma)$ is a valued field. The subring $V$ is convex in $R$, and the theory of real closed fields expanded by a predicate for a proper non-trivial convex subring, such as $V$, is complete; it has a quantifier elimination in a language with a binary relation symbol for $v(x) \leq v(y)$ [7]. The theory is *weakly o-minimal*, in the sense of [10], namely that in any model any definable subset of the field id a finite union of convex sets. Some initial model theory in this setting is developed in [31]. It follows from [1] that if $M$ is an o-minimal expansion of a non-archimedean real closed field, and $V$ is a predicate for a proper non-trivial convex subring, then $(M, V)$ is weakly o-minimal.

Finally, recall that a complete theory $T$ has the *independence property* if for some $m, n$ there is a formula $\phi(x, y)$, where $l(x) = m$ and $l(y) = n$, and some $M \models T$ containing $\{a_i : i \in \omega\} \subset M^m$, such that for any $S \subseteq \omega$ there is $b_S \in M^n$ with $M \models \phi(a_i, b_S)$ if and only if $i \in S$. A complete theory without the independence property is called an *NIP theory*. By a theorem of Shelah [44, Theorem 4.11], if $T$ has the independence property then there is a formula $\phi(x, y)$ as above, but with $m = 1$, witnessing it.

**Proposition 2.1.3** *If $T$ is strongly minimal, o-minimal or more generally weakly o-minimal, $P$-minimal or $C$-minimal, then $T$ is NIP.*

*Proof.* Exercise. It suffices to consider formulas $\phi(x, y)$ with $l(x) = 1$, and the description of one-variable definable sets in the definition of minimality then suffices.

See also [37] for more on the independence property.

# 3 Imaginaries in ACVF

**Definition 3.0.4** A complete theory $T$ has *elimination of imaginaries* (EI) if for all $M \models T$, $n > 0$ and $\emptyset$-definable equivalence relations $E$ on $M^n$, there is $m > 0$ and an $\emptyset$-definable function $f_E : M^n \to M^m$ such that for all $x, y \in M^n$, $E(x, y) \Leftrightarrow f(x) = f(y)$.

By an *imaginary* we mean an equivalence class of some $\emptyset$-definable equivalecne relation.

The idea here is the following. Suppose $R \subset M^n \times M^t$ is a $\emptyset$-definable relation, and for $a \in M^n$ let $R_a := \{y \in M^t : M \models R(a, y)\}$. Thus, $R_a \subset M^t$, so $R$ may be viewed as a *definable family* of definable subsets of $M^t$, with $x$ a parameter for $R_x$. Write $E_R xy$ if and only if $R_x = R_y$. Thus, the imaginary $x/E_R$ *codes* the definable set $R_x$, and we write $x/E_R = \ulcorner R_x \urcorner$. The code is unique up to interdefinability over $\emptyset$. Given EI, there is $m > 0$ and an $\emptyset$-definable $f : M^n \to M^m$ such that $f(x) = f(y)$ if and only if $R_x = R_y$. Put $Z = Im(f)$. We may now replace the relation $R$ by $R' \subset Z \times M^t$, with

$R(x,y) \Leftrightarrow R'(f(x),y)$. Then $R$ and $R'$ define the same family of sets, but with respect to $R'$ each set in the family has a unique parameter.

Elimination of imaginaries was introduced by Poizat [38]. It is a key property in stability theory, and in the fine structural analysis of theories. It is always possible to ensure elimination of imaginaries by working in $M^{\rm eq}$, and this is the usual convention in stability theory. However, in the process we lose control of definability. One therefore typically aims to prove elimination of imaginaries in $M$, or by adding to $M$ very specific, well-understood, sorts from $M^{\rm eq}$.

It is easily seen that if $T$ is a complete theory, and $\mathcal{U}$ is a sufficiently saturated model of $T$, then $T$ has elimination of imaginaries if and only if, for each imaginary $e \in \mathcal{U}^{\rm eq}$, $e \in {\rm dcl}({\rm dcl}(e) \cap \mathcal{U})$. This suggests a natural weakening: the theory $T$ has *weak elimination of imaginaries* (weak EI) if for any sufficiently saturated $\mathcal{U} \models T$ and imaginary $e$ of $\mathcal{U}$, $e \in {\rm dcl}({\rm acl}(e) \cap \mathcal{U})$.

**Example 3.0.5** 1. The theory of an infinite pure set does not have elimination of imaginaries, since it is not possible to code a finite set of size bigger than one by a tuple: for example we cannot code the equivalence relation $\{x_1, x_2\} = \{y_1, y_2\}$. This theory does have weak EI. Similar assertions hold for vector spaces, in the language of modules.

2. For a real closed field $R$ (or for any o-minimal structure which expands an ordered field) one can prove elimination of imaginaries by showing that for any $\emptyset$-definable equivalence relation $E$ on $R^n$, there is an $\emptyset$-definable function $f : R^n \to R^n$ which picks a representative of each $E$-class. This holds because any definable set in 1-variable can be defined with canonical choice of parameters (e.g. the endpoints of an interval), and because there are definable Skolem functions; for the latter the main point is that with the additive group structure we may canonically choose the midpoint of a bounded interval.

3. The theory of algebraically closed fields has elimination of imaginaries. As a first observation, we can code the finite *set* $\{a_1, \ldots, a_n\}$ by the coefficients of the polynomial $\Pi_{i=1}^n (X - a_i)$ (a construction not available in (1) above). A slight extension of this codes finite sets of *tuples*. There are then two alternative approaches: an algebraic method, which rests on quantifier elimination and the fact that any affine variety has a unique smallest field of definition. And a purely model-theoretic argument of Pillay (see Section 3.2 of [33]) which suffices for weak elimination of imaginaries in any strongly minimal set.

4. Recall the theory ACFA of algebraically closed fields with a generic automorphism, investigated in [6]. It is shown in [6] that the completions of ACFA have EI. The proof uses the Independence Theorem for simple theories.

For an extensive review of a number of variants on elimination of imaginaries, see [5].

**Remark 3.0.6** The theory ACVF does not have elimination of imaginaries to the sort $K$. Indeed, $k$ and $\Gamma$ are *quotients* of $K$, and easy dimension arguments show they cannot be coded. (Exercise). Furthermore, even parsed in the language $L_{k,\Gamma}$ with three sorts $K, k, \Gamma$, ACVF does not have EI – see [16].

The main theorem of [14] is that once one adds certain specific sorts (coset spaces) from $K^{\mathrm{eq}}$ to ACVF, EI holds. The sorts are described in this section. The following section contains a very rough account of the proof.

