Imaginaries in algebraically closed valued fields

David Lippel   Dugald Macpherson

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1 Introduction

These notes are intended to accompany the tutorial series 'Model theory of algebraically closed valued fields' in the Workshop 'An introduction to recent applications of model theory', Cambridge March 29–April 8, 2005. They do not contain any new results, except for a slightly new method of exposition, due to Lippel, of parts of the proof of elimination of imaginaries, in Sections 8 and 9. They form an exposition of [4], and, much more briefly, [5]. These papers are also surveyed in [6].

The plan of the tutorial series is as follows: the first lecture and a half will be devoted to background on algebraically closed valued fields, the meaning of 'elimination of imaginaries', with examples, and description of the sorts in which algebraically closed valued fields eliminate imaginaries. The next two and a half lectures will be devoted to (parts of) the proof of this theorem. The final session will be on stable domination and orthogonality to the value group.

2 Elimination of imaginaries

Definition 2.1 *A complete theory T has* elimination of imaginaries *if for each $M \models T$, each $n > 0$, and each $\emptyset$-definable equivalence relation E on M^n, there is $m > 0$ and an $\emptyset$-definable function $f_E : M^n \to M^m$ such that for all $x, y \in M^n$, we have $E(x, y) \leftrightarrow f(x) = f(y)$.*

An *imaginary* of a structure M is an equivalence class of an $\emptyset$-definable equivalence relation on a power of M. The idea of this definition is that for $a \in M^n$, $f(a)$ provides a code for the E-class of A.

Given a complete theory T over a language L, we can form a multisorted language L^{eq}, a complete L^{eq}-theory T^{eq}, and extend each $M \models T$ to a structure $M^{\mathrm{eq}} \models T^{\mathrm{eq}}$ as follows. For each n and $\emptyset$-definable equivalence relation E on M^n, add a sort interpreted in M^{eq} by M^n/E, and a function symbol for the natural map $f : M^n \to M^n/E$. The theory T^{eq} has elimination of imaginaries.

In stability theory, the move from M to M^{eq} is harmless (e.g. it preserves stability). Thus, one typically works in M^{eq}, so has elimination of imaginaries. This ensures, for example, existence of canonical bases for types. However, when dealing with fine questions of definability for particular algebraic structures, e.g. on the structure of definable groups, it can be essential to work with definable quotients. Here, it is helpful to prove elimination of imaginaries for M itself, or for M expanded by as few sorts from M^{eq} as possible, as in [4].

In the absence of field structure one often cannot code finite sets, and can only hope for *weak* elimination of imaginaries. A theory T with large saturated model $\mathcal{U}$ has *weak elimination of imaginaries* if for every imaginary $e \in \mathcal{U}^{\mathrm{eq}}$ there is a tuple a of $\mathcal{U}$ such that $a \in \mathrm{acl}(e)$ and $b \in \mathrm{dcl}(a)$. Here, as throughout thes notes, $\mathrm{acl}(A)$ denotes the set of elements, possibly imaginary, which satisfy a formula (e.g. of L^{eq})) over A which has finitely many realisations; and $\mathrm{dcl}(A)$ is the set of elements which satisfy a formula over A with a *unique* realisation.

Example 2.2 1. The theory of an infinite pure set does not have full elimination of imaginaries, since it is not possible to code an unordered pair by a tuple: more formally, we cannot code classes of the equivalence relation E where $(a_1, a_2)E(b_1, b_2)$ if and only if $\{a_1, a_2\} = \{b_1, b_2\}$. For the same reason, the theory of an infinite dimensional vector space over a finite field F_q (in say the language of F_q-modules) does not have elimination of imaginaries. Both theories have weak elimination of imaginaries (as does the expansion of the vector space by a non-degenerate alternating bilinear form). There are various ways to prove such results. See for example [2], or [3] for a different approach, in which it is proved via the *small index property.*

2. For a real closed field R (or for any o-minimal structure which expands an ordered field) one can prove elimination of imaginaries by showing that for any $\emptyset$-definable equivalence relation E on R^n, there is an $\emptyset$-definable function $f : R^n \to R^n$ which picks a representative of each E-class. This holds because any definable set in 1-variable can be defined with canonical choice of parameters (e.g. the endpoints of an interval), and because there are definable Skolem functions; for the latter the main point is that with the additive group structure we may canonically choose the midpoint of a bounded interval.

3. The theory of algebraically closed fields has elimination of imaginaries. As a first observation, we can code the finite *set* $\{a_1, \ldots, a_n\}$ by the coefficients of the polynomial $\Pi_{i=1}^n (X - a_i)$ (a construction not available in (1) above). A slight extension of this codes finite sets of *tuples.* There are then two alternative approaches: an algebraic method, which rests on quantifier elimination and the fact that any affine variety has a unique smallest field of definition. And a purely model-theoretic argument of Pillay (see Section 3.2 of [18]) which suffices for weak elimination of imaginaries in any strongly minimal set.

Suppose that $\mathcal{U}$ is any structure, assumed to be 'sufficiently saturated'. Given a definable relation R, we say that $e \in \mathcal{U}^{\mathrm{eq}}$ is a *code for* R provided for each $\sigma \in \mathrm{Aut}(\mathcal{U})$, we have $\sigma(R) = R$ if and only if $\sigma(e) = e$. Each definable relation R has a code in $\mathcal{U}^{\mathrm{eq}}$: if R is defined by the formula $\phi(x, c)$, let E be the equivalence relation $y_1 E y_2 \Leftrightarrow \mathcal{U} \models \forall x(\phi(x, y_1) \leftrightarrow \phi(x, y_2))$; then, c/E is a code for R. Note that any two codes for R are interdefinable.

If we can find a tuple from $\mathcal{U}$ that is a code for R, we say that R is *coded in*

$\mathcal{U}$. Let $e' \in \mathcal{U}^{\mathrm{eq}}$ be any code for R. Then R is coded in $\mathcal{U}$ if e' is interdefinable with a tuple from $\mathcal{U}$, i.e. $e' \in \mathrm{dcl}(\mathrm{dcl}(e') \cap \mathcal{U})$.

Proposition 2.3 *Assume that $\mathcal{U}$ has two definable elements. Then, $T = Th(\mathcal{U})$ eliminates imaginaries if and only if every definable relation on $\mathcal{U}$ is coded in $\mathcal{U}$.*

Sometimes, Definition 2.1 is called *uniform* elimination of imaginaries, and the condition of Proposition 2.3 is called elimination of imaginaries; see, for example [8]. The distinction is often blurred because many natural theories have a supply of definable elements.

Since any two codes for R are interdefinable, we abuse notation by writing $\ulcorner R \urcorner$ for some (or any) code. Thus, a relation R is coded in $\mathcal{U}$ if and only if R is definable over $\mathrm{dcl}(\ulcorner R \urcorner) \cap \mathcal{U}$.

Our strategy for elimination of imaginaries starts with the following simple observation, which we successively refine below.

For an extensive review of a number of variants on elimination of imaginaries, see [1].

3 Valued fields

Let F be a field, and Γ an ordered abelian group. A *valuation* to Γ is a function $v : F \to \Gamma \cup \{\infty\}$ such that

(i) $v(x) = \infty$ if and only if $x = 0$,

(ii) $v(xy) = v(x) + v(y)$, and

(iii) $v(x + y) \geq \mathrm{Min}\{v(x), v(y)\}$.

(Here, $\gamma < \infty$ for $\gamma \in \Gamma$, and $+$ extends naturally to $\Gamma \cup \{\infty\}$).

If (F, Γ, v) is a valued field, then $\mathcal{O} := \{x \in F : v(x) \geq 0\}$ is a subring of F, the *valuation ring*. It has a unique maximal ideal $\mathcal{M}$, namely $\{x \in F : v(x) > 0\}$. The quotient $\mathcal{O}/\mathcal{M}$ is a field, the *residue field*.

Example 3.1 Let p be a prime. Define a valuation v_p on the rational field $\mathbb{Q}$, with values in $\mathbb{Z}$: for $r \in \mathbb{Q}$, write $r = p^n a/b$ ($n \in \mathbb{Z}$, p coprime to a, b). Put $v_p(r) = n$.

From this valuation, we may define a norm satisfying the *ultrametric* inequality ($|ab| \leq \mathrm{Max}\{|a|, |b|\}$): put $|r| = p^{-v_p(r)}$. This is a metric, and the standard completion procedure yields the p-adic field $\mathbb{Q}_p$, which inherits a valuation to $\mathbb{Z}$.

Model-theoretically, there is an issue of how to present a valued field (K, v, Γ). Perhaps the most natural formalism is to have a sort for K (endowed with the language $(+, -, \times, 0, 1)$ of rings), a sort for $\Gamma \cup \{\infty\}$, viewed as an abelian group with an extra point (usually forgotten), and a function symbol for v. An alternative is to work with a single sort K, but add a predicate for the valuation ring, or a binary predicate $\mathrm{Div}(x, y)$ which expresses $v(x) \leq v(y)$. All of these presentations have the same universe of interpretable sets.

Observe that if char(K) has prime characteristic p, then char(k) $= p$. However, if char(K) $= 0$, then k can have characteristic 0 or p.

We shall let $L_{\rm rings}$ be the language $(+, -, \times, 0, 1)$ of rings, and $L_{\rm rob}$ consist of this together with the binary predicate Div (interpreted as above). Write $L_{\rm val}$ for the language above with sorts $K, \Gamma \cup \{\infty\}$.

