# MATH5835M Statistical Computing
# Exercise Sheet 4 (answers)


`http://www1.maths.leeds.ac.uk/~voss/2018/MATH5835M/`
Jochen Voss, J.Voss@leeds.ac.uk

2018/19, semester 1

**Answer 13.**

a) Since $X \sim \mathcal{N}(0,1)$, we have $\mathbb{E}(X^2) = \mathrm{Var}(X^2) + \mathbb{E}(X)^2 = 1 + 0^1 = 1$, and thus

$$\mathbb{E}\big(1 - X^2/2\big) = 1 - \frac{1}{2}\mathbb{E}(X^2) = 1 - \frac{1}{2} = \frac{1}{2}.$$

b) Using Taylor expansion around $x = 0$ we find

$$\cos(x) \approx 1 + 0 \cdot x + \frac{1}{2}(-1)x^2 = 1 - \frac{x^2}{2}.$$

A control variates estimate can be obtained by using the analytical result from part a to get an exact value for the expectation of $1 - X^2/2$, and then to estimate the (smaller) difference between this and the expectation of $\cos(X)$ using Monte Carlo:

$$\begin{aligned}
\mathbb{E}\big(\cos(X)\big) &= \mathbb{E}\big(1 - \frac{X^2}{2}\big) + \mathbb{E}\Big(\cos(X) - 1 + \frac{X^2}{2}\Big) \\
&= \frac{1}{2} + \mathbb{E}\Big(\cos(X) - 1 + \frac{x^2}{2}\Big) \\
&\approx \frac{1}{2} + \frac{1}{N}\sum_{j=1}^{N}\Big(\cos(X_j) - 1 + \frac{X_j^2}{2}\Big).
\end{aligned}$$

**Answer 14.** The procedure described in the question generates $X \sim \mathcal{U}[a,b]$, and thus $X$ has density

$$f(x) = \frac{1}{b-a}1_{[a,b]}(x).$$

The value $X$ is accepted (in step ii), if $f(X) < Y$, where $Y \sim \mathcal{U}[c,d]$, *i.e.* with probability $p(X)$, where

$$p(x) = P\big(Y > f(x)\big) = \frac{d - f(x)}{d - c}.$$

From lecture we know then, that accepted samples have density

$$\varphi(x) \propto f(x)p(x) = \frac{1}{b-a}1_{[a,b]}(x)\frac{d-f(x)}{d-c} \propto \big(d - f(x)\big)1_{[a,b]}(x).$$

We can find the normalising constant by integrating:

$$Z = \int_{-\infty}^{\infty}\big(d - f(x)\big)1_{[a,b]}(x)\,dx = \int_a^b \big(d - f(x)\big)\,dx.$$

Thus, the numbers the algorithm outputs have density

$$\varphi(x) = \frac{1}{Z}\big(d - f(x)\big)1_{[a,b]}(x) = \frac{d - f(x)}{\int_a^b \big(d - f(x)\big)\,dx}1_{[a,b]}(x).$$

**Answer 15.** The initial distribution is the distribution of $X_0$, and thus is the standard normal distribution $\mathcal{N}(0,1)$, with density

$$\varphi(x) = \frac{1}{\sqrt{2\pi}}\exp\big(-x^2/2\big)$$

for all $x \in \mathbb{R}$. The transition density is the the density of $X_{n+1}$, given $X_n = x$. Since $X_{n+1} \sim \mathcal{N}(0,1)$, irrespective of the value of $X_n$, we find the transition density as

$$p(x,y) = \varphi(y)$$

for all $x, y \in \mathbb{R}$.

**Answer 16.** This question anticipates results we later covered in the chapter about bootstrap estimates.

a) $P(Y = 1) = P\big(I \in \{2, 4\}\big) = 2/6 = 1/3.$

b) We have

$$\begin{aligned}\mathbb{E}(Y) &= \mathbb{E}(x_I) \\ &= 3P(I = 1) + 1P(I = 2) + 3P(I = 3) + 1P(I = 4) + 5P(I = 5) + 2P(I = 6) \\ &= \big(3 + 1 + 4 + 1 + 5 + 2\big)\frac{1}{6} \\ &= \frac{16}{6}.\end{aligned}$$

c) We have

$$\mathbb{E}(x_I^2) = \big(3^2 + 1^2 + 4^2 + 1^2 + 5^2 + 2^2\big)\frac{1}{6} = \frac{56}{6}$$

and thus

$$\mathrm{Var}(Y) = \mathrm{Var}(x_I^2) = \mathbb{E}(x_I^2) - \mathbb{E}(x_I)^2 = \frac{56}{6} - \left(\frac{16}{6}\right)^2 = \frac{20}{9}.$$