By an *$n$-lattice* I shall mean a free rank $n$ $R$-submodule of $K^n$. It will have the form $Rv_1 \oplus \ldots \oplus Rv_n$, where $v_1, \ldots, v_n \in K$ are linearly independent over $R$; and any such tuple $(v_1, \ldots, v_n)$, that is, any ordered $R$-basis of $K^n$, determines an $n$-lattice. The group $\mathrm{GL}_n(K)$ is transitive on the set of such ordered bases, so acts transitively on the set of $n$-lattices. One such lattice is $R^n$, determined by the standard ordered basis $(e_1, \ldots, e_n)$. The stabiliser of $R^n$ is just $\mathrm{GL}_n(R)$ (Exercise), the group of $n \times n$ matrices over $R$ which are invertible over $R$. Thus, the space of $n$-lattices is naturally in bijection with the coset space $\mathrm{GL}_n(K)/\mathrm{GL}_n(R)$. We may view this space as a quotient, by an $\emptyset$-definable equivalence relation $E_{S_n}$, of an $\emptyset$-definable subset $X$ of $K^{n^2}$; here $X$ is the set of ordered bases of $K^n$, and $E_{S_n}$ is the equivalence relation 'generates the same $n$-lattice'. This space is a sort of $K^{\mathrm{eq}}$, and will be denoted by $S_n$. Thus, each $s \in S_n$ is a *code* for an $n$-lattice.

Observe that $S_1$ is naturally in bijection with $\Gamma$: indeed, $\mathrm{GL}_1(K)/\mathrm{GL}_1(R) = K^*/U \cong \Gamma$, where $U$ is the group of units of $R$. Thus, $S_1$ has a natural group structure, but this is not the case for $S_n$ with $n > 1$.

An arbitrary (infinite) closed ball $B$ not containing 0 is coded by an element of $S_2$. Indeed, $\{1\} \times B \subset K^2$ generates an $R$-module $L$. One finds (Exercise)

(i) $B = L \cap (\{1\} \times K)$, and

(ii) $L$ is a 2-lattice.

The space $S_n$ can also be viewed as a space of codes for the sort $B_n(K)/B_n(R)$, where $B_n(K)$ is the group of upper-trianguar matrices in $\mathrm{GL}_n(K)$; this is because every $n$-lattice in $K^n$ has a 'triangular' basis. This presentation of $S_n$ is heavuly used in [14] and [15].

Suppose $L$ is an $n$-lattice, and let $\mathcal{M}L := \{ax : x \in L, a \in \mathcal{M}\}$. Then $\mathcal{M}L$ is an $R$-submodule of $L$, and the quotient, $\mathrm{red}(L) := L/\mathcal{M}L$, has the structure of an $n$-dimensional vector space over $R/\mathcal{M} = k$.

If $L$ is a 1-lattice, then there are $a, \gamma$ such that $L$ has the form $\gamma R := aR$ where $a \in K$ with $v(a) = \gamma \in \Gamma$. Now,

$$\mathrm{red}(L) = \gamma R/\gamma\mathcal{M} = \{\gamma\mathcal{M} + a : v(a) = \gamma\} = \{B_{>\gamma}(a) : v(a) = \gamma\}.$$

Then $\mathrm{red}(L)$ is just a torsor (a one-dimensional affine space) for the residue field $k$, so is strongly minimal.

The set $\bigcup_{\ulcorner L \urcorner \in S_n} \mathrm{red}(L)$ is a uniformly definable family of subsets of $K^n$. Define $T_n$ to be the set of codes for this family. We may view $T_n$ as a quotient, by an equivalence relation $E_{T_n}$, of the set

$$X_n = \{(x, y_1, \ldots, y_n) \in K^{n(n+1)} : y_1, \ldots, y_n \text{ linearly independent}, x \in Ry_1 + \ldots + Ry_n\}.$$

Here, $(x, y_1, \ldots, y_n) E_{T_n} (x', y'_1 \ldots, y'_n)$ if and only if $Ry_1 + \ldots + Ry_n = Ry'_1 + \ldots + Ry'_n = L$, say, and $x + \mathcal{M}L = x' + \mathcal{M}L$. There is an $\emptyset$-definable surjection $\pi_n : T_n \to S_n$, where $\pi_n(x, y_1, \ldots, y_n)/E_{T_n} \mapsto (y_1, \ldots, y_n)/E_{S_n}$.

We have

$$T_1 = \{\ulcorner B_{>v(a)}(a)\urcorner : a \in K^*\} = \{a(1+\mathcal{M}) : a \in K^*\} = K^*/(1+\mathcal{M}) = RV.$$

In particular, as for $S_1$, there is an $\emptyset$-definable group structure on $T_1$.

For $m = 1, \ldots, n$ let $B_{n,m}(k)$ be the set of elements of $B_n(k)$ whose $m^{th}$ column has a 1 in the $m^{th}$ entry and other entries 0, and let $B_{n,m}(R)$ be the set of matrices in $B_n(R)$ which reduce modulo $\mathcal{M}$ to a member of $B_{n,m}(k)$. Also put $B_{n,0}(R) = B_n(R)$. Then each $B_{n,m}(R)$ is a group, and we may identify $T_n$ with the union of coset spaces $\bigcup_{m=0}^{n} B_n(K)/B_{n,m}(R)$.

The language $L_{\mathcal{G}}$ is a multisorted language with sorts $K, k, \Gamma, S_n, T_n$ (for $n \geq 1$), and we refer to these as the sorts $\mathcal{G}$ (for *geometric* sorts). Formally, $k$ and $\Gamma$ are redundant, as $\Gamma$ is identifiable with $S_1$ and $k$ with $\mathrm{red}(R) \subset T_1$, but they are included for convenience. We do not specify the symbols of the language, but the $\emptyset$-definable sets are exactly the $\emptyset$-interpretable sets of an algebraically closed valued field. An appropriate QE result with these sorts is given in [14, Theorem 3.1.2]. We view ACVF as a theory in $L_{\mathcal{G}}$, and $\mathcal{U}$ as a sufficently saturated model in this language.

**Theorem 3.0.7 ([14])** *ACVF has elimination of imaginaries in $L_{\mathcal{G}}$.*

It is shown in Section 3.5 that to obtain elimination of imaginaries for ACVF, one needs this level of complexity of sorts; at any rate, no finite subcollection of these sorts suffices for EI.

Analogous results are known for certain other valued fields: EI is proved for RCVF to the same sorts by Mellor in [34]. For $\mathbb{Q}_p$, EI holds in the sorts $K, S_n$, by a result of Hrushovski and Martin [20]. In the proof, a model of $\mathrm{Th}(\mathbb{Q}_p)$ is embedded in an algebraically closed valued field, and the proof uses both EI for ACVF, and aspects of its proof.

This last result has a beautiful application. If $G$ is a finitely generated nilpotent group, let $R_n(G)$ be the set of irreducible complex $n$-dimensional characters of $G$. There is a natural equivalence relation $\sim$ on $R_n(G)$: put $\sigma_1 \sim \sigma_2$ if there is a linear character $\chi$ of $G$ with $\sigma_1 = \chi\sigma_2$. It was shown by Lubotzky and Magid [29] that $a_n(G) = |R_n(G)/\sim|$ is always finite. Using EI for $\mathbb{Q}_p$, together with results of Denef on $p$-adic integrals, Hrushovski and Martin prove the following.