4 Algebraically closed valued fields

Suppose K is an algebraically closed field with a surjective valuation $v : K \to \Gamma$, where Γ is non-trivial. Write R for the valuation ring, $\mathcal{M}$ for the maximal ideal, and k for the residue field. Write res for the map $R \to k$.

The ordered group Γ is divisible: indeed, for $\gamma \in \Gamma$ there is $a \in K$ with $v(a) = \gamma$. Find $b \in K$ with $b^n = a$. Then $nv(b) = \gamma$.

Likewise, k is algebraically closed. To see this, let $p(x) = \Sigma_{i=0}^{n} \alpha_i x^i$ be a polynomial over k. For each i choose $a_i \in R$ with $\text{res}(a_i) = \alpha_i$, chosen so that $a_i = 0$ if $\alpha_i = 0$. As K is algebraically closed, there is $b \in K$ with $\Sigma a_i b^i = 0$. It can now be checked that $b \in R$, and res(b) is a root of $p(x)$.

Example 4.1 1. Let k be an algebraically closed field, and Γ a divisible ordered abelian group. With t an indeterminate, let $K := k((t^\Gamma))$ denote the set of all formal expressions $f = \Sigma a_\gamma t^\gamma$, where $a_\gamma \in k$, $\gamma \in \Gamma$, and the *support* $\text{supp}(f) = \{\gamma \in \Gamma : a_\gamma \neq 0\}$, is well-ordered. Addition and multiplication are defined as for ordinary power series, the well-ordering of support ensuring that the sum in the definition of multiplication is finite. Now there is a surjective valuation $v : K \to \Gamma$ with $v(f) = \text{Min}(\text{supp}(f))$, and the residue field is k. The field K is algebraically closed, with char(K) $=$ char(k),

This construction is related to that of *Puiseux series*. With k as above, we let L be the field of Puiseux series

$$L = k \cup k((t^{\mathbb{Z}})) \cup k((t^{\mathbb{Z}/2!})) \cup \ldots \cup k((t^{\mathbb{Z}/n!})) \ldots$$

where $\mathbb{Z}/n!$ denotes the group of rationals $a/n!$ where $a \in \mathbb{Z}$. Clearly $k \subset L \subset k((t^{\mathbb{Q}}))$. The field L is henselian, its residue field is k, and its value group is $\mathbb{Q}$. If char(k) $= 0$, then L is algebraically closed, so elementarily equivalent to $k((t^{\mathbb{Q}}))$. If char(k) $= p$, however, then the polynomial $y^p - y = t^{-1}$ has no solution in L, so L is not algebraically closed. This is an example of how the Ax-Kochen/Ershov principles do not work smoothly in characteristic p.

2. For another example, recall that by Chevalley's place extension theorem, if $K < L$ is an extension of fields, and v is a valuation on K, then v extends to a valuation on L. Thus, we may obtain an algebraically closed valued field by extending the valuation on $\mathbb{Q}_p$ (with value group $\mathbb{Z}$) to a valuation on its algebraic closure $\tilde{\mathbb{Q}}_p$ (with value group $\mathbb{Q}$). More commonly, one goes one step further, to $\mathbb{C}_p$, the completion of $\tilde{\mathbb{Q}}_p$ with respect to the $\mathbb{R}$-valued norm from a valuation; $\mathbb{C}_p$ is also algebraically closed (characteristic 0, with residue characteristic p).

Theorem 4.2 (Robinson) *Let* (K, v, Γ) *be an algebraically closed valued field.*

(i) $\mathrm{Th}(K)$ *is model-complete when parsed in the language* L_{val}*, and has quantifier elimination in* L_{rob}*.*

(ii) The completions of K *are determined by the pair* $(\mathrm{char}(K), \mathrm{char}(k))$*.*

From quantifier elimination, it is easy to verify that the model theoretic algebraic closure operation in an algebraically closed valued field is the same as in the pure field, i.e. is field-theoretic algebraic closure. However, definable closure is modified. Also, it is easily checked that the value group is *o-minimal*, in the sense that any subset of Γ, definable with parameters from K, is a finite union of intervals and singletons. Likewise, the residue field is *strongly minimal.*

In fact, we can prove more. If M is a structure, and A is a set a-definable in M^{eq}, then A is *stably embedded* if, for any r and any definable set D of M^{eq}, the set $D \cap A^r$ is definable over Aa (uniformly in the parameters which define D, so the condition carries through to elementary extensions).

Let $L_{\Gamma k}$ be a 3-sorted language for algebraically closed valued fields, with a sort for K (equipped with the usual language of rings), a sort for $\Gamma \cup \{\infty\}$, with the ordered abelian group language, a sort for the residue field k, again with the language of rings, and a map $\mathrm{Res} : K^2 \to k$ given by $\mathrm{Res}(x, y) = \mathrm{res}(xy^{-1})$.

Proposition 4.3 (2.1.1, 2.1.3 of [4]) *(i) Algebraically closed valued fields have quantifier-elimination in* $L_{\Gamma k}$*.*

(ii) If (K, v, Γ) *is an algebraically closed valued field, then the value group* Γ *and residue field* k *are stably embedded.*

If $\gamma \in \Gamma \cup \{\infty\}$ and $a \in K$, define $B_{>\gamma}(a) := \{x \in K : v(x - a) > \gamma\}$ and $B_{\geq\gamma}(a) := \{x \in K : v(x - a) \geq \gamma\}$, respectively called *open* and *closed balls* of radius γ with centre a. By basic properties of valuations, if two balls have non-trivial intersection, then one contains the other. We occasionally regard K as a ball of radius $-\infty$. Observe that a singleton is a closed ball of radius ∞, and, by the ultrametric inequality, any point in a ball is a centre of it. The collection of balls of finite radius forms a (uniformly definable) basis for a topology on K, with respect to which K is a topological field. The terms *open* and *closed* are misleading – all balls of finite radius are open and closed in this topology. As an easy consequence of quantifier elimination in L_{rob}, we obtain

Proposition 4.4 (3.26 of [10]) *Let* K *be an algebraically closed valued field. Then any definable subset of* K *is a boolean combination of balls. Furthermore, this Boolean combination can be chosen in a canonical way.*

For canonicity, one expresses the definable set as a disjoint union of 'trivially nested Swiss cheeses' – see [10] (or Theorem 2.1.2 of [4]).

We remark that the intersection properties of the balls allow one to identify a valued field with a set of maximal chains in a certain treelike partial order (a semilinear order). This point of view is developed in [17] and [6], where it is shown that algebraically closed valued fields are precisely the 'C-minimal' fields (an analogue of an o-minimal ordered field); indeed, C-minimality follows from the last proposition. The combinatorics of trees provides a useful way of thinking about the genericity discussed in Section 6.

In these notes we focus on elimination of imaginaries. Algebraically closed valued fields do not have elimination of imaginaries just to the sort K. Indeed, k and Γ are interpretable but not definable in K. (As noted by Holly, to see non-definability, one can use a dimension argument, using that algebraic closure has the exchange property.) It is not sufficient to add Γ and k as sorts, for one cannot eliminate the equivalence relations on K^2 defined by $B_{\geq v(x-y)}(x) = B_{\geq v(x'-y')}(x')$; this equivalence relation expresses that the smallest closed ball containing the pair $\{x, y\}$ is the same as the smallest closed ball containing $\{x', y'\}$, so the classes code closed balls. This equivalence relation can be eliminated by adding a sort from M^{eq} for all closed balls, and similarly one could add a sort for all open balls to eliminate a related equivalence relation. Holly ([9], [10]) began an investigation of whether this would suffice, and originally the authors thought that it would. In fact, however, one needs infinitely many sorts for elimination of imaginaries: see Proposition 10.1 below.

We introduce the additional sorts needed for elimination of imaginaries. Let S_n be the set of all free rank n R-submodules of K^n, that is, R-lattices in K^n. This can be regarded as a quotient of a 0-definable equivalence relation. Formally, let

$$U_n := \{(x_1, \ldots, x_n) \in K^n : x_1, \ldots, x_n \text{ generate a free } R\text{-module}\},$$

and define an equivalence relation E_n on U_n, putting $E(\bar{x}, \bar{y})$ if $\bar{x}, \bar{y}$ generate the same R-lattice. Then U_n/E_n is canonically identified with S_n. We shall have a sort for each S_n ($n \geq 1$).

We can regard S_n as a coset space. Indeed, $\mathrm{GL}_n(K)$ acts transitively on the set U above. The standard basis of K^n generates the R-lattice R^n, and the stabiliser in $\mathrm{GL}_n(K)$ of this lattice is exactly $\mathrm{GL}_n(R)$. Thus, S_n can be identified with the coset space $\mathrm{GL}_n(K)/\mathrm{GL}_n(R)$. If $B_n(K)$ denotes the group of invertible upper-triangular $n \times n$ matrices, then S_n can also be identified with $B_n(K)/B_n(R)$. This is essentially because the embedding of a lattice into K^n gives it a triangular basis.

Observe that S_1 can be identified with Γ, for given $x \in K \setminus \{0\}$, the free R-modules Rx is $\{x \in K : v(x) \geq \gamma\}$. Thus, S_1 is just the set of closed balls of finite radius containing 0.

There is a canonical way of identifying *any* closed ball with a member of S_2. Indeed, given a closed ball B not containing 0_K, let A be the lattice in K^2 generated by $\{1\} \times B$. Then $\{1\} \times B = A \cap (\{1\} \times K)$, and hence, a code for B is interdefinable over $\emptyset$ with a code for A. (In fact, the ball $B = B_{\geq v(\gamma)}(a)$ is just a torsor, i.e. 1-dimensional affine space for the lattice $R\gamma = B_{\geq v(\gamma)}(0)$ of S_1, and in general, any torsor of a member of S_n has code interdefinable with a code for a member of S_{n+1} – see 2.2.6 of [4].)

