MATH5835M Statistical Computing
Exercise Sheet 5

http://www1.maths.leeds.ac.uk/~voss/2018/MATH5835M/
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This does not count towards your final mark, the questions are for self-study only. We will discuss the answers to these questions in the lecture on 13th December.

Exercise 17. Let $(X_j)_{j\in\mathbb{N}}$ be a sequence of i.i.d. random variables and $f\colon \mathbb{R} \to \mathbb{R}$ a function. Assume $\mathbb{E}\bigl(f(X_j)\bigr) = \mu$ and $\mathrm{Var}\bigl(f(X_j)\bigr) = \sigma^2$ for all $j \in \mathbb{N}$. The central limit theorem states that, for large N, the random variable

$$Y_N = \frac{1}{\sqrt{N}} \sum_{j=1}^N \frac{f(X_j) - \mu}{\sigma}$$

is approximately $\mathcal{N}(0,1)$-distributed.

a) Write the Monte-Carlo estimator

$$Z_N = \frac{1}{N} \sum_{j=1}^N f(X_j)$$

in terms of Y_N.

b) Derive the approximate distribution of Z_N for large N.

c) Show that

$$P\Bigl(\mu \in \bigl[Z_N - \frac{1.96\,\sigma}{\sqrt{N}}, Z_N + \frac{1.96\,\sigma}{\sqrt{N}}\bigr]\Bigr) \approx 0.95$$

for large N.

Exercise 18. Assume that we want to use MCMC to study the distribution with density

$$\pi(x) = \frac{1}{Z} \cdot \begin{cases} \frac{\sin(x)^2}{x^2}, & \text{if } x \in [-3\pi, 3\pi], \text{ and} \\ 0 & \text{else,} \end{cases}$$

where Z is the normalisation constant.

a) Assuming that the proposals Y_j are constructed as $Y_j = X_{j-1} + \varepsilon_j$ where $\varepsilon_j \sim \mathcal{N}(0, \sigma^2)$ is independent of $X_1, \ldots, X_{j-1}$, determine the corresponding transition densities $p(x,y)$ and acceptance probabilities $\alpha(x,y)$.

b) Let $X \sim \pi$. Using the Random Walk Metropolis algorithm from part (a), with $\sigma = 6$, determine the expectations $\mathbb{E}(X^k)$ for $k = 1,2,3,4$ and the probability $P\bigl(|X| \leq 2\bigr)$.

c) For $\sigma = 1, 6, 36$, use the output of the Random Walk Metropolis algorithm to estimate the lag k auto-correlations $\rho_k = \mathrm{Corr}(X_j, X_{j+k})$ for $k = 1, 2, \ldots, 100$. Create plots of the auto-correlations ρ_k as a function of k.

d) Use the results from part (c) to determine, for each value of σ considered, a sample size N such that the MCMC estimate for $\mathbb{E}(X^4)$ has root-mean squared error smaller than 0.1.

Exercise 19. Let $X_1, \dots, X_n$ be i.i.d. with $\mathbb{E}(X_i) = \mu$ and $\operatorname{Var}(X_i) = \sigma^2$. Let X^* be a bootstrap sample drawn from $X = (X_1, \dots, X_n)$. In lectures we have learned that

$$\operatorname{Var}(X^*) = \mathbb{E}\big(\operatorname{Var}^*_X(X^*)\big) + \operatorname{Var}\big(\mathbb{E}^*_X(X^*)\big).$$

a) Determine $\mathbb{E}^*_X(X^*)$ and $\operatorname{Var}^*_X(X^*)$ in terms of $X_1, \dots, X_n$ and n.

b) Determine $\mathbb{E}\big(\operatorname{Var}^*_X(X^*)\big)$ and $\operatorname{Var}\big(\mathbb{E}^*_X(X^*)\big)$ in terms of μ, σ^2, and n.

c) Explain the behaviour of $\mathbb{E}\big(\operatorname{Var}^*_X(X^*)\big)$ and $\operatorname{Var}\big(\mathbb{E}^*_X(X^*)\big)$ for $n = 1$ and for $n \to \infty$.