**Theorem 3.0.8** *Let $G$ be a finitely generated nilpotent group Then, for any prime $p$, the zeta function $\Sigma_{n=0}^{\infty} a_{p^n} t^n$ is a rational function.*

The paper [20] also uses $p$-adic EI to reprove other rationality results (of Grunewald, Segal, Smith, du Sautoy) for zeta functions in groups.

*Problem.* Prove EI, to the sorts $K, S_n, T_n$, for the rigid analytic expansions of ACVF studied by Lipshitz and Robinson.

# 4 Proof of Elimination of Imaginaries

The key problems with EI in ACVF are:

(i) there is no really good independence theory, beyond that furnished by algebraic closure (which is just field-theoretic algebraic closure, as in ACF);

(i) balls do not have canonical parameters – every element of a ball is a centre of it; likewise a lattice $L$ has no canonical basis.

The proof has several key ingredients, and some of these are relevant also to Sections 5 and 6 below. Below, $\mathcal{U}$ denotes a large sufficiently saturated model of ACVF. Sets $A, B, C$, and models $M$, are always assumed to be small relative to the degree of saturation of $\mathcal{U}$, and to live in $\mathcal{U}$.

## 4.1 The stable part of ACVF

Let $C$ be any parameter set. Define $VS_{k,C}$ to be the multi-sorted structure with a sort $\mathrm{red}(s)$ for each $C$-definable $n$-lattice with code $s \in S_n$. Thus, $\mathrm{red}(s)$ is an $n$-dimensional vector space over $k$. Note that $R$ is a 1-lattice, and so $\mathrm{red}(R) = k$ is a sort of $VS_{k,C}$. We suppose that $VS_{k,C}$ has, as its $\emptyset$-definable relations, just the $C$-definable relations on products of powers of the sorts.

Recall that if $T$ is a complete theory, and $D_1, D_2$ are definable sets in a sufficiently saturated model of $T^{\mathrm{eq}}$, then $D_2$ is $D_1$-internal if there is finite $F$ such that $D_2 \subset \mathrm{dcl}(D_1 \cup F)$. Now for each $s \in S_n$, $\mathrm{red}(s)$ is $k$-internal: choose as $F$ a basis of $\mathrm{red}(s)$. It follows easily that $VS_{k,C}$ is stable, and any finite union of its sorts has finite Morley rank. Each of the sorts is stably embedded.

Following Shelah, in any model $M$ of a complete theory $T$, we say that a $C$-definable set $D$ is *stable* if it is stably embedded, and is stable as a structure equipped by the $C$-definable relations on $D$ (in the sense of the ambient theory $T$). It turns out that the following (and other) conditions are equivalent.

(i) $D$ is stable.

(ii) For any formula $\phi(x_1, \ldots, x_n, y)$ which implies $\bigwedge_{i=1}^{n} D(x_i)$, if $\phi$ is parsed as $\phi(x_1, \ldots x_n, y)$, then $\phi$ is a stable formula.

(iii) If $\lambda > |T| + |C|$ with $\lambda = \lambda^{\aleph_0}$ and $B \supseteq C$ with $|B| = \lambda$, then there are at most $\lambda$ 1-types over $B$ realised in $D$.

**Theorem 4.1.1** *(ACVF) Let $D$ be a $C$-definable subset of $K^{\mathrm{eq}}$. Then the following are equivalent.*

*(i) $D$ is stable.*

*(ii) $D$ is $k$-internal.*

*(iii) $D \subset \mathrm{dcl}(C \cup VS_{k,C})$.*

**Theorem 4.1.2** *$VS_{k,C}$ has elimination of imaginaries.*

*Sketch Proof.* The family of vector spaces in $VS_{k,C}$ is closed (up to canonical isomorphisms) under duals and tensor and exterior products.

It is easy to reduce to coding definable subsets of a single vector space $V = \mathrm{red}(s)$. Via the identification $V \leftrightarrow k^n$ (over a basis) there is a notion of Zariski closed subset of $V$, which is independent of the choice of basis. By quantifier elimination in ACF, EI now reduces to coding Zariski closed subsets of $V$. A Zariski closed set is determined by the ideal vanishing on it in $k[X_1, \ldots, X_n]$, and this is identified with an ideal in the symmetric power of $V^*$. The problem reduces to coding a subspace of $S^m(V) := k \oplus V^* \oplus \Sigma_{i=2}^m \mathrm{Sym}^i(V^*)$. This is done using exterior powers.

## 4.2 Description of 1-types

We shall work in this setion in the sort $K$ of ACVF, though the arguments below actually work more generally, for 'unary types'.

Work over a base set $C$ of parameters. For convenience we suppose $C = \mathrm{acl}^{\mathrm{eq}}(C)$. Let $a \in K$. Define

$$\mathcal{B}_C(a) := \{B : a \in B, B \text{ a ball definable over } C\}.$$

Then $\mathcal{B}_C(a)$ is a chain of balls, ordered by inclusion. Also define $Loc_C(a) := \bigcap(B : B \in \mathcal{B}_C(a))$. Then $Loc_C(a)$ is a $C$-definable subset of $\mathcal{U}$ containing $a$.

Note that if $\mathcal{B}_C(a)$ contains a smallest ball, then $Loc_C(a)$ equals this ball. Otherwise, $Loc_C(a)$ is just an infinitely definable subset of $K$. Any $C$-definable ball, or the intersection of any chain of $C$-definable balls, has the form $Loc_C(a)$ for some $a \in K$.

We shall say that $b \in K$ is *generic* in $Loc_C(a)$, if $Loc_C(b) = Loc_C(a)$, that is, if any $C$-definable ball which contains $b$ also contains $a$. More generally, if $C \subset D$, we say $a$ is generic in $Loc_C(a)$ over $D$, and write $a \mathop{\smile\!\!\!\!\!\vert}\,^g_C D$, if $Loc_C(a) = Loc_D(a)$.

**Lemma 4.2.1** *Suppose $C \subset D$.*

*(i) $Loc_C(a)$ has a unique generic type; that is, if $b$ is generic over $C$ in $Loc_C(a)$, then $a \equiv_C b$.*

*(ii) $\mathrm{tp}(a/C)$ has a generic extension over $D$.*

*(ii) if $a \mathop{\smile\!\!\!\!\!\vert}\,^g_C D$ and $a' \mathop{\smile\!\!\!\!\!\vert}\,^g_C D$ and $\mathrm{tp}(a/C) = \mathrm{tp}(a'/C)$, then $\mathrm{tp}(a/D) = \mathrm{tp}(a'/D)$.*

*(iii) If $B$ is a ball of $\mathcal{U}$, then the generic type over $\mathcal{U}$ of $B$ is definable over $\ulcorner B \urcorner$.*

*Proof.* Exercise. Parts (i) and (ii) follow from the fact that any definable subset of $K$ is a boolean combination of balls, so if $\mathrm{tp}(a/C) \neq \mathrm{tp}(a'/C)$ then some ball contains just one of $a, a'$.