If A is an R-lattice in K^n, then $\mathcal{M}A$ is an R-submodule of A, and the quotient $\mathrm{red}(A) := A/\mathcal{M}A$ carries (definably over a code for A) the structure of an n-dimensional k-vector space. Define $T_n := \bigcup\{\mathrm{red}(A) : A \in S_n\}$, the union of a set of n-dimensional k-spaces. This can also be regarded as a quotient of an $\emptyset$-definable equivalence relation, and as a coset space. See Section 2.4 of [4] for more of this. One identifies T_n with $\bigcup_{i=1}^m B_n(K)/B_{n,m}(R)$, where $B_{n,m}(R)$ is the subgroup of $B_n(R)$ consisting of matrices which, when reduced modulo $\mathcal{M}$, have mth column consisting of a 1 on the diagonal and zeros elsewhere.

An important point in certain arguments is that if M is an algebraically closed valued field, and A is an M-definable lattice, then M contains an R-basis of A. Hence, there is an M-definable R-module isomorphism $A \to R^n$, and in many arguments we may replace A by R^n. Likewise, if $V = \mathrm{red}(A)$ then there is an M-definable k-basis of V, and hence an M-definable isomorphism $V \to k^n$.

The sorts in which we work are K, S_n, T_n ($n \geq 1$). We call these the ACVF sorts. Just as Γ is identifiable with S_1, k is identifiable with $\mathrm{red}(R) \subset T_1$. Like S_1, the sort T_1 carries the structure of a group: the ball $B_{>v(a)}(a)$ is just the coset $a(1+\mathcal{M})$ in the multiplicative group K^*, so T_1 is in bijection with $K^*/(1+\mathcal{M})$.

The main theorem of these notes is the following.

Theorem 4.5 *Let T be a complete theory of algebraically closed valued fields. Then T has elimination of imaginaries to the sorts K, S_n, T_n ($n \geq 1$).*

For the rest of these notes, $\mathcal{U}$ will denote a large sufficiently saturated model of ACVF, so a structure in the sorts K, S_n, T_n (≥ 1). We shall use K to denote the field sort of $\mathcal{U}$, so the field sort of an elementary submodel M will be $M \cap K$. Likewise, Γ will denote the value group of $\mathcal{U}$, and k the residue field of $\mathcal{U}$. If $A \subset \mathcal{U}^{\mathrm{eq}}$ then $\Gamma(A) := \mathrm{dcl}(A) \cap \Gamma$ and $k(A) := \mathrm{dcl}(A) \cap k$. Occasionally, it is convenient to emphasise that we are referring to these 'home' sorts K, S_n, T_n (for $n \geq 1$) rather than the whole of $\mathcal{U}^{\mathrm{eq}}$. We refer to them collectively as G (the 'geometric' sorts), and for $A \subset \mathcal{U}^{\mathrm{eq}}$ we write $\mathrm{acl}_G(A)$ to mean the elements of these sorts (rather than arbitrary) imaginaries) which lie in $\mathrm{acl}(A)$; likewise for $\mathrm{dcl}_G(A)$.

5 The k-internal sorts

We turn next to the stable part of $\mathcal{U}$.

Definition 5.1 *A definable set D of $\mathcal{U}$ is k-internal if there is finite F such that $D \subset \mathrm{dcl}(k \cup F)$.*

The following lemma has a fairly elementary proof (Lemma 2.6.2 of [4]).

Lemma 5.2 *Let C be a parameter set (possibly consisting of imaginaries), and $D \subset \mathcal{U}^l$ be C-definable. Then the following are equivalent.*

(i) D is k-internal.

(ii) D (expanded by predicates for the C-definable relations) has finite Morley rank.

(iii) D (expanded by C-definable structure as in (ii)) does not have the strict order property.

(iv) D is (after permutation of coordinates) contained in a finite union of sets $\mathrm{red}(A_1) \times \ldots \times \mathrm{red}(A_n) \times F$*, where the* A_i *are* $\mathrm{acl}(C)$*-definable lattices, and* F *is a* C*-definable finite set of tuples from* $\mathcal{U}$*.*

We now let $\mathrm{Int}_{k,C}$ be the many sorted structure whose sorts are the k-vector spaces $\mathrm{red}(A)$ where A is a C-definable lattice. Each such vector space is equipped with its k-vector space structure, and we also endow $\mathrm{Int}_{k,C}$ with all the C-definable relations. For any C-definable lattice A, there is finite $E \subset \mathrm{red}(A)$ such that $\mathrm{red}(A) \subset \mathrm{dcl}(k \cup E)$: just choose E to be a k-basis of $\mathrm{red}(A)$. Since k is stably embedded, it follows easily from this that $\mathrm{Int}_{k,C}$ is stably embedded, and is stable.

Proposition 5.3 *Let C be a set of imaginary parameters.*
(i) $\mathrm{Int}_{k,C}$ *has elimination of imaginaries.*
(ii) If D is a C-definable k-internal subset of $\mathcal{U}^{\mathrm{eq}}$ *then* $D \subset \mathrm{dcl}(C \cup \mathrm{Int}_{k,C})$*.*

Sketch Proof. (i) We must show that any definable subset of $\mathrm{Int}_{k,C}$ is coded by a tuple of $\mathrm{Int}_{k,C}$. The first step is to reduce to coding definable subsets of a single C-definable lattice. For this, note that if $U \subset \mathrm{red}(A_1)^{i_1} \times \ldots \times \mathrm{red}(A_k)^{i_k}$ is C-definable, then U is interdefinable over C with some definable subset of $\mathrm{red}(A_1^1 \times \ldots \times A_k^{i_k})$.

Next, it can be shown that the collection of sorts in $\mathrm{Int}_{k,C}$ is closed under certain natural operations: exterior powers, duals, and tensors. One uses this to show that if $A \in S_n$, then any subspace of the k-vector space $\mathrm{red}(A)$ is coded in $k \cup \bigcup_{n \geq 1} T_n$. For this, note that an l-dimensional subspace of $\mathrm{red}(A)$ is coded by a 1-dimensional subspace of $\bigwedge^l(\mathrm{red}(A))$, and the latter is identifiable with $\mathrm{red}(\bigwedge^l(A))$; also, $\bigwedge^l(A) \in S_N$, where $N = \binom{n}{l}$. Thus, it suffices to code 1-dimensional subspaces of $\mathrm{red}(A)$ in $k \cup \bigcup_{n \geq 1} T_n$. For details, see Lemma 2.6.4 of [4].

Finally, suppose that Y is a definable subset of $V = \mathrm{red}(A)$, where $A \in S_n$ is C-definable. We may talk of Zariski closed subsets of V: for given a basis of V, we can identify V with k^n, and the corresponding notion of 'Zariski closed' is independent of the choice of basis. Since any definable subset of V is a Boolean combination of Zariski closed sets which are definable over the same parameters, we may suppose that Y is Zariski closed. Now given any fixed basis of V there is a corresponding dual basis of the dual space V^*. This enables us to identify the polynomial ring $k[X_1, \ldots, X_n]$ with the ring $S(V) = k \oplus V^* \oplus \Sigma_{i=2}^{\infty} \mathrm{Sym}^i(V^*)$, where $\mathrm{Sym}^i(V^*)$ is the ith symmetric power of V^*. Elements of $S(V)$ induce functions $V \to k$ independently of the choice of basis, and the ideal I in $S(V)$ which vanishes on Y is independent of the choice of basis, and determines Y. Let $I_m = I \cap (k \oplus V^* \oplus \Sigma_{i=2}^{m} \mathrm{Sym}^i(V^*))$. It suffices to show that each I_m is coded in $\mathrm{Int}_{k,C}$ and for this one must code its pullback in $T^m(V) = k \oplus V^* \oplus \Sigma_{i=2}^{m} \otimes^i (V^*)$. This pullback is a subspace of $T_m(V)$. By the closure properties of $\mathrm{Int}_{k,C}$ mentioned above, and the coding of subspaces, this pullback is coded by a tuple from $\mathrm{Int}_{k,C}$.

We omit the proof of (ii).

If M is a model, and A is an M-definable rank n lattice, then, as noted in Section 4, there is an M-definable isomorphism of k-vector spaces between $\mathrm{red}(A)$ and k^n. Hence, in this case, we do not need the full structure of $\mathrm{Int}_{k,M}$ – the residue field k itself would suffice.

6 1-types of field elements

We describe here 1-types in the field sort over any set C of (possibly imaginary) parameters. First, we shall say that $U \subset K$ is a *$C-\infty$-definable ball* if U is one of: a C-definable closed ball, a C-definable open ball, or the intersection of a chain of C-definable balls, where the chain has no least element.

Lemma 6.1 *Suppose $C \subset K^{\mathrm{eq}}$ with $C = \mathrm{acl}(C)$. Let U be a $C-\infty$-definable ball. Suppose $a, b \in U$ (so a, b are field elements) and no C-definable sub-ball of U contains a or b. Then $\mathrm{tp}(a/C) = \mathrm{tp}(b/C)$.*

Proof. Immediate from Proposition 4.4.

Definition 6.2 *Let $C \subset K^{\mathrm{eq}}$ and U be a $\mathrm{acl}(C)-\infty$-definable ball. If $a \in U$, then a is* generic in U over C *if a lies in no $\mathrm{acl}(C)$-definable proper sub-ball of U.*

Lemma 6.3 *Let $C = \mathrm{acl}(C) \subset K^{\mathrm{eq}}$, and U be a $C-\infty$-definable ball. Then*
(i) if a, b are generic in U over C, then $\mathrm{tp}(a/C) = \mathrm{tp}(b/C)$, and
(ii) if $a \in U$ then a realises the generic type over C of a unique $C-\infty$-definable sub-ball of U.

