# 8 MATH3733

## 8.1 Conditional distributions for random vectors

**Definition 1** *1. Let a n-dimensional random vector* $(X_1, \dots X_n)$ *have a joint (n-dimensional) density* $p(x_1, \dots, x_n)$*. Denote* $Y = (X_1, \dots, X_{n-1})$*. We call a conditional density of* $X_n$ *given* $Y$ *the function*

$$p_{X_n|Y}(x_n|Y) := \frac{p(X_1, \dots, X_{n-1}, x_n)}{p_Y(X_1, \dots, X_{n-1})}.$$

*2. Let a n-dimensional random vector* $(X_1, \dots X_n)$ *be discrete one with a finite set of possible values of all coordinates. Denote* $Y = (X_1, \dots, X_{n-1})$*. Conditional distribution of* $X_n$ *given* $Y = (X_1, \dots, X_{n-1})$ *is function (here* $y_m$ *is an* $(n-1)$*-tuple)*

$$P(X = x_k|Y) \mid_{Y=y_m} := \frac{P(X = x_k, Y = y_m)}{P(Y = y_m)}, \quad m = 1, \dots, n,$$

*or equivalently,*

$$P(X = x_k|Y) := \sum_{m=1}^{n} \frac{P(X = x_k, Y = y_m)}{P(Y = y_m)} 1(Y = y_m).$$

## 8.2 Conditional expectations

**Definition 2** *1.* **A conditional expectation of** $X$ **given** $Y = (X_1, \dots, X_{n-1})$ *is*

$$E(X|Y) = \int_{-\infty}^{\infty} x p_{X|Y}(x|Y)\, dx \qquad \text{(continuous case)}$$

*(that is, integration with respect to the conditional density), and*

$$E(X|Y)|_{Y=y_m} = \sum_k x_k P(X = x_k|Y = y_m), \quad m = 1, \dots, n \qquad \text{(discrete case)};$$

*otherwise, the latter formula can be expressed by*

$$E(X|Y) = \sum_{k,m} x_k P(X = x_k|Y = y_m) 1(Y = y_m), \quad m = 1, \dots, n.$$

*2.* **A conditional expectation of** $X_n$ **given information** $\mathcal{F}^X_{n-1}$ *is*

$$E(X_n|\mathcal{F}^X_{n-1}) := E(X_n|Y) \equiv E(X_n|X_1, \dots, X_{n-1}).$$

## 8.3 Markov process and martingale

**Definition 3** *We call process* $(X_t,\ t = 0, \delta, 2\delta, \ldots)$ **Markov** *(or* **markovian***) if for any available* $t$*, the conditional probability of the event* $\{X_t \le x\}$ **given all the past** *of the process* $X$ *equals to the (again conditional) probability of this event* **given only the last value of the process***, that is, given* $X_{t-\delta}$*:*

$$P(X_t \le x|\mathcal{F}^X_{t-1}) = P(X_t \le x|X_{t-1}).$$

Remind that the information $F^X_{t-1}$ coincides with the vector $(X_0, X_1, \ldots, X_{t-1})$, so that one could rewrite the definition above as

$$P(X_t \le x|X_1, \ldots, X_{t-1}) = P(X_t \le x|X_{t-1}).$$

**Definition 4** *Markov process* $(X_t,\ t = 0, \delta, 2\delta, \ldots)$ *is a* **martingale** *if for any* $t < t'$*,*

$$E|X_t| < \infty \quad \text{and} \quad E(X_{t'}|X_t) = X_t, \quad \text{or} \quad E(X_{t'} - X_t|X_t) = 0.$$

**Definition 5** *Markov process* $(X_t,\ t = 0, \delta, 2\delta, \ldots)$ *is a* **discounted martingale with the rate** $r$ *if* $\underline{\exp(-rt)X_t \text{ is a martingale}}$*, i.e. for any* $t < t'$*,*

$$E|X_t| < \infty \quad \text{and} \quad \exp(-rt')E(X_{t'}|X_t) = \exp(-rt)X_t.$$

## 8.4 General CRR model

This is a continuation of two-steps CRR model: a discrete time model with time $t = 0, \delta, 2\delta, \ldots$ with a bank account $B_t$ and a stock $S_t$. The interest rate is $r \ge 0$, i.e. $B_\delta = B_0 \exp(r\delta)$. The stock has only one *node* at time 0 with value $S_0$, two nodes at time $\delta$, with values $S^1_\delta > S^2_\delta$, three nodes at time $2\delta$, $S^1_{2\delta} > S^2_{2\delta} > S^3_{2\delta}$, and so on.

There are probabilities $p_0$ & $1 - p_0$ to move from $S_0$ to $S^1_\delta$ or to $S^2_\delta$ correspondingly, probabilities $p_\delta(S^1_\delta)$, $1 - p_\delta(S^1_\delta)$, $p_\delta(S^2_\delta)$, and $1 - p_\delta(S^2_\delta)$ to move from $S_\delta$ to $S_{2\delta}$, etc. From each node at time $t$ the price $S$ can move only to two possible nodes at time $t + \delta$.

For any possible value $S_t$, we will keep notations $S^\pm_t$ to denote two possible values for the stock price at $t + \delta$ (in [BR]): $S_{now}, S_{up/down}, f_{now}, f_{up/down}$ are used). To avoid arbitrage possibilities, we assume for any $t < T$,

$$\exp(-r\delta)S^-_t < S_t < \exp(-r\delta)S^+_t.$$

The method how to value the call option for any time $t$:

1. firstly to **use the one-step model** to value the option at $t = T - \delta$;
2. secondly, after we did it and know the values of $C$ at each node for time $t = T - \delta$, use again the same one-step model to find the price at $T - 2\delta$;
3. and so on, by induction, which leads finally to the value $\boxed{C_0 = e^{-rT}\tilde{E}C_T.}$