For part (iii), observe that if $B$ is a closed ball $B_{\geq\gamma}(a)$ then for any formula $\phi(x, y)$, there is $n_\phi$ such that $\phi(x, c) \in p$ if and only if $\phi(x, c)$ contains all but at most $n_\phi$ open sub-balls of $B_{\geq\gamma}(a)$ of radius $\gamma$.

## 4.3 Proof of Theorem 3.0.7

The key to the proof of Theorem 3.0.7 is the following lemma.

**Lemma 4.3.1** *Let $\mathcal{U}$ be a sufficiently saturated multi-sorted structure, with a sort $D$ such that $\mathcal{U} \subset (\mathcal{U} \cap D)^{\text{eq}}$. Suppose that for each sort $S$ of $\mathcal{U}$, every definable one-variable partial function $f : D \to \mathcal{U}$ is coded. Then* $\text{Th}(\mathcal{U})$ *has EI.*

*Proof.* We prove by induction on $n$ that each definable $R \subset D^n$ is coded. To start the induction, observe that any $R \subset D$ is coded by $\text{id}_R$.

For the inductive step, suppose $R \subset D^{n+1}$ is definable, let $\pi : D^{n+1} \to D$ be the projection to the first coordinate, and put $Y = \pi_1(R)$. For each $y \in Y$, the fibre $R_y \subseteq D^n$, so by induction $R(y)$ is coded by some tuple $h(y)$. By compactness, $h$ is definable, and we may write $Y = Y_1 \cup \ldots \cup Y_t$ such that $h|_{Y_i}$ is a map to a specific tuple of sorts (different for different $i$). Then by assumption $h|_{Y_i}$ is coded, and $(\ulcorner h|_{Y_1} \urcorner, \ldots, \ulcorner h|_{Y_t} \urcorner)$ codes $R$.

We mention three further ingredients in the proof of EI for ACVF.

(1) Definable $R$-submodules of $K^n$ are coded in $\mathcal{G}$.

(2) Definable functions $\Gamma \to \mathcal{G}$ are coded. In addition, *germs* of such functions, e.g. on the definable type immediately above some $\gamma_0 \in \Gamma$, are coded in $\mathcal{G}$.

(3) Finite sets are coded. To be precise, finite sets of tuples from $\mathcal{G}$ are coded in $\mathcal{G}$.

To prove EI for ACVF in $L_{\mathcal{G}}$, we now suppose $f : K \to \mathcal{G}$ is a definable function, and aim to code $\ulcorner f \urcorner$ in $\mathcal{G}$. So let $B := \text{acl}(\ulcorner f \urcorner) \cap \mathcal{G}$. We must show $\ulcorner f \urcorner \in \text{dcl}(B)$; this suffices, by (3). Roughly speaking, we aim to code the restriction to $f$ on each complete type over $B$, and then piece these codes together using compactness.

Each such (non-algebraic) complete type is the generic type of some $Loc_B(a)$. This is the generic type of a closed ball, of an open ball, or of the intersection of a chain of balls. The key is the closed ball case. Here one uses definable modules (so uses (1) above), and also uses the notion of *strong code* (see section 5). It is shown that the germ of $f$ on the generic type of a closed ball has a strong code. Given this, the generic type os an open ball, or of a chain of balls, can be approximated from inside by closed balls. The argument here uses (2) above.

# 5 Stable Domination

We sketch here the theory developed in Part I of [15]. Stable domination seems to be the key to the role of stability theory in ACVF.

## 5.1 Invariant types

Initially, we work with an arbitrary complete theory $T$ over a language $L$. Recall that a type $p$ over a model $M$ is *definable* over $C \subseteq M$ if for each formula $\phi(x, y)$ there is a formula $d_p x\phi(y)$ over $C$ such that for any $a \in M$, $d_p x\phi(a)$ holds if and only if $\phi(x, a) \in p$.

We shall work over a large sufficiently saturated and homogeneous model $\mathcal{U}$ of $T$. Let $C \subset \mathcal{U}$ be small. The group $\mathrm{Aut}(\mathcal{U}/C)$ acts on the space of types over $\mathcal{U}$. We say that a type $p \in S(\mathcal{U})$ is $\mathrm{Aut}(\mathcal{U}/C)$*-invariant* (or $C$-invariant, or just *invariant*, if $C$ is understood) if $p$ is fixed by $\mathrm{Aut}(\mathcal{U}/C)$. This is equivalent to saying that $p$ does not split over $C$, in the sense of Shelah [44]; that is, if $a_1, a_2 \in \mathcal{U}$ and $a_1 \equiv_C a_2$ then for any formula $\phi(x, y)$, $\phi(x, a_1) \in p$ if and only if $\phi(x, a_2) \in p$.

Observe that any $C$-definable type over $\mathcal{U}$ is $\mathrm{Aut}(\mathcal{U}/C)$-invariant. There are invariant types which are not definable. For example, in the theory of dense linear orders without endpoints, let $\mathcal{U}$ be a large saturated model, and let $(c_i : i \in \omega)$ be an increasing sequence with no greatest element, and $C := \{c_i : i \in \omega\}$. There is an $\mathrm{Aut}(\mathcal{U}/C)$-invariant but not definable 1-type over $\mathcal{U}$ which contains all the formulas $c_i < x$, and the formula $x < d$ for any $d \in \mathcal{U}$ with $c_i < d$ for all $i \in \omega$.

*Exercise.* Verify this, and verify that in an o-minimal (or weakly o-minimal) theory, any type over any parameter set has an invariant extension.

We mention one frequently used fact: any $\mathrm{Aut}(\mathcal{U})/C)$-invariant type which is definable is $C$-definable. This is a consequence of compactness.

We often view an $\mathrm{Aut}(\mathcal{U}/C)$-invariant type $p$ as a functor $D \mapsto p|D$, where $D$ ranges through subsets of $\mathcal{U}$ containing $C$.

**Example 5.1.1** (ACVF) The generic type over $\mathcal{U}$ of a $C$-definable ball is $C$-definable, so $C$-invariant. The generic type of a chain of balls with no least element is $C$-invariant, but not $C$-definable.