Proof. (i) Again, this is immediate from Proposition 4.4.
(ii) Let $\{U_i : i \in I\}$ be the set of C-sub-balls of U which contain a. This set of balls is totally ordered by inclusion (as they are not disjoint), and a realises the generic type of the $C-\infty$-definable ball $\bigcap(U_i : i \in I)$.

Definition 6.4 *Suppose that $C, B \subset K^{\mathrm{eq}}$ and $C = \mathrm{acl}(C) \subset \mathrm{dcl}(B)$. If $a \in K$, then a is* generically independent *from B over C, written $a \downarrow^g_C B$, if either $a \in \mathrm{acl}(C)$, or, if U is the $C-\infty$-definable ball (given by Lemma 6.3(ii)) such that a is generic in U over C, then a is also generic in U over $C \cup B$.*

Thus, $a \downarrow^g_C B$ means that $B \cup C$ doesn't pin a down more finely than C does, by forcing it into a ball of smaller radius. Compare the algebraic-geometric notion of generic, that a is generic in the C-definable variety V over $C \cup B$ if a does not lie in any $C \cup B$-definable proper subvariety of C.

Lemma 6.5 *Suppose $C = \mathrm{acl}(C) \subset \mathrm{dcl}(B)$, and p is a 1-type in the field sort over C. Then p has a unique extension q over B such that if $a \models q$, then $a \downarrow^g_C B$.*

Proof. We may suppose that p is non-algebraic. Then by Lemma 6.3, p is the generic type of a $C - \infty$-definable ball U. The type q has to be the (unique) generic type of U over $C \cup B$ (and this exists).

Definition 6.6 *Let* $C = \mathrm{acl}(C)$ *and* $a \in K$. *We say* $\mathrm{tp}(a/C)$ is orthogonal to Γ*, and write* $\mathrm{tp}(a/C) \perp \Gamma$*, if, for any algebraically closed field* M *such that* $C \subset \mathrm{dcl}(M)$ *and* $a \downarrow^g_C M$*, we have* $\Gamma(M) = \Gamma(Ma)$.

Lemma 6.7 *Let* $C = \mathrm{acl}(C)$ *and* $a \in K$. *Then the following are equivalent.*
(i) a *is generic over* C *in a* C*-definable closed ball.*
(ii) $\mathrm{tp}(a/C) \perp \Gamma$.

Proof. (i) $\Rightarrow$ (ii) Suppose that a is generic over C in the C-definable closed ball U, and that (ii) is false. Then there is a model M with $a \downarrow^g_C M$ and some $\gamma \in \Gamma(Ma) \setminus \Gamma(M)$. Thus, there is an M-definable partial function $f : U \to \Gamma$ with $f(a) = \gamma$, whose domain is a definable set D containing generic elements of U over C. If V is an open sub-ball of U of the same radius, consisting of generic elements of U (i.e. $V \in \mathrm{red}(U)$), then f is not constant on U: indeed, otherwise, f would induce a definable partial function from the strongly minimal set $\mathrm{red}(U)$ to the o-minimal set Γ, non-constant on generic elements of $\mathrm{red}(U)$, and this is clearly impossible. Thus, for generic $V \in \mathrm{red}(U)$, $\{f(x) : x \in V\}$ is an infinite definable subset of Γ, so a finite union of intervals and singletons of the latter. By considering the endpoints of these intervals, we obtain corresponding definable functions from $\mathrm{red}(U)$ to Γ. Since the latter must be generically constant, we may choose $\gamma \in \Gamma$ such that the fibre $f^{-1}(\gamma)$, a definable set, meets each generic element of $\mathrm{red}(U)$ in a proper non-empty subset. This contradicts Proposition 4.4.

(ii) $\Rightarrow$ (i). Suppose that (i) is false. Then, by Lemma 6.3(ii), a is generic in a C-∞-ball U which is an open C-ball or in the intersection of a chain of balls with no least element. There is a model M containing C, and containing some $b \in U$. By replacing M by a conjugate over C, we may suppose (using also Lemma 6.5) that $a \downarrow^g_C M$. However, now $v(a - b) \in \Gamma(Ma) \setminus \Gamma(M)$, so (ii) is false.

We remark that the generic type of a C-definable ball is definable over C. However, if U is a C-∞-definable ball which is not definable, then the generic type of U over $\mathcal{U}$ is not definable (see Lemma 2.3.8 of [4]).

7 Unary sets and unary codes

By the last section, 1-types of field elements are easily described, and by iterating the $\downarrow^g$ relation, one can easily define a notion $(a_1, \ldots, a_n) \downarrow^g_C B$, where $a_1, \ldots, a_n \in K$ (see Definition 11.2 below). However, in ACVF the S_n and T_n are also sorts, and it is convenient to define $s \downarrow^g_C B$ for $s \in S_n \cup T_n$. Also, in the proof of elimination of imaginaries, we need to code functions $\Gamma \to S_n$ and $\Gamma \to T_n$.

A *definable 1-module* is an R-module (in K^{eq}) which is definably isomorphic to a quotient of one definable R-submodule of K by another. Examples are $\gamma R/\delta R$, $\gamma R/\delta\mathcal{M}$, $K/\delta R$, etc., where $\delta \geq \gamma$. More generally, a *definable 1-torsor* is a definable torsor (i.e. 1-dimensional affine space) of a definable 1-module. Thus, a definable 1-torsor is essentially a ball, or for some γ and ball B, the set of all open sub-balls (or of all closed sub-balls) of B of radius γ. Just as in the last section we considered ∞-definable balls, meaning the intersection of a chain of balls possibly with no least element, so we talk of an ∞-*definable* 1-torsor: this is a coset of a 1-module which is ∞-definable, i.e. is the intersection of a chain of sdefinable 1-modules. The analogy with balls enables us to talk of a *generic element*, over parameters C, of a (possibly ∞-definable) 1-torsor, and most of the last section carries through. In fact, it is developed in the slightly more general setting of *unary sets*. These are 1-torsors, possibly ∞-definable, and subsets of Γ of the form (γ, ∞). There is a notion of generic element (over C) of a $C-\infty$ definable unary set. It can be shown that if $C = \mathrm{acl}(C)$ and a is an element of a C-definable unary set U, then a realises the generic type of a unique $C-\infty$-definable unary subset of U.

If $e \in \mathcal{U}^{\mathrm{eq}}$, then a sequence $(a_1, \ldots, a_m)$ of $\mathcal{U}^{\mathrm{eq}}$ is a *unary code* for e if $\mathrm{dcl}(e) = \mathrm{dcl}(a_1, \ldots, a_m)$ and for each $i = 1, \ldots, m$, a_i is an element of a unary set defined over $\mathrm{dcl}(a_j : j < i)$. The main point of this section is the following, which enables us to treat arbitrary elements of S_n and T_n as sequences of elements of unary sets, and thereby define a notion of genericity 'stepwise' (see Definition 11.2 and the remarks before it).

Proposition 7.1 (2.3.10 of [5]) *Let $s \in \mathcal{U}$. Then s has a unary code whose elements lie in $\mathcal{U}$.*

The idea of the last proposition, treating an element of S_n or T_n stepwise, is also used in the description of definable functions $\Gamma \to \mathcal{U}$; also in the proof that definable functions $\Gamma \to \mathcal{U}$ (and their germs below any $\gamma_0 \in \Gamma$) are coded in the sorts of $\mathcal{U}$. See Section 2.4, and Proposition 3.3.4 of [4] for details. These results are essential in the proof of elimination of imaginaries, for coding of (germs of) functions on open balls and ∞-definable balls.

8 A strategy for elimination of imaginaries

In this section, we discuss a general strategy for elimination of imaginaries, as adapted from [4]. Let $\mathcal{U}$ be any sufficiently saturated multi-sorted structure. Let $\mathcal{D}$ be a *dominant sort* of $\mathcal{U}$; this means that $\mathcal{D}^{\mathrm{eq}} = \mathcal{U}^{\mathrm{eq}}$. When specializing to $ACVF$, we take $\mathcal{U}$ to be the structure defined above, with sorts K, S_n, T_n (for $n \geq 1$), and we take $\mathcal{D}$ to be the field sort K. In this section, we distinguish between definable subsets of $\mathcal{U}$ (one free variable) and definable relations on $\mathcal{U}$ (more than one). For emphasis, we often specify a subset by $X \subset \mathcal{U}^1$.

Lemma 8.1 (Remark 3.2.2 of [4]) *$\mathcal{U}$ has elimination of imaginaries if and only if the graph of every definable function $f : \mathcal{D}^1 \to \mathcal{U}^1$ is coded.*

The next lemma says that under certain hypotheses, the job of coding functions is "localized."

Lemma 8.2 *Suppose that all definable subsets of $\mathcal{D}^1$ are coded in $\mathcal{U}$. Then, in order to code the graph of the definable function $f : \mathcal{D}^1 \to \mathcal{U}^1$ it suffices to show that*

$$\textit{for each } a \in \mathcal{D}^1,\ f(a) \in \mathrm{dcl}(a\,\mathrm{dcl}(\ulcorner f \urcorner) \cap \mathcal{U}). \tag{1}$$

Proof. For each $a \in \mathcal{D}^1$, let g_a be a function that is definable over $\mathrm{dcl}(\ulcorner f \urcorner) \cap \mathcal{U}$ so that $f(a) = g_a(a)$; the existence of g_a is guaranteed by (1). Note that $X_a = \{x \in \mathcal{D}^1 : g_a(x) = f(x)\}$ is $\ulcorner f \urcorner$-definable and is coded in $\mathcal{U}$, by hypothesis. Thus, X_a is definable over $\mathrm{dcl}(\ulcorner f \urcorner) \cap \mathcal{U}$. By compactness, $\mathcal{D}^1$ is covered by finitely many sets of the form X_a, and we can piece together the various functions g_a to get a definition of f over $\mathrm{dcl}(\ulcorner f \urcorner) \cap \mathcal{U}$.

