MATH5835M Statistical Computing
Exercise Sheet 3 (answers)

`http://www1.maths.leeds.ac.uk/~voss/2018/MATH5835M/`

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Answer 9.

a) From lectures we know $\mathbb{E}\big(\tilde Z_N\big) = \mathbb{E}\big(f(X)\big)$, where $X \sim \mathcal{N}(0, \sigma^2)$. Writing this as an integral, we find

$$\mathbb{E}\big(\tilde Z_N\big) = \int_{-\infty}^{\infty} f(x) \, \frac{1}{\sqrt{2\pi\sigma^2}} \exp(-x^2/2\sigma^2) \, dx.$$

b) To cancel the density of the normal distribution in the preceding integral, we can choose

$$Z_N = \frac{1}{N} \sum_{j=1}^{N} \sqrt{2\pi\sigma^2} \exp(X_j^2/2\sigma^2) f(X_j),$$

i.e. $g(x) = \sqrt{2\pi\sigma^2} \exp(x^2/2\sigma^2) f(x)$. This estimator has $\mathbb{E}\big(Z_N\big) = \int_{-\infty}^{\infty} f(x)\, dx$ as required.

c) Since Z_N is an unbiased estimator, we have

$$\begin{aligned}
\mathrm{MSE}(Z_N) &= \mathrm{Var}(Z_n) \\
&= \frac{1}{N} \mathrm{Var}\Big(\sqrt{2\pi\sigma^2} \exp(X^2/2\sigma^2) f(X) \Big) \\
&= \frac{2\pi\sigma^2}{N} \mathbb{E}\Big(\exp(X^2/\sigma^2) f(X)^2 \Big) - \frac{1}{N} \Big(\int_{-\infty}^{\infty} f(x)\, dx \Big)^2 \\
&= \frac{2\pi\sigma^2}{N} \int_{-\infty}^{\infty} f(x)^2 \exp(x^2/2\sigma^2)\, dx - \frac{1}{N} \Big(\int_{-\infty}^{\infty} f(x)\, dx \Big)^2.
\end{aligned}$$

Answer 10. We can apply our general result about rejection sampling, where the acceptance probability if $p(x) = 1_A(x)$. From lectures we know that accepted samples then follow the density given by

$$\varphi(x) = \frac{1}{Z} p(x) f(x) = \frac{1}{Z} 1_A(x) f(x)$$

for all $x \in \mathbb{R}^d$, where

$$Z = \int p(x) f(x)\, dx = \int 1_A(x) f(x)\, dx = \int_A f(x)\, dx = P(X \in A).$$

Using the definition of a density, we know that we can integrate over φ to get the probability that an accepted sample falls into a given set $B \subseteq \mathbb{R}^d$:

$$\begin{aligned}
P\big(X \in B \,\big|\, X \text{ accepted}\big) &= \int_B \varphi(x)\, dx \\
&= \int_B \frac{1}{Z} 1_A(x) f(x)\, dx \\
&= \frac{1}{Z} \int_{A\cap B} f(x)\, dx \\
&= \frac{P(X \in A \cap B)}{P(X \in A)} \\
&= P\big(X \in B \,\big|\, X \in A\big).
\end{aligned}$$

This completes the proof.

Answer 11. In a transition matrix, the rows must sum to 1 and therefore we need $\alpha_1 = 0.6$, $\alpha_2 = 0.1$ and $\alpha_3 = 0.6$. From lectures we know that the condition for π to be a stationary distribution is $\pi^\top P = \pi^\top$, *i.e.*

$$(\pi_1 \ \pi_2 \ \pi_3) \begin{pmatrix} 0.4 & 0.6 & 0.0 \\ 0.3 & 0.1 & 0.6 \\ 0.0 & 0.6 & 0.4 \end{pmatrix} = (\pi_1 \ \pi_2 \ \pi_3). \tag{1}$$

Together with the condition that π is a probability vector, we get a system of four equations:

$$\begin{aligned} \frac{4}{10}\pi_1 + \frac{3}{10}\pi_2 &= \pi_1, \\ \frac{6}{10}\pi_1 + \frac{1}{10}\pi_2 + \frac{6}{10}\pi_3 &= \pi_2, \\ \frac{6}{10}\pi_2 + \frac{4}{10}\pi_3 &= \pi_3, \\ \pi_1 + \pi_2 + \pi_3 &= 1. \end{aligned}$$

We have four equations for three unknowns, so one of the equations is redundant. Leaving out any of the first three equations, we can solve this system to get

$$\pi_1 = 1/5, \quad \pi_2 = 2/5, \quad \pi_3 = 2/5.$$

An alternative way to obtain the same solution is to observe the fact that equation (1) implies that π is an eigenvector of $P^\top$ with eigenvalue 1. Using R we get the same result as above:

```
> P <- matrix(c(.4, .3, 0, .6, .1, .6, 0, .6, .4), 3, 3)
> P
     [,1] [,2] [,3]
[1,]  0.4  0.6  0.0
[2,]  0.3  0.1  0.6
[3,]  0.0  0.6  0.4
> eigen(t(P))
eigen() decomposition
$values
[1]  1.0 -0.5  0.4

$vectors
          [,1]       [,2]          [,3]
[1,] 0.3333333  0.2672612  7.071068e-01
[2,] 0.6666667 -0.8017837 -3.561232e-16
[3,] 0.6666667  0.5345225 -7.071068e-01

> pi <- eigen(t(P))$vectors[,1]
> pi / sum(pi)
[1] 0.2 0.4 0.4
```

Answer 12. Rather than working with the definition of a stationary density directly, for an AR(1) process it is easier to use the fact that all X_k are normally distributed, and to just find the mean and variance which make the process stationary. From this we can then get the required density.

Assume that $X_{k-1} \sim \mathcal{N}(\mu, \sigma^2)$. Then

$$\mathbb{E}(X_k) = \mathbb{E}\big(\alpha X_{k-1} + \varepsilon_k\big) = \alpha \mathbb{E}(X_{k-1}) + \mathbb{E}(\varepsilon_k) = \alpha\mu + 0 = \alpha\mu$$

and

$$\mathrm{Var}(X_k) = \mathrm{Var}\big(\alpha X_{k-1} + \varepsilon_k\big) = \alpha^2 \mathrm{Var}(X_{k-1}) + \mathrm{Var}(\varepsilon_k) = \alpha^2\sigma^2 + 1.$$

If the process is stationary, X_k has the same distribution as X_{k-1}, and in particular has the same mean and variance. From this we get the two equations $\mu = \alpha\mu$ and $\sigma^2 = \alpha^2\sigma^2 + 1$. Solving these equations, we find $\mu = 0$ and $\sigma^2 = 1/(1 - \alpha^2)$. Thus the stationary distribution of the process is $\mathcal{N}\big(0, 1/(1-\alpha^2)\big)$, with density

$$\pi(x) = \sqrt{\frac{1-\alpha^2}{2\pi}} \exp\Big(-\frac{1}{2}(1-\alpha^2)x^2\Big).$$