**Theorem 5.1.2** *Let $\mathcal{U} \models ACVF$, in the sorts $\mathcal{G}$.*

*(i) If $C = \mathrm{acl}(C) \subset \mathcal{U}$, then any 1-type over $C$ in the sort $K$ has an invariant extension.*

*(ii) If $C = \mathrm{acl}(C)$, then* any *type over $C$ (in any sort) has an invariant extension.*

The proof of (i) is easy, and it holds in any $C$-minimal structure (working over $\mathrm{acl}^{\mathrm{eq}}(C)$). The proof of Theorem 5.1.2(ii) given in [15] is rather intricate, and depends on much of [14]. However, it is shown in [22, Proposition 2.13] that the conclusion (ii) follows from the existence part of (i) in *any* one-sorted NIP theory (assuming $C = \mathrm{acl}^{\mathrm{eq}}(C)$).

## 5.2 Definition of stable domination

We shall assume that the complete theory $T$ has elimination of imaginaries, and that $\mathcal{U} \models T$ is sufficiently saturated. Below, we shall also assume $C = \mathrm{acl}(C)$, though this assumption is not needed for some of the definitions and results.

**Definition 5.2.1** We denote by $\mathrm{St}_C$ the many-sorted structure consisting of all $C$-definable stable sets, in the sense of Section 4.1. We endow $\mathrm{St}_C$ with all $C$-definable relations on products of the sorts. If $A \subset \mathcal{U}$ put $\mathrm{St}_C(A) := \mathrm{dcl}(CA) \cap \mathrm{St}_C$; likewise, if $a$ is a tuple, possibly infinite, then $\mathrm{St}_C(a) := \mathrm{dcl}(Ca) \cap \mathrm{St}_C$.

Note that the elements of $C$ are one-element sorts of $\mathrm{St}_C$, named by constants. In ACVF, by Theorem 4.1.1 $\mathrm{St}_C$ is essentially the same as $VS_{k,C}$, though it is not formally the same. If $C$ is a model of ACVF, then $\mathrm{St}_C$ is essentially just $k$.

**Definition 5.2.2** We say that $\mathrm{tp}(a/C)$ is *stably dominated* if, for any $b$, if $\mathrm{St}_C(a) \perp_C \mathrm{St}_C(b)$ (where $\perp$ is just forking independence in the stable structure $\mathrm{St}_C$), then

$$\mathrm{tp}(b/\mathrm{St}_C(a)) \vdash \mathrm{tp}(b/Ca).$$

This definition takes some unravelling. Formally, it means that for any $b'$, if $b' \equiv_{\mathrm{St}_C(a)} b$, then $b' \equiv_{Ca} b$. By easy automorphism arguments (*Exercise*: check this), the condition is equivalent to the assertion that $\mathrm{tp}(a/\mathrm{St}_C(b)) \vdash \mathrm{tp}(a/Cb)$.

We describe another presentation. We shall view $\mathrm{St}_C(a)$ as an infinite tuple, with respect to some indexing. Consider the map $f : a \mapsto \mathrm{St}_C(a)$ on $\mathrm{tp}(a/C)$. Let $D$ be any definable set, in the sort containing $a$. Put $d := \ulcorner D \urcorner$. We say a fibre $X$ of $f$ is *generic* if, for any $a \in X$, $\mathrm{St}_C(a) \perp_C \mathrm{St}_C(d)$. Then $\mathrm{tp}(a/C)$ is stably dominated if and only if, either $D$ contains all generic fibres of $f$, or $D$ is disjoint from all generic fibres of $f$.

Also, $\mathrm{tp}(a/C)$ is stably dominated if and only if, for any $b$ such that $\mathrm{St}_C(a) \perp_C \mathrm{St}_C(b)$, $\mathrm{tp}(a/\mathrm{St}_C(a))$ has a unique extension over $C\,\mathrm{St}_C(a)b$.

*Exercise.* Verify these equivalences.

We list some basic properties of stable domination.

**Theorem 5.2.3** *Let* $C = \mathrm{acl}(C)$.

*(i) [15, 3.12] If $p \in S(\mathcal{U})$ with $p|C$ stably dominated, then $p$ is $C$-definable, so has an* $\mathrm{Aut}(\mathcal{U}/C)$*-invariant extension; in fact, $p$ has a* unique $\mathrm{Aut}(\mathcal{U}/C)$*-invariant extension.*

*(ii) [15, 4.1] If $p \in S(\mathcal{U})$ is* $\mathrm{Aut}(\mathcal{U}/C)$*-invariant and $C \subset B \subset \mathcal{U}$ and $p|C$ is stably dominated, then $p|B$ is stably dominated.*

*(iii) [15, 4.9] ('Descent') If $p, q$ are* $\mathrm{Aut}(\mathcal{U}/C)$*-invariant, $b \models q|C$ and $p|Cb$ is stably dominated, then $p|C$ is stably dominated.*

*(iv) [15, 6.11]If* $\mathrm{tp}(a/C)$ *and* $\mathrm{tp}(b/Ca)$ *are stably dominated, so is* $\mathrm{tp}(ab/C)$.

In (iii), we do not know if the assumption that $q$ is invariant is needed. The proof of (iii) is intricate.

Suppose $p$ is a $C$-definable type over $\mathcal{U}$, and $f_a$ is an $a$-definable function defined on realisations of $p$. Consider the equivalence relation $\sim$: $a \sim a'$ iff for every $x \models p$, $f_a(x) = f_{a'}(x)$. As $p$ is definable, this is a $C$-definable equivalence relation, defined by the formula $E(y, y')$: $d_p x(f_y(x) = f_{y'}(x))$. Thus, the $\sim$-class of $a$ is an imaginary, called the $p$-germ of $f_a$. More generally, we say that two functions $f, g$ defined on $p$ have the same $p$-germ if $f(x) = g(x)$ for all $x \models p$. We say that the $p$-germ $e$ of $f_a$ is *strong* over $C$ if there is a $Ce$-definable function $g$ defined on $p$ and having the same germ as $f_a$ on $p$. It is well-known that in a stable theory, $p$-germs are strong.

The following result is used in the proof of (iv) above, and in Theorem 6.2.4 below. Similar ideas are used in the proof of EI for ACVF (the proof uses the fact that the germ of a definable function on the generic type of a closed ball is strong).

**Theorem 5.2.4 ([15])** *Let $C = \mathrm{acl}(C)$, let $p$ be a $C$-definable type over $\mathcal{U}$ with $p|C$ stably dominated, and let $f$ be a definable function defined on realisations of $p$. Then the $p$-germ of $f$ is strong over $C$. Also, if $f(a) \in \mathrm{St}_{Ca}$ for $a \models p$, then the code for the $p$-germ of $f$ is in $\mathrm{St}_C$.*

**Remark 5.2.5** In [22, Section 3], Hrushovski and Pillay consider a generalisation of stable domination. Let $T$ be NIP, $\mathcal{U} \models T$, and $p$ an $\mathrm{Aut}(\mathcal{U}/C)$-invariant type over $\mathcal{U}$, where $C = \mathrm{acl}^{\mathrm{eq}}(C)$. Say that $p$ is *generically stable* over $C$ if it is definable (and hence $C$-definable) and is finitely satisfiable in any model containing $C$. This is equivalent to several other conditions, such as that for any small $B \supset C$, $p$ is the unique global non-forking extension of $p|B$. Any stably dominated type (over $C = \mathrm{acl}^{\mathrm{eq}}(C)$) is generically stable.