The final refinement of our strategy is to separate the coding of finite sets from the general coding. For $e \in \mathcal{U}^{\mathrm{eq}}$, we define the *$\mathcal{U}$-coordinates* of e to be $\mathrm{coord}(e) = \mathrm{acl}(e) \cap \mathcal{U}$.

Proposition 8.3 *Suppose that for each $n \geq 1$, every finite subset of $\mathcal{U}^n$ is coded in $\mathcal{U}$. Then, for $e \in \mathcal{U}^{\mathrm{eq}}$, we have $e \in \mathrm{dcl}(\mathrm{dcl}(e) \cap \mathcal{U})$ if and only if $e \in \mathrm{dcl}(\mathrm{coord}(e))$.*

Using the proposition, we can state the strategy in its final form.

Theorem 8.4 *Let $\mathcal{U}$ be any sufficiently saturated multi-sorted structure; let $\mathcal{D}$ be a dominant sort. $\mathcal{U}$ has elimination of imaginaries if and only if the following three properties hold*

1. *For each $n \geq 1$, every finite subset of $\mathcal{U}^n$ is coded in $\mathcal{U}$.*
2. *Each definable subset of $\mathcal{D}^1$ is coded in $\mathcal{U}$.*
3. *Each definable function $f : \mathcal{D}^1 \to \mathcal{U}^1$ satisfies the condition*

$$\textit{for each } a \in \mathcal{D}^1, f(a) \in \mathrm{dcl}(a\,\mathrm{coord}(\ulcorner f \urcorner)). \tag{2}$$

Proof. It suffices to show that elimination of imaginaries is implied by the three conditions in the theorem. Let $f : \mathcal{D}^1 \to \mathcal{U}^1$ be any definable function. By modifying the argument for Lemma 8.2 in a straightforward way, we conclude that f is definable over $\mathrm{coord}(\ulcorner f \urcorner)$. Thus, $\ulcorner f \urcorner \in \mathrm{dcl}(\mathrm{coord}(\ulcorner f \urcorner))$. By Proposition 8.3, $\ulcorner f \urcorner \in \mathrm{dcl}(\mathrm{dcl}(\ulcorner f \urcorner \cap \mathcal{U})$. Equivalently, f is coded in $\mathcal{U}$. By Lemma 8.1, $\mathcal{U}$ eliminates imaginaries.

9 Elimination of imaginaries for ACVF

We return now to the specific setting of ACVF, so here $\mathcal{U}$ is the particular multisorted structure for ACVF defined in Section 4, and the dominant sort $\mathcal{D}$ is the field sort K. We give an overview of how one verifies the hypotheses of Theorem 8.4; details can be found in [4].

9.1 Coding definable subsets of K

Holly showed how to code definable subsets of K in terms of codes for balls [9]. (Holly works in equi-characteristic 0; §3.4 of [4] contains an argument that works in all characteristics.) Proposition 4.4 says that every definable subset X of K has a canonical decomposition as a (finite) boolean combination of balls. Specifically, X can be written canonically as a finite disjoint union of "swiss cheeses", i.e. sets of the form $B \setminus (E_1 \cup \ldots \cup E_l)$, where B is a ball and the E_i's are proper sub-balls. Each such swiss cheese is coded by $e = (\ulcorner B \urcorner, \{\ulcorner E_1 \urcorner, \ldots, \ulcorner E_l \urcorner\})$. In fact, the set $\{\ulcorner E_1 \urcorner, \ldots, \ulcorner E_l \urcorner\}$ can itself be coded by a tuple of balls [9]; thus, we may take e to be a tuple of balls. Because X has a canonical decomposition as a disjoint union of swiss cheeses, X is coded by a set $d = \{e_1, \ldots, e_n\}$ of codes for swiss cheeses; again, d is coded by a tuple of balls. Thus, we have coded X by a tuple of balls. Recall from Section 4 that closed balls are coded by elements of S_2 and open balls are coded by elements of T_2, so the coding of X can be carried out in $\mathcal{U}$.

9.2 Coding of finite sets

Let X be a finite subset of tuples from $\mathcal{U}$. We want to find a tuple in $\mathcal{U}$ that is a code for X. We may assume that X is a subset of $\mathcal{S}^l$ for some fixed sort $\mathcal{S}$ and some $l \geq 1$. A few cases are easy. For example, If $\mathcal{S}$ is the field K, we can code X as discussed in Example 2.2, and if $\mathcal{S}$ is $S_0 = \Gamma$, we can use the ordering to code X. As remarked above, the case that X consists of tuples of balls is handled in [9] (in equi-characteristic 0). Theorem 3.4.1 of [4] gives a complete proof (in all characteristics) that the finite set X is coded. The proof is a delicate simultaneous induction on $|X|$, l and n, where n is the subscript of the sort $\mathcal{S}$ (i.e. $\mathcal{S}$ is either S_n or T_n). The argument use the fact that finite subsets of $\mathrm{Int}_{k,C}$ are coded in $\mathrm{Int}_{k,C}$, which is a consequence of Proposition 5.3.

9.3 Coding functions

Here, we outline how to verify condition (2) in Theorem 8.4. Fix a definable function $f : K \to \mathcal{U}$. We write $f_c(x)$ when we want to display the parameters $c \in K^l$ that are used to define f. We may assume that the codomain of f is a single sort $\mathcal{S}$ of $\mathcal{U}$. Take $a \in K$. We want to show that $f(a) \in \mathrm{dcl}(a \,\mathrm{coord}(\ulcorner f \urcorner))$. Consider $p(x) = \mathrm{tp}(a/\mathrm{acl}(\ulcorner f \urcorner))$. From Lemma 6.1 above, we know that the set of realizations of p is the gneric type of an $\mathrm{acl}(\ulcorner f \urcorner)$-$\infty$-definable ball U. Notice that the restriction $p|\mathrm{coord}(\ulcorner f \urcorner)$ describes the same ball (because balls are coded in $\mathcal{U}$). We break into three cases: U is a definable closed ball, U is a definable open ball, and U is an intersection of a chain of definable balls. Most of the work is for the case of a closed ball; we handle the other cases by approximating U with closed balls.

9.3.1 U is a definable closed ball, [4, 3.3.2]

In this case, p is orthogonal to the value group (Lemma 6.7). We may assume that $a \downarrow^g_{\mathrm{coord}(\ulcorner f \urcorner)} c$, i.e. a is generic in U over $\mathrm{coord}(\ulcorner f \urcorner)c$. Let σ be an automorphism fixing $a\,\mathrm{coord}(\ulcorner f \urcorner)$. Let $c' = \sigma(c)$. We want to show that $f_c(a) = f_{c'}(a)$. First, we consider a special case.

Claim 9.1 *If $a \downarrow^g_{\mathrm{coord}(\ulcorner f \urcorner)} cc'$, then $f_c(a) = f_{c'}(a)$.*

Proof sketch for claim: Choose a model $M \prec \mathcal{U}$ so that $\mathrm{coord}(\ulcorner f \urcorner) \cup \{c, c'\} \subset M$ and $a \downarrow^g_{\mathrm{coord}(\ulcorner f \urcorner)} M$. (Such a model exists by Lemmas 6.1 and 6.5. Choose any model $M' \supset \mathrm{coord}(\ulcorner f \urcorner) \cup \{c, c'\}$ and choose a' generic in U so that $a' \downarrow_{\mathrm{coord}(\ulcorner f \urcorner)} M'$. Since a' and a have the same type over $\mathrm{coord}(\ulcorner f \urcorner) \cup \{c, c'\}$, we can find M by applying an automorphism.) Notice that the type $\mathrm{tp}(a/M)$ is definable over $\mathrm{coord}(\ulcorner f \urcorner)$, because for each M-definable ball U', the formula $x \in U'$ lies in $\mathrm{tp}(a/M)$ if and only if $U \subseteq U'$. Let $q(x, y) = \mathrm{tp}(af(a)/M)$; here, y is a variable for $\mathcal{S}$, the codomain of f. Since $f(a)$ is definable from a over $\ulcorner f \urcorner$, it is easy to see that the type q is definable over $\mathrm{acl}(\ulcorner f \urcorner)$. To prove the claim, it suffices to show that q is actually definable over $\mathrm{coord}(\ulcorner f \urcorner)$, because then the set $\{z \in K(M) : f_z(a) = f(a)\}$ is definable over $\mathrm{coord}(\ulcorner f \urcorner)$, so c' lies in this set.

