MATH5835M Statistical Computing
Exercise Sheet 3

http://www1.maths.leeds.ac.uk/~voss/2018/MATH5835M/
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This does not count towards your final mark, the questions are for self-study only. We will discuss the answers to these questions in the lecture on 29th November.

Exercise 9. In this question, we will derive a Monte Carlo method to estimate the value of an integral over the whole real line. For this, let $f\colon \mathbb{R} \to \mathbb{R}$ be given. Our aim is to estimate $\int_{-\infty}^{\infty} f(x)\,dx$.

a) Let $X_j \sim \mathcal{N}(0, \sigma^2)$ be i.i.d. and

$$\tilde Z_N = \frac{1}{N} \sum_{j=1}^N f(X_j).$$

Determine the expectation $\mathbb{E}(\tilde Z_N)$, as an integral.

b) In terms of f, find a function $g\colon \mathbb{R} \to \mathbb{R}$ such that

$$Z_N = \frac{1}{N} \sum_{j=1}^N g(X_j)$$

has expectation $\mathbb{E}(Z_N) = \int_{-\infty}^{\infty} f(x)\,dx$.

c) Determine $\mathrm{MSE}(Z_N)$.

Exercise 10. Let X be a random vector with density $f\colon \mathbb{R}^d \to [0, \infty)$ and let $A \subseteq \mathbb{R}^d$. Consider the following algorithm: a sequence of independent samples $X_n \sim f$, *i.e.* a sequence of i.i.d. copies of X, is generated. Each sample X_n is accepted, if and only $X_n \in A$. Show that the distribution of the accepted samples is the conditional distribution of X, conditioned on $X \in A$.

Exercise 11. For $\alpha_1, \alpha_2, \alpha_3 \in \mathbb{R}$ consider the matrix

$$P = \begin{pmatrix} 0.4 & \alpha_1 & 0.0 \\ 0.3 & \alpha_2 & 0.6 \\ 0.0 & \alpha_3 & 0.4 \end{pmatrix}.$$

For which values of $\alpha_1, \alpha_2, \alpha_3$ is P a transition matrix? Find a probability vector $\pi \in \mathbb{R}^3$ which is a stationary distribution of the Markov chain with transition matrix P.

Exercise 12. For $\alpha \in (-1, 1)$, consider the AR(1) process given by

$$X_k = \alpha X_{k-1} + \varepsilon_k$$

for all $k \in \mathbb{N}$ and $X_0 = 0$, where $\varepsilon_k \sim \mathcal{N}(0, 1)$ are i.i.d. This process is a Markov Chain with state space $\mathbb{R}$. Write down the transition density $p(x, y)$ and find the stationary density $\pi(x)$ for this Markov Chain.