If $p$ is an $\mathrm{Aut}(\mathcal{U}/C)$-invariant type, then a *Morley sequence* for $p$ is a sequence $(a_i : i \in \omega)$ such that $a_{i+1} \models p|C \cup \{a_j : j \leq i\}$ for each $i$. Such a sequence is indiscernible over $C$. Hrushovski and Pillay show that if $p$ is generically stable in an NIP theory, then any Morley sequence in $p$ over $C$ is indiscernible as a set over $C$.

# 6 More on stable domination

## 6.1 Stable domination in ACVF

Below, when we work in ACVF, we work in the sorts $\mathcal{G}$ over which EI holds; so 'definable' and 'interpretable' are interchangeable.

**Definition 6.1.1** In ACVF, let $C = \mathrm{acl}(C)$, and $p \in S(C)$. We say $p \perp \Gamma$ if, for any $\mathrm{Aut}(\mathcal{U}/C)$-invariant extension $p'$ of $p$, and any model $M \supseteq C$, if $a \models p'|M$ then $\mathrm{dcl}_\Gamma(Ma) = \mathrm{dcl}_\Gamma(M)$.

This looks slightly different to the definition given in [15], which involves $\mathop{\smile\!\!\!\!\!\!|}^{g}$, but it is equivalent. By o-minimality of $\Gamma$, the condition $\mathrm{dcl}_\Gamma(Ma) = \mathrm{dcl}_\Gamma(M)$ is equivalent also to saying that $\mathrm{tp}(a/M)$ has a unique extension over $\Gamma$. Thus, we could have parsed the above definition by saying that for every $C$-invariant extension $p'$ of $p$ over $\mathcal{U}$, $p'$ has a unique extension over $\mathcal{U} \cup \Gamma$.

**Theorem 6.1.2** *(ACVF) Let $C = \mathrm{acl}(C)$, $p \in S(C)$. Then the following are equivalent.*
*(i) $p \perp \Gamma$*
*(ii) $p$ is stably dominated.*

**Example 6.1.3** Let $p$ be the generic type of the valuation ring $R$, over any parameter set $C$. Then $p$ is stably dominated.

Indeed, if $a \models p$, then $\mathrm{res}(a)$ is transcendental in $k$ over $\mathrm{acl}(C) \cap k$. For any $B \supset C$, put $\mathrm{res}(B) := \mathrm{dcl}(B) \cap k$. Then there is a unique complete type $p'$

over $B$ such that $p'$ contains $x \in R$ and implies $\mathrm{res}(x) \downarrow_C \mathrm{res}(B)$; for otherwise, there would be a formula $\phi(x)$ over $B$ which meets each set $\mathcal{M} + a$, for generic $a \in R$, in a proper non-empty set, contradicting $C$-minimality: the formula $\phi(x)$ would not define a boolean combination of balls. Thus, $\mathrm{tp}(a/C\,\mathrm{res}(B))$ implies $\mathrm{tp}(a/B)$ so $p$ is stably dominated.

Recall the following, from Section 1.

**Definition 6.1.4** (i) An extension of valued fields $(F, v) < (F', v')$ is *immediate*, if, under the natural embeddings, the two valued fields have the same value groups and residue fields.

(ii) A valued field is *maximally complete* if it has no proper non-trivial immediate extensions.

For us, the key fact about maximally complete valued fields is the following result of Baur [4]. Proofs of the following results are given in Section 12 of [15].

**Theorem 6.1.5 (Baur)** *Let $(F, v) < (L, v)$ be an extension of valued fields, with $F$ maximally complete. Let $U$ be a finite-dimensional $F$-subspace of $L$. Then $U$ has a basis $\{u_1, \dots, u_n\}$ such that for any $a_1, \dots, a_n \in F$, $v(a_1u_1 + \dots + a_nu_n) = \mathrm{Min}\{v(a_iu_i) : 1 \leq i \leq n\}$.*

Using this, one can show the following.

**Proposition 6.1.6** *Let $C \leq A, B$ be algebraically closed valued fields, with $C$ maximally complete. Suppose that the value groups $\Gamma(C)$ and $\Gamma(A)$ are equal, and that the residue fields $k(A)$ and $k(B)$ are linearly disjoint over $k(C)$. Then if $A' \equiv_C A$ and $k(A')$ is also linearly disjoint from $k(B)$ over $k(C)$, then $A \equiv_B A'$.*

**Theorem 6.1.7 ([15])** *(ACVF) Let $M$ be a maximally complete model of ACVF, in the sorts $\mathcal{G}$. Then for any $a$, $\mathrm{tp}(a/M \cup \mathrm{dcl}_\Gamma(Ma))$ is stably dominated.*

We mention also the following result over maximally complete fields, not restricted to the ACVF context.

**Proposition 6.1.8** *Let $F$ be a maximally complete valued field, $F \subset A = \mathrm{dcl}(A)$. Let $M$ be an extension of $F$ with the residue fields $k(A)$, $k(M)$ linearly disjoint over $k(F)$, and with the value groups $\Gamma(A) \cap \Gamma(M) = \Gamma(F)$. Then $\mathrm{tp}(M/F, k(A), \Gamma(A))$ implies $\mathrm{tp}(M/A)$.*

In recent work [21], Hrushovski investigates definable groups in ACVF. The context is rather more general, that of a *metastable theory.* Hrushovski's paper is currently being revised, and the definitions below may not be in quite final form. Below, I may have misunderstood some issues in [21]. In particular, I brush under the carpet the distinction between 'definable' and '$*$-definable' in arguments below. Essentially, a $*$-definable set is an $\infty$-definable set in infinitely many variables, and (in a sufficiently saturated model) can be viewed as an inverse limit of definable sets. In particular, it is hyperdefinable.

**Definition 6.1.9** Assume $T$ is a complete multi-sorted theory with a specific sort $\Gamma$. We say $T$ is *metastable* if:

(1) $\Gamma$ is stably embedded;

(2) no infinite definable subset of $\Gamma^{\mathrm{eq}}$ is stable;

(3) any type over $C = \mathrm{acl}^{\mathrm{eq}}(C)$ has a $C$-invariant extension; and

(4) for any partial type $p$ over a base $C_0$, there is $C_1 \supseteq C_0$ such that if $a \models p$, $\mathrm{tp}(a/C_1 \cup \mathrm{dcl}_\Gamma(C_1 a))$ is stably dominated.