Let us consider the special case that the codomain $\mathcal{S}$ is K, the field sort. Our first task is to find a simple set of formulas so that q is implied by its restriction to this set of formulas. First, since y is a field variable, we can use quantifier elimination to conclude that $q(x, y)$ is determined by its restriction to atomic and negated atomic L_{rob}-formulas, i.e. formulas of the form

$$v(P(x,y,z)) \leq v(Q(x,y,z)) \quad (3)$$
$$P(x,y,z) = 0, \quad (4)$$

and their negations, where $P, Q \in \mathbb{Z}[x, y, z]$. Because q is orthogonal to the value group, $\mathrm{dcl}(af(a)M) \cap \Gamma = \Gamma(M)$. In particular, the formula $P(x, y, b) = 0$ is in q if and only if $v(P(x, y, b)) \geq v(d)$ is in q for each $d \in K(M)$; hence, we can dispense with formulas of form 4 in favor of formulas of form 3. Furthermore, for each tuple b from $K \cap M$, there is $d \in K \cap M$ so that $v(P(x, y, b)) = v(d)$ lies in q. Thus, $v(P(x, y, b)) \leq v(Q(x, y, b))$ is in q if and only if both $v(P(x, y, b)) \leq v(d)$ and $v(d) \leq v(Q(x, y, b))$ are. Therefore, q is determined by its restriction to formulas of the form

$$v(P(x,y,z)) \leq v(w) \quad (5)$$

and their negations. Finally, by multiplying $P(x, y, z)$ by an element of $K(M)$ with value $-v(w)$, we can reduce any instance of (5) to an instance of

$$v(P(x,y,z)) \leq 0. \quad (6)$$

Therefore, q is determined by its restriction to formulas of type (6) and negations. We may assume that z appears linearly in P, i.e. $P(x,y,z) = \sum_{\alpha,\beta} z_{\alpha,\beta} x^\alpha y^\beta$; moreover, we may assume that P has no constant term.

In order to show that q is definable over $\mathrm{coord}(\ulcorner f \urcorner)$, it suffices to show that for each formula $\delta(x,y,z)$ which is a negation (6), the δ-definition of q is definable over $\mathrm{coord}(\ulcorner f \urcorner)$. So, consider the set $Z = \{z : v(P(x,y,z)) > 0\}$. Notice that Z is a definable R-module, since z is linear in $P(x,y,z)$. We now invoke an important lemma from [4].

Lemma 9.2 (2.6.5 [4]) *Every definable R-submodule of K^n is coded in $\mathcal{U}$.*

Since Z is coded, we know that Z is definable over $\mathrm{coord}(\ulcorner f \urcorner)$, as desired.

To complete the proof sketch, it remains to consider the other possibilities for the codomain f. We briefly indicate the idea of the arguments here; for more detail see the proof of Theorem 3.3.2 in [4]. If $\mathcal{S} = S_n$, we choose a basis $b \in K^{n^2}$ for the lattice $f(a)$ so that b is sequentially generic over M (see §3.1 of [4]). The key feature of b is that $\mathrm{tp}(b/Mf(a))$ is definable over $f(a)$. Combining this fact with the fact that $\mathrm{tp}(af(a)/M)$ is definable over $\mathrm{acl}(\ulcorner f \urcorner)$, we can conclude that $\mathrm{tp}(ab/M)$ is definable over $\mathrm{acl}(\ulcorner f \urcorner)$. This allows us to essentially repeat the argument above, replacing $f(a)$ with b everywhere. On the other hand, if $\mathcal{S} = T_n$, then $f(a) \in \mathrm{red}(s)$ for some $s \in S_n$ definable from $f(a)$. Fixing a generic basis b for s gives a corresponding basis $\mathrm{red}(b)$ for $\mathrm{red}(s)$ and a representation of $f(a)$ as a k-linear combination of $\mathrm{red}(b)$ with coefficient tuple α. Then, $\mathrm{tp}(ab\alpha/M)$ is definable over $\mathrm{acl}(\ulcorner f \urcorner)$. By extending the argument in the paragraph above to allow variables ranging over the residue field, one shows that $\mathrm{tp}(ab\alpha/M)$ is definable over $\mathrm{coord}(\ulcorner f \urcorner)$. This completes the proof sketch for the claim.

Remark 9.3 *Use the same notation as in the proof sketch above. Take $c_1, c_2 \in M$. We say that* f_{c_1} and f_{c_2} have the same germ on p *if $f_{c_1}(a) = f_{c_2}(a)$. Because $tp(a/M)$ is a definable type, this is a definable equivalence relation on pairs c_1, c_2 from M. The* germ of f on p *is the equivalence class of c. The claim says exactly that the germ of f on p is coded in $\mathcal{U}$.*

Now we sketch how one shows that $f_c(a) = f_{c'}(a)$ in the general case. The basic idea is to find $c'' \models \mathrm{tp}(c/\mathrm{coord}(\ulcorner f \urcorner))$ so that both

$$a \downarrow^g_{\mathrm{coord}(\ulcorner f \urcorner)} cc'' \quad \text{and} \quad a \downarrow^g_{\mathrm{coord}(\ulcorner f \urcorner)} c'c''. \tag{7}$$

Once we have found such a c'', we can apply the lemma to conclude that $f_c(a) = f_{c''}(a) = f_{c'}(a)$. If we knew enough about the properties of sequential independence at this point, we could choose c'' so that $c'' \downarrow_{\mathrm{acl}(\ulcorner f \urcorner)} acc'$, and then derive (7) from transitivity and symmetry of sequential independence (see Proposition12.6; recall that $\mathrm{tp}(a/\mathrm{acl}(\ulcorner f \urcorner))$ is orthogonal to Γ). However, the various properties of sequential independence are worked out in [5] *under the assumption that $\mathcal{U}$ eliminates imaginaries.* Sequential independence is analyzed via the idea of stable domination, and the smooth development of the theory of stable domination requires elimination of imaginaries. Fortunately, (7) can be achieved directly; essentially, we find c'' by applying the techniques of stable domination in their simplest instance. Suppose that U is the closed ball $B_{\geq\gamma}(b)$. For $x \in U$, write $\mathrm{red}(x)$ for the ball $B_{>\gamma}(x)$. Write $\mathrm{red}(U)$ for the set $\{\mathrm{red}(x) : x \in U\}$. Then, $\mathrm{red}(U)$ is in definable bijection with the residue field k, so $\mathrm{red}(U)$ is a strongly minimal set. (Notice that the definable bijection between $\mathrm{red}(U)$ and k is not canonical.) One can check that for any set of parameters D, we have $a \downarrow^{g}_{\mathrm{acl}(\ulcorner f \urcorner)} D$ if and only if $\mathrm{red}(a) \downarrow^{\mathrm{red}(U)}_{\mathrm{acl}(\ulcorner f \urcorner)} D$, where the latter independence is simply stable independence in $\mathrm{red}(U)$. Because $\mathrm{red}(a) \notin \mathrm{acl}(\ulcorner f \urcorner)$, $\mathrm{red}(a) \downarrow^{\mathrm{red}(U)}_{\mathrm{acl}(\ulcorner f \urcorner)} D$ if and only if $\mathrm{red}(a) \notin \mathrm{acl}(\ulcorner f \urcorner)D)$. Thus, it suffices to choose c'' so that $\mathrm{red}(a) \notin \mathrm{acl}(\ulcorner f \urcorner cc'') \cup \mathrm{acl}(\ulcorner f \urcorner c'c'')$. Claim 2 in the proof of [4, 3.3.2] shows how to find such a c''.

Remark 9.4 *Let p be any definable type and let f be a definable function on the realizations of p. If e is a code for the germ of f on p, then e is a* strong code *if there is a e-definable function g so that $f(x) = g(x)$ lies in p. In other words, e is a strong code if "locally" f is definable over e. When p is the generic type in a definable closed ball U, the argument sketched in the paragraph above shows that any code for a germ on p is actually a strong code. But not all codes are strong: see Remark 3.3.1 in [4] for an example where p is the generic type in an* open *ball, and f is a function such that the germ of f on p has no strong code. This example poses no problem for the proof of elimination of imaginaries: by Theorem 8.4, we need to show that locally f is definable over* $\mathrm{coord}(\ulcorner f \urcorner)$*; we actually do not care whether f is locally definable over a code for the germ.*

9.4 U is a definable open ball or U is an intersection of definable balls [4, 3.3.6]

10 Necessity of the sorts

It seems that for algebraically closed valued fields, one could not prove elimination of imaginaries down to a much simpler collection of sorts than K, the S_n, and the T_n. In particular, the following holds (Proposition 3.5.1 of [4]).

Proposition 10.1 *Let $n \in \mathbb{N}$ with $n > 1$. Then the theory of an algebraically closed valued field does not admit elimination of imaginaries down to the sorts K, S_m $(m \geq 1)$ and T_m $(m \leq n)$.*

Proof (sketch). We suppose that K is a large and saturated algebraically closed valued field with residue field k. For a parameter set C, let $\mathrm{Int}^n_{k,C}$ be the many sorted substructure of $\mathrm{Int}_{k,C}$ (see Section 5) consisting of sorts $\mathrm{red}(s)$ where $s \in S_m$ for some $m \leq n$. The first step is to show that there is a (small) parameter set C and some $s \in \mathrm{dcl}(C) \cap S_{n+2}$ such that $\mathrm{red}(s)$ is not a subset of $\mathrm{dcl}(C \cup \mathrm{Int}_{k,C})$. This follows immediately from the following two assertions:

(i) the group induced on $\mathrm{Int}^n_{k,C}$ by $\mathrm{Aut}(K/k \cup C)$ is soluble of derived length at most n;

(ii) for *any* $m \geq 1$, there is $s \in S_m$ such that if $C = \mathrm{dcl}_G(s)$, then $\mathrm{Aut}(K/k \cup C)$ induces on $\mathrm{red}(s)$ a soluble group of derived length $m-1$.

To see (i), observe that if $s \in S_n$ and $V = \mathrm{red}(s)$, then the embedding of s in S_n means that a natural $\mathrm{Aut}(K/k \cup C)$-invariant filtration is induced on V, and hence the group induced on V by $\mathrm{Aut}(K/k \cup C)$ embeds in the soluble group $B_n(k)$ of upper triangular matrices.

For (ii), one chooses a lattice s generated by the rows of a lower unitriangular matrix $B = (b_{ij})$, where

$$0 > v(b_{21}) > \ldots > v(b_{m1}) > v(b_{32}) > \ldots > v(b_{m2}) > \ldots > v(b_{m,m-1}).$$

It suffices then to show that the group induced by $\mathrm{Aut}(K/k \cup C)$ on $\mathrm{red}(s)$ includes the group $U_m(k)$ of upper unitriangular matrices.

If $m = n+2$ and s is chosen as in (ii), it is rather easy to find $v \in \mathrm{red}(s)$ and some $\sigma \in \mathrm{Aut}(K/k \cup \mathrm{acl}(C))$ such that σ moves v but fixes $\mathrm{Int}^n_{k,C}$ pointwise. It follows that v has no code in the sorts of the proposition.

11 Sequential independence, invariant extensions, orthogonality to Γ

We sketch here some developments from [5], which are used in [11] and [12].

Let T be a complete theory, $\mathcal{U} \models T$ be saturated, and $C \subset \mathcal{U}$ be a small parameter set. The group $\mathrm{Aut}(\mathcal{U})$, and hence also $\mathrm{Aut}(\mathcal{U}/C)$ acts on the set of types over $\mathcal{U}$.

Definition 11.1 *In the above situation, if p is a type over C and q is an extension of p over $\mathcal{U}$, we say that q is an* invariant extension *of p if $\mathrm{Aut}(\mathcal{U}/C)$ fixes q.*

If the type p over C is definable over C, then it has an invariant extension q: for each formula $\phi(x, y)$, let $d\phi(y)$ be the corresponding defining formula; then for $a \in \mathcal{U}$, $\phi(x, a) \in q$ if and only if $d\phi(a)$ holds. If T is a stable theory and $C = \mathrm{acl}(C)$, then, essentially because any type over C is definable over C, any type over C has an $\mathrm{Aut}(\mathcal{U}/C)$-invariant extension; in fact, it has a unique such extension (the unique non-forking extension). Thus, an invariant extension is something like a non-forking extension.

Invariant extensions are investigated in [14] under the name 'strongly determined type'. There, it is shown, for example, that in an o-minimal (or weakly o-minimal) theory, any type over C has an $\mathrm{Aut}(\mathcal{U}/C)$-invariant extension. However, we caution that there is an error in Proposition 2.1 of [14]: to prove that all types over algebraically closed sets have invariant extensions, it does not suffice to do it for 1-types, and a counterexample is given in [15].

In this and the next section, if p is an $\mathrm{Aut}(\mathcal{U})$-invariant type over $\mathcal{U}$, and $C \subseteq B \subset \mathcal{U}$, then $p|B$ denotes the restriction of p to B.

In algebraically closed valued fields, it is rather easy to see that if $C = \mathrm{acl}(C)$ and p is a 1-type in the field sort over C, then p has an $\mathrm{Aut}(\mathcal{U}/C)$-invariant extension q: indeed, p is the generic type of a $C-\infty$-definable ball, and we may take q to be the generic type of this ball over $\mathcal{U}$. A similar argument holds for types in unary sets. Using this iteratively, along with a rather delicate argument for algebraic 1-types in the iterations, it can be shown that in ACVF, if $C = \mathrm{acl}(C)$, then any type (in the sorts G) over C has an $\mathrm{Aut}(\mathcal{U}/C)$-invariant extension over $\mathcal{U}$. In general there may be many such.

The notion of g-independence, used in the proof of EI, can be iterated, to obtain a rudimentary notion of 'sequential' independence in ACVF. To avoid technicalities, we just define it in the field sort; using unary codes, it extends to all sorts of G.

Definition 11.2 *Let A, B, C be sets of parameters from $\mathcal{U}$, and $a = (a_1, \ldots, a_n) \in K^n$ with $A = \mathrm{acl}(Ca)$. We write $a \downarrow^g_C B$, or $A \downarrow^g_C B$ via a, read A* is sequentially independent from B over C via *a, if, for each $i = 1, \ldots, n$, we have $a_i \downarrow_{\mathrm{acl}(C \cup \{a_j : j < i\})} B$.*

This notion is not symmetric, and crucially depends on the *ordering* of the tuple a. However, its advantages are that, in ACVF, any type has a sequentially independent extension over $\mathcal{U}$, and (clearly) the notion is monotonic and transitive on the right: if $C \subset B \subset D$, then $a \downarrow^g_C D$ if and only if $a \downarrow^g_C B$ and $a \downarrow^g_B D$. Furthermore, if $\mathrm{tp}(Aa/C) = \mathrm{tp}(A'a'/C)$ and $\mathrm{tp}(B/C) = \mathrm{tp}(B'/C)$, then $\mathrm{tp}(AaB/C) = \mathrm{tp}(Aa'B'/C)$. If $C = \mathrm{acl}(C)$ and $a \downarrow^g_C \mathcal{U}$, then $\mathrm{tp}(a/\mathcal{U})$ is $\mathrm{Aut}(\mathcal{U}/C)$-invariant; in fact, this is how one proves existence of invariant extensions. Similar results to these hold in o-minimal (or weakly o-minimal) theories, with an appropriate definition of g-independence for 1-types (e.g 'left-generic').

Using sequential independence, we can define a natural notion of *orthogonality* to the value group, extending Definition 6.6 to arbitrary types.

Definition 11.3 *If $C = \mathrm{acl}(C)$, and $a = (a_1, \ldots, a_n) \in G$ is unary, we say $\mathrm{tp}(a/C)$ is* orthogonal *to Γ, written $\mathrm{tp}(a/C) \perp \Gamma$, if for any model M such that $C \subseteq \mathrm{dcl}(M)$ and $a \downarrow^g_C M$, we have $\Gamma(M) = \Gamma(Ma)$.*

When this notion was defined, the intuition was that it captured types which behaved like stable types. This is indeed true, but in the end it turned out more helpful to develop the theory in the more general setting of *stable domination.*

12 Stable Domination

The theory in this section is developed in Part A of [5]. We work in this section in a sufficiently saturated model $\mathcal{U}$ of a complete theory with elimination of imaginaries. Let C be a (small) set of parameters. We denote by St_C the multi-sorted structure whose sorts are the C-definable sets which are stable and stably embedded. The $\emptyset$-definable relations on St_C are those induced by C-definable relations on powers of $\mathcal{U}$. If a is a tuple (possibly infinite) from

$\mathcal{U}$, then $\mathrm{St}_C(a) := \mathrm{dcl}(Ca) \cap \mathrm{St}_C$, and likewise we sometimes write $\mathrm{St}_C(A)$ for $A \subset \mathcal{U}$. Usually, we have in mind some enumeration of $\mathrm{St}_C(a)$, so can talk of $\mathrm{tp}(\mathrm{St}_C(a))$. Sometimes $\mathrm{St}_C(A)$ (or $\mathrm{St}_C(a)$) is abbreviated as A^{st} (respectively, a^{st}).

Definition 12.1 *In the above situation, we say that* $\mathrm{tp}(a/C)$ *is* stably dominated *(or* stably dominated over C*), if, for any possibly infinite tuple* b*,* if $a^{st} \downarrow b^{\mathrm{st}}$ *(in the sense of stable independence in the stable structure* St_C*),* then $\mathrm{tp}(b/a^{\mathrm{st}}) \vdash \mathrm{tp}(b/Ca)$*. Also, we write* $a \downarrow_C^{\mathrm{dom}} b$ *if* $a^{\mathrm{st}} \downarrow b^{\mathrm{st}}$ *and* $\mathrm{tp}(b/a^{\mathrm{st}}) \vdash \mathrm{tp}(b/Ca)$*.*

Remark. (i) An easy argument with automorphisms gives that if $a^{\mathrm{st}} \downarrow b^{\mathrm{st}}$ then $\mathrm{tp}(b/Ca^{\mathrm{st}}) \vdash \mathrm{tp}(b/Ca)$ if and only if $\mathrm{tp}(a/Cb^{\mathrm{st}}) \vdash \mathrm{tp}(a/Cb)$. Hence, by symmetry of stable independence, $a \downarrow_C^{\mathrm{dom}} b$ if and only if $b \downarrow_C^{\mathrm{dom}} a$.

(ii) If $p = \mathrm{tp}(a/C)$ is stably dominated, then it extends to an $\mathrm{acl}(C)$-definable type over $\mathcal{U}$, by definability of types in stable theories and Beth's theorem. Hence, $\mathrm{tp}(a/\mathrm{acl}(C))$ has an $\mathrm{Aut}(\mathcal{U}/\mathrm{acl}(C))$-invariant extension q over $\mathcal{U}$, and in fact a unique such. If $a' \models q|B$, then $a' \downarrow_C^{\mathrm{dom}} B$.

(iii) If $\mathrm{tp}(a/C)$ is stably dominated over C, then $\mathrm{tp}(a/\mathrm{acl}(C))$ is stably dominated over $\mathrm{acl}(C)$.

Part (i) of the following theorem is completely straightforward (3.1 of [5]). However, (ii) (Theorem 3.9 of [5]) has a rather intricate proof, and we do not know whether the assumption that q is invariant is needed. The types involved can be infinitary.

Theorem 12.2 *Suppose* $C = \mathrm{acl}(C)$ *and* p *is an* $\mathrm{Aut}(\mathcal{U}/C)$*-invariant type over* $\mathcal{U}$*.*

(i) If $C \subset B \subset \mathcal{U}$ *and* $p|C$ *is stably dominated over* C*, then* $p|B$ *is stably dominated over* B*.*

(ii) If q *is also an* $\mathrm{Aut}(\mathcal{U}/C)$*-invariant type over* $\mathcal{U}$*,* $B \models q$*, and* $p|B$ *is stably dominated, then* $p|C$ *is stably dominated.*

Recall from Section 9 the notion of *germ* of a function on a definable type, and *strong code* for a germ. It was noted in [4, 3.3.2] that, in ACVF, if p is the generic type of a closed ball, and f is a definable function defined on p, then the germ of f on p has a strong code. It turns out that this, a phenomenon of stable theories, always holds for stably dominated types.

Theorem 12.3 (Theorem 5.1 of [5]) *Let* p_0 *be a stably dominated type over* $C = \mathrm{acl}(C)$*, with invariant extension* p *over* $\mathcal{U}$*. Let* f *be a definable function whose domain contains the set of realisations of* $\mathcal{U}$*. Then the* p*-germ of* f *has a strong code over* C*. Furthermore, if* $f(a) \in \mathrm{St}_{Ca}$ *for all* $a \models p$*, then the* p*-germ for* f *lies in* St_C*.*

From the above theorem, one rapidly obtains the following.

Proposition 12.4 *(a) Suppose $C \subset B$, $\mathrm{tp}(A/C)$ is stably dominated, and $A \downarrow_C^{\mathrm{dom}} B$. Then*

(i) $\mathrm{St}_C(AB) = \mathrm{St}_C(A^{\mathrm{st}} B^{\mathrm{st}})$*,*

(ii) if also $\mathrm{tp}(B/C)$ is stably dominated, then $\mathrm{St}_B(A) = \mathrm{dcl}(BA^{\mathrm{st}}) \cap \mathrm{St}_B$.

(b) Suppose $C \subset B$, and $\mathrm{tp}(A/C)$, $\mathrm{tp}(B/CA)$ are both stably dominated. Then $\mathrm{tp}(AB/C)$ is stably dominated.

We now revert to ACVF. In ACVF, working over any parameter set C, the structure St_C is essentially the same as $\mathrm{Int}_{k,C}$: indeed, by Proposition 5.3(ii), any element of any k-internal C-definable set is coded by a tuple of $\mathrm{Int}_{k,C}$; and by Theorem 4.5 and Lemma 5.2, any C-definable stable, stably embedded set is k-internal. The sorts of $\mathrm{Int}_{k,C}$ are C-definable k-vector spaces, and if C is a model M then any such vector space has a basis in M, so is identifiable over M with k^n (for some n). Thus, St_M may be identified with k. Any sort of $\mathrm{Int}_{k,C}$ has finite Morley rank, and in fact, in a natural sense, if a is a finite tuple then any sequence enumerating $\mathrm{St}_C(a)$ has finite Morley rank. Making heavy use of Theorem 12.2, one can show

Theorem 12.5 (8.7 of [5]) *If $C = \mathrm{acl}(C)$ and a is a finite tuple, then $\mathrm{tp}(a/C)$ is stably dominated if and only if it is orthogonal to Γ.*

From this and the symmetry properties of $\downarrow^{\mathrm{dom}}$, we obtain

Proposition 12.6 (8.9 of [5]) *If $C = \mathrm{acl}(C)$, a, b are finite tuples, and $A = \mathrm{acl}(Ca)$, $B = \mathrm{acl}(Cb)$, and $\mathrm{tp}(a/C) \perp \Gamma$, then the following are equivalent.*

(i) $\mathrm{St}_C(A) \downarrow \mathrm{St}_C(B)$ *(in* St_C*);*

(ii) $A \downarrow_C^{\mathrm{dom}} B$*;*

(iii) $A \downarrow_C^{g} B$ *via* some *generating sequence, i.e. via a' for* some *a' with $C = \mathrm{acl}(Ca')$;*

(iv) $A \downarrow_C^{g} B$ *via* every *generating sequence;*

(v) $B \downarrow_C^{\mathrm{dom}} A$*;*

(vi) $B \downarrow_C^{g} A$ *via some generating sequence;*

(vii) $B \downarrow_C^{g} A$ *via every generating sequence.*

Finally, we mention some more algebraic results, for ACVF. First, a valued field is *maximally complete* if it has no proper immediate extension; that is, if every proper valued field extension increases either the value group or the residue field. Obvious examples are power series fields, in the sense of Example 4.1, and $\mathbb{Q}_p$ (not algebaically closed).

Proposition 12.7 (10.10 of [5]) *Let C be a maximally complete algebraically closed valued field, and let L, M be algebraically closed valued fields containing C. Let $h : \Gamma(L) \to \Gamma$ be an embedding, with $h(\Gamma(L) \cap \Gamma(M) = \Gamma(C)$. Then*

(i) Up to conjugacy over $M \cup h(\Gamma(L))$, there is a unique valued field N and valued field embedding $f : L \to N$ over C such that f induces h, $k(f(L))$ is independent from $k(M)$ over $k(C)$, and $\langle f(L), M\rangle = N$ (as fields).

(ii) With f, N as in (i), $\Gamma(N)$ is equal of the subgroup of Γ generated by $h(\Gamma(L))$ and $\Gamma(M)$.

The above proposition has a purely algebraic proof. However, the next theorem makes substantial use of the machinery developed in [5].

Theorem 12.8 (10.12 of [5]) *Suppose that C is a maximally complete algebraically closed valued field, a is a finite tuple from $\mathcal{U}$, and $A = \mathrm{acl}(Ca)$. Then* $\mathrm{tp}(A/C \cup \Gamma(Ca))$ *is stably dominated.*

Hrushovski has used 12.7 and 12.8 in more recent work. First, in [12], he develops a technique for proving elimination of imaginaries for several henselian valued fields, both with divisible value group (as in ACVF) and with discrete value group. One requires sorts S_n and T_n as in ACVF, and may require certain fairly carefully described sorts internal to the residue field, and sorts needed to eliminate imaginaries of the value group. Examples handled include ultraproducts (over primes) of fields $\mathbb{Q}_p$ (so with pseudofinite residue field), and power series fields $\mathbb{C}[[t]]$. A key idea in the proof is to work in a model of ACVF as a kind of universal domain, so that, working in the language in which ACVF eliminates quantifiers, partial types consisting of quantifier-free formulas are stably dominated. Maximally complete fields play a key role, via 12.7 and 12.8. And, whereas in Section 9 we had to code definable functions on any type (of field elements) we only have to code certain kinds of functions ('quasi-translations') on a set of types which is dense in the Stone space. In fact, it seems that this gives a slightly different treatment of EI for ACVF itself. For, over a base set C of parameters, the generic types of open and closed C-definable balls are dense in the space of all 1-types over C in the field sort. So close analysis of functions on non-definable balls is not needed, but Proposition 12.7 is.

In [11], Hrushovski has developed a theory of groups with stably dominated generic types. In the context of ACVF, a natural example is $\mathrm{SL}_n(R)$ (but $\mathrm{SL}_n(K)$ does *not* have a stably dominated generic type). Theorem 12.8 plays an important role.

References

[1] E. Casanovas, R. Farré, 'Weak forms of elimination of imaginaries', Math. Logic Quart. 50 (2004), 126–140.

[2] G. Cherlin, E. Hrushovski, *Finite structures with few types*, Ann. Math. Studies No. 152, Princeton Univ. Press, Princeton, 2003.

[3] D. Evans, 'The small index property for infinite dimensional classical groups', J. Alg. 136 (1991), 248–264.

[4] D. Haskell, E. Hrushovski, H.D. Macpherson, 'Definable sets in algebraically closed valued fields. Part I: elimination of imaginaries', preprint.

[5] D. Haskell, E. Hrushovski, H.D. Macpherson, 'Definable sets in algebraically closed valued fields. Part II: stable domination and independence', preprint.

[6] D. Haskell, H.D. Macpherson, 'Cell decompositions of C-minimal structures', Ann. Pure Appl. Logic 66 (1994), 113–162.

[7] D. Haskell, H.D. Macpherson, 'Definable sets in valued fields', to appear in

[8] W. Hodges, *Model Theory*, Cambridge University Press, 1993.

[9] J.E. Holly, 'Prototypes for definable subsets of algebraically closed valued fields', J. Symb. Logic 62 (1997), 1093–1141.

[10] J.E. Holly, 'Canonical forms for definable subsets of algebraically closed and real closed valued fields', J. Symb. Logic 60 (1995), 843–860.

[11] E. Hrushovski, 'Valued fields, metastable groups', draft March 2004.

[12] E. Hrushovski, 'Imaginaries in valued fields', draft December 2004.

[13] E. Hrushovski, B. Martin, 'Zeta functions from definable equivalence relations', preprint.

[14] A. Ivanov, H.D. Macpherson, 'Strongly determined types', Ann. Pure Appl. Logic 99 (1999), 197–230.

[15] A. Ivanov, 'Strongly determined types and G-compactness', preprint.

[16] A.J. Macintyre, K. McKenna, L. van den Dries, 'Elimination of quantifiers in algebraic structures', Adv. in Math. 47 (1983), 74–87.

[17] H.D. Macpherson, C. Steinhorn, 'On variants of o-minimality', Ann. Pure Appl. Logic 79 (1996), 165–209.

[18] D. Marker, *Model theory: an introduction*, Graduate Texts in Mathematics, Springer, New York, 2002.

[19] A. Robinson, *Complete theories*, North-Holland, Amsterdam, 1956.