# Permutation groups without irreducible elements

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Dedicated to the memory of Rüdiger Göbel

**Abstract**

We call a non-identity element of a permutation group irreducible if it cannot be written as a product of non-identity elements of disjoint support. We show that it is indeed possible for a sublattice subgroup of $\mathrm{Aut}(\mathbb{R}, \leq)$ to have no irreducible elements and still be transitive on the set of pairs $\alpha < \beta$ in $\mathbb{R}$. This answers a question raised in "The first-order theory of $\ell$-permutation groups", a Conference talk by the first author.

$G \restriction H$

Let $(\Omega, \leqslant)$ be a totally ordered set and $G$ be a subgroup of $\mathrm{Aut}(\Omega, \leqslant)$. Let 1 be the identity element of $\mathrm{Aut}(\Omega, \leq)$ and $g \in G \setminus \{1\}$. Then $g$ is said to be *irreducible* if $g = g_1 g_2$ with $g_1, g_2 \in G$ and $\mathrm{supp}(g_1) \cap \mathrm{supp}(g_2) = \emptyset$ implies $g_1 = 1$ or $g_2 = 1$. Note that if $G = \mathrm{Aut}(\Omega, \leq)$, then $g \in G$ is irreducible if and only if $g$ has a single supporting interval; *i. e.*, there is $\sigma \in \mathrm{supp}(g)$ such that the convexification in $\Omega$ of $\{\sigma g^n \mid n \in \mathbb{Z}\}$ is $\mathrm{supp}(g)$. We prove:

**Theorem A.** *There is an $\ell$-subgroup of* $\mathrm{Aut}(\mathbb{R}, \leq)$ *that is transitive on ordered pairs $\alpha < \beta$ and has no irreducible elements.*

Here, an *$\ell$-subgroup* of $\mathrm{Aut}(\mathbb{R}, \leq)$ is a subgroup $G$ of $\mathrm{Aut}(\mathbb{R}, \leq)$ such that $g_+ \in G$ whenever $g \in G$, where $\alpha g_+ := \alpha g$ if $\alpha g \geq \alpha$ and $\alpha g_+ = \alpha$ if $\alpha g \leq \alpha$

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($\alpha \in \mathbb{R}$). In particular, $G$ is a lattice-ordered group where $f \vee g = (fg^{-1} \vee 1)g$ and $f \wedge g = (f^{-1} \vee g^{-1})^{-1}$. For background on ordered permutation groups and $\ell$-groups see [1].

*Proof of Theorem* A. Let $g \in \mathrm{Aut}(\mathbb{R}, \leq)$. We say that $g$ has period $n \in \mathbb{Z}_+$ if $(\alpha + n)g = \alpha g + n$ for all $\alpha \in \mathbb{R}$. Let

$$G := \{g \in \mathrm{Aut}(\mathbb{R}, \leq) \mid (\exists m \in \mathbb{Z}_+)(g \text{ has period } m)\}.$$

Then $G$ is transitive on ordered pairs $\alpha < \beta$ in $\mathbb{R}$ and it is easily checked that $(G, \mathbb{R})$ is an $\ell$-permutation group. Obviously, if $f \in G$ fixes no point in $\mathbb{R}$, then it must be irreducible. So $G$ has irreducible elements. On the other hand, if $g \in G$ has period $m$ and is not the identity but fixes $\alpha_0 \in \mathbb{R}$ (and so fixes $\alpha_0 + km$ for all $k \in \mathbb{Z}$), define $g_1, g_2 \in G$, each with periods $2m$, as follows:

$$g_1(x) = \begin{cases} g(x) & \text{if } x \in [\alpha_0 + 2km, \alpha_0 + (2k+1)m), \quad k \in \mathbb{Z} \\ x & \text{if } x \in [\alpha_0 + (2k+1)m, \alpha_0 + (2k+2)m), \quad k \in \mathbb{Z} \end{cases}$$

$$g_2(x) = \begin{cases} g(x) & \text{if } x \in [\alpha_0 + (2k+1)m, \alpha_0 + (2k+2)m), \quad k \in \mathbb{Z} \\ x & \text{if } x \in [\alpha_0 + 2km, \alpha_0 + (2k+1)m), \quad k \in \mathbb{Z} \end{cases} .$$

Then $g_1$ and $g_2$ have disjoint supports and $g = g_1 g_2$, so $g$ is reducible. Thus if $H := \{g \in G \mid 0g = 0\}$, then $H$ has no irreducible elements. Now $H$ acts faithfully on $\mathbb{R}_+$ and $(H \restriction \mathbb{R}_+, \mathbb{R}_+)$ (the permutation group induced by $H$ on $\mathbb{R}_+$) is an $\ell$-permutation group that is transitive on ordered pairs $\alpha < \beta$ in $\mathbb{R}_+$. Consequently we obtain an $\ell$-permutation group $(H^*, \mathbb{R})$ that is transitive on pairs $\alpha < \beta$ in $\mathbb{R}$ and has no irreducible elements. For let $\varphi : \mathbb{R} \longrightarrow \mathbb{R}_+$ be an order-preserving bijection between $\mathbb{R}$ and $\mathbb{R}_+$ and $h^* \in \mathrm{Aut}(\mathbb{R}, \leq)$ be given by $\alpha h^* = (\alpha\varphi)h\varphi^{-1}$ ($\alpha \in \mathbb{R}$, $h \in H$). Then the desired properties transfer from $H$ (acting on $\mathbb{R}_+$) to $H^* = \{h^* : h \in H\}$ (acting on $\mathbb{R}$). □

The above proof can similarly be adapted to $\ell$-permutation groups $(L, \mathbb{Q})$ that are transitive on pairs $\alpha < \beta$ in $\mathbb{Q}$ and have no irreducible elements.

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# References

[1] A. M. W. Glass, *Ordered Permutation Groups*, London Math. Soc. Lecture Notes Series **55**, Cambridge University Press, Cambridge, 1981.

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