**Remark 6.1.10** 1. ACVF is metastable. Here $\Gamma$ is the value group, so (1) follows from Proposition 2.0.5, (2) is obvious, and (3) and (4) come from Theorems 5.1.2 and 6.1.7 respectively. It is shown in [15] that the valued field $\mathbb{C}((t))$ is also metastable. Likewise, the theory DCVF of differentially closed valued fields (introduced by Scanlon [42]) is metastable. See [15, Ch. 16].

2. For trivial reasons, any o-minimal (or just weakly o-minimal) theory is metastable.

3. Condition (3) may not be needed. It is included to ensure that 'Descent' (Theorem 5.2.3(iii)) is available.

4. In some places, Hrushovski assumes additional conditions $(FD)$ and $(FD)_\omega$. These are conditions which hold in ACVF. They include an assumption that $\Gamma$ is o-minimal, and that, for any definable set $D$, there is an upper bound on the Morley rank of any image of it in $\mathrm{St}_C$ under a definable function, and a similar upper bound for the o-minimal dimension of its images in $\Gamma^{\mathrm{eq}}$. The condition $(FD)_\omega$ includes also a kind of 'isolated types are dense' condition.

## 6.2 Generically metastable groups

Hrushovski considers groups which are definable (or $\infty$-definable, or $*$-definable) in a theory $T$ with elimination of imaginaries, with sufficiently saturated model $\mathcal{U}$. Some notion of generic type is needed. So suppose $G$ is a group in $T$, in one of the above senses. Let $C$ be a parameter set, and $p$ a $C$-definable type, of elements of $G$. For $a \in G$, $p$ has translates $ap$, $pa$, which are definable over $C \cup \{a\}$.

**Definition 6.2.1** In the above setting, $p$ is *left-generic* if, for any $B = \mathrm{acl}(B)$ over which $p$ is defined, and any $b \in G$, $pb$ is defined over $B$. *Right-generic* is defined analogously.

We say $p$ is *symmetric* if, whenever $q$ is a definable type, $p, q$ both defined over $C$, then

$$b \models q|C \text{ and } a \models p|\,\mathrm{acl}(Cb) \Rightarrow b \models q|\,\mathrm{acl}(Ca).$$

The following lemma yields a basic theory of symmetric left generics.

**Lemma 6.2.2** *(i) Any symmetric left generic is right generic.*

*(ii) Any two symmetric left generics differ by a left translation.*

*(iii) $G$ has an $\infty$-definable subgroup $G^o$ of bounded index, with a unique symmetric left generic.*

It is easily shown that any stably dominated left generic is symmetric. We call such a type 'generic'.

**Definition 6.2.3** A group $G$ (definable, $\infty$-definable, $*$-definable over $C$) is *generically metastable* if it has a stably dominated generic type over $C$.

The following theorem is from [21] (see also Theorem 6.13 of [15].

**Theorem 6.2.4 (Hrushovski)** *Let $G$ be a generically metastable group over $C$ with a translation-invariant stably dominated generic type $p$.*

*(i) There is a $*$-definable (over $C$) stable group $\mathfrak{g}$ and a $*$-definable (over $C$) homomorphism $\phi : G \to \mathfrak{g}$ such that $p$ is stably dominated over $C$ by $\phi$: that is, if $a \models p|C$, and $b$ is any tuple, then* $\mathrm{tp}(b/C \cup \phi(a)) \vdash \mathrm{tp}(b/C \cup a)$.

*(ii) In ACVF, if $G$ is definable and $C = \mathrm{acl}(C)$, then $\phi$ and $\mathfrak{g}$ are definable.*

*(iii) Given any other such homomorphism $\phi' : G \to \mathfrak{g}'$, with $\mathfrak{g}'$ stable, there is $\psi : \mathfrak{g} \to \mathfrak{g}'$ such that $\phi' = \psi \circ \phi$.*

This theorem is stated slightly differently in [15, Theorem 6.13]. The assertion there is that if $G$ has a translation-invariant stably dominated generic type $p$ then there are *definable* stable groups $\mathfrak{g}_i$ and homomorphisms $\phi_i : G \to \mathfrak{g}_i$ (for $i \in I$) such that $p$ is stably dominated by $(\phi_i : i \in I)$.

*Sketch Proof of (i).* For $a \models p$, let $\theta(a)$ be an enumeration of $\mathrm{St}_C(a)$. For each such $a$, define $f_a$ on realisations of $p$ by $f_a(b) = \theta(ab)$. The $p$-germ $e$ of $f_a$ lies in $\mathrm{St}_C$ (Theorem 5.2.4), and so, as $e$ is $Ca$-definable, we have $e \in \theta(a)$. Also, as $e$ is a strong germ (again by 5.2.4)), there is $f'_{\theta(a)}$ defined over $\theta(a)$ and having the same $p$-germ as $f_a$. Now let $b \models p|Ca$ and $d = f_a(b)$. Then $d \in \mathrm{St}_C$, so as $\mathrm{St}_C$ is stably embedded, we have $\mathrm{tp}(\theta(a), d/C\theta(b)) \vdash \mathrm{tp}(\theta(a), d/Cb)$. Hence, as $d \in \mathrm{dcl}(C, \theta(a), b)$, we have $d \in \mathrm{dcl}(C, \theta(a), \theta(b))$. Thus, there is a $C$-definable function $F$ such that $d = F(\theta(a), \theta(b))$.

There is a 'group chunk' theorem proved in [21], in the very general setting of left-generics; however, we are applying it in the stable structure $\mathrm{St}_C$, so all we really need is a $*$-definable version of the usual group chunk theorem from [18] or [39], say. The function $F$ is generically associative on $p' := \mathrm{tp}(\theta(a)/C)$, and has the other required properties. Hence, $p'$ is the generic type of a group $\mathfrak{g}$ in $\mathrm{St}_C$ (so a stable group), with the group operation extending $F$. Also, $\theta$ is generically a group homomorphism, so extends to a $*$-definable group homomorphism.

**Example 6.2.5** In ACVF, let $G = (R, +)$, and work over $\emptyset$. The generic type $p$ of $G$ is just the generic type of $R$ viewed as a closed ball. Then $p$ is stably dominated by the residue map; see Example 6.1.3. Thus, we may take $\mathfrak{g} := (k, +)$, and $\phi$ as the residue map.

More generally, if $G = \mathrm{SL}_n(R)$, then $\mathfrak{g} := \mathrm{SL}_n(k)$ and $\phi$ is the natural reduction map which reduces the entries of each matrix in $\mathrm{SL}_n(R)$ modulo $\mathcal{M}$.

The results in [21] go much further than this. Generalising the last example, Hrushovski proves

**Theorem 6.2.6** *(ACVF) Let $H$ be an affine algebraic group, and let $G$ be a generically metastable definable subgroup of $H$. Then, for some algebraic group scheme $H_1$ over $R$, $G$ is isomorphic to $H_1(R)$.*

There is in [21] a nice theory of abelian groups definable in ACVF (or in any metastable theory). The methods developed are also used to prove that in ACVF, any infinite definable field is definably isomorphic to the given valued field, or to its residue field.

# References


[1] Y. Baisalov, B. Poizat, 'Paires de structures o-minimales', J. Symb. Logic 63 (1998), 570–578.

[2] S. Basarab, 'Relative elimination of quantifiers for Henselian valued fields', Ann. Pure Appl. Logic 53 (1991) 51-74.

[3] S. Basarab, F-V Kuhlmann, 'An isomorphism theorem for Heneslian algebraic extensions of valued fields', Man. Math. 77 (1992), 113–126.

[4] W. Baur, 'Die Theorie der Paare reell abgeschlossner Körper', Logic and algorithmic (in honour of E. Specker), Monographie No. 30 de L'Enseignement Mathématique, Université de Genève, Geneva 1982, 25-34.

[5] E. Casanovas, R. Farré, 'Weak forms of elimination of imaginaries', Math. Logic Quart. 50 (2004), 126–140.

[6] Z. Chatzidakis, E. Hrushovski, 'Model theory of difference fields', Trans. Amer. Math. Soc. 351 (1999), 2997–3051.

[7] G. Cherlin, M. Dickmann, "Real closed rings II: model theory', Ann. Pure Appl. Logic 25 (1983), 213–231.

[8] R. Cluckers, F. Loeser, '$b$-minimality', preprint.

[9] J. Denef, L. van den Dries, '$p$-adic and real subanalytic sets', Ann. Math. 128 (1988), 79–138.

[10] M. Dickmann, 'Elimination of quantifiers for ordered valuation rings', J. Symb. Logic 52 (1987), 116–128.

[11] L. van den Dries, D. Haskell, H.D. Macpherson, 'One-dimensional $p$-adic subanalytic sets', J. London Math. Soc. (2) 59 (1999), 1–20.

[12] D. Haskell, H.D. Macpherson, 'Cell decompositions of $C$-minimal structures', Ann. Pure Appl. Logic 66 (1994), 113–162.

[13] D. Haskell, H.D. Macpherson, 'A version of o-minimality for the $p$-adics', J. Symb. Logic 62 (1997), 1075–1092.

[14] D. Haskell, E. Hrushovski, H. D. Macpherson, 'Definable sets in algebraically closed valued fields: elimination of imaginaries', J. reine und angew. Math. 597 (2006), 175–236.

[15] D. Haskell, E, Hrushovski, H.D. Macpherson, *Stable domination and independence in algebraically closed valued fields*, Lecture Notes in Logic, Cambridge Unversity Press, 2008.

[16] J.E. Holly, *Definable equivalence relations and disc spaces of algebraically closed valued fields'*, PhD thesis, University of Illinois, 1992.

[17] J.E. Holly, 'Canonical forms for definable subsest of algebraically closed and real closed valued fields', J. Symb. Logic 60 (1995), 843–860.

[18] E. Hrushovski, 'Unidimensional theories are superstable', Ann. Pure Appl. Logic 50(1990), 117–138.

[19] E. Hrushovski, D. Kazhdan, 'Integration in valued fields', Algebraic Geometry and Number Theory, Progr. Math.vol. 253, Birkhäuser Boston, Boston, MMA, 2006, 261–405.

[20] E. Hrushovski, B. Martin, 'Zeta functions from definable equivalence relations', http://arxiv.org/abs/math.LO/0701011/

[21] E. Hrushovski, 'Valued fields, metastable groups', preprint.

[22] E. Hrushovski, A. Pillay, 'On NIP and invariant measures', preprint.

[23] I. Kaplanskiy, 'Maximal fields with valuations I', Duke Math. J., 9 (1942), 303–321.

[24] I. Kaplansky, 'Maximal fields with valuations. II', Duke Math. J. 12 (1945), 243–248.

[25] W. Krull, 'Allgemeine Bewertungstheorie', J. für Mathematik, 167 (1932), 160–196.

[26] S. Lang, *Algebra* (2nd. Ed.), Addison-Wesley, 1984.

[27] L. Lipshitz, 'Rigid subanalytic sets', Amer. J. Math. 115 (1993), 77–108

[28] L. Lipshitz, Z. Robinson, 'One-dimensional fibers of rigid subanalytic sets', J. Symb. Logic, 63 (1998), 83–88.

[29] A. Lubotzky, A.R. Magid, 'Varieties of representations of finitely generated groups', Mem. Amer. Math. Soc. 58 (1985), no. 336.

[30] F. Maalouf, 'Construction d'un groupe dans les structures $C$-minimales', preprint.

[31] H.D. Macpherson, D. Marker, C. Steinhorn, 'Weakly o-minimal structures and real closed fields', Trans. Amer. Math. Soc. 352 (2000), 5435–5483.

[32] H.D. Macpherson, C. Steinhorn, 'On variants of o-minimality', Ann. Pure Appl. Logic 79 (1996), 165–209.

[33] D. Marker, *Model Theory: An Introduction*, Graduate Texts in Mathematics, Springer, 2002.

[34] T. Mellor, 'Imaginaries in real closed valued fields', Ann. Pure Appl. Logic 139 (2006), 230–279.

[35] J. Pas, 'Uniform $p$-adic cell decomposition and local zeta functions', J. reine und angew. Math., 399 (1989), 137–172.

[36] J. Pas, 'On the angular component modulo $p$', J. Symb. Logic 55 (1990), 1125–1129.

[37] B. Poizat, *Cours de Theorie des Modeles: Nur al mantiq wal ma'arifah (A Course in Model Theory: an Introduction to Contemporary Mathematical Logic*, Universitext, Springer-Verlag, New York 2000 (translated from the French by M. Klein and revised by the author).

[38] B. Poizat, 'Une théorie du Galois imaginaire', J. Symb. Logic, 48 (1983)1151–1170.

[39] B. Poizat, *Groupes Stables*, Nur al-Mantiq wal-Ma'riah 1987. Translated as *Stable Groups*, Amer. Math. Soc., Providence, RI, 2001.

[40] P. Ribenboim, *The Theory of Classical Valuations*, Springer Monographs in Mathematics, 1998.

[41] A. Robinson, *Complete Theories*, North-Holland, Amsterdam, 1956.

[42] T. Scanlon, 'A model complete theory of valued $D$-fields', J. Symb. Logic 65 (2000), 1758–1784.

[43] T. Scanlon, 'Quantifier elimination for the relative Frobenius', in *Valuation theory and its applications, Vol. II*, Conference Proceedings of the International Conference on Valuation Theory (Saskatoon 1999), AMS, Providence, 2003, 323–352.

[44] S. Shelah, *Classiifcation Theory* (2nd Ed.), North-Holland, Amsterdam, 1990.

[45] P. Simonetta, 'On non-abelian $C$-minimal groups', Ann. Pure Appl. Logic 122 (2003